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Research Article

A Regularity Criterion for the Navier-Stokes Equations in the Multiplier Spaces

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We exhibit a regularity condition concerning the pressure gradient for the Navier-Stokes equations in a special class. It is shown that if the pressure gradient belongs to $L^{2/(2-r)}((0,T);\mathcal{M}(\dot{H}^r(\mathbb{R}^3) \to \dot{H}^{-r}(\mathbb{R}^3)))$, where $\mathcal{M}(\dot{H}^r(\mathbb{R}^3) \to \dot{H}^{-r}(\mathbb{R}^3))$ is the multipliers between Sobolev spaces whose definition is given later for $0 < r < 1$, then the Leray-Hopf weak solution to the Navier-Stokes equations is actually regular.

1. Introduction

Consider the Navier-Stokes equations in $\mathbb{R}^3$:

$$\begin{gathered}
\partial_t u + u \cdot \nabla u - \Delta u + \nabla p = 0, \qquad (x,t) \in \mathbb{R}^3 \times (0,\infty), \\
\operatorname{div} u = 0, \qquad (x,t) \in \mathbb{R}^3 \times (0,\infty), \\
u(x,0) = u_0(x), \qquad x \in \mathbb{R}^3,
\end{gathered} \tag{1.1}$$

where $u = u(x,t)$ is the velocity field, $p = p(x,t)$ is the scalar pressure, and $u_0(x)$ with div $u_0 = 0$ in the sense of distribution is the initial velocity field. For simplicity, we assume that the external force has a scalar potential and is included into the pressure gradient.

In the famous paper, Leray [1] and Hopf [2] constructed a weak solution u of (1.1) for arbitrary $u_0 \in L^2(\mathbb{R}^3)$ with $\operatorname{div} u_0 = 0$. The solution is called the Leray-Hopf weak solution. Regularity of such Leray-Hopf weak solutions is one of the most significant open problems in mathematical fluid mechanics. We note here that there are partial regularity results from Scheffer and from Caffarelli et al., see [3, 4] and references therein. Besides, more work was pioneered by Serrin [5] and extended and improved by Giga [6], Struwe [7, 8], and Zhou [9]. Further results can be found in [10–16] and references therein.

Introducing the class $L^{\gamma}((0,T);L^{\alpha}(\mathbb{R}^3))$, Serrin [5] showed that if we have a Leray-Hopf weak solution u belonging to $L^{\gamma}((0,T);L^{\alpha}(\mathbb{R}^3))$ with the exponents α and γ satisfying $2/\gamma+3/\alpha<1$, $2<\gamma<\infty$, $3<\alpha<\infty$, then the solution $u(x,t)\in C^{\infty}((0,T)\times\mathbb{R}^3)$, while the limit case $2/\gamma+3/\alpha=1$ was shown much later by Sohr [17] (see also [18]).

Regularity results including assumptions on the pressure gradient have been given by Zhou [15], and it was extended later by Struwe [8] to any dimension $n\geq 3$. It is shown that if the gradient of pressure $\nabla p\in L^{\alpha}((0,T);L^{q}(\mathbb{R}^3))$ with $2/\alpha+3/q\leq 3$, then the corresponding weak solution is actually strong. For the recent work on the regularity problem containing the pressure, velocity field, and the quotient of pressure-velocity, we refer to [19–21] for details.

The purpose of this short paper is to establish a regularity criterion in terms of the pressure gradient for weak solutions to the Navier-Stokes equations in the class $L^{2/(2-r)}((0,T);\mathcal{M}(\dot{H}^r \to \dot{H}^{-r}))$. This work is motivated by the recent results [22, 23] on the Navier-Stokes equations. It is an unusual, larger space considered in the current paper than $L^{3/2r}$ (the following Lemma 2.3) and possesses more information. Obviously, the present result extends some previous ones. For more facts concerning regularity of weak solutions, we refer the readers to the celebrated papers [24–30].

2. Preliminaries

We recall the definition of the multiplier space, which was introduced in [31] (see also [32, 33]). The space $\mathcal{M}(\dot{H}^r(\mathbb{R}^3)\to\dot{H}^{-r}(\mathbb{R}^3))$ of pointwise multipliers, which map $\dot{H}^r$ into $\dot{H}^{-r}$, is defined in the following way.

Definition 2.1. For $0\leq r<3/2$, $\mathcal{M}(\dot{H}^r(\mathbb{R}^3)\to\dot{H}^{-r}(\mathbb{R}^3))$ is a Banach space of all distributions f on $\mathbb{R}^3$ such that there exists a constant C such that for all $u\in\mathfrak{D}(\mathbb{R}^3)$ we have $fu\in\dot{H}^{-r}(\mathbb{R}^3)$ and

$$\|fu\|_{\dot{H}^{-r}}\leq C\|u\|_{\dot{H}^{r}}, \tag{2.1}$$

where we denote by $\dot{H}^r(\mathbb{R}^3)$ the completion of the space $C_0^{\infty}(\mathbb{R}^3)$ with respect to the norm $\|u\|_{\dot{H}^r}=\|(-\Delta)^{r/2}u\|_{L^2}$ and denote by $\mathfrak{D}(\mathbb{R}^3)$ the Schwarz class.

The norm of $\mathcal{M}(\dot{H}^r(\mathbb{R}^3)\to\dot{H}^{-r}(\mathbb{R}^3))$ is given by the operator norm of pointwise multiplication

$$\|f\|_{\mathcal{M}(\dot{H}^r(\mathbb{R}^3)\to\dot{H}^{-r}(\mathbb{R}^3))}=\sup\left\{\|fu\|_{\dot{H}^{-r}}:\|u\|_{\dot{H}^r}\leq 1,u\in\mathfrak{D}\left(\mathbb{R}^3\right)\right\}. \tag{2.2}$$

Remark 2.2. Equivalently, we will say that $f\in\mathcal{M}(\dot{H}^r(\mathbb{R}^3)\to\dot{H}^{-r}(\mathbb{R}^3))$ if and only if the inequality

$$\left|\langle fu,v\rangle\right|\leq C\|u\|_{\dot{H}^r}\|v\|_{\dot{H}^r} \tag{2.3}$$

holds for all $u,v\in\mathfrak{D}(\mathbb{R}^3)$.

Lemma 2.3. *Let $0 \le r < 3/2$. Then the following embedding:*

$$L^{3/2r}\left(\mathbb{R}^3\right) \subset \mathcal{M}\left(\dot{H}^r\left(\mathbb{R}^3\right) \longrightarrow \dot{H}^{-r}\left(\mathbb{R}^3\right)\right) \tag{2.4}$$

holds.

Proof. Indeed, let $f \in L^{3/2r}(\mathbb{R}^3)$. By using the following well-known Sobolev embedding:

$$L^q\left(\mathbb{R}^3\right) \subset \dot{H}^{-r}\left(\mathbb{R}^3\right) \tag{2.5}$$

with $3/q = 3/2 + r$, we have by Hölder's inequality

$$\begin{aligned} \|fg\|_{\dot{H}^{-r}} &\le C\|fg\|_{L^q} \le C\|f\|_{L^{3/2r}}\|g\|_{L^\sigma} \\ &\le C\|f\|_{L^{(3/2r)}}\|g\|_{\dot{H}^r} \quad \left(\dot{H}^r\left(\mathbb{R}^3\right) \subset L^\sigma\left(\mathbb{R}^3\right)\right), \end{aligned} \tag{2.6}$$

where $1/\sigma = 1/2 - r/3$. Then, it follows that

$$\|f\|_{\mathcal{M}(\dot{H}^r \to \dot{H}^{-r})} = \sup_{\|g\|_{\dot{H}^r} \le 1} \|fg\|_{\dot{H}^{-r}} \le C\|f\|_{L^{3/2r}}. \tag{2.7}$$

This completes the proof. □

Example 2.4. Due to the well-known inequality

$$\left\| \frac{u}{|x|} \right\|_{L^2} \le 2\|\nabla u\|_{L^2}, \tag{2.8}$$

we see that $|x|^{-2} \in \mathcal{M}(\dot{H}^1(\mathbb{R}^3) \rightarrow \dot{H}^{-1}(\mathbb{R}^3))$.

Indeed, since the functions of class $C_0^\infty(\mathbb{R}^3)$ are dense in $\dot{H}^1(\mathbb{R}^3)$ in the norm $\|\cdot\|_{\dot{H}^r(\mathbb{R}^3)}$, suppose $u, v \in C_0^\infty(\mathbb{R}^3)$. Then by virtue of the Cauchy-Schwarz inequality we obtain

$$\begin{aligned} \left|\left\langle |x|^{-2}u, v\right\rangle\right| &\le \left\| \frac{u}{|x|} \right\|_{L^2} \left\| \frac{v}{|x|} \right\|_{L^2} \\ &\le 4\|\nabla u\|_{L^2}\|\nabla v\|_{L^2}, \end{aligned} \tag{2.9}$$

and thus for $u, v \in \mathfrak{D}$, $\|u\|_{\dot{H}^r} \le 1$, and $\|v\|_{\dot{H}^r} \le 1$

$$\left\| |x|^{-2} \right\|_{\mathcal{M}(\dot{H}^r \to \dot{H}^{-r})} = \sup_{u \in \mathfrak{D}, \|u\|_{\dot{H}^r} \le 1} \left\| |x|^{-2}u \right\|_{\dot{H}^{-r}} = \sup_{u,v \in \mathfrak{D}} \left|\left\langle |x|^{-2}u, v\right\rangle\right| \le 4 < \infty. \tag{2.10}$$

3. Regularity Theorem

Now we state our result as following.

Theorem 3.1. *Let* $u_0 \in L^2(\mathbb{R}^3) \cap L^q(\mathbb{R}^3)$ *for some* $q \geq 3$ *and* $\nabla \cdot u_0 = 0$ *in the sense of distributions. Suppose that* $u(t,x)$ *is a Leray-Hopf solution of* (1.1) *in* $[0,T)$. *If the pressure gradient satisfies*

$$\nabla p \in L^{2/(2-r)}\left((0,T); \mathcal{M}\left(\dot{H}^r \longrightarrow \dot{H}^{-r}\right)\right) \quad \text{with } 0 < r < 1, \tag{3.1}$$

then $u(t,x)$ *is a regular solution in the sense that*

$$u \in C^\infty\left([0,T] \times \mathbb{R}^3\right). \tag{3.2}$$

Proof. In order to prove this result, we have to do a priori estimates for the Navier-Stokes equations and then show that the solution satisfies the well-known Serrin regularity condition. Multiply both sides of the first equation of (1.1) by $4u|u|^2$ and integrate by parts to obtain (see, e.g., [30])

$$\frac{d}{dt}\|u(\cdot,t)\|_{L^4}^4 - 4\int_{\mathbb{R}^3} (\Delta u) \cdot u|u|^2 dx = -4\int_{\mathbb{R}^3} \nabla p \cdot u|u|^2 dx, \tag{3.3}$$

for $t \in (0,T)$. Then we have

$$\frac{d}{dt}\|u(\cdot,t)\|_{L^4}^4 + 2\left\|\nabla|u|^2\right\|_{L^2}^2 \leq -4\int_{\mathbb{R}^3} \nabla p \cdot u|u|^2 dx, \tag{3.4}$$

where we have used

$$4\int_{\mathbb{R}^3} (\Delta u) \cdot u|u|^2 dx \leq -2\int_{\mathbb{R}^3} \left|\nabla|u|^2\right|^2 dx. \tag{3.5}$$

Let us estimate the integral

$$I = \int_{\mathbb{R}^3} \nabla p \cdot u|u|^2 dx \tag{3.6}$$

on the right-hand side of (3.4). By the Hölder inequality and the Young inequality, we have

$$\begin{aligned}
I &\leq \|\nabla p \cdot u\|_{\dot{H}^{-r}} \left\||u|^2\right\|_{\dot{H}^r} \\
&\leq \|\nabla p\|_{\mathcal{M}(\dot{H}^r \to \dot{H}^{-r})} \|u\|_{\dot{H}^r} \left\||u|^2\right\|_{\dot{H}^r} \\
&\leq C\|\nabla p\|_{\mathcal{M}(\dot{H}^r \to \dot{H}^{-r})} \|u\|_{L^2}^{1-r} \|\nabla u\|_{L^2}^r \left\||u|^2\right\|_{L^2}^{1-r} \left\|\nabla|u|^2\right\|_{L^2}^r \\
&\leq C\|\nabla p\|_{\mathcal{M}(\dot{H}^r \to \dot{H}^{-r})} \left\||u|^2\right\|_{L^2}^{1-r} \left\|\nabla|u|^2\right\|_{L^2}^r
\end{aligned}$$

$$\begin{aligned}
&\leq C(\epsilon)\|\nabla p\|_{\mathcal{M}(\dot{H}^r\to\dot{H}^{-r})}^{2/(2-r)}\left\||u|^2\right\|_{L^2}^{2(1-r)/(2-r)}+\epsilon\left\|\nabla|u|^2\right\|_{L^2}^2 \\
&= C(\epsilon)\|\nabla p\|_{\mathcal{M}(\dot{H}^r\to\dot{H}^{-r})}^{2/(2-r)}\|u\|_{L^4}^{4\xi}+\epsilon\left\|\nabla|u|^2\right\|_{L^2}^2,
\end{aligned} \tag{3.7}$$

where $\xi=(1-r)/(2-r)<1$; we have used the inequality

$$\|w\|_{\dot{H}^r}\leq C\|w\|_{L^2}^{1-r}\|\nabla w\|_{L^2}^r \tag{3.8}$$

and the Young inequality with ϵ:

$$ab\leq \epsilon a^p+C(\epsilon)b^q \quad (a,b>0,\epsilon>0),\left(\frac{1}{p}+\frac{1}{q}=1\right), \tag{3.9}$$

for $C(\epsilon)=(\epsilon p)^{-q/p}q^{-1}$. Hence by (3.4) and the above inequality, we derive

$$\frac{d}{dt}\|u(\cdot,t)\|_{L^4}^4+(2-\epsilon)\left\|\nabla|u|^2\right\|_{L^2}^2\leq C(\epsilon)\|\nabla p\|_{\mathcal{M}(\dot{H}^r\to\dot{H}^{-r})}^{2/(2-r)}\|u\|_{L^4}^{4\xi}. \tag{3.10}$$

Now by Gronwall's lemma (see for instance in [28, Lemma 2]), we have

$$\begin{aligned}
\|u(\cdot,t)\|_{L^4}^4 &\leq C\left[\|u(0)\|_{L^4}^4+\left(\int_0^t\|\nabla p\|_{\mathcal{M}(\dot{H}^r\to\dot{H}^{-r})}^{2/(2-r)}d\tau\right)^{1/(1-\xi)}\right] \\
&= C\left[\|u(0)\|_{L^4}^4+\left(\|\nabla p\|_{L^{2/(2-r)}((0,T);\mathcal{M}(\dot{H}^r\to\dot{H}^{-r}))}^{2/(2-r)}\right)^{1/(1-\xi)}\right].
\end{aligned} \tag{3.11}$$

Due to the integrability of the pressure gradient, it follows that

$$u\in L^\infty\left(0,T;L^4\left(\mathbb{R}^3\right)\right). \tag{3.12}$$

Consequently u falls into the well-known Serrin's regularity framework. Therefore, the smoothness of u follows immediately. This completes the proof of Theorem 3.1. □

Remark 3.2. By a strong solution we mean a weak solution of the Navier-Stokes equation such that

$$u\in L^\infty\left((0,T);H^1\right)\cap L^2\left((0,T);H^2\right). \tag{3.13}$$

It is wellknown that strong solutions are regular (we say classical) and unique in the class of weak solutions.

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