Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2012, Article ID 678536, 12 pages
doi:10.1155/2012/678536

Research Article

Exponential Stability of Impulsive Stochastic Functional Differential Systems

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Received 2 January 2012; Revised 25 February 2012; Accepted 26 February 2012

Academic Editor: Josef Diblík

This paper is concerned with stabilization of impulsive stochastic delay differential systems. Based on the Razumikhin techniques and Lyapunov functions, several criteria on pth moment and almost sure exponential stability are established. Our results show that stochastic functional differential systems may be exponentially stabilized by impulses.

1. Introduction

In the past decades, many authors have obtained various results of deterministic functional differential systems (see [1–6] and the references therein). But it is well known that there are many stochastic factors in the realistic environment, and it is necessary to consider stochastic models. In fact, stochastic functional differential systems (SFDSs) have received more attention in recent years. The properties of SFDSs including stability have been studied in [7–10], which can be widely used in science and engineering (see [11] and the references therein). Furthermore, besides stochastic effects, impulsive effects likewise exist in many evolution processes in which system states change abruptly at certain moments of time, involving such fields as medicine and biology, economics, mechanics, electronics, and telecommunications, and so forth. The impulsive control theory comes to play an important role in science and industry [12]. So the stability investigation of impulsive stochastic differential systems (ISDSs) and impulsive stochastic functional differential systems (ISFDSs) is interesting to many authors [13–20].

Recently, the Razumikhin-type asymptotical stability theorems for ISFESs were established [21, 22]. However, little work has been done on generally exponential stability of ISFESs [23, 24]. In this paper, stability criteria for impulsive stochastic function differential systems are investigated by Razumikhin technique and Lyapunov functions. It is shown that an unstable stochastic delay system can be successfully stabilized by impulses and the results can be easily applied.

2. Preliminaries

Throughout this paper, unless otherwise specified, let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, P)$ be a complete probability space with a filtration $\{\mathcal{F}_t\}_{t\geq 0}$ satisfying the usual conditions (i.e., it is right continuous and $\mathcal{F}_0$ contains all P-null sets). $w(t) = (w_1(t), w_2(t), \ldots, w_d(t))^T$ means a d-dimensional Brownian motion defined on this probability space. R denotes the set of real numbers, R_+ is the set of nonnegative real numbers, and R^n denotes the n-dimensional real space equipped with Euclidean norm $|\cdot|$. If A is a vector or matrix, its transpose is denoted by A^T and its operator norm is denoted by $\|A\| = \sup\{|Ax| : |x| = 1\}$. Moreover, let $\tau > 0$ and denote by $C([-\tau, 0]; R_+)$ the family of continuous functions from $[-\tau, 0]$ to R_+. Let N denote the set of positive integers, that is, $N = \{1, 2, \ldots\}$.

For $-\infty < a < b < +\infty$, a function from $[a, b]$ to R^n is called piecewise continuous, if the function has at most a finite number of jump discontinuities on $(a, b]$, which is continuous from the right for all points in $[a, b)$. Given $\tau > 0$, $PC([-\tau, 0]; R^n)$ denotes the family of piecewise continuous functions from $[-\tau, 0]$ to R^n. A norm on $PC([-\tau, 0]; R^n)$ is defined as $\|\phi\| = \sup_{-\tau\leq s\leq 0}|\phi(s)|$ for $\phi \in PC([-\tau, 0]; R^n)$.

For $p > 0$ and $t \geq 0$, let $PC^p_{\mathcal{F}_t}([-\tau, 0]; R^n)$ denote the family of all $\mathcal{F}_t$-measurable $PC([-\tau, 0]; R^n)$-value random variables ϕ such that $\sup_{-\tau\leq\theta\leq 0} E|\phi(\theta)|^p < \infty$ and $PC^b_{\mathcal{F}_t}([-\tau, 0]; R^n)$ denote the family of $PC([-\tau, 0]; R^n)$-value random variables that are bounded and $\mathcal{F}_t$-measurable.

In this paper, we consider the following ISFDS:

$$\begin{gathered} dx(t) = f(x_t, t)dt + g(x_t, t)dw(t), \quad t \neq t_k,\ t \geq t_0, \\ \Delta x(t_k) = I_k\left(x_{t_k^-}, t_k\right), \quad k \in N, \\ x_{t_0} = \xi, \end{gathered} \tag{2.1}$$

where the initial value $\xi \in PC^b_{\mathcal{F}_{t_0}}([-\tau, 0]; R^n)$, $x(t) = (x_1(t), \ldots, x_n(t))^T$, x_t is regarded as a $PC([-\tau, 0]; R^n)$-value process and $x_t(\theta) = x(t+\theta)$, $\theta \in [-\tau, 0]$. Similarly, x_{t^-} is defined by $x_{t^-}(\theta) = x(t+\theta)$, $\theta \in [-\tau, 0)$ and $x_{t^-}(0) = \lim_{s\to t^-} x(s)$. Both $f : PC^b_{\mathcal{F}_t}([-\tau, 0]; R^n) \times R_+ \to R^n$ and $g : PC^b_{\mathcal{F}_t}([-\tau, 0]; R^n) \times R_+ \to R^{n\times d}$ are Borel measurable, and $I_k : PC^b_{\mathcal{F}_t}([-\tau, 0]; R^n) \times R_+ \to R^n$ represents the impulsive perturbation of x at time t_k. The fixed moments of impulse times t_k satisfy $0 \leq t_0 < t_1 < \cdots < t_k < \cdots$, $t_k \to \infty$ (as $k \to \infty$), $\Delta x(t_k) = x(t_k) - x(t_k^-)$. Moreover, f, g, and I_k are assumed to satisfy necessary assumptions so that, for any initial data $\xi \in PC^b_{\mathcal{F}_{t_0}}([-\tau, 0]; R^n)$, system (2.1) has a unique global solution, denoted by $x(t; t_0, \xi)$ (e.g., see [25] for existence and uniqueness results for general impulsive hybrid stochastic delay systems including (2.1)). For the purpose of stability in this note, we also assume the $f(0, t) \equiv 0$, $g(t, 0) \equiv 0$ and $I_k(0, t) \equiv 0$ for all $t \geq t_0$, $k \in N$, then system (2.1) admits a trivial solution.

Definition 2.1. The trivial solution of system (2.1) is said to be pth ($p > 0$) moment exponentially stable if there is a pair of positive constants λ, C such that

$$E|x(t; t_0, \xi)|^p \leq C\|\xi\|^p e^{-\lambda(t-t_0)}, \quad t \geq t_0, \tag{2.2}$$

for all $\xi \in PC^b_{\mathcal{F}_{t_0}}([-\tau, 0]; R^n)$. When $p = 2$, it is usually said to be exponentially stable in mean square. It follows from (2.2) that

$$\limsup_{t \to \infty} \frac{1}{t} \log E|x(t; t_0, \xi)|^p \le -\lambda. \tag{2.3}$$

The left-hand side of (2.3) is called the pth moment Lyapunov exponent of the solution.

Definition 2.2. The trivial solution of system (2.1) is said to be almost exponentially stable if there is a pair of positive constants λ, C such that for $t \ge t_0$

$$|x(t; t_0, \xi)|^p \le C\|\xi\| e^{-\lambda(t-t_0)}, \quad \text{a.s.}, \tag{2.4}$$

for all $\xi \in PC^b_{\mathcal{F}_{t_0}}([-\tau, 0]; R^n)$. It follows from (2.4) that

$$\limsup_{t \to \infty} \frac{1}{t} \log|x(t; t_0, \xi)| \le -\lambda. \tag{2.5}$$

The left-hand side of (2.5) is called the Lyapunov exponent of the solution.

Definition 2.3. Let $C^{2,1}(R^n \times [t_0, \infty); R_+)$ denote the family of all nonnegative functions $V(x,t)$ on $R^n \times [t_0 - \tau, \infty)$ that are continuously twice differential in x and once in t. If $V \in C^{2,1}(R^n \times [t_0, \infty); R_+)$, define the operator $\mathcal{L}V : PC([-\tau, 0]; R^n) \times [t_0, \infty) \to R$ for system (2.1) by

$$\mathcal{L}V(x_t, t) = V_t(x, t) + V_x(x, t) f(x_t, t) + \frac{1}{2}\text{trace}\left[g^T(x_t, t) V_{xx}(x, t) g(x_t, t)\right], \tag{2.6}$$

where $V_t(x,t) = \partial V(x,t)/\partial t$, $V_x(x,t) = (\partial V(x,t)/\partial x_1, \dots, \partial V(x,t)/\partial x_n)$, $V_{xx}(x,t) = (\partial^2 V(x,t)/\partial x_i \partial x_j)_{n\times n}$.

3. Main Results

In this section, we will establish some criteria on the pth moment exponential stability and almost exponential stability for system (2.1) by using the Razumikhin technique and Lyapunov functions. We begin with the following lemma, which concerns with the continuity of $EV(x(t), t)$.

Lemma 3.1. *Let $V(x,t) \in C^{2,1}(R^n \times [t_0, \infty); R_+)$, and let $x(t)$ be a solution of system* (2.1). *If there exists $c > 0$ such that $V(x,t) \le c|x|^p$, then $EV(x(t), t)$ is continuous on $[t_{k-1}, t_k)$, $k \in N$.*

Proof. By the Itô formula,

$$V(x(t), t) = V(x(t_{k-1}), t_{k-1}) + \int_{t_{k-1}}^{t} \mathcal{L}V(x_s, s) ds + \int_{t_{k-1}}^{t} V_x(x(s), s) g(x_s, s) dw(s) \tag{3.1}$$

for all $t \in [t_{k-1}, t_k)$, where $k \in N$. Since $x_{t_{k-1}} \in PC^b_{\mathcal{F}_{t_{k-1}}}([-\tau, 0]; R^n)$, we can find an integer l_0 such that $\|x_{t_{k-1}}\| < l_0$ a.s. For any integer $l > l_0$, define the stopping time

$$\rho_l = \inf\{t \in [t_{k-1}, t_k) : |x(t)| \geq l\}, \tag{3.2}$$

where $\inf \emptyset = \infty$ as usual. Since $x(t)$ is continuous on $[t_{k-1}, t_k)$, $|x(t)|$ is also continuous on $[t_{k-1}, t_k)$. Clearly, $\rho_l \to \infty$ a.s. as $l \to \infty$. Moreover, it has $EV(x(t_{k-1}), t_k) \leq cl_0$, following from $x_{t_{k-1}} \in PC^b_{\mathcal{F}_{t_{k-1}}}([-\tau, 0]; R^n)$. It then follows from the definition of ρ_l above that

$$EV\left(x(t'_l), t'_l\right) = EV(x(t_{k-1}), t_{k-1}) + E\left[\int_{t_{k-1}}^{t'_l} \mathcal{L}V(x_s, s)ds\right], \tag{3.3}$$

where $t'_l = t \wedge \rho_l$. So, letting $l \to \infty$, by the dominated convergence theorem and Fubini's theorem, we have

$$\begin{aligned} EV(x(t), t) &= EV(x(t_{k-1}), t_{k-1}) + E\left[\int_{t_{k-1}}^{t} \mathcal{L}V(x_s, s)ds\right] \\ &= EV(x(t_{k-1}), t_{k-1}) + \int_{t_{k-1}}^{t} E[\mathcal{L}V(x_s, s)]ds, \end{aligned} \tag{3.4}$$

for $t \in [t_{k-1}, t_k)$. This implies that $EV(x(t), t)$ is continuous on $[t_{k-1}, t_k)$, $k \in N$. □

Theorem 3.2. *Let $V \in C^{2,1}(R^n \times [t_0 - \tau, \infty); R_+)$ and $u : [t_0, \infty) \to R_+$ be a piecewise continuous function. Suppose there exist some positive constants p, c_1, c_2, and λ such that*

(i) *for all $(x, t) \in R^n \times [t_0 - \tau, \infty)$,*

$$c_1|x|^p \leq V(x, t) \leq c_2|x|^p, \tag{3.5}$$

(ii) *for all $k \in N$, and $\phi \in PC^p_{\mathcal{F}_t}([-\tau, 0]; R^n)$,*

$$EV\left(\phi(0^-) + I(t_k, \phi), t_k\right) \leq d_k EV\left(\phi(0^-), t_k^-\right), \tag{3.6}$$

where $0 < d_k < \exp\{-\lambda(t_{k+1} - t_k) - \int_{t_k}^{t_{k+1}} u(s)ds\}$,

(iii) *for all $t \geq t_0$, $t \neq t_k$, $k \in N$ and $\phi \in PC^p_{\mathcal{F}_t}([-\tau, 0]; R^n)$,*

$$E\left[\mathcal{L}V(\phi, t)\right] \leq u(t)EV(\phi(0), t) \tag{3.7}$$

whenever

$$EV(\phi, t + \theta) < qEV(\phi(0), t), \quad \theta \in [-\tau, 0], \tag{3.8}$$

where $q > \max_{k\in N}\{d_k^{-1}e^{\lambda\tau}\} \vee \exp\{\int_{t_0}^{t_1} u(s)ds\}$.

Then the trivial solution of system (2.1) is pth moment exponentially stable and its pth moment Lyapunov exponent is not greater than $-\lambda$.

Proof. Given any initial data $\xi \in PC^b_{\mathcal{F}_{t_0}}([-\tau,0];R^n)$, the global solution $x(t;t_0,\xi) = x(t)$ of (2.1) is written as $x(t)$ in this proof. Without loss of generality, assume that the initial date ξ is nontrivial so that $x(t)$ is not a trivial solution. Choose M such that

$$c_2 e^{\lambda(t_1-t_0)+\int_{t_0}^{t_1} u(s)ds} < M < c_2 q e^{\lambda(t_1-t_0)}. \tag{3.9}$$

Then it follows from condition (i) and (3.9) that

$$EV(x(t),t) \le c_2\|\xi\|^p < M\|\xi\|^p e^{-\lambda(t_1-t_0)}, \quad t \in [t_0-\tau, t_0]. \tag{3.10}$$

In the following, we will show that

$$EV(x(t),t) \le M\|\xi\|^p e^{-\lambda(t_k-t_0)}, \quad t \in [t_{k-1}, t_k),\ k \in N. \tag{3.11}$$

In order to do so, we first prove that

$$EV(x(t),t) \le M\|\xi\|^p e^{-\lambda(t_1-t_0)}, \quad t \in [t_0, t_1). \tag{3.12}$$

If (3.12) is not true, then there exist some $t \in [t_0,t_1)$ such that $EV(x(t),t) > M\|\xi\|^p e^{-\lambda(t_1-t_0)}$. Set $t^* = \inf\{t \in [t_0,t_1) : EV(x(t),t) > M\|\xi\|^p e^{-\lambda(t_1-t_0)}\}$. Then $t^* \in (t_0,t_1)$ and also, by the continuity of $EV(x(t),t)$) (see Lemma 3.1),

$$EV(x(t),t) < EV(x(t^*),t^*) = M\|\xi\|^p e^{-\lambda(t_1-t_0)}, \quad t \in [t_0-\tau, t_*). \tag{3.13}$$

In view of (3.10), define $t_* = \sup\{t \in [t_0-\tau,t^*) : EV(x(t),t) \le c_2\|\xi\|^p\}$. Then $t_* \in [t_0,t^*)$ and, by the continuity of $EV(x(t),t)$,

$$EV(x(t),t) > EV(x(t_*),t_*) = c_2\|\xi\|^p, \quad t \in (t_*, t^*]. \tag{3.14}$$

Now in view of (3.9), (3.13), and (3.14), one has, for $t \in [t_*,t^*]$ and $\theta \in [-\tau,0]$,

$$EV(x(t+\theta),t+\theta) \le M\|\xi\|^p e^{-\lambda(t_1-t_0)} < qEV(x(t_*),t_*) \le qEV(x(t),t). \tag{3.15}$$

By the Razumikhin-type condition (iii),

$$E[\mathcal{L}V(x_t,t)] \le u(t)EV(x(t),t), \quad \forall t \in [t_*, t^*]. \tag{3.16}$$

Applying Itô formula and by (3.16), one obtains that

$$EV(x(t^*),t^*) \le EV(x(t_*),t_*) + \int_{t_*}^{t^*} u(s)EV(x(s),s)ds. \tag{3.17}$$

Finally, by (3.9), (3.13), (3.14), and the Gronwall inequality,

$$\begin{aligned} EV(x(t^*), t^*) &\le EV(x(t_*), t_*) e^{\int_{t_*}^{t^*} u(s)ds} \le c_2 \|\xi\|^p e^{\int_{t_0}^{t_1} u(s)ds} \\ &< M\|\xi\|^p e^{-\lambda(t_1 - t_0)} = EV(x(t^*), t^*), \end{aligned} \tag{3.18}$$

which is a contradiction. So inequality (3.12) holds and (3.11) is true for $k = 1$.

Now assume that

$$EV(x(t), t) \le M\|\xi\|^p e^{-\lambda(t_k - t_0)}, \quad \forall t \in [t_{k-1}, t_k),\ k \in N, \tag{3.19}$$

for all $k \le m$, where $k, m \in N$. We proceed to show that

$$EV(x(t), t) \le M\|\xi\|^p e^{-\lambda(t_{m+1} - t_0)}, \quad \forall t \in [t_m, t_{m+1}). \tag{3.20}$$

Suppose (3.20) is not true, set $\bar{t} = \inf\{t \in [t_m, t_{m+1}) : EV(x(t), t) > M\|\xi\|^p e^{-\lambda(t_{m+1} - t_0)}\}$. By condition (ii) and (3.20), we know

$$EV(x(t_m), t_m) \le d_m EV\left(x\left(t_m^-\right), t_m^-\right) \le d_m M\|\xi\|^p e^{-\lambda(t_m - t_0)} < M\|\xi\|^p e^{-\lambda(t_{m+1} - t_0)}. \tag{3.21}$$

From this, together with $EV(x(t), t)$ being continuous on $t \in [t_m, t_{m+1})$, we know that $\bar{t} \in (t_m, t_{m+1})$ and

$$EV(x(t), t) < EV\left(x\left(\bar{t}\right), \bar{t}\right) = M\|\xi\|^p e^{-\lambda(t_{m+1} - t_0)}, \quad \forall t \in \left[t_m, \bar{t}\right). \tag{3.22}$$

Define $\underline{t} = \sup\{t \in [t_0, \bar{t}] : EV(x(t), t) \le d_m M\|\xi\|^p e^{-\lambda(t_m - t_0)}\}$, then $\underline{t} \in [t_m, \bar{t})$ and

$$EV(x(t), t) > EV\left(x\left(\underline{t}\right), \underline{t}\right) = d_m M\|\xi\|^p e^{-\lambda(t_m - t_0)}, \quad \forall t \in \left(\underline{t}, \bar{t}\right]. \tag{3.23}$$

For $t \in [\underline{t}, \bar{t}]$ and $\theta \in [-\tau, 0]$, when $t + \theta \ge t_m$, then (3.22) and (3.23) imply that

$$\begin{aligned} EV(x(t+\theta), t+\theta) &\le M\|\xi\|^p e^{-\lambda(t_{m+1} - t_0)} < M\|\xi\|^p e^{-\lambda(t + \theta - t_0)} \\ &\le M e^{\lambda\tau}\|\xi\|^p e^{-\lambda(t - t_0)} \le M e^{\lambda\tau}\|\xi\|^p e^{-\lambda(t_m - t_0)} \\ &\le q EV\left(x\left(\underline{t}\right), \underline{t}\right). \end{aligned} \tag{3.24}$$

If $t + \theta < t_m$ for some $\theta \in [-\tau, 0)$, we assume that, without loss of generality, $t + \theta \in [t_l, t_{l+1})$ for some $l \in N,\ l \le m - 1$, then from (3.19) and (3.23),

$$\begin{aligned} EV(x(t+\theta), t+\theta) &\le M\|\xi\|^p e^{-\lambda(t_{l+1} - t_0)} < M\|\xi\|^p e^{-\lambda(t + \theta - t_0)} \\ &\le M e^{\lambda\tau}\|\xi\|^p e^{-\lambda(t - t_0)} \le M e^{\lambda\tau}\|\xi\|^p e^{-\lambda(t_m - t_0)} \\ &\le q EV\left(x\left(\underline{t}\right), \underline{t}\right). \end{aligned} \tag{3.25}$$

Therefore,

$$EV(x(t+\theta),t+\theta) < qEV(x(t),t), \quad t \in \left[\underline{t},\bar{t}\right],\ \theta \in [-\tau,0]. \tag{3.26}$$

Then, it follows from condition (iii) that

$$E[\mathcal{L}V(x_t,t)] \le u(t)EV(x(t),t), \quad \forall t \in \left[\underline{t},\bar{t}\right]. \tag{3.27}$$

Combining Itô formula with (3.27), we can check that

$$EV\left(x\left(\bar{t},\bar{t}\right)\right) \le EV(x(\underline{t}),\underline{t}) + \int_{\bar{t}}^{\underline{t}} u(s)EV(x(s),s)ds. \tag{3.28}$$

Finally, by (3.22), (3.23), and the Gronwall inequality,

$$\begin{aligned} EV\left(x\left(\bar{t}\right),\bar{t}\right) &\le EV(x(\underline{t}),\underline{t})e^{\int_{\underline{t}}^{\bar{t}} u(s)ds} \le EV(x(\underline{t}),\underline{t})e^{\int_{t_m}^{t_{m+1}} u(s)ds} \\ &= d_m M\|\xi\|^p e^{-\lambda(t_m-t_0)} e^{\int_{t_m}^{t_{m+1}} u(s)ds} < EV\left(x\left(\bar{t}\right),\bar{t}\right), \end{aligned} \tag{3.29}$$

which is a contradiction. So inequality (3.20) holds. By mathematical induction, we obtain that (3.11) holds for all $k \in N$. Furthermore, from condition (i), we have

$$E|x(t)|^p \le \frac{c_1}{c_2} M\|\xi\|^p e^{-\lambda(t_k-t_0)} \le \frac{c_1}{c_2} M\|\xi\|^p e^{-\lambda(t-t_0)}, \quad t \in [t_{k-1},t_k),\ k \in N, \tag{3.30}$$

which implies

$$E\|x\|^p \le \frac{c_1}{c_2} M\|\xi\|^p e^{-\lambda(t-t_0)}, \quad t \ge t_0, \tag{3.31}$$

that is, system (2.1) is pth moment exponentially stable. The proof is complete. □

Remark 3.3. If $u(t) \equiv c > 0$, then Theorem 3.1 of [23] follows from Theorem 3.2 immediately.

Theorem 3.4. *Let $V \in C^{2,1}(R^n \times [t_0-\tau,\infty); R_+)$, and let $u : [t_0,\infty) \to R_+$ be a piecewise continuous function. Suppose there exist some positive constants $p,\ c_1,\ c_2$, and λ such that*

(i) *for all $(x,t) \in R^n \times [t_0-\tau,\infty)$,*

$$c_1|x|^p \le V(x,t) \le c_2|x|^p, \tag{3.32}$$

(ii) *for all $k \in N$ and $\phi \in PC^p_{\mathcal{F}_t}([-\tau,0];R^n)$,*

$$EV(\phi(0^-) + I(t_k,\phi)) \le \rho d_k EV(\phi(0^-),t_k^-), \tag{3.33}$$

where $0 < \rho < \max\{e^{-\lambda(t_{m+1}-t_m)}\}$ and $d_k > 0$ with $\widehat{d} = \sup_{n\in N}\Pi_{k=1}^n d_k < \infty$,

(iii) *for all* $t \geq t_0,\ t \neq t_k,\ k \in N$, *and* $\phi \in PC^p_{\mathcal{F}_t}([-\tau, 0]; R^n)$,

$$E[\mathcal{L}V(\phi, t)] \leq u(t)EV(\phi(0), t) \tag{3.34}$$

whenever

$$EV(\phi, t+\theta) < qEV(\phi(0), t), \quad \theta \in [-\tau, 0], \tag{3.35}$$

where $q > (\rho^{-1}e^{\lambda\tau}) \vee (\rho^{-1}e^{\lambda\tau}\widehat{d}^{-1})$.

Then the trivial solution of system (2.1) is pth moment exponentially stable and its pth moment Lyapunov exponent is not greater than $-\lambda$.

Proof. Given any initial data $\xi \in PC^b_{\mathcal{F}_{t_0}}([-\tau, 0]; R^n)$, the global solution $x(t; t_0; \xi) = x(t)$ of (2.1) is written as $x(t)$ in this proof. Without loss of generality, assume that the initial date ξ is nontrivial so that $x(t)$ is not a trivial solution. Choose M such that

$$c_2 e^{\lambda(t_1 - t_0) + \int_{t_0}^{t_1} u(s)ds} < M < c_2 q e^{\lambda(t_1 - t_0)}. \tag{3.36}$$

Then it follows from condition (i) and (3.36) that

$$EV(x(t), t) \leq c_2\|\xi\|^p < M\|\xi\|^p e^{-\lambda(t_1 - t_0)}, \quad t \in [t_0 - \tau, t_0]. \tag{3.37}$$

In the following, we will show that

$$EV(x(t), t) \leq M_k\|\xi\|^p e^{-\lambda(t_k - t_0)}, \quad t \in [t_{k-1}, t_k), \tag{3.38}$$

where $k \in N$ and M_k is defined as $M_1 = M$ and $M_k = M\Pi_{1\leq l\leq k-1}d_l$. Similarly, as the proof in Theorem 3.2, one can prove that

$$EV(x(t), t) \leq M\|\xi\|^p e^{-\lambda(t_1 - t_0)}, \quad t \in [t_0, t_1). \tag{3.39}$$

Now assume that

$$EV(x(t), t) \leq M_k\|\xi\|^p e^{-\lambda(t_k - t_0)}, \quad \forall t \in [t_{k-1}, t_k),\ k \in N, \tag{3.40}$$

for all $k \leq m$, where $k, m \in N$. We proceed to show that

$$EV(x(t), t) \leq M_{m+1}\|\xi\|^p e^{-\lambda(t_{m+1} - t_0)}, \quad \forall t \in [t_m, t_{m+1}). \tag{3.41}$$

Suppose (3.41) is not true, set $\bar{t} = \inf\{t \in [t_m, t_{m+1}) : EV(x(t), t) > M_k\|\xi\|^p e^{-\lambda(t_{m+1}-t_0)}\}$. By condition (ii),

$$\begin{aligned} EV(x(t_m), t_m) &\leq \rho d_m EV\left(x(t_m^-), t_m^-\right) \leq \rho M_{m+1}\|\xi\|^p e^{-\lambda(t_m - t_0)} \\ &< M_{m+1}\|\xi\|^p e^{-\lambda(t_{m+1}-t_0)}. \end{aligned} \tag{3.42}$$

From this, together with $EV(x(t), t)$ being continuous on $t \in [t_m, t_{m+1})$, we know that $\bar{t} \in (t_m, t_{m+1})$ and

$$EV(x(t), t) < EV\left(x\left(\bar{t}\right), \bar{t}\right) = M_{m+1}\|\xi\|^p e^{-\lambda(t_{m+1}-t_0)}, \quad \forall t \in \left[t_m, \bar{t}\right). \tag{3.43}$$

Define $\underline{t} = \sup\{t \in [t_0, \bar{t}] : EV(x(t), t) \leq \rho M_{m+1}\|\xi\|^p e^{-\lambda(t_m - t_0)}\}$, then $\underline{t} \in [t_m, \bar{t})$ and

$$EV(x(t), t) > EV\left(x\left(\underline{t}\right), \underline{t}\right) = \rho M_{m+1}\|\xi\|^p e^{-\lambda(t_m - t_0)}, \quad \forall t \in \left(\underline{t}, \bar{t}\right]. \tag{3.44}$$

For $t \in [\underline{t}, \bar{t}]$ and $\theta \in [-\tau, 0]$, when $t + \theta \geq t_m$, then (3.44) implies that

$$\begin{aligned} EV(x(t+\theta), t+\theta) &\leq M_{m+1}\|\xi\|^p e^{-\lambda(t_{m+1}-t_0)} \\ &= \rho^{-1} e^{-\lambda(t_{m+1}-t_m)} EV\left(x\left(\underline{t}\right), \underline{t}\right) \\ &< qEV\left(x\left(\underline{t}\right), \underline{t}\right). \end{aligned} \tag{3.45}$$

If $t+\theta < t_m$ for some $\theta \in [-\tau, 0)$, we assume that, without loss of generality, $t + \theta \in [t_l, t_{l+1})$ for some $l \in N,\ l \leq m-1$, then from (3.41) and (3.44), we obtain

$$\begin{aligned} EV(x(t+\theta), t+\theta) &\leq M_{l+1}\|\xi\|^p e^{-\lambda(t_{l+1}-t_0)} < M_{l+1}\|\xi\|^p e^{-\lambda(t+\theta-t_0)} \\ &\leq M_{l+1} e^{\lambda\tau}\|\xi\|^p e^{-\lambda(t-t_0)} \leq \rho^{-1} e^{\lambda\tau} \frac{M_{l+1}}{M_{m+1}} EV\left(x\left(\underline{t}\right), \underline{t}\right) \\ &= \rho^{-1} e^{\lambda\tau} \widehat{d}^{-1} EV\left(x\left(\underline{t}\right), \underline{t}\right) \leq qEV\left(x\left(\underline{t}\right), \underline{t}\right). \end{aligned} \tag{3.46}$$

Therefore,

$$EV(x(t+\theta), t+\theta) < qEV(x(t), t), \quad t \in \left[\underline{t}, \bar{t}\right],\ \theta \in [-\tau, 0]. \tag{3.47}$$

The rest of the proof is similar to that of Theorem 3.2 and omitted here. □

Remark 3.5. Let $\overline{u}$ and δ be positive constants. Assume that the conditions of Theorem 3.4 hold, function $u : [t_0, \infty) \to R_+$ satisfies $\int_t^{t+\delta} u(s)ds \leq \overline{u}\delta$ and $\sup_{k\in\mathbb{N}}\{t_k - t_{k-1}\} = \delta < -(\ln\rho/(\lambda + \overline{u}))$. Then Theorem 3.1 of [24] follows immediately.

Remark 3.6. It is not strictly required by condition (ii) of Theorem 3.4 that each impulse contributes to stabilize the system, as long as the overall contribution of the impulses are stabilizing. Without these d_k (i.e., $d_k \equiv 1$), it is required that each impulse is a stabilizing factor ($\rho < 1$), which is more restrictive.

Remark 3.7. It is clear that Theorems 3.2 and 3.4 allow the continuous dynamics of system (2.1) to be unstable, since the function $u(t)$, which characterizes the changing rate of $V(x(t),t)$ at t, is assumed to be nonnegative. Theorems 3.2 and 3.4 show that an unstable stochastic delay system can be successfully stabilized by impulses.

The following theorems show that the trivial solutions of system (2.1) are also almost surely exponentially stable, under some additional conditions.

Assumption 3.8. Suppose the impulsive instances t_k satisfy

$$\sup_{k\in N}\{t_k - t_{k-1}\} < \infty, \qquad \inf_{k\in N}\{t_k - t_{k-1}\} > 0. \tag{3.48}$$

Assumption 3.9. Assume that there is a constant $L > 0$ such that, for all $(\phi,t) \in PC^p_{\mathcal{F}_t}([-\tau,0];R^n) \times [t_0,\infty)$,

$$E\left[\left|f(\phi,t)\right|^p + \left|g(\phi,t)\right|^p\right] \le L \sup_{-\tau\le\theta\le 0} E\left|\phi(\theta)\right|. \tag{3.49}$$

Lemma 3.10 (see [23]). *Let $p \ge 1$, and let Assumptions 3.8 and 3.9 hold. Then* (3.31) *implies that, for all $t \ge t_0$,*

$$|x(t;\xi,t_0)| \le Ce^{-(\lambda/p)(t-t_0)}\|\xi\|^p \quad a.s., \tag{3.50}$$

where C is a positive constant. In other words, under Assumptions 3.8 and 3.9, the pth moment exponential stability implies the almost exponential stability for system (2.1).

By using Theorems 3.2 and 3.4 and Lemma 3.10, it is easy to show the following conclusions.

Theorem 3.11. *Suppose that $p \ge 1$, Assumptions 3.8 and 3.9 and the same conditions as in Theorem 3.2 hold. Then the trivial solution of system* (2.1) *is also almost surely exponentially stable, with its Lyapunov exponent not greater than $-\lambda/p$.*

Theorem 3.12. *Suppose that $p \ge 1$, Assumptions 3.8 and 3.9 and the same conditions as in Theorem 3.4 hold. Then the trivial solution of system* (2.1) *is also almost surely exponentially stable, with its Lyapunov exponent not greater than $-\lambda/p$.*

4. An Example

Example 4.1. Consider a scalar ISDDs of the form

$$\begin{gathered} dx(t) = x(t)dt + \frac{1}{4}\sqrt{x^2(t) + x^2(t-2)}\,dw(t), \qquad t \ne t_k,\ t \ge t_0, \\ \Delta x(t_k) = -0.4x(t_k^-), \qquad k \in N. \end{gathered} \tag{4.1}$$

It is easy to check that the corresponding system without impulses is not mean square exponentially stable. In fact, if $V(x,t) = x^2$, then it follows from the Itô formula that

$E[\mathcal{L}V(x(t), x(t-2), t)] \geq 2E|x(t)|^2 = 2EV(x(t), t)$. This leads to $E|x(t)|^2 = EV(x(t), t) \geq EV(x(0), 0)e^{2t} = E|x(0)|^2 e^{2t}$ for all $t \geq 0$. But, in the following, we will show that system (4.1) is mean square exponentially stable and almost exponentially stable.

If $V(x(t), t) = x^2$, then condition (i) of Theorem 3.2 holds with $c_1 = c_2 = 1, p = 2$, and condition (ii) holds with $d_k = 0.36$. By calculating, we have $E[\mathcal{L}V(x(t), x(t-2), t)] \leq (33/16)EV(x(t), t) + (1/16)EV(x(t-2), t)$. By taking $q = 5$, $\lambda = 0.5$, and $t_k - t_{k-1} = 0.3$, it is easy to verify that condition (iii) of Theorem 3.2 is satisfied, which means system (4.1) is mean square exponentially stable. Applying Theorem 3.11, we can derive that system (4.1) is almost exponentially stable.

Acknowledgments

The authors are grateful to Editor Professor Josef Diblík and anonymous referees for their helpful comments and suggestions which have improved the quality of this paper. This work is supported by Natural Science Foundation of China (no. 10771001), Research Fund for Doctor Station of Ministry of Education of China (no. 20113401110001, no. 20103401120002), TIAN YUAN Series of Natural Science Foundation of China (no. 11126177), Key Natural Science Foundation (no. KJ2009A49), Talent Foundation (no. 05025104) of Anhui Province Education Department, 211 Project of Anhui University (no. KJJQ1101), Anhui Provincial Nature Science Foundation (no. 090416237, no. 1208085QA15), and Foundation for Young Talents in College of Anhui Province (no. 2012SQRL021).

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