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Research Article

On Asymptotically Quasi-ϕ-Nonexpansive Mappings in the Intermediate Sense

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A projection iterative process is investigated for the class of asymptotically quasi-ϕ-nonexpansive mappings in the intermediate sense. Strong convergence theorems of common fixed points of a family of asymptotically quasi-ϕ-nonexpansive mappings in the intermediate sense are established in the framework of Banach spaces.

1. Introduction

Fixed point theory as an important branch of nonlinear analysis theory has been applied in many disciplines, including economics, image recovery, mechanics, quantum physics, and control theory; see, for example, [1–4]. The theory itself is a beautiful mixture of analysis, topology, and geometry. During the four decades, many famous existence theorems of fixed points were established; see, for example, [5–13]. However, from the standpoint of real world applications it is not only to know the existence of fixed points of nonlinear mappings but also to be able to construct an iterative algorithm to approximate their fixed points. The computation of fixed points is important in the study of many real world problems, including inverse problems; for instance, it is not hard to show that the split feasibility problem and the convex feasibility problem in signal processing and image reconstruction can both be formulated as a problem of finding fixed points of certain operators, respectively; see [14–16] for more details and the reference therein. Iterative methods play an important role in the computation of fixed points of nonlinear mappings. Indeed, many well-known problems can be studied by using algorithms which are iterative in their nature.

In this paper, we introduced a new class of nonlinear mappings: asymptotically quasi-ϕ-nonexpansive mappings in the intermediate sense and considered the problem of approximating a common fixed point of a family of the mappings based on a projection iterative process.

The organization of this paper is as follows. In Section 2, we provide some necessary preliminaries. In Section 3, strong convergence of a projection iterative algorithm is obtained in a reflexive, strictly convex, and smooth Banach space such that both E and E^* enjoy Kadec-Klee property. Some corollaries as the immediate results of main results are given.

2. Preliminaries

Let H be a real Hilbert space, C a nonempty subset of H, and $T : C \to C$ a mapping. The symbol $F(T)$ stands for the fixed point set of T. Recall the following. T is said to be *nonexpansive* if and only if

$$\|Tx - Ty\| \leq \|x - y\|, \quad \forall x, y \in C. \tag{2.1}$$

T is said to be *quasi-nonexpansive* if and only if $F(T) \neq \emptyset$, and

$$\|p - Ty\| \leq \|p - y\|, \quad \forall p \in F(T),\ \forall y \in C. \tag{2.2}$$

We remark here that a nonexpansive mapping with a nonempty fixed point set is quasi-nonexpansive: however, the inverse may be not true. See the following example [17].

Example 2.1. Let $H = \mathbb{R}^1$ and define a mapping by $T : H \to H$ by

$$Tx = \begin{cases} \dfrac{x}{2} \sin \dfrac{1}{x}, & \text{if } x \neq 0, \\ 0, & \text{if } x = 0. \end{cases} \tag{2.3}$$

Then T is quasi-nonexpansive but not nonexpansive.

T is said to be *asymptotically nonexpansive* if and only if there exists a sequence $\{\mu_n\} \subset [0, \infty)$ with $\mu_n \to 0$ as $n \to \infty$ such that

$$\|T^n x - T^n y\| \leq (1 + \mu_n) \|x - y\|, \quad \forall x, y \in C,\ \forall n \geq 1. \tag{2.4}$$

It is easy to see that a nonexpansive mapping is an asymptotically nonexpansive mapping with the sequence $\{1\}$. The class of asymptotically nonexpansive mappings was introduced by Goebel and Kirk [7]. Since 1972, a host of authors have studied the convergence of iterative algorithms for such a class of mappings.

T is said to be *asymptotically quasi-nonexpansive* if and only if $F(T) \neq \emptyset$, and there exists a sequence $\{\mu_n\} \subset [0, \infty)$ with $\mu_n \to 0$ as $n \to \infty$ such that

$$\|p - T^n y\| \leq (1 + \mu_n) \|p - y\|, \quad \forall p \in F(T),\ \forall y \in C,\ \forall n \geq 1. \tag{2.5}$$

It is easy to see that a quasi-nonexpansive mapping is an asymptotically quasi-nonexpansive mapping with the sequence $\{1\}$.

T is said to be *asymptotically nonexpansive in the intermediate sense* if and only if it is continuous, and the following inequality holds:

$$\limsup_{n\to\infty}\sup_{x,y\in C}\left(\left\|T^n x - T^n y\right\| - \|x-y\|\right) \le 0. \tag{2.6}$$

T is said to be *asymptotically quasi-nonexpansive in the intermediate sense* if and only if $F(T) \ne \emptyset$ and the following inequality holds:

$$\limsup_{n\to\infty}\sup_{p\in F(T),y\in C}\left(\left\|p - T^n y\right\| - \|p-y\|\right) \le 0. \tag{2.7}$$

The class of mappings which are asymptotically nonexpansive in the intermediate sense was considered by Bruck et al. [18]. It is worth mentioning that the class of mappings which are asymptotically nonexpansive in the intermediate sense may not be Lipschitz continuous. However, asymptotically nonexpansive mappings are Lipschitz continuous.

In what follows, we always assume that E is a Banach space with the dual space E^*. Let C be a nonempty, closed, and convex subset of E. We use the symbol J to stand for the *normalized duality mapping* from E to 2^{E^*} defined by

$$Jx = \left\{f^* \in E^* : \langle x, f^*\rangle = \|x\|^2 = \left\|f^*\right\|^2\right\}, \quad \forall x \in E, \tag{2.8}$$

where $\langle\cdot,\cdot\rangle$ denotes the generalized duality pairing of elements between E and E^*. It is well known that if E^* is strictly convex, then J is single valued; if E^* is reflexive and smooth, then J is single valued and demicontinuous; see [19] for more details and the references therein.

It is also well known that if D is a nonempty, closed, and convex subset of a Hilbert space H, and $P_C : H \to D$ is the metric projection from H onto D, then P_D is nonexpansive. This fact actually characterizes Hilbert spaces and consequently, it is not available in more general Banach spaces. In this connection, Alber [20] introduced a generalized projection operator in Banach spaces which is an analogue of the metric projection in Hilbert spaces.

Recall that a Banach space E is said to be *strictly convex* if $\|(x+y)/2\| < 1$ for all $x, y \in E$ with $\|x\| = \|y\| = 1$, and $x \ne y$. It is said to be *uniformly convex* if $\lim_{n\to\infty}\|x_n - y_n\| = 0$ for any two sequences $\{x_n\}$ and $\{y_n\}$ in E such that $\|x_n\| = \|y_n\| = 1$ and $\lim_{n\to\infty}\|(x_n + y_n)/2\| = 1$. Let $U_E = \{x \in E : \|x\| = 1\}$ be the unit sphere of E. Then the Banach space E is said to be *smooth* provided $\lim_{t\to 0}((\|x+ty\|-\|x\|)/t)$ exists for all $x, y \in U_E$. It is also said to be *uniformly smooth* if the limit is attained uniformly for all $x, y \in U_E$.

Recall that a Banach space E enjoys *Kadec-Klee property* if for any sequence $\{x_n\} \subset E$ and $x \in E$ with $x_n \rightharpoonup x$, and $\|x_n\| \to \|x\|$, then $\|x_n - x\| \to 0$ as $n \to \infty$. For more details on Kadec-Klee property, the readers can refer to [19, 21] and the references therein. It is well known that if E is a uniformly convex Banach spaces, then E enjoys Kadec-Klee property.

Let E be a smooth Banach space. Consider the functional defined by

$$\phi(x, y) = \|x\|^2 - 2\langle x, Jy\rangle + \left\|y\right\|^2, \quad \forall x, y \in E. \tag{2.9}$$

Notice that, in a Hilbert space H, (2.9) is reduced to $\phi(x,y) = \|x-y\|^2$ for all $x,y \in H$. The *generalized projection* $\Pi_C : E \to C$ is a mapping that assigns to an arbitrary point $x \in E$, the minimum point of the functional $\phi(x,y)$; that is, $\Pi_C x = \overline{x}$, where $\overline{x}$ is the solution to the following minimization problem:

$$\phi(\overline{x}, x) = \min_{y\in C} \phi(y,x). \tag{2.10}$$

The existence and uniqueness of the operator Π_C follow from the properties of the functional $\phi(x,y)$ and the strict monotonicity of the mapping J; see, for example, [19, 20]. In Hilbert spaces, $\Pi_C = P_C$. It is obvious from the definition of the function ϕ that

$$(\|y\| - \|x\|)^2 \leq \phi(y,x) \leq (\|y\| + \|x\|)^2, \quad \forall x,y \in E, \tag{2.11}$$

$$\phi(x,y) = \phi(x,z) + \phi(z,y) + 2\langle x-z, Jz - Jy\rangle, \quad \forall x,y,z \in E. \tag{2.12}$$

Remark 2.2. If E is a reflexive, strictly convex, and smooth Banach space, then, for all $x,y \in E$, $\phi(x,y) = 0$ if and only if $x = y$. It is sufficient to show that if $\phi(x,y) = 0$, then $x = y$. From (2.11), we have $\|x\| = \|y\|$. This implies that $\langle x, Jy\rangle = \|x\|^2 = \|Jy\|^2$. From the definition of J, we see that $Jx = Jy$. It follows that $x = y$; see [20] for more details.

Next, we recall the following.

(1) A point p in C is said to be an *asymptotic fixed point* of T [22] if and only if C contains a sequence $\{x_n\}$ which converges weakly to p such that $\lim_{n\to\infty}\|x_n - Tx_n\| = 0$. The set of asymptotic fixed points of T will be denoted by $\widetilde{F}(T)$.

(2) T is said to be *relatively nonexpansive* if and only if

$$\widetilde{F}(T) = F(T) \neq \emptyset, \quad \phi(p,Tx) \leq \phi(p,x), \quad \forall x \in C,\ \forall p \in F(T). \tag{2.13}$$

The asymptotic behavior of relatively nonexpansive mappings was studied in [23, 24].

(3) T is said to be *relatively asymptotically nonexpansive* if and only if

$$\widetilde{F}(T) = F(T) \neq \emptyset, \quad \phi(p,T^n x) \leq (1+\mu_n)\phi(p,x), \quad \forall x \in C,\ \forall p \in F(T),\ \forall n \geq 1, \tag{2.14}$$

where $\{\mu_n\} \subset [0,\infty)$ is a sequence such that $\mu_n \to 1$ as $n \to \infty$.

Remark 2.3. The class of relatively asymptotically nonexpansive mappings was first considered in Su and Qin [25]; see also, Agarwal et al. [26], and Qin et al. [27].

(4) T is said to be *quasi-ϕ-nonexpansive* if and only if

$$F(T) \neq \emptyset, \quad \phi(p,Tx) \leq \phi(p,x), \quad \forall x \in C,\ \forall p \in F(T). \tag{2.15}$$

(5) T is said to be *asymptotically quasi-ϕ-nonexpansive* if and only if there exists a sequence $\{\mu_n\} \subset [0,\infty)$ with $\mu_n \to 0$ as $n \to \infty$ such that

$$F(T) \neq \emptyset, \quad \phi(p,T^n x) \leq (1+\mu_n)\phi(p,x), \quad \forall x \in C,\ \forall p \in F(T),\ \forall n \geq 1. \tag{2.16}$$

Remark 2.4. The class of quasi-ϕ-nonexpansive mappings and the class of asymptotically quasi-ϕ-nonexpansive mappings were first considered in Zhou et al. [28]; see also Qin et al. [29], Qin, and Agarwal [30], Qin et al. [31], Qin et al. [32], and Qin et al. [33].

Remark 2.5. The class of quasi-ϕ-nonexpansive mappings and the class of asymptotically quasi-ϕ-nonexpansive mappings are more general than the class of relatively nonexpansive mappings and the class of relatively asymptotically nonexpansive mappings. Quasi-ϕ-nonexpansive mappings and asymptotically quasi-ϕ-nonexpansive do not require $F(T) = \widetilde{F}(T)$.

Remark 2.6. The class of quasi-ϕ-nonexpansive mappings and the class of asymptotically quasi-ϕ-nonexpansive mappings are generalizations of the class of quasi-nonexpansive mappings and the class of asymptotically quasi-nonexpansive mappings in Banach spaces.

In this paper, based on asymptotically (quasi-) nonexpansive mappings in the intermediate sense which was first considered by Bruck et al. [18], we introduce and consider the following new nonlinear mapping: asymptotically (quasi-) ϕ-nonexpansive mappings in the intermediate sense.

(6) T is said to be an *asymptotically ϕ-nonexpansive mapping in the intermediate sense* if and only if

$$\limsup_{n\to\infty} \sup_{x,y\in C} \left(\phi(T^n x, T^n y) - \phi(x,y)\right) \leq 0. \tag{2.17}$$

(7) T is said to be an *asymptotically quasi-ϕ-nonexpansive mapping in the intermediate sense* if and only if $F(T) \neq \emptyset$, and

$$\limsup_{n\to\infty} \sup_{p\in F(T),x\in C} \left(\phi(p, T^n x) - \phi(p,x)\right) \leq 0. \tag{2.18}$$

Remark 2.7. The class of asymptotically (quasi-) ϕ-nonexpansive mappings in the intermediate sense is a generalization of the class of asymptotically (quasi-) nonexpansive mappings in the intermediate sense in the framework of Banach spaces.

Let $E = \mathbb{R}^1$ and $C = [0,1]$. Define the following mapping $T : C \to C$ by

$$Tx = \begin{cases} \dfrac{1}{2}x, & x \in \left[0, \dfrac{1}{2}\right], \\ 0, & x \in \left(\dfrac{1}{2}, 1\right]. \end{cases} \tag{2.19}$$

Then T is an asymptotically ϕ-nonexpansive mapping in the intermediate sense with the fixed point set $\{0\}$. We also have the following:

$$
\begin{aligned}
\phi(T^n x, T^n y) = \left|T^n x - T^n y\right|^2 = \frac{1}{2^{2n}}|x-y|^2 \le |x-y|^2 = \phi(x,y), \quad \forall x,y \in \left[0,\frac{1}{2}\right], \\
\phi(T^n x, T^n y) = \left|T^n x - T^n y\right|^2 = 0 \le |x-y|^2 = \phi(x,y), \quad \forall x,y \in \left(\frac{1}{2},1\right], \\
\phi(T^n x, T^n y) &= \left|T^n x - T^n y\right|^2 \\
&= \left|\frac{1}{2^n}x - 0\right|^2 \\
&\le \left(\frac{1}{2^n}|x-y| + \frac{1}{2^n}|y|\right)^2 \\
&\le \left(|x-y| + \frac{1}{2^n}\right)^2 \\
&\le |x-y|^2 + \xi_n \\
&= \phi(x,y) + \xi_n, \quad \forall x \in \left[0,\frac{1}{2}\right],\ \forall y \in \left(\frac{1}{2},1\right],
\end{aligned}
\tag{2.20}
$$

where $\xi_n = 1/2^{2n} + 1/2^{n-1}$. Hence, we have

$$\phi(T^n x, T^n y) \le \phi(x,y) + \xi_n, \quad \forall x,y \in [0,1]. \tag{2.21}$$

(8) The mapping T is said to be *asymptotically regular* on C if and only if

$$\lim_{n\to\infty} \sup_{x\in C}\left\{\left\|T^{n+1}x - T^n x\right\|\right\} = 0. \tag{2.22}$$

In order to prove our main results, we also need the following lemmas.

Lemma 2.8 (see [20]). *Let C be a nonempty, closed, and convex subset of a smooth Banach space E, and $x \in E$. Then $x_0 = \Pi_C x$ if and only if*

$$\left\langle x_0 - y, Jx - Jx_0\right\rangle \ge 0, \quad \forall y \in C. \tag{2.23}$$

Lemma 2.9 (see [20]). *Let E be a reflexive, strictly, convex, and smooth Banach space, C a nonempty, closed, and convex subset of E, and $x \in E$. Then*

$$\phi(y, \Pi_C x) + \phi(\Pi_C x, x) \le \phi(y,x), \quad \forall y \in C. \tag{2.24}$$

3. Main Results

Theorem 3.1. *Let E be a reflexive, strictly convex, and smooth Banach space such that both E and E^* have Kadec-Klee property. Let C be a nonempty, bounded closed, and convex subset of E. Let Δ be an index set, and $T_i : C \to C$ a closed asymptotically quasi-ϕ-nonexpansive mapping in the intermediate sense, for every $i \in \Delta$. Assume that $\bigcap_{i\in\Delta} F(T_i)$ is nonempty, and T_i is asymptotically regular, for every $i \in \Delta$. Let $\{x_n\}$ be a sequence generated in the following manner:*

$$\begin{gathered}
x_0 \in E, \quad \text{chosen arbitrarily}, \\
C_{(1,i)} = C, \\
C_1 = \bigcap_{i\in\Delta} C_{(1,i)}, \\
x_1 = \Pi_{C_1} x_0, \\
C_{(n+1,i)} = \left\{ u \in C_{(n,i)} : \phi\left(x_n, T_i^n x_n\right) \le 2\left\langle x_n - u, Jx_n - JT_i^n x_n \right\rangle + \xi_{(n,i)} \right\}, \\
C_{n+1} = \bigcap_{i\in\Delta} C_{(n+1,i)}, \\
x_{n+1} = \Pi_{C_{n+1}} x_0, \quad \forall n \ge 1,
\end{gathered} \tag{3.1}$$

where $\xi_{(n,i)} = \max\{0, \sup_{p\in F(T_i), x\in C}(\phi(p, T_i^n x) - \phi(p, x))\}$. Then $\{x_n\}$ converges strongly to $\Pi_{\bigcap_{i\in\Delta} F(T_i)} x_0$, where $\Pi_{\bigcap_{i\in\Delta} F(T_i)}$ stands for the generalized projection from E onto $\bigcap_{i\in\Delta} F(T_i)$.

Proof. The proof is split into the following 5 steps.

Step 1. It show that $\bigcap_{i\in\Delta} F(T)$ is closed and convex.

Since T_i is closed, we can easily conclude that $F(T_i)$ is closed. The proof that $\bigcap_{i\in\Delta} F(T)$ is closed. We only prove that $\bigcap_{i\in\Delta} F(T_i)$ is convex. Let $p_{1,i}, p_{2,i} \in F(T_i)$ and $p_i = t_i p_{1,i} + (1-t_i) p_{2,i}$, where $t_i \in (0, 1)$, for every $i \in \Delta$. We see that $p_i = T_i p_i$. Indeed, we see from the definition of T_i that

$$\begin{gathered}
\phi\left(p_{1,i}, T_i^n p_i\right) \le \phi\left(p_{1,i}, p_i\right) + \xi_{(n,i)}, \\
\phi\left(p_{2,i}, T_i^n p_i\right) \le \phi\left(p_{(2,i)}, p_i\right) + \xi_{(n,i)}.
\end{gathered} \tag{3.2}$$

In view of (2.12), we obtain (3.2) that

$$\begin{gathered}
\phi\left(p_i, T_i^n p_i\right) \le 2\left\langle p_i - p_{1,i}, Jp_i - J\left(T_i^n p_i\right)\right\rangle + \xi_{(n,i)}, \\
\phi\left(p_i, T_i^n p_i\right) \le 2\left\langle p_i - p_{2,i}, Jp_i - J\left(T_i^n p_i\right)\right\rangle + \xi_{(n,i)}.
\end{gathered} \tag{3.3}$$

Multiplying t_i and $(1-t_i)$ on the both sides of (3.3), respectively, yields that $\phi(p_i, T_i^n p_i) \le \xi_{(n,i)}$. This implies that

$$\lim_{n\to\infty} \phi\left(p_i, T_i^n p_i\right) = 0. \tag{3.4}$$

In light of (2.11), we arrive at

$$\lim_{n\to\infty}\|T_i^n p_i\| = \|p_i\|. \tag{3.5}$$

It follows that

$$\lim_{n\to\infty}\|J(T_i^n p_i)\| = \|Jp_i\|. \tag{3.6}$$

Since E^* is reflexive, we may, without loss of generality, assume that $J(T_i^n p_i) \rightharpoonup e^{*,i} \in E^*$. In view of the reflexivity of E, we have $J(E) = E^*$. This shows that there exists an element $e^i \in E$ such that $Je^i = e^{*,i}$. It follows that

$$\begin{aligned}\phi(p_i, T_i^n p_i) &= \|p_i\|^2 - 2\langle p_i, J(T_i^n p_i)\rangle + \|T_i^n p_i\|^2\\ &= \|p_i\|^2 - 2\langle p_i, J(T_i^n p_i)\rangle + \|J(T_i^n p_i)\|^2.\end{aligned} \tag{3.7}$$

Taking $\liminf_{n\to\infty}$ on the both sides of the equality above, we obtain that

$$\begin{aligned}0 &\geq \|p_i\|^2 - 2\left\langle p_i, e^{*,i}\right\rangle + \left\|e^{*,i}\right\|^2\\ &= \|p_i\|^2 - 2\left\langle p_i, Je^i\right\rangle + \left\|Je^i\right\|^2\\ &= \|p_i\|^2 - 2\left\langle p_i, Je^i\right\rangle + \left\|e^i\right\|^2\\ &= \phi\left(p_i, e^i\right).\end{aligned} \tag{3.8}$$

This implies that $p_i = e^i$, that is, $Jp_i = e^{*,i}$. It follows that $J(T_i^n p_i) \rightharpoonup Jp_i \in E^*$. In view of Kadec-Klee property of E^*, we obtain from (3.6) that $\lim_{n\to\infty}\|J(T_i^n p_i) - Jp_i\| = 0$. Since $J^{-1} : E^* \to E$ is demicontinuous, we see that $T_i^n p_i \rightharpoonup p_i$. By virtue of Kadec-Klee property of E, we see from (3.5) that $T_i^n p_i \to p_i$ as $n \to \infty$. Hence $T_i T_i^n p_i = T_i^{n+1} p_i \to p_i$, as $n \to \infty$. In view of the closedness of T_i, we can obtain that $p_i \in F(T_i)$, for every $i \in \Delta$. This shows, for every $i \in \Delta$, that $F(T_i)$ is convex. This proves that $\bigcap_{i\in\Delta} F(T_i)$ is convex. This completes Step 1.

Step 2. It show that C_n is closed and convex, $\forall n \geq 1$.

It suffices to show, for any fixed but arbitrary $i \in \Delta$, that $C_{n,i}$ is closed and convex, for every $n \geq 1$. This can be proved by induction on n. It is obvious that $C_{(1,i)} = C$ is closed and convex. Assume that $C_{(j,i)}$ is closed and convex for some $j \geq 1$. We next prove that $C_{(j+1,i)}$ is closed and convex for the same j. This completes the proof that C_n is closed and convex. The closedness of $C_{(j+1,i)}$ is clear. We only prove the convexness. Indeed, $\forall a_i, b_i \in C_{(j+1,i)}$, we see that $a_i, b_i \in C_{(j,i)}$, and

$$\begin{aligned}\phi\left(x_j, T_i^j x_j\right) &\leq 2\left\langle x_j - a_i, Jx_j - J\left(T_i^j x_j\right)\right\rangle + \xi_{(j,i)},\\ \phi\left(x_j, T_i^j x_j\right) &\leq 2\left\langle x_j - b_i, Jx_j - J\left(T_i^j x_j\right)\right\rangle + \xi_{(j,i)}.\end{aligned} \tag{3.9}$$

In view of (3.9), we find that

$$\phi\left(x_j, T_i^j x_j\right) \le 2\left\langle x_j - c_i, Jx_j - J\left(T_i^j x_j\right)\right\rangle + \xi_{(j,i)}, \tag{3.10}$$

where $c_i = t_i a_i + (1-t_i) b_i \in C_{(j,i)}$, $t_i \in (0,1)$. It follows that $C_{(j+1,i)}$ is convex. This in turn implies that $C_n = \bigcap_{i\in\Lambda} C_{(n,i)}$ is closed, and convex. This completes Step 2.

Step 3. It show that $\bigcap_{i\in\Delta} F(T_i) \subset C_n$, $\forall n \ge 1$.

It is obvious that $\bigcap_{i\in\Delta} F(T_i) \subset C = C_1$. Suppose that $\bigcap_{i\in\Delta} F(T_i) \subset C_j$ for some $j \ge 1$. For any $u \in \bigcap_{i\in\Delta} F(T_i) \subset C_j$, we see that

$$\phi\left(u, T_i^j x_j\right) \le \phi(u, x_j) + \xi_{(j,i)}. \tag{3.11}$$

On the other hand, we obtain from (2.12) that

$$\phi\left(u, T_i^j x_j\right) = \phi(u, x_j) + \phi\left(x_j, T_i^j x_j\right) + 2\left\langle u - x_j, Jx_j - J\left(T_i^j x_j\right)\right\rangle. \tag{3.12}$$

Combining (3.11) with (3.12), we arrive at

$$\phi\left(x_j, T_i^j x_j\right) \le 2\left\langle x_j - u, Jx_j - JT_i^j x_j\right\rangle + \xi_{(j,i)}, \tag{3.13}$$

which implies that $u \in C_{(j+1,i)}$. This proves that $\bigcap_{i\in\Delta} F(T_i) \subset C_n$, $\forall n \ge 1$. This completes Step 3.

Step 4. It show that $x_n \to \overline{x}$, where $\overline{x} \in \bigcap_{i\in\Delta} F(T_i)$, as $n \to \infty$.

Since $\{x_n\}$ is bounded and the space is reflexive, we may assume that $x_n \rightharpoonup \overline{x}$. Since C_n is closed and convex, we see that $\overline{x} \in C_n$. On the other hand, we see from the weakly lower semicontinuity of the norm that

$$\begin{aligned}
\phi(\overline{x}, x_0) &= \|\overline{x}\|^2 - 2\langle \overline{x}, Jx_0\rangle + \|x_0\|^2 \\
&\le \liminf_{n\to\infty}\left(\|x_n\|^2 - 2\langle x_n, Jx_0\rangle + \|x_0\|^2\right) \\
&= \liminf_{n\to\infty} \phi(x_n, x_0) \\
&\le \limsup_{n\to\infty} \phi(x_n, x_0) \\
&\le \phi(\overline{x}, x_0),
\end{aligned} \tag{3.14}$$

which implies that $\phi(x_n, x_0) \to \phi(\overline{x}, x_0)$ as $n \to \infty$. Hence, $\|x_n\| \to \|\overline{x}\|$ as $n \to \infty$. In view of Kadec-Klee property of E, we see that $x_n \to \overline{x}$ as $n \to \infty$. On the other hand, we see from $x_{n+1} = \Pi_{C_{n+1}} x_0 \in C_{n+1}$ that $x_{n+1} \in C_{(n+1,i)}$. It follows that

$$\phi\left(x_n, T_i^n x_n\right) \le 2\left\langle x_n - x_{n+1}, Jx_n - JT_i^n x_n\right\rangle + \xi_{(n,i)}, \tag{3.15}$$

from which it follows that $\phi(x_n, T_i^n x_n) \to 0$ as $n \to \infty$. In view of (2.11), we see that $\|x_n\| - \|T_i^n x_n\| \to 0$ as $n \to \infty$. This in turn implies that $\|T_i^n x_n\| \to \|\overline{x}\|$ as $n \to \infty$. Hence,

$\|J(T_i^n x_n)\| \to \|J\overline{x}\|$ as $n \to \infty$. This shows that $\{J(T_i^n x_n)\}$ is bounded. Since E is reflexive, we see that E^* is also reflexive. We may, without loss of generality, assume that $J(T_i^n x_n) \rightharpoonup f^{*,i} \in E^*$. In view of the reflexivity of E, we have $J(E) = E^*$. This shows that there exists an element $f^i \in E$ such that $Jf^i = f^{*,i}$. It follows that

$$\begin{aligned}\phi(x_n, T_i^n x_n) &= \|x_n\|^2 - 2\langle x_n, J(T_i^n x_n)\rangle + \left\|T_i^n x_x\right\|^2 \\ &= \|x_n\|^2 - 2\langle x_n, J(T_i^n x_n)\rangle + \left\|J(T_i^n x_n)\right\|^2.\end{aligned} \tag{3.16}$$

Taking $\liminf_{n\to\infty}$ on the both sides of the equality above, we obtain that

$$\begin{aligned}0 &\geq \|\overline{x}\|^2 - 2\langle \overline{x}, f^{*,i}\rangle + \left\|f^{*,i}\right\|^2 \\ &= \|\overline{x}\|^2 - 2\langle \overline{x}, Jf^i\rangle + \left\|Jf^i\right\|^2 \\ &= \|\overline{x}\|^2 - 2\langle \overline{x}, Jf^i\rangle + \left\|f^i\right\|^2 \\ &= \phi\left(\overline{x}, f^i\right).\end{aligned} \tag{3.17}$$

This implies that $\overline{x} = f^i$, that is, $J\overline{x} = f^{*,i}$. It follows that $J(T_i^n x_n) \rightharpoonup J\overline{x} \in E^*$. In view of Kadec-Klee property of E^*, we obtain that $\lim_{n\to\infty}\|J(T_i^n x_n) - J\overline{x}\| = 0$. Since $J^{-1} : E^* \to E$ is demicontinuous, we see that $T_i^n x_n \rightharpoonup \overline{x}$. In the light of Kadec-Klee property of E, we see that $T_i^n x_n \to \overline{x}$, $\forall i \in \Delta$, as $n \to \infty$. On the other hand, we have

$$\left\|T_i^{n+1} x_n - \overline{x}\right\| \leq \left\|T_i^{n+1} x_n - T_i^n x_n\right\| + \left\|T_i^n x_n - \overline{x}\right\|. \tag{3.18}$$

It follows from the asymptotic regularity of T that $T_i^{n+1} x_n \to \overline{x}$ as $n \to \infty$. That is, $T_i T_i^n x_n \to \overline{x}$. From the closedness of T_i, we obtain that $\overline{x} = T_i\overline{x}$. This implies that $\overline{x} \in \bigcap_{i\in\Delta} F(T_i)$.

Step 5. It show that $\overline{x} = \Pi_{\bigcap_{i\in\Delta} F(T_i)} x_0$.

In view of $x_n = \Pi_{C_n} x_0$, we see from Lemma 2.8 that

$$\langle x_n - z, Jx_0 - Jx_n\rangle \geq 0, \quad \forall z \in C_n. \tag{3.19}$$

Since $\bigcap_{i\in\Delta} F(T_i) \subset C_n$, we arrive at

$$\langle x_n - w, Jx_0 - Jx_n\rangle \geq 0, \quad \forall w \in \bigcap_{i\in\Delta} F(T_i). \tag{3.20}$$

Letting $n \to \infty$ in the above, we arrive at

$$\langle \overline{x} - w, Jx_0 - J\overline{x}\rangle \geq 0, \quad \forall w \in \bigcap_{i\in\Delta} F(T_i). \tag{3.21}$$

It follows from Lemma 2.8 that $\overline{x} = \Pi_{\bigcap_{i\in\Delta} F(T_i)} x_0$. This completes the proof of Theorem 3.1. □

Remark 3.2. The space in Theorem 3.1 can be applicable to L^p, $p > 1$. Since the class of asymptotically quasi-ϕ-nonexpansive mappings includes the class of asymptotically quasi-ϕ-nonexpansive mappings as a special case, we see that Theorem 3.1 still holds for the class of asymptotically quasi-ϕ-nonexpansive mappings.

For a single mapping, we can easily conclude the following.

Corollary 3.3. *Let E be a reflexive, strictly convex, and smooth Banach space such that both E and E^* have Kadec-Klee property. Let C be a nonempty, bounded, closed, and convex subset of E. Let $T : C \to C$ be a closed asymptotically quasi-ϕ-nonexpansive mapping in the intermediate sense. Assume that $F(T)$ is nonempty, and T is asymptotically regular. Let $\{x_n\}$ be a sequence generated in the following manner:*

$$\begin{gathered}
x_0 \in E, \quad \text{chosen arbitrarily}, \\
C_1 = C, \\
x_1 = \Pi_{C_1} x_0, \\
C_{n+1} = \{u \in C_n : \phi(x_n, T^n x_n) \le 2\langle x_n - u, Jx_n - JT^n x_n \rangle + \xi_n\}, \\
x_{n+1} = \Pi_{C_{n+1}} x_0, \quad \forall n \ge 1,
\end{gathered} \tag{3.22}$$

where $\xi_n = \max\{0, \sup_{p\in F(T), x\in C}(\phi(p, T^n x) - \phi(p, x))\}$. Then $\{x_n\}$ converges strongly to $\Pi_{F(T)} x_0$, where $\Pi_{F(T)}$ stands for the generalized projection from E onto $F(T)$.

In the framework of Hilbert spaces, Theorem 3.1 is reduced to the following.

Corollary 3.4. *Let C be a nonempty, bounded, closed, and convex subset of a Hilbert space E. Let Δ be an index set, and $T_i : C \to C$ a closed asymptotically quasi-nonexpansive mapping in the intermediate sense, for every $i \in \Delta$. Assume that $\bigcap_{i\in\Delta} F(T_i)$ is nonempty, and T_i is asymptotically regular, for every $i \in \Delta$. Let $\{x_n\}$ be a sequence generated in the following manner:*

$$\begin{gathered}
x_0 \in E, \quad \text{chosen arbitrarily}, \\
C_{(1,i)} = C, \\
C_1 = \bigcap_{i\in\Delta} C_{(1,i)}, \\
x_1 = P_{C_1} x_0, \\
C_{(n+1,i)} = \left\{u \in C_{(n,i)} : \left\|x_n - T_i^n x_n\right\|^2 \le 2\left\langle x_n - u, x_n - T_i^n x_n \right\rangle + \xi_{(n,i)}\right\}, \\
C_{n+1} = \bigcap_{i\in\Delta} C_{(n+1,i)}, \\
x_{n+1} = P_{C_{n+1}} x_0, \quad \forall n \ge 1,
\end{gathered} \tag{3.23}$$

where $\xi_{(n,i)} = \max\{0, \sup_{p\in F(T_i), x\in C}(\|p - T_i^n x\|^2 - \|p - x\|^2)\}$. Then $\{x_n\}$ converges strongly to $P_{\bigcap_{i\in\Delta} F(T_i)} x_0$, where $P_{\bigcap_{i\in\Delta} F(T_i)}$ stands for the metric projection from E onto $\bigcap_{i\in\Delta} F(T_i)$.

For a single mapping, we can easily conclude the following.

Corollary 3.5. *Let C be a nonempty, bounded, closed, and convex subset of a Hilbert space E. Let T be a closed asymptotically quasi-nonexpansive mapping in the intermediate sense. Assume that $F(T)$ is nonempty, and T is asymptotically regular. Let $\{x_n\}$ be a sequence generated in the following manner:*

$$\begin{gathered}
x_0 \in E, \quad \text{chosen arbitrarily}, \\
C_1 = C, \\
x_1 = P_{C_1} x_0, \\
C_{n+1} = \left\{ u \in C_n : \|x_n - T^n x_n\|^2 \leq 2\langle x_n - u, x_n - T^n x_n \rangle + \xi_n \right\}, \\
C_{n+1} = \bigcap_{i \in \Delta} C_{(n+1,i)}, \\
x_{n+1} = P_{C_{n+1}} x_0, \quad \forall n \geq 1,
\end{gathered} \tag{3.24}$$

where $\xi_n = \max\{0, \sup_{p \in F(T), x \in C}(\|p - T^n x\|^2 - \|p - x\|^2)\}$. Then $\{x_n\}$ converges strongly to $P_{F(T)} x_0$, where $P_{F(T)}$ stands for the metric projection from E onto $F(T)$.

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