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Research Article

The Local and Global Existence of Solutions for a Generalized Camassa-Holm Equation

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A nonlinear generalization of the Camassa-Holm equation is investigated. By making use of the pseudoparabolic regularization technique, its local well posedness in Sobolev space $H^S(R)$ with $s > 3/2$ is established via a limiting procedure. Provided that the initial value u_0 satisfies the sign condition and $u_0 \in H^s(R)$ $(s > 3/2)$, it is shown that there exists a unique global solution for the equation in space $C([0,\infty); H^s(R)) \cap C^1([0,\infty); H^{s-1}(R))$.

1. Introduction

Camassa and Holm [1] employed the Hamiltonian method to derive a completely integrable shallow water wave model

$$u_t - u_{txx} + 2ku_x + 3uu_x = 2u_x u_{xx} + uu_{xxx}, \tag{1.1}$$

which was alternatively established as a water wave equation in [2–4]. Equation (1.1) also models wave current interaction [5], while Dai [6] derived it as a model in elasticity (see [7]). In addition, it was pointed out in Lakshmanan [8] that the Camassa-Holm equation (1.1) could be relevant to the modeling of tsunami waves (see Constantin and Johnson [9]).

After the birth of the Camassa-Holm equation (1.1), many works have been carried out to probe its dynamic properties. For $k = 0$, (1.1) has travelling wave solutions of the form $ce^{-|x-ct|}$, called peakons, which describes an essential feature of the travelling waves of largest amplitude (see [10–14]). For $k > 0$, its solitary waves are stable solitons [15]. It is shown in [16–18] that the inverse spectral or scattering approach is a powerful tool to handle the Camassa-Holm equation and analyze its dynamics. It is worthwhile to mention that (1.1) gives rise to geodesic flow of a certain invariant metric on the Bott-Virasoro group [19–21], and this geometric illustration leads to a proof that the least action principle holds. Xin and Zhang [22] proved the global existence of the weak solution in the energy space $H^1(R)$ without any sign conditions on the initial value, and the uniqueness of this weak solution is obtained under some assumptions on the solution [23]. Coclite et al. [24] extended the analysis presented in [22, 23] and obtained many useful dynamic properties to other partial differential equations (see [25–28] for an alternative approach). Li and Olver [29] established the local well posedness in the Sobolev space $H^s(R)$ with $s > 3/2$ for (1.1) and gave conditions on the initial data that lead to finite time blowup of certain solutions. It is shown in Constantin and Escher [30] that the blowup occurs in the form of breaking waves, namely, the solution remains bounded but its slope becomes unbounded in finite time. For other methods to handle the problems relating to various dynamic properties of the Camassa-Holm equation and other shallow water equations, the reader is referred to [31–39] and the references therein.

Motivated by the work in Hakkaev and Kirchev [33] to investigate the generalization forms of the Camassa-Holm equation with high-order nonlinear terms, we study the following generalized Camassa-Holm equation:

$$u_t - u_{txx} + ku^m u_x + (m+3)u^{m+1}u_x = (m+2)u^m u_x u_{xx} + u^{m+1}u_{xxx}, \tag{1.2}$$

where $m \geq 0$ is a natural number and $k \geq 0$. Obviously, (1.2) reduces to (1.1) if we set $m = 0$. As the Camassa-Holm equation (1.1) has been discussed by many mathematicians, we let the natural number $m \geq 1$ in this paper.

The objective of this paper is to study (1.2). Its local well posedness of solutions in the Sobolev space $H^s(R)$ with $s > 3/2$ is developed by using the pseudoparabolic regularization method. Provided that $(1 - \partial_x^2)u_0 + k/2(m+1) \geq 0$ and $u_0 \in H^s$ $(s > 3/2)$, the existence and uniqueness of the global solutions are established in space $C([0,\infty); H^s(R)) \bigcap C^1([0,\infty); H^{s-1}(R))$. It should be mentioned that the existence and uniqueness of global strong solutions for the nonlinear generalized Camassa-Holm models like (1.2) have never been investigated in the literatures.

2. Main Results

The space of all infinitely differentiable functions $\phi(t,x)$ with compact support in $[0,+\infty) \times R$ is denoted by C_0^∞. $L^p = L^p(R)(1 \leq p < +\infty)$ is the space of all measurable functions h such that $\|h\|_{L^p}^p = \int_R |h(t,x)|^p dx < \infty$. We define $L^\infty = L^\infty(R)$ with the standard norm $\|h\|_{L^\infty} = \inf_{m(e)=0} \sup_{x \in R \setminus e} |h(t,x)|$. For any real number s, $H^s = H^s(R)$ denotes the Sobolev space with the norm defined by

$$\|h\|_{H^s} = \left(\int_R \left(1 + |\xi|^2\right)^s \left|\widehat{h}(t,\xi)\right|^2 d\xi \right)^{1/2} < \infty, \tag{2.1}$$

where $\widehat{h}(t,\xi) = \int_R e^{-ix\xi} h(t,x) dx$.

For $T > 0$ and nonnegative number s, $C([0,T); H^s(R))$ denotes the Frechet space of all continuous H^s-valued functions on $[0,T)$. We set $\Lambda = (1 - \partial_x^2)^{1/2}$. For simplicity, throughout this paper, we let c denote any positive constant which is independent of parameter ε.

We consider the Cauchy problem of (1.2), which has the equivalent form

$$\begin{aligned} u_t - u_{txx} &= -\frac{k}{m+1}\left(u^{m+1}\right)_x - \frac{m+3}{m+2}\left(u^{m+2}\right)_x + \frac{1}{m+2}\partial_x^3\left(u^{m+2}\right) \\ &\quad - (m+1)\partial_x\left(u^m u_x^2\right) + u^m u_x u_{xx}, \quad k \geq 0,\ m \geq 1, \\ u(0,x) &= u_0(x). \end{aligned} \tag{2.2}$$

Now, we give our main results for problem (2.2).

Theorem 2.1. *Suppose that the initial function $u_0(x)$ belongs to the Sobolev space $H^s(R)$ with $s > 3/2$. Then there is a $T > 0$, which depends on $\|u_0\|_{H^s}$, such that there exists a unique solution $u(t,x)$ of the problem* (2.2) *and*

$$u(t,x) \in C([0,T]; H^s(R)) \bigcap C^1\left([0,T]; H^{s-1}(R)\right). \tag{2.3}$$

Theorem 2.2. *Let $u_0(x) \in H^s$, $s > 3/2$ and $(1-\partial_x^2)u_0 + k/2(m+1) \geq 0$ for all $x \in R$. Then problem* (2.2) *has a unique solution satisfying that*

$$u(t,x) \in C([0,\infty); H^s(R)) \bigcap C^1\left([0,\infty); H^{s-1}(R)\right). \tag{2.4}$$

3. Local Well-Posedness

In order to prove Theorem 2.1, we consider the associated regularized problem

$$\begin{aligned} u_t - u_{txx} + \varepsilon u_{txxxx} &= -\frac{k}{m+1}\left(u^{m+1}\right)_x - \frac{m+3}{m+2}\left(u^{m+2}\right)_x + \frac{1}{m+2}\partial_x^3\left(u^{m+2}\right) \\ &\quad - (m+1)\partial_x\left(u^m u_x^2\right) + u^m u_x u_{xx}, \\ u(0,x) &= u_0(x), \end{aligned} \tag{3.1}$$

where the parameter ε satisfies $0 < \varepsilon < 1/4$.

Lemma 3.1. *Let r and q be real numbers such that $-r < q \leq r$. Then*

$$\begin{aligned} \|uv\|_{H^q} &\leq c\|u\|_{H^r}\|v\|_{H^q}, \quad \text{if } r > \frac{1}{2}, \\ \|uv\|_{H^{r+q-1/2}} &\leq c\|u\|_{H^r}\|v\|_{H^q}, \quad \text{if } r < \frac{1}{2}. \end{aligned} \tag{3.2}$$

This lemma can be found in [34, 40].

Lemma 3.2. *Let $u_0(x) \in H^s(R)$ with $s > 3/2$. Then the Cauchy problem* (3.1) *has a unique solution $u(t,x) \in C([0,T]; H^s(R))$ where $T > 0$ depends on $\|u_0\|_{H^s(R)}$. If $s \geq 2$, the solution $u \in C([0,+\infty); H^s)$ exists for all time.*

Proof. Assuming that $D = (1 - \partial_x^2 + \varepsilon\partial_x^4)^{-1}$, we know that $D : H^s \to H^{s+4}$ is a bounded linear operator. Applying the operator D on both sides of the first equation of system (3.1) and then integrating the resultant equation with respect to t over the interval $(0, t)$ lead to

$$\begin{aligned} u(t,x) = u_0(x) + \int_0^t D\Big[&-\frac{k}{m+1}\left(u^{m+1}\right)_x - \frac{m+3}{m+2}\left(u^{m+2}\right)_x + \frac{1}{m+2}\partial_x^3\left(u^{m+2}\right) \\ &-(m+1)\partial_x\left(u^m u_x^2\right) + u^m u_x u_{xx}\Big] dt. \end{aligned} \tag{3.3}$$

Suppose that both u and v are in the closed ball $B_{M_0}(0)$ of radius M_0 about the zero function in $C([0,T]; H^s(R))$ and A is the operator in the right-hand side of (3.3). For any fixed $t \in [0,T]$, we get the following:

$$\begin{aligned} &\Bigg\| \int_0^t D\Big[-\frac{k}{m+1}\left(u^{m+1}\right)_x - \frac{m+3}{m+2}\left(u^{m+2}\right)_x + \frac{1}{m+2}\partial_x^3\left(u^{m+2}\right) \\ &\qquad -(m+1)\partial_x\left(u^m u_x^2\right) + u^m u_x u_{xx}\Big] dt \\ &\quad - \int_0^t D\Big[-\frac{k}{m+1}\left(v^{m+1}\right)_x - \frac{m+3}{m+2}\left(v^{m+2}\right)_x + \frac{1}{m+2}\partial_x^3\left(v^{m+2}\right) \\ &\qquad -(m+1)\partial_x\left(v^m v_x^2\right) + v^m v_x v_{xx}\Big] dt \Bigg\|_{H^s} \\ &\le TC_1\Bigg(\sup_{0\le t\le T}\left\|u^{m+1} - v^{m+1}\right\|_{H^s} + \sup_{0\le t\le T}\left\|u^{m+2} - v^{m+2}\right\|_{H^s} \\ &\qquad + \sup_{0\le t\le T}\left\|D\partial_x\left[u^m u_x^2 - v^m v_x^2\right]\right\|_{H^s} + \sup_{0\le t\le T}\|D[u^m u_x u_{xx} - v^m v_x v_{xx}]\|_{H^s}\Bigg), \end{aligned} \tag{3.4}$$

where C_1 may depend on ε. The algebraic property of $H^{s_0}(R)$ with $s_0 > 1/2$ derives

$$\begin{aligned} \left\|u^{m+2} - v^{m+2}\right\|_{H^s} &= \left\|(u-v)\left(u^{m+1} + u^m v + \cdots + u v^m + v^{m+1}\right)\right\|_{H^s} \\ &\le \|(u-v)\|_{H^s}\sum_{j=0}^{m+1}\|u\|_{H^s}^{m+1-j}\|v\|_{H^s}^{j} \\ &\le M_0^{m+1}\|(u-v)\|_{H^s}, \end{aligned} \tag{3.5}$$

$$\begin{aligned} \left\|u^{m+1} - v^{m+1}\right\|_{H^s} &\le M_0^{m}\|(u-v)\|_{H^s}, \\ \left\|D\partial_x\left(u^m u_x^2 - v^m v_x^2\right)\right\|_{H^s} &\le \left\|D\partial_x\left[u^m\left(u_x^2 - v_x^2\right)\right]\right\|_{H^s} + \left\|D\partial_x\left[v_x^2(u^m - v^m)\right]\right\|_{H^s} \\ &\le C\left(\left\|u^m\left(u_x^2 - v_x^2\right)\right\|_{H^{s-1}} + \left\|v_x^2(u^m - v^m)\right\|_{H^{s-1}}\right) \\ &\le CM_0^{m+1}\|u - v\|_{H^s}. \end{aligned} \tag{3.6}$$

Using the first inequality of Lemma 3.1, we have

$$\begin{aligned}
\|D[u^m u_x u_{xx} - v^m v_x v_{xx}]\|_{H^s} &= \left\| \frac{1}{2} D\left[u^m \left(u_x^2\right)_x - v^m \left(v_x^2\right)_x\right] \right\|_{H^s} \\
&\le \frac{1}{2}\left(\left\| D\left[u^m \left(u_x^2 - v_x^2\right)_x\right] \right\|_{H^s} + \left\| D\left[\left(v_x^2\right)_x (u^m - v^m)\right] \right\|_{H^s} \right) \\
&\le C\left(\left\| u^m \left(u_x^2 - v_x^2\right)_x \right\|_{H^{s-2}} + \left\| \left(v_x^2\right)_x (u^m - v^m) \right\|_{H^{s-2}} \right) \\
&\le C\left(\|u^m\|_{H^s} \left\| u_x^2 - v_x^2 \right\|_{H^{s-1}} + \left\| v_x^2 \right\|_{H^{s-1}} \|u^m - v^m\|_{H^s} \right) \\
&\le C M_0^{m+1} \|u - v\|_{H^s},
\end{aligned} \tag{3.7}$$

where C may depend on ε. From (3.5)–(3.7), we obtain that

$$\|Au - Av\|_{H^s} \le \theta \|u - v\|_{H^s}, \tag{3.8}$$

where $\theta = TC_2(M_0^m + M_0^{m+1})$ and C_2 is independent of $0 < t < T$. Choosing T sufficiently small such that $\theta < 1$, we know that A is a contraction. Applying the above inequality yields that

$$\|Au\|_{H^s} \le \|u_0\|_{H^s} + \theta \|u\|_{H^s}. \tag{3.9}$$

Choosing T sufficiently small such that $\theta M_0 + \|u_0\|_{H^s} < M_0$, we deduce that A maps $B_{M_0}(0)$ to itself. It follows from the contraction-mapping principle that the mapping A has a unique fixed-point u in $B_{M_0}(0)$.

For $s \ge 2$, using the first equation of system (3.1) derives

$$\frac{d}{dt} \int_R \left(u^2 + u_x^2 + \varepsilon u_{xx}^2\right) dx = 0, \tag{3.10}$$

from which we have the conservation law

$$\int_R \left(u^2 + u_x^2 + \varepsilon u_{xx}^2\right) dx = \int_R \left(u_0^2 + u_{0x}^2 + \varepsilon u_{0xx}^2\right) dx. \tag{3.11}$$

The proof of the global existence result is a routine argument by using (3.11) (see Xin and Zhang [22]). □

Lemma 3.3 (Kato and Ponce [41]). *If $r \ge 0$, then $H^r \bigcap L^\infty$ is an algebra. Moreover*

$$\|uv\|_r \le c(\|u\|_{L^\infty} \|v\|_r + \|u\|_r \|v\|_{L^\infty}), \tag{3.12}$$

where c is a constant depending only on r.

Lemma 3.4 (Kato and Ponce [41]). *Let $r > 0$. If $u \in H^r \bigcap W^{1,\infty}$ and $v \in H^{r-1} \bigcap L^\infty$, then*

$$\|[\Lambda^r, u]v\|_{L^2} \le c\left(\|\partial_x u\|_{L^\infty} \left\|\Lambda^{r-1} v\right\|_{L^2} + \|\Lambda^r u\|_{L^2}\|v\|_{L^\infty}\right). \tag{3.13}$$

Lemma 3.5. *Let $s \ge 2$, and the function $u(t,x)$ is a solution of problem* (3.1) *and the initial data $u_0(x) \in H^s$. Then the following inequality holds:*

$$\begin{aligned}\|u\|_{H^1}^2 &\le \int_R \left(u^2 + u_x^2 + \varepsilon u_{xx}^2\right) dx \\ &= \int_R \left(u_0^2 + u_{0x}^2 + \varepsilon u_{0xx}^2\right) dx.\end{aligned} \tag{3.14}$$

For $q \in (0, s-1]$, there is a constant c independent of ε such that

$$\begin{aligned}\int_R \left(\Lambda^{q+1} u\right)^2 dx \le{}& \int_R \left[\left(\Lambda^{q+1} u_0\right)^2 + \varepsilon(\Lambda^q u_{0xx})^2\right] dx \\ &+ c\int_0^t \|u\|_{H^{q+1}}^2 \left(\left(\|u\|_{L^\infty}^{m-1} + \|u\|_{L^\infty}^m\right)\|u_x\|_{L^\infty} + \|u\|_{L^\infty}^{m-1}\|u_x\|_{L^\infty}^2\right) d\tau.\end{aligned} \tag{3.15}$$

For $q \in [0, s-1]$, there is a constant c independent of ε such that

$$(1-2\varepsilon)\|u_t\|_{H^q} \le c\|u\|_{H^{q+1}}\left(\left(\|u\|_{L^\infty}^{m-1} + \|u\|_{L^\infty}^m\right)\|u\|_{H^1} + \|u\|_{L^\infty}^m\|u_x\|_{L^\infty} + \|\|\|u\|_{L^\infty}^{m-1}\|u_x\|_{L^\infty}^2\right). \tag{3.16}$$

Proof. The inequality $\|u\|_{H^1}^2 \le \int_R (u^2 + u_x^2) dx$ and (3.11) derives (3.14).

Using $\partial_x^2 = -\Lambda^2 + 1$ and the Parseval equality gives rise to

$$\begin{aligned}\int_R \Lambda^q u \Lambda^q \partial_x^3\left(u^{m+2}\right) dx ={}& -(m+2)\int_R \left(\Lambda^{q+1} u\right)\Lambda^{q+1}\left(u^{m+1}u_x\right) dx \\ &+ (m+2)\int_R (\Lambda^q u)\Lambda^q\left(u^{m+1}u_x\right) dx.\end{aligned} \tag{3.17}$$

For $q \in (0, s-1]$, applying $(\Lambda^q u)\Lambda^q$ to both sides of the first equation of system (3.1) and integrating with respect to x by parts, we have the identity

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\int_R \left((\Lambda^q u)^2 + (\Lambda^q u_x)^2 + \varepsilon(\Lambda^q u_{xx})^2\right) dx \\ &= -\frac{k}{m+1}\int_R (\Lambda^q u)\Lambda^q\left(u^{m+1}\right)_x dx - (m+2)\int_R (\Lambda^q u)\Lambda^q\left(u^{m+1}u_x\right) dx \\ &\quad - \int_R \left(\Lambda^{q+1} u\right)\Lambda^{q+1}\left(u^{m+1}u_x\right) dx + (m+1)\int_R (\Lambda^q u_x)\Lambda^q\left(u^m u_x^2\right) dx \\ &\quad + \int_R \Lambda^q u \Lambda^q (u^m u_x u_{xx}) dx.\end{aligned} \tag{3.18}$$

We will estimate the terms on the right-hand side of (3.18) separately. For the second term, by using the Cauchy-Schwartz inequality and Lemmas 3.3 and 3.4, we have

$$\begin{aligned}\left|\int_R (\Lambda^q u)\Lambda^q\left(u^{m+1}u_x\right)dx\right| &= \left|\int_R (\Lambda^q u)\left[\Lambda^q\left(u^{m+1}u_x\right) - u^{m+1}\Lambda^q u_x\right]dx + \int_R (\Lambda^q u)u^{m+1}\Lambda^q u_x dx\right| \\ &\leq c\|u\|_{H^q}\left((m+1)\|u\|_{L^\infty}^m\|u_x\|_{L^\infty}\|u\|_{H^q} + \|u_x\|_{L^\infty}\|u\|_{L^\infty}^m\|u\|_{H^q}\right) \\ &\quad + \frac{m+1}{2}\|u\|_{L^\infty}^m\|u_x\|_{L^\infty}\|\Lambda^q u\|_{L^2}^2 \\ &\leq c\|u\|_{H^q}^2\|u\|_{L^\infty}^m\|u_x\|_{L^\infty}.\end{aligned} \tag{3.19}$$

Similarly, for the first term in (3.18), we have

$$\left|\int_R (\Lambda^q u)\Lambda^q(u^m u_x)dx\right| \leq c\|u\|_{H^q}^2\|u\|_{L^\infty}^{m-1}\|u_x\|_{L^\infty}. \tag{3.20}$$

Using the above estimate to the third term yields that

$$\left|\int_R \left(\Lambda^{q+1}u\right)\Lambda^{q+1}\left(u^{m+1}u_x\right)dx\right| \leq c\|u\|_{H^{q+1}}^2\|u\|_{L^\infty}^m\|u_x\|_{L^\infty}. \tag{3.21}$$

For the fourth term, using the Cauchy-Schwartz inequality and Lemma 3.3, we obtain that

$$\begin{aligned}\left|\int_R (\Lambda^q u_x)\Lambda^q\left(u^m u_x^2\right)dx\right| &\leq \|\Lambda^q u_x\|_{L^2}\left\|\Lambda^q\left(u^m u_x^2\right)\right\|_{L^2} \\ &\leq c\|u\|_{H^{q+1}}\left(\|u^m u_x\|_{L^\infty}\|u_x\|_{H^q} + \|u_x\|_{L^\infty}\|u^m u_x\|_{H^q}\right) \\ &\leq c\|u\|_{H^{q+1}}^2\|u_x\|_{L^\infty}\|u\|_{L^\infty}^m,\end{aligned} \tag{3.22}$$

in which we have used $\|u^m u_x\|_{H^q} \leq c\|(u^{m+1})_x\|_{H^q} \leq c\|u\|_{L^\infty}^m\|u\|_{H^{q+1}}$.

For the last term in (3.18), using $u^m(u_x^2)_x = (u^m u_x^2)_x - (u^m)_x u_x^2$ results in

$$\begin{aligned}\left|\int_R (\Lambda^q u)\Lambda^q(u^m u_x u_{xx})dx\right| &\leq \left|\int_R \Lambda^q u_x\Lambda^q\left(u^m u_x^2\right)dx\right| + \int_R \Lambda^q u\Lambda^q\left[(u^m)_x u_x^2\right]dx \\ &= K_1 + K_2.\end{aligned} \tag{3.23}$$

For K_1, it follows from (3.22) that

$$K_1 \leq c\|u\|_{H^{q+1}}^2\|u_x\|_{L^\infty}\|u\|_{L^\infty}^m. \tag{3.24}$$

For K_2, applying Lemma 3.3 derives

$$\begin{aligned} K_2 &\le c\|u\|_{H^q}\left\|(u^m)_x u_x^2\right\|_{H^q} \\ &\le c\|u\|_{H^q}\left(\|(u^m)_x\|_{L^\infty}\left\|u_x^2\right\|_{H^q} + \|(u^m)_x\|_{H^q}\left\|u_x^2\right\|_{L^\infty}\right) \\ &\le c\|u\|_{H^{q+1}}^2\left(\|u\|_{L^\infty}^{m-1}\|u_x\|_{L^\infty}^2\right). \end{aligned} \tag{3.25}$$

It follows from (3.19)–(3.25) that there exists a constant c depending only on m such that

$$\frac{1}{2}\frac{d}{dt}\int_R\left[(\Lambda^q u)^2 + (\Lambda^q u_x)^2 + \varepsilon(\Lambda^q u_{xx})^2\right]dx \le c\|u\|_{H^{q+1}}^2\left(\|u_x\|_{L^\infty}\|u\|_{L^\infty}^m + \|u\|_{L^\infty}^{m-1}\|u_x\|_{L^\infty}^2\right). \tag{3.26}$$

Integrating both sides of the above inequality with respect to t results in (3.15).

To estimate the norm of u_t, we apply the operator $(1-\partial_x^2)^{-1}$ to both sides of the first equation of system (3.1) to obtain the equation

$$\begin{aligned} (1-\varepsilon)u_t - \varepsilon u_{txx} = \left(1-\partial_x^2\right)^{-1}\Bigg[&-\varepsilon u_t - k u^m u_x - \frac{m+3}{m+2}\left(u^{m+2}\right)_x \\ &+\frac{1}{m+2}\partial_x^3\left(u^{m+2}\right) - (m+1)\partial_x\left(u^m u_x^2\right) + u^m u_x u_{xx}\Bigg]. \end{aligned} \tag{3.27}$$

Applying $(\Lambda^q u_t)\Lambda^q$ to both sides of (3.27) for $q \in [0, s-1]$ gives rise to

$$\begin{aligned} &(1-\varepsilon)\int_R(\Lambda^q u_t)^2 dx + \varepsilon\int_R(\Lambda^q u_{xt})^2 dx \\ &\quad= \int_R(\Lambda^q u_t)\Lambda^{q-2}\Bigg[-\varepsilon u_t + \partial_x\left(-\frac{k}{m+1}u^{m+1} - \frac{m+3}{m+2}u^{m+2} + \frac{1}{m+2}\partial_x^2 u^{m+2} - (m+1)u^m u_x^2\right) \\ &\qquad\qquad + u^m u_x u_{xx}\Bigg]d\tau. \end{aligned} \tag{3.28}$$

For the right hand of (3.28), we have

$$\begin{aligned} &\left|\int_R(\Lambda^q u_t)\Lambda^{q-2}(-\varepsilon u_t)dx\right| \le \varepsilon\|u_t\|_{H^q}^2, \\ &\left|\int_R(\Lambda^q u_t)\left(1-\partial_x^2\right)^{-1}\Lambda^q\partial_x\left(-\frac{k}{m+1}u^{m+1} - \frac{m+3}{m+2}u^{m+2} - (m+1)u^m u_x^2\right)dx\right| \\ &\quad\le c\|u_t\|_{H^q}\Bigg(\int_R\left(1+\xi^2\right)^{q-1}\Bigg[\int_R\Bigg[-\frac{k}{m+1}\widehat{u^m}(\xi-\eta)\widehat{u}(\eta) - \frac{m+3}{m+2}\widehat{u^{m+1}}(\xi-\eta)\widehat{u}(\eta) \\ &\qquad\qquad -(m+1)\widehat{u^m u_x}(\xi-\eta)\widehat{u_x}(\eta)\Bigg]d\eta\Bigg]^2\Bigg)^{1/2} \\ &\quad\le c\|u_t\|_{H^q}\|u\|_{H^1}\|u\|_{H^{q+1}}\left(\|u\|_{L^\infty}^{m-1} + \|u\|_{L^\infty}^m\right). \end{aligned} \tag{3.29}$$

Since

$$\int (\Lambda^q u_t)\left(1-\partial_x^2\right)^{-1}\Lambda^q\partial_x^2\left(u^{m+1}u_x\right)dx = -\int (\Lambda^q u_t)\Lambda^q\left(u^{m+1}u_x\right)dx \\ +\int (\Lambda^q u_t)\left(1-\partial_x^2\right)^{-1}\Lambda^q\left(u^{m+1}u_x\right)dx, \tag{3.30}$$

using Lemma 3.3, $\|u^{m+1}u_x\|_{H^q} \le c\|(u^{m+2})_x\|_{H^q} \le c(m+2)\|u\|_{L^\infty}^{m+1}\|u\|_{H^{q+1}}$ and $\|u\|_{L^\infty} \le \|u\|_{H^1}$, we have

$$\begin{aligned} \left|\int (\Lambda^q u_t)\Lambda^q\left(u^{m+1}u_x\right)dx\right| &\le c\|u_t\|_{H^q}\left\|u^{m+1}u_x\right\|_{H^q} \\ &\le c\|u_t\|_{H^q}\|u\|_{L^\infty}^m\|u\|_{H^1}\|u\|_{H^{q+1}}, \\ \left|\int (\Lambda^q u_t)\left(1-\partial_x^2\right)^{-1}\Lambda^q\left(u^{m+1}u_x\right)dx\right| &\le c\|u_t\|_{H^q}\|u\|_{L^\infty}^m\|u\|_{H^1}\|u\|_{H^{q+1}}. \end{aligned} \tag{3.31}$$

Using the Cauchy-Schwartz inequality and Lemmas 3.1 and 3.3 yields that

$$\begin{aligned} \left|\int_R (\Lambda^q u_t)\left(1-\partial_x^2\right)^{-1}\Lambda^q(u^m u_x u_{xx})dx\right| &\le c\|u_t\|_{H^q}\|u^m u_x u_{xx}\|_{H^{q-2}} \\ &\le c\|u_t\|_{H^q}\left\|u^m\left(u_x^2\right)_x\right\|_{H^{q-2}} \\ &\le c\|u_t\|_{H^q}\left\|\left[u^m\left(u_x^2\right)\right]_x-(u^m)_x u_x^2\right\|_{H^{q-2}} \\ &\le c\|u_t\|_{H^q}\left(\left\|u^m u_x^2\right\|_{H^{q-1}}+\left\|(u^m)_x u_x^2\right\|_{H^{q-2}}\right) \\ &\le c\|u_t\|_{H^q}\left(\left\|u^m u_x^2\right\|_{H^q}+\left\|(u^{\mathrm{m}})_x u_x^2\right\|_{H^q}\right) \\ &\le c\|u_t\|_{H^q}\|u\|_{H^{q+1}}\left(\|u\|_{L^\infty}^m\|u_x\|_{L^\infty}+\|u\|_{L^\infty}^{m-1}\|u_x\|_{L^\infty}^2\right), \end{aligned} \tag{3.32}$$

in which we have used (3.25).

Applying (3.29)–(3.32) into (3.28) yields the inequality

$$(1-2\varepsilon)\|u_t\|_{H^q} \le c\|u\|_{H^{q+1}}\left(\left(\|u\|_{L^\infty}^{m-1}+\|u\|_{L^\infty}^m\right)\|u\|_{H^1} \\ +\|u\|_{L^\infty}^m\|u_x\|_{L^\infty}+\|u\|_{L^\infty}^{m-1}\|u_x\|_{L^\infty}^2\right) \tag{3.33}$$

for a constant $c > 0$. This completes the proof of Lemma 3.5. □

Remark 3.6. In fact, letting $\varepsilon = 0$ in problem (3.1), (3.14), (3.15), and (3.16) are still valid.

Setting $\phi_\varepsilon(x) = \varepsilon^{-1/4}\phi(\varepsilon^{-1/4}x)$ with $0 < \varepsilon < 1/4$ and $u_{\varepsilon 0} = \phi_\varepsilon \star u_0$, we know that $u_{\varepsilon 0} \in C^\infty$ for any $u_0 \in H^s,\ s > 0$. From Lemma 3.2, it derives that the Cauchy problem

$$\begin{aligned} u_t - u_{txx} + \varepsilon u_{txxxx} = &-\frac{k}{m+1}\left(u^{m+1}\right)_x - \frac{m+3}{m+2}\left(u^{m+2}\right)_x + \frac{1}{m+2}\partial_x^3\left(u^{m+2}\right) \\ &- (m+1)\partial_x\left(u^m u_x^2\right) + u^m u_x u_{xx}, \\ u(0,x) = u_{\varepsilon 0}(x),&\ \ x \in R \end{aligned} \tag{3.34}$$

has a unique solution $u_\varepsilon(t,x) \in C^\infty([0,\infty); H^\infty)$.

Furthermore, we have the following.

Lemma 3.7. *For $s > 0,\ u_0 \in H^s$, it holds that*

$$\|u_{\varepsilon 0x}\|_{L^\infty} \le c\|u_{0x}\|_{L^\infty}, \tag{3.35}$$

$$\|u_{\varepsilon 0}\|_{H^q} \le c \quad \text{if } q \le s, \tag{3.36}$$

$$\|u_{\varepsilon 0}\|_{H^q} \le c\varepsilon^{(s-q)/4} \quad \text{if } q > s, \tag{3.37}$$

$$\|u_{\varepsilon 0} - u_0\|_{H^q} \le c\varepsilon^{(s-q)/4} \quad \text{if } q \le s, \tag{3.38}$$

$$\|u_{\varepsilon 0} - u_0\|_{H^s} = o(1), \tag{3.39}$$

where c is a constant independent of ε.

The proof of Lemma 3.7 can be found in [38].

Remark 3.8. For $s \ge 1$, using $\|u_\varepsilon\|_{L^\infty} \le c\|u_\varepsilon\|_{H^{1/2+}} \le c\|u_\varepsilon\|_{H^1}$, $\|u_\varepsilon\|_{H^1}^2 \le c\int_R (u_\varepsilon^2 + u_{\varepsilon x}^2)dx$, (3.14), (3.36), and (3.37), we know that

$$\begin{aligned} \|u_\varepsilon\|_{L^\infty}^2 \le c\|u_\varepsilon\|_{H^1} &\le c\int_R \left(u_{\varepsilon 0}^2 + u_{\varepsilon 0x}^2 + \varepsilon u_{\varepsilon 0xx}^2\right)dx \\ &\le c\left(\|u_{\varepsilon 0}\|_{H^1}^2 + \varepsilon\|u_{\varepsilon 0}\|_{H^2}^2\right) \\ &\le c\left(c + c\varepsilon \times \varepsilon^{(s-2)/2}\right) \\ &\le c_0, \end{aligned} \tag{3.40}$$

where c_0 is independent of ε.

Lemma 3.9. *If $u_0(x) \in H^s(R)$ with $s \ge 1$ such that $\|u_{0x}\|_{L^\infty} < \infty$. Let $u_{\varepsilon 0}$ be defined as in system* (3.34)*. Then there exist two positive constants T and c, which are independent of ε, such that the solution u_ε of problem* (3.34) *satisfies $\|u_{\varepsilon x}\|_{L^\infty} \le c$ for any $t \in [0,T)$.*

Proof. Using notation $u = u_\varepsilon$ and differentiating both sides of the first equation of problem (3.34) or (3.27) with respect to x give rise to

$$\begin{aligned}(1-\varepsilon)u_{tx} - \varepsilon u_{txxx} + \frac{1}{m+2}\partial^2\left(u^{m+2}\right) - \left(m+\frac{1}{2}\right)\left(u^m u_x^2\right) \\ = \frac{k}{m+1}u^{m+1} + u^{m+2} - \Lambda^{-2}\left[\varepsilon u_{tx} + \frac{k}{m+1}u^{m+1} + u^{m+2} + \left(m+\frac{1}{2}\right)\left(u^m u_x^2\right)\right. \\ \left. + \frac{1}{2}\partial_x\left[(u^m)_x u_x^2\right]\right].\end{aligned} \tag{3.41}$$

Letting $p > 0$ be an integer and multiplying the above equation by $(u_x)^{2p+1}$ and then integrating the resulting equation with respect to x yield the equality

$$\begin{aligned}&\frac{1-\varepsilon}{2p+2}\frac{d}{dt}\int_R (u_x)^{2p+2}dx - \varepsilon\int_R (u_x)^{2p+1}u_{txxx}dx + \frac{p-m}{2p+2}\int_R (u_x)^{2p+3}u^m dx \\ &\quad = \int_R (u_x)^{2p+1}\left(\frac{k}{m+1}u^{m+1} + u^{m+2}\right)dx \\ &\qquad - \int_R (u_x)^{2p+1}\Lambda^{-2}\left[\varepsilon u_{tx} + \frac{k}{m+1}u^{m+1} + u^{m+2} + \left(m+\frac{1}{2}\right)\left(u^m u_x^2\right) + \frac{1}{2}\partial_x\left[(u^m)_x u_x^2\right]\right]dx.\end{aligned} \tag{3.42}$$

Applying the Hölder's inequality yields that

$$\begin{aligned}\frac{1-\varepsilon}{2p+2}\frac{d}{dt}\int_R (u_x)^{2p+2}dx \le &\left\{\varepsilon\left(\int_R |u_{txxx}|^{2p+2}dx\right)^{1/(2p+2)} + \left(\int_R \left|u^{m+1}\right|^{2p+2}dx\right)^{1/(2p+2)}\right. \\ &\left. + \left(\int_R \left|u^{m+2}\right|^{2p+2}dx\right)^{1/(2p+2)} + \left(\int_R |G|^{2p+2}dx\right)^{1/(2p+2)}\right\} \\ &\times\left(\int_R |u_x|^{2p+2}dx\right)^{(2p+1)/(2p+2)} + \left|\frac{p-m}{2p+2}\right|\|u_x\|_{L^\infty}\|u\|_{L^\infty}^m\int_R |u_x|^{2p+2}dx\end{aligned} \tag{3.43}$$

or

$$\begin{aligned}(1-\varepsilon)\frac{d}{dt}\left(\int_R (u_x)^{2p+2}dx\right)^{1/(2p+2)} \le &\left\{\varepsilon\left(\int_R |u_{txxx}|^{2p+2}dx\right)^{1/(2p+2)} + \left(\int_R \left|u^{m+1}\right|^{2p+2}dx\right)^{1/(2p+2)}\right. \\ &\left. + \left(\int_R \left|u^{m+2}\right|^{2p+2}dx\right)^{1/(2p+2)} + \left(\int_R |G|^{2p+2}dx\right)^{1/(2p+2)}\right\} \\ &+ \left|\frac{p-m}{2p+2}\right|\|u_x\|_{L^\infty}\|u\|_{L^\infty}^m\left(\int_R |u_x|^{2p+2}dx\right)^{1/(2p+2)},\end{aligned} \tag{3.44}$$

where

$$G=\Lambda^{-2}\left[\varepsilon u_{tx}+\frac{k}{m+1}u^{m+1}+u^{m+2}+\left(m+\frac{1}{2}\right)\left(u^m u_x^2\right)+\frac{1}{2}\partial_x\left[(u^m)_x u_x^2\right]\right]. \tag{3.45}$$

Since $\|f\|_{L^p}\to\|f\|_{L^\infty}$ as $p\to\infty$ for any $f\in L^\infty\bigcap L^2$, integrating both sides of (3.44) with respect to t and taking the limit as $p\to\infty$ result in the estimate

$$\begin{aligned}(1-\varepsilon)\|u_x\|_{L^\infty}&\le(1-\varepsilon)\|u_{0x}\|_{L^\infty}\\&\quad+\int_0^t\left[\varepsilon\|u_{txxx}\|_{L^\infty}+c\left(\|u\|_{L^\infty}^{m+1}+\|u\|_{L^\infty}^{m+2}+\|G\|_{L^\infty}\right)+\frac{1}{2}\|u\|_{L^\infty}^m\|u_x\|_{L^\infty}^2\right]d\tau.\end{aligned} \tag{3.46}$$

Using the algebraic property of $H^{s_0}(R)$ with $s_0>1/2$ and (3.40) yields that

$$\|u\|_{L^\infty}^{m+2}\le c\|u\|_{H^1}^{m+2}\le c, \tag{3.47}$$

$$\begin{aligned}\|G\|_{L^\infty}&\le c\|G\|_{H^{1/2+}}\\&=c\left\|\Lambda^{-2}\left[\varepsilon u_{tx}+u^{m+1}+u^{m+2}+\left(m+\frac{1}{2}\right)\left(u^m u_x^2\right)+\frac{1}{2}\partial_x\left[(u^m)_x u_x^2\right]\right]\right\|_{H^{1/2+}}\\&\le c\left(\left\|\Lambda^{-2}u_{xt}\right\|_{H^{1/2+}}+\left\|\Lambda^{-2}\left(u^m u_x^2\right)\right\|_{H^{1/2+}}+\left\|\Lambda^{-2}\partial_x\left[(u^m)_x u_x^2\right]\right\|_{H^{1/2+}}\right)+c\\&\le c\left(\|u_t\|_{L^2}+\left\|u^m u_x^2\right\|_{H^0}+\left\|(u^m)_x u_x^2\right\|_{H^0}\right)+c\\&\le c\left(\|u_t\|_{L^2}+\|u\|_{L^\infty}^m\|u_x\|_{L^\infty}\|u\|_{H^1}+\|u_x\|_{L^\infty}^2\|u\|_{L^\infty}^{m-1}\|u\|_{H^1}\right)+c\\&\le c\left(\|u_t\|_{L^2}+\|u_x\|_{L^\infty}+\|u_x\|_{L^\infty}^2\right)+c\\&\le c\left(1+\|u_x\|_{L^\infty}^2\right),\end{aligned} \tag{3.48}$$

where we have used (3.16) and (3.40). Using (3.48), we have

$$\int_0^t\|G\|_{L^\infty}d\tau\le c\int_0^t\left(1+\|u_x\|_{L^\infty}^2\right)d\tau, \tag{3.49}$$

where c is a constant independent of ε. Moreover, for any fixed $r\in(1/2,1)$, there exists a constant c_r such that $\|u_{txxx}\|_{L^\infty}\le c_r\|u_{txxx}\|_{H^r}\le c_r\|u_t\|_{H^{r+3}}$. Using (3.16) and (3.40) yields that

$$\|u_{txxx}\|_{L^\infty}\le c\|u\|_{H^{r+4}}\left(1+\|u_x\|_{L^\infty}^2\right). \tag{3.50}$$

Making use of the Gronwall's inequality to (3.15) with $q=r+3$, $u=u_\varepsilon$ and (3.40) gives rise to

$$\|u\|_{H^{r+4}}^2\le\left(\int_R\left(\Lambda^{r+4}u_0\right)^2+\varepsilon\left(\Lambda^{r+3}u_{0xx}\right)^2\right)\exp\left[c\int_0^t\left(1+\|u_x\|_{L^\infty}^2\right)d\tau\right]. \tag{3.51}$$

From (3.36), (3.37), (3.50), and (3.51), one has

$$\|u_{txxx}\|_{L^\infty} \le c\varepsilon^{(s-r-4)/4}\left(1+\|u_x\|_{L^\infty}^2\right)\exp\left[c\int_0^t\left(1+\|u_x\|_{L^\infty}^2\right)d\tau\right]. \tag{3.52}$$

For $\varepsilon < 1/4$, it follows from (3.46), (3.49), and (3.52) that

$$\begin{aligned}\|u_x\|_{L^\infty} &\le \|u_{0x}\|_{L^\infty}\\ &\quad + c\int_0^t\left[\varepsilon^{(s-r)/4}\left(1+\|u_x\|_{L^\infty}^2\right)\exp\left(c\int_0^\tau\left(1+\|u_x\|_{L^\infty}^2\right)d\varsigma\right)+1+\|u_x\|_{L^\infty}^2\right]d\tau.\end{aligned} \tag{3.53}$$

It follows from the contraction mapping principle that there is a $T > 0$ such that the equation

$$\begin{aligned}\|W\|_{L^\infty} &= \|u_{0x}\|_{L^\infty}\\ &\quad + c\int_0^t\left[\left(1+\|W\|_{L^\infty}^2\right)\exp\left(c\int_0^\tau\left(1+\|W\|_{L^\infty}^2\right)d\varsigma\right)+1+\|W\|_{L^\infty}^2\right]d\tau\end{aligned} \tag{3.54}$$

has a unique solution $W \in C[0,T]$. Using the Theorem presented at page 51 in Li and Olver [29] or Theorem II in section I.1 presented in [42] yields that there are constants $T > 0$ and $c > 0$, which are independent of ε, such that $\|u_x\|_{L^\infty} \le W(t)$ for arbitrary $t \in [0,T]$, which leads to the conclusion of Lemma 3.9. □

Lemma 3.10 (Li and Olver [29]). *If u and f are functions in $H^{q+1} \cap \{\|u_x\|_{L^\infty} < \infty\}$, then*

$$\left|\int_R \Lambda^q u \Lambda^q (uf)_x dx\right| \le \begin{cases} c_q\|f\|_{H^{q+1}}\|u\|_{H^q}^2, & q \in \left(\dfrac{1}{2},1\right],\\ c_q\Big(\|f\|_{H^{q+1}}\|u\|_{H^q}\|u\|_{L_\infty} & \\ \quad +\|f_x\|_{L_\infty}\|u\|_{H^q}^2+\|f\|_{H^q}\|u\|_{H^q}\|u_x\|_{L_\infty}\Big), & q \in (0,\infty). \end{cases} \tag{3.55}$$

Lemma 3.11. *For $u, v \in H^s(R)$ with $s > 3/2$, $w = u - v$, $q > 1/2$, and a natural number n, it holds that*

$$\left|\int_R \Lambda^s w \Lambda^s\left(u^{n+1}-v^{n+1}\right)_x dx\right| \le c\left(\|w\|_{H^s}\|w\|_{H^q}\|v\|_{H^{s+1}}+\|w\|_{H^s}^2\right). \tag{3.56}$$

The proof of this Lemma can be found in [38].

Lemma 3.12. *For problem* (3.34)*,* $s > 3/2$ *and* $u_0 \in H^s(R)$*, there exist two positive constants* c *and* M*, which are independent of* ε*, such that the following inequalities hold for any sufficiently small* ε *and* $t \in [0, T)$

$$\|u_\varepsilon\|_{H^s} \le M e^{ct}, \tag{3.57}$$

$$\|u_\varepsilon\|_{H^{s+k_1}} \le \varepsilon^{-k_1/4} M e^{ct}, \quad k_1 > 0, \tag{3.58}$$

$$\|u_{\varepsilon t}\|_{H^{s+k_1}} \le \varepsilon^{-(k_1+1)/4} M e^{ct}, \quad k_1 > -1. \tag{3.59}$$

Proof. If $s > 3/2$, $u_0 \in H^s$, we obtain that

$$\begin{gathered} u_0 \in H^{s_1} \quad \text{with } 1 \le s_1 \le \frac{3}{2}, \\ \|u_{0x}\|_{L^\infty} \le c\|u_{0x}\|_{H^{1/2+}} \le c\|u_0\|_{H^s} \le c. \end{gathered} \tag{3.60}$$

From Lemma 3.9, we know that there exist two constants T and c (both independent of ε) such that

$$\|u_{\varepsilon x}\|_{L^\infty} \le c \quad \text{for any } t \in [0, T). \tag{3.61}$$

Applying the inequality (3.15) with $q + 1 = s$ and the bounded property of solution u (see (3.40) and (3.60)), we have

$$\begin{aligned} \int_R (\Lambda^s u_\varepsilon)^2 dx &\le \int_R \left[(\Lambda^s u_{\varepsilon 0})^2 + \varepsilon\left(\Lambda^{s-1} u_{\varepsilon 0xx}\right)^2 \right] dx + c\int_0^t \|u_\varepsilon\|_{H^s}^2 d\tau, \\ &= A + c\int_0^t \|u_\varepsilon\|_{H^s}^2 d\tau, \end{aligned} \tag{3.62}$$

where

$$\begin{aligned} A &= \int_R \left[(\Lambda^s u_{\varepsilon 0})^2 + \varepsilon\left(\Lambda^{s-1} u_{\varepsilon 0xx}\right)^2 \right] dx \le \|u_{\varepsilon 0}\|_{H^s}^2 + \|u_{\varepsilon 0}\|_{H^{s+1}}^2 \\ &\le c + c\varepsilon\varepsilon^{-1/2} \le 2c, \end{aligned} \tag{3.63}$$

in which we have used (3.36) and (3.37).

From (3.61) and (3.62) and using the Gronwall's inequality, we get the following:

$$\|u_\varepsilon\|_{H^s} \le 2c e^{ct}, \tag{3.64}$$

from which we know that (3.57) holds.

In a similar manner, for $q + 1 = s + k_1$ and $k_1 > 0$, applying (3.40) and (3.60) to (3.15), we have

$$\|u_\varepsilon\|_{H^{s+k_1}}^2 \le \left(c\varepsilon^{-k_1/2} + c\varepsilon^{-(k_1+1)/2}\varepsilon \right) + c\int_0^t \|u_\varepsilon\|_{H^{s+k_1}}^2 d\tau, \tag{3.65}$$

which results in (3.58) by using Gronwall's inequality.

From (3.16), for $q = s + k_1$, we have

$$(1-2\varepsilon)\|u_{\varepsilon t}\|_{H^{s+k_1}} \le c\|u_\varepsilon\|_{H^{s+k_1+1}}, \tag{3.66}$$

which leads to (3.59) by (3.58). □

Lemma 3.13. *If $1/2 < q < \min\{1, s-1\}$ and $s > 3/2$, then for any functions w, f defined on R, it holds that*

$$\left|\int_R \Lambda^q w \Lambda^{q-2}(wf)_x dx\right| \le c\|w\|_{H^q}^2 \|f\|_{H^q}, \tag{3.67}$$

$$\left|\int_R \Lambda^q w \Lambda^{q-2}(w_x f_x)_x dx\right| \le c\|w\|_{H^q}^2 \|f\|_{H^s}. \tag{3.68}$$

The proof of this lemma can be found in [38].

Our next step is to demonstrate that u_ε is a Cauchy sequence. Let u_ε and u_δ be solutions of problem (3.34), corresponding to the parameters ε and δ, respectively, with $0 < \varepsilon < \delta < 1/4$, and let $w = u_\varepsilon - u_\delta$. Then w satisfies the problem

$$\begin{aligned}
&(1-\varepsilon)w_t - \varepsilon w_{xxt} + (\delta-\varepsilon)(u_{\delta t} + u_{\delta xxt}) \\
&\quad = \left(1-\partial_x^2\right)^{-1}\Bigg[-\varepsilon w_t + (\delta-\varepsilon)u_{\delta t} - \frac{k}{m+1}\partial_x\left(u_\varepsilon^{m+1} - u_\delta^{m+1}\right) - \partial_x\left(u_\varepsilon^{m+2} - u_\delta^{m+2}\right) \\
&\qquad - \partial_x\left[\partial_x\left(u_\varepsilon^{m+1}\right)\partial_x w + \partial_x\left(u_\varepsilon^{m+1} - u_\varepsilon^{m+1}\right)\partial_x u_\delta\right] \\
&\qquad + \left[u_\varepsilon^m u_{\varepsilon x} u_{\varepsilon xx} - u_\delta^m u_{\delta x} u_{\delta xx}\right]\Bigg] - \frac{1}{m+2}\partial_x\left(u_\varepsilon^{m+2} - u_\delta^{m+2}\right),
\end{aligned} \tag{3.69}$$

$$w(x,0) = w_0(x) = u_{\varepsilon 0}(x) - u_{\delta 0}(x). \tag{3.70}$$

Lemma 3.14. *For $s > 3/2$, $u_0 \in H^s(R)$, there exists $T > 0$ such that the solution u_ε of (3.34) is a Cauchy sequence in $C([0,T]; H^s(R)) \bigcap C^1([0,T]; H^{s-1}(R))$.*

Proof. For q with $1/2 < q < \min\{1, s-1\}$, multiplying both sides of (3.69) by $\Lambda^q w \Lambda^q$ and then integrating with respect to x give rise to

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_R \left[(1-\varepsilon)(\Lambda^q w)^2 + \varepsilon(\Lambda^q w_x)^2\right]dx \\
&\quad = (\varepsilon-\delta)\int_R (\Lambda^q w)[(\Lambda^q u_{\delta t}) + (\Lambda^q u_{\delta xxt})]dx \\
&\qquad - \varepsilon\int_R \Lambda^q w \Lambda^{q-2} w_t dx + (\delta-\varepsilon)\int_R \Lambda^q w \Lambda^{q-2} u_{\delta t} dx \\
&\qquad - \frac{1}{m+2}\int_R (\Lambda^q w)\Lambda^q\left(u_\varepsilon^{m+2} - u_\delta^{m+2}\right)_x dx \\
&\qquad - \frac{k}{m+1}\int_R \Lambda^q w \Lambda^{q-2}\left(u_\varepsilon^{m+1} - u_\delta^{m+1}\right)_x dx
\end{aligned}$$

$$
\begin{aligned}
&-\int_R \Lambda^q w \Lambda^{q-2}\left(u_\varepsilon^{m+2}-u_\delta^{m+2}\right)_x dx \\
&-\int_R \Lambda^q w \Lambda^{q-2}\left[\partial_x\left(u_\varepsilon^{m+1}\right)\partial_x w\right]_x dx \\
&-\int_R \Lambda^q w \Lambda^{q-2}\left[\partial_x\left(u_\varepsilon^{m+1}-u_\delta^{m+1}\right)\partial_x u_\delta\right]_x dx \\
&+\int_R \Lambda^q w \Lambda^{q-2}\left[u_\varepsilon^m u_{\varepsilon x}u_{\varepsilon xx}-u_\delta^{\mathrm{m}} u_{\delta x}u_{\delta xx}\right]dx.
\end{aligned}
\tag{3.71}
$$

It follows from the Schwarz inequality that

$$
\begin{aligned}
&\frac{d}{dt}\int\left[(1-\varepsilon)(\Lambda^q w)^2+\varepsilon(\Lambda^q w_x)^2\right]dx \\
&\quad\le c\Big\{\|\Lambda^q w\|_{L^2}\Big[(\delta-\varepsilon)(\|\Lambda^q u_{\delta t}\|_{L^2}+\|\Lambda^q u_{\delta xxt}\|_{L^2}) \\
&\qquad\qquad +\varepsilon\left\|\Lambda^{q-2}w_t\right\|_{L^2}+(\delta-\varepsilon)\left\|\Lambda^{q-2}u_{\delta t}\right\|_{L^2}\Big] \\
&\qquad +\left|\int_R \Lambda^q w \Lambda^q\left(u_\varepsilon^{m+2}-u_\delta^{m+2}\right)_x dx\right|\left|\int \Lambda^q w \Lambda^{q-2}\left(u_\varepsilon^{m+1}-u_\delta^{m+1}\right)_x dx\right| \\
&\qquad +\left|\int \Lambda^q w \Lambda^{q-2}\left(u_\varepsilon^{m+2}-u_\delta^{m+2}\right)_x dx\right|+\left|\int_R \Lambda^q w \Lambda^{q-2}\left[\partial_x\left(u_\varepsilon^{m+1}\right)\partial_x w\right]_x dx\right| \\
&\qquad +\left|\int_R \Lambda^q w \Lambda^{q-2}\left[\partial_x\left(u_\varepsilon^{m+1}-u_\delta^{m+1}\right)\partial_x u_\delta\right]_x dx\right| \\
&\qquad +\left|\int_R \Lambda^q w \Lambda^{q-2}\left[u_\varepsilon^m u_{\varepsilon x}u_{\varepsilon xx}-u_\delta^m u_{\delta x}u_{\delta xx}\right]dx\right|\Big\}.
\end{aligned}
\tag{3.72}
$$

Using the first inequality in Lemma 3.10, we have

$$
\begin{aligned}
\left|\int_R \Lambda^q w \Lambda^q\left(u_\varepsilon^{m+2}-u_\delta^{m+2}\right)_x dx\right| &= \left|\int_R \Lambda^q w \Lambda^q (w g_{m+1})_x dx\right| \\
&\le c\|w\|_{H^q}^2\|g_{m+1}\|_{H^{q+1}},
\end{aligned}
\tag{3.73}
$$

where $g_{m+1}=\sum_{j=0}^{m+1} u_\varepsilon^{m+1-j}u_\delta^j$. For the last three terms in (3.72), using Lemmas 3.1 and 3.13, $1/2<q<\min\{1,s-1\}$, $s>3/2$, the algebra property of H^{s_0} with $s_0>1/2$ and (3.40), we have

$$
\left|\int_R \Lambda^q w \Lambda^{q-2}\left(\partial_x\left(u_\varepsilon^{m+1}\right)\partial_x w\right)_x dx\right| \le c\|w\|_{H^q}^2\|u_\varepsilon\|_{H^s}^{m+1},
\tag{3.74}
$$

$$
\begin{aligned}
&\left|\int_R \Lambda^q w \Lambda^{q-2}\left(\partial_x\left(u_\varepsilon^{m+1}-u_\delta^{m+1}\right)\partial_x u_\delta\right)_x dx\right| \\
&\quad\le c\|w\|_{H^q}\|u_\delta\|_{H^s}\left\|u_\varepsilon^{m+1}-u_\delta^{m+1}\right\|_{H^q} \\
&\quad\le c\|w\|_{H^q}^2\|u_\delta\|_{H^s},
\end{aligned}
$$

$$\begin{aligned}
&\left|\int_R \Lambda^q w \Lambda^{q-2}\left[u_\varepsilon^m u_{\varepsilon x} u_{\varepsilon xx} - u_\delta^m u_{\delta x} u_{\delta xx}\right] dx\right| \\
&\quad \leq c\|w\|_{H^q} \left\| \left(u_\varepsilon^m - u_\delta^m\right)\left(u_{\varepsilon x}^2\right)_x + u_\delta^m \left[u_{\varepsilon x}^2 - u_{\delta x}^2\right]_x \right\|_{H^{q-2}} \\
&\quad \leq c\|w\|_{H^q} \left(\left\| \left(u_\varepsilon^m - u_\delta^m\right)\left(u_{\varepsilon x}^2\right)_x \right\|_{H^{q-1}} + \left\| u_\delta^m \left[u_{\varepsilon x}^2 - u_{\delta x}^2\right]_x \right\|_{H^{q-2}} \right) \\
&\quad \leq c\|w\|_{H^q} \left(\left\| u_\varepsilon^m - u_\delta^m \right\|_{H^q} \left\| \left(u_{\varepsilon x}^2\right)_x \right\|_{H^{q-1}} + \left\| u_\delta^m \right\|_{H^s} \left\| \left[u_{\varepsilon x}^2 - u_{\delta x}^2\right]_x \right\|_{H^{q-2}} \right) \\
&\quad \leq c\|w\|_{H^q} \left(\|w\|_{H^q} \left\| g_{m-1} \right\|_{H^q} \|u\|_{H^s}^2 + \left\| u_\delta^m \right\|_{H^s} \|u_{\varepsilon x} + u_{\delta x}\|_{H^q} \|w\|_{H^q} \right) \\
&\quad \leq c\|w\|_{H^q}^2 \left(\left\| g_{m-1} \right\|_{H^q} \|u\|_{H^s}^2 + \left\| u_\delta^m \right\|_{H^s} \|u_{\varepsilon x} + u_{\delta x}\|_{H^q} \right).
\end{aligned} \tag{3.75}$$

Using (3.67), we derives that the inequality

$$\begin{aligned}
\left|\int_R \Lambda^q w \Lambda^{q-2}\left(u_\varepsilon^{m+2} - u_\delta^{m+2}\right)_x dx\right| &= \left|\int_R \Lambda^q w \Lambda^{q-2}\left(w g_{m+1}\right)_x dx\right| \\
&\leq c\left\| g_{m+1} \right\|_{H^q} \|w\|_{H^q}^2
\end{aligned} \tag{3.76}$$

holds for some constant c, where $g_{m+1} = \sum_{j=0}^{m+1} u_\varepsilon^{m+1-j} u_\delta^j$. Using the algebra property of H^q with $q > 1/2$, $q+1 < s$ and Lemma 3.12, we have $\|g_m\|_{H^{q+1}} \leq c$ for $t \in (0, \widetilde{T}]$. Then it follows from (3.57)–(3.59) and (3.73)–(3.76) that there is a constant c depending on $\widetilde{T}$ such that the estimate

$$\frac{d}{dt}\int_R \left[(1-\varepsilon)(\Lambda^q w)^2 + \varepsilon(\Lambda^q w_x)^2\right] dx \leq c\left(\delta^\gamma \|w\|_{H^q} + \|w\|_{H^q}^2\right) \tag{3.77}$$

holds for any $t \in [0, \widetilde{T})$, where $\gamma = 1$ if $s \geq 3+q$ and $\gamma = (1+s-q)/4$ if $s < 3+q$. Integrating (3.77) with respect to t, one obtains the estimate

$$\begin{aligned}
\frac{1}{2}\|w\|_{H^q}^2 &= \frac{1}{2}\int_R (\Lambda^q w)^2 dx \\
&\leq \int_R \left[(1-\varepsilon)(\Lambda^q w)^2 + \varepsilon(\Lambda^q w)^2\right] dx \\
&\leq \int_R \left[(\Lambda^q w_0)^2 + \varepsilon(\Lambda^q w_{0x})^2\right] dx + c\int_0^t \left(\delta^\gamma \|w\|_{H^q} + \|w\|_{H^q}^2\right) d\tau.
\end{aligned} \tag{3.78}$$

Applying the Gronswall inequality, (3.37) and (3.39) yields that

$$\|u\|_{H^q} \leq c\delta^{(s-q)/4} e^{ct} + \delta^\gamma \left(e^{ct} - 1\right), \tag{3.79}$$

for any $t \in [0, \widetilde{T})$.

Multiplying both sides of (3.69) by $\Lambda^s w \Lambda^s$ and integrating the resultant equation with respect to x, one obtains that

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_R \left[(1-\varepsilon)(\Lambda^s w)^2 + \varepsilon(\Lambda^s w_x)^2\right]dx \\
&\quad = (\varepsilon-\delta)\int_R (\Lambda^s w)[(\Lambda^s u_{\delta t}) + (\Lambda^s u_{\delta xxt})]dx \\
&\quad\quad - \varepsilon\int_R \Lambda^s w \Lambda^{s-2} w_t dx + (\delta-\varepsilon)\int_R \Lambda^s w \Lambda^{s-2} u_{\delta t} dx \\
&\quad\quad - \frac{k}{m+1}\int_R (\Lambda^s w)\Lambda^s\left(u_\varepsilon^{m+1} - u_\delta^{m+1}\right)_x dx \\
&\quad\quad - \frac{1}{m+2}\int_R (\Lambda^s w)\Lambda^s\left(u_\varepsilon^{m+2} - u_\delta^{m+2}\right)_x dx \\
&\quad\quad - \int_R \Lambda^s w \Lambda^{s-2}\left(u_\varepsilon^{m+2} - u_\delta^{m+2}\right)_x dx \\
&\quad\quad - \int_R \Lambda^s w \Lambda^{s-2}\left[\partial_x\left(u_\varepsilon^{m+1}\right)\partial_x w\right]_x dx \\
&\quad\quad - \int_R \Lambda^s w \Lambda^{s-2}\left[\partial_x\left(u_\varepsilon^{m+1} - u_\delta^{m+1}\right)\partial_x u_\delta\right]_x dx \\
&\quad\quad + \int_R \Lambda^s w \Lambda^{s-2}\left[u_\varepsilon^m u_{\varepsilon x} u_{\varepsilon xx} - u_\delta^m u_{\delta x} u_{\delta xx}\right]dx.
\end{aligned} \tag{3.80}$$

From Lemma 3.13, we have

$$\left|\int_R \Lambda^s w \Lambda^{s-2}\left(u_\varepsilon^{m+2} - u_\delta^{m+2}\right)_x dx\right| \le c_3 \left\|g_{m+1}\right\|_{H^s} \|w\|_{H^s}^2. \tag{3.81}$$

From Lemma 3.11, it holds that

$$\left|\int_R \Lambda^s w \Lambda^s\left(u_\varepsilon^{m+2} - u_\delta^{m+2}\right)_x dx\right| \le c\left(\|w\|_{H^s}\|w\|_{H^q}\|u_\delta\|_{H^{s+1}} + \|w\|_{H^s}^2\right). \tag{3.82}$$

Using the Cauchy-Schwartz inequality and the algebra property of H^{s_0} with $s_0 > 1/2$, for $s > 3/2$, we have

$$\begin{aligned}
\left|\int_R \Lambda^s w \Lambda^{s-2}\left[\partial_x\left(u_\varepsilon^{m+1}\right)\partial_x w\right]_x dx\right| &= \left|\int_R \Lambda^q w \Lambda^{s-2}\left[\partial_x\left(u_\varepsilon^{m+1}\right)\partial_x w\right]_x dx\right| \\
&\le c\|\Lambda^s w\|_{L^2}\left\|\Lambda^{s-2}\left[\partial_x\left(u_\varepsilon^{m+1}\right)\partial_x w\right]_x\right\|_{L^2} \\
&\le c\|w\|_{H^q}\left\|\partial_x\left(u_\varepsilon^{m+1}\right)\partial_x w\right\|_{H^{s-1}} \\
&\le c\|u_\varepsilon\|_{H^s}^{m+1}\|w\|_{H^s}^2 \\
\left|\int_R \Lambda^s w \Lambda^{s-2}\left[\partial_x\left(u_\varepsilon^{m+1} - u_\delta^{m+1}\right)\partial_x u_\delta\right]_x dx\right| &\le c\|w\|_{H^s}\left\|\Lambda^{s-2}\left[\partial_x\left(u_\varepsilon^{m+1} - u_\delta^{m+1}\right)\partial_x u_\delta\right]_x\right\|_{L^2} \\
&\le c\|u_\delta\|_{H^s}\left\|g_m\right\|_{H^s}\|w\|_{H^s}^2,
\end{aligned} \tag{3.83}$$

$$
\begin{aligned}
&\left|\int_R \Lambda^s w \Lambda^{s-2}\left[u_\varepsilon^m u_{\varepsilon x} u_{\varepsilon xx} - u_\delta^m u_{\delta x} u_{\delta xx}\right] dx\right| \\
&\quad \le c\|w\|_{H^s}\left(\left\|\left(u_\varepsilon^m - u_\delta^m\right)\left(u_{\varepsilon x}^2\right)_x\right\|_{H^{s-2}} + \left\|u_\delta^m\left[u_{\varepsilon x}^2 - u_{\delta x}^2\right]_x\right\|_{H^{s-2}}\right) \\
&\quad \le c\|w\|_{H^s}\left(\left\|\left(u_\varepsilon^m - u_\delta^m\right)\right\|_{H^s}\left\|\left(u_{\varepsilon x}^2\right)_x\right\|_{H^{s-2}} + \left\|u_\delta^m\right\|_{H^s}\left\|\left[u_{\varepsilon x}^2 - u_{\delta x}^2\right]_x\right\|_{H^{s-2}}\right) \\
&\quad \le c\|w\|_{H^s}^2,
\end{aligned}
\tag{3.84}
$$

in which we have used Lemma 3.1 and the bounded property of $\|u_\varepsilon\|_{H^s}$ and $\|u_\delta\|_{H^s}$ (see Lemma 3.12). It follows from (3.80)–(3.84) and (3.57)–(3.59) and (3.79) that there exists a constant c depending on m such that

$$
\begin{aligned}
&\frac{d}{dt}\int_R\left[(1-\varepsilon)(\Lambda^s w)^2 + \varepsilon(\Lambda^s w_x)^2\right]dx \\
&\quad \le 2\delta\left(\|u_{\delta t}\|_{H^s} + \|u_{\delta xxt}\|_{H^s} + \left\|\Lambda^{s-2} w_t\right\|_{L^2} + \left\|\Lambda^{s-2} u_{\delta t}\right\|\right)\|w\|_{H^s} \\
&\qquad + c\left(\|w\|_{H^s}^2 + \|w\|_{H^q}\|w\|_{H^s}\|u_\delta\|_{H^{s+1}}\right) \\
&\quad \le c\left(\delta^{\gamma_1}\|w\|_{H^s} + \|w\|_{H^s}^2\right),
\end{aligned}
\tag{3.85}
$$

where $\gamma_1 = \min(1/4, (s-q-1)/4) > 0$. Integrating (3.85) with respect to t leads to the estimate

$$
\begin{aligned}
\frac{1}{2}\|w\|_{H^s}^2 &\le \int_R\left[(1-\varepsilon)(\Lambda^s w)^2 + \varepsilon(\Lambda^s w_x)^2\right]dx \\
&\le \int_R\left[(\Lambda^s w_0)^2 + \varepsilon(\Lambda^s w_{0x})^2\right]dx + c\int_0^t\left(\delta^{\gamma_1}\|w\|_{H^s} + \|w\|_{H^s}^2\right)d\tau.
\end{aligned}
\tag{3.86}
$$

It follows from the Gronwall inequality and (3.86) that

$$
\begin{aligned}
\|w\|_{H^s} &\le \left(2\int_R\left[(\Lambda^s w_0)^2 + \varepsilon(\Lambda^s w_{0x})^2\right]dx\right)^{1/2} e^{ct} + \delta^{\gamma_1}\left(e^{ct}-1\right) \\
&\le c_1\left(\|w_0\|_{H^s} + \delta^{3/4}\right)e^{ct} + \delta^{\gamma_1}\left(e^{ct}-1\right),
\end{aligned}
\tag{3.87}
$$

where c_1 is independent of ε and δ.

Then (3.39) and the above inequality show that

$$
\|w\|_{H^s} \longrightarrow 0 \quad \text{as } \varepsilon \longrightarrow 0,\ \delta \rightarrow 0. \tag{3.88}
$$

Next, we consider the convergence of the sequence $\{u_{\varepsilon t}\}$. Multiplying both sides of (3.69) by $\Lambda^{s-1}w_t\Lambda^{s-1}$ and integrating the resultant equation with respect to x, we obtain

$$\begin{aligned}
&(1-\varepsilon)\|w_t\|^2_{H^{s-1}} + \frac{1}{m+2}\int_R \left(\Lambda^{s-1}w_t\right)\Lambda^{s-1}\left(u_\varepsilon^{m+2}-u_\delta^{m+2}\right)_x dx \\
&\quad + \int_R \left[-\varepsilon\left(\Lambda^{s-1}w_t\right)\left(\Lambda^{s-1}w_{xxt}\right) + (\delta-\varepsilon)\left(\Lambda^{s-1}w_t\right)\Lambda^{s-1}(u_{\delta t}+u_{\delta xxt})\right]dx \\
&\qquad = \int_R \left(\Lambda^{s-1}w_t\right)\Lambda^{s-3}\Bigg[-\varepsilon w_t + (\delta-\varepsilon)u_{\delta t} - \frac{k}{m+1}\partial_x\left(u_\varepsilon^{m+1}-u_\delta^{m+1}\right) - \partial_x\left(u_\varepsilon^{m+2}-u_\delta^{m+2}\right) \\
&\qquad\qquad - \partial_x\left[\partial_x\left(u_\varepsilon^{m+1}\right)\partial_x w + \partial_x\left(u_\varepsilon^{m+1}-u_\varepsilon^{m+1}\right)\partial_x u_\delta\right] \\
&\qquad\qquad + \left[u_\varepsilon^m u_{\varepsilon x}u_{\varepsilon xx} - u_\delta^m u_{\delta x}u_{\delta xx}\right]\Bigg]dx.
\end{aligned} \tag{3.89}$$

It follows from (3.57)–(3.60) and the Schwartz inequality that there is a constant c depending on $\widetilde{T}$ and m such that

$$(1-\varepsilon)\|w_t\|^2_{H^{s-1}} \le c\left(\delta^{1/2} + \|w\|_{H^s} + \|w\|_{s-1}\right)\|w_t\|_{H^{s-1}} + \varepsilon\|w_t\|^2_{H^{s-1}}. \tag{3.90}$$

Hence,

$$\begin{aligned}
\frac{1}{2}\|w_t\|^2_{H^{s-1}} &\le (1-2\varepsilon)\|w_t\|^2_{H^{s-1}} \\
&\le c\left(\delta^{1/2} + \|w\|_{H^s} + \|w\|_{H^{s-1}}\right)\|w_t\|_{H^{s-1}},
\end{aligned} \tag{3.91}$$

which results in

$$\frac{1}{2}\|w_t\|_{H^{s-1}} \le c\left(\delta^{1/2} + \|w\|_{H^s} + \|w\|_{H^{s-1}}\right). \tag{3.92}$$

It follows from (3.79) and (3.88) that $w_t \to 0$ as $\varepsilon, \delta \to 0$ in the H^{s-1} norm. This implies that u_ε is a Cauchy sequence in the spaces $C([0,T); H^s(R))$ and $C([0,T); H^{s-1}(R))$, respectively. The proof is completed. □

Proof of Theorem 2.1. We consider the problem

$$\begin{aligned}
(1-\varepsilon)u_t - \varepsilon u_{txx} &= \left(1-\partial_x^2\right)^{-1}\Bigg[-\frac{k}{m+1}\left(u^{m+1}\right)_x - \frac{m+3}{m+2}\left(u^{m+2}\right)_x + \frac{1}{m+2}\partial_x^3\left(u^{m+2}\right) \\
&\qquad -(m+1)\partial_x\left(u^m u_x^2\right) + u^m u_x u_{xx}\Bigg], \\
u(0,x) &= u_{\varepsilon 0}(x).
\end{aligned} \tag{3.93}$$

Letting $u(t,x)$ be the limit of the sequence u_ε and taking the limit in problem (3.93) as $\varepsilon \to 0$, from Lemma 3.14, it is easy to see that u is a solution of the problem

$$\begin{aligned} u_t &= \left(1-\partial_x^2\right)^{-1}\left[-\frac{k}{m+1}\left(u^{m+1}\right)_x - \frac{m+3}{m+2}\left(u^{m+2}\right)_x + \frac{1}{m+2}\partial_x^3\left(u^{m+2}\right)\right. \\ &\qquad \left. -(m+1)\partial_x\left(u^m u_x^2\right) + u^m u_x u_{xx}\right], \\ u(0,x) &= u_0(x), \end{aligned} \tag{3.94}$$

and hence u is a solution of problem (3.94) in the sense of distribution. In particular, if $s \geq 4$, u is also a classical solution. Let u and v be two solutions of (3.94) corresponding to the same initial data u_0 such that $u, v \in C([0,T); H^s(R))$. Then $w = u - v$ satisfies the Cauchy problem

$$\begin{aligned} w_t &= \left(1-\partial_x^2\right)^{-1}\left\{\partial_x\left[-\frac{k}{m+1}wg_m - \frac{m+3}{m+2}wg_{m+1} + \frac{1}{m+2}\partial_x^2\left(wg_{m+1}\right)\right.\right. \\ &\qquad \left. -\partial_x\left(u^{m+1}\right)\partial_x w - \partial_x\left(u^{m+1} - v^{m+1}\right)\partial_x v\right] \\ &\qquad \left. + u^m u_x u_{xx} - v^m v_x v_{xx}\right\}, \\ w(0,x) &= 0. \end{aligned} \tag{3.95}$$

For any $1/2 < q < \min\{1, s-1\}$, applying the operator $\Lambda^q w \Lambda^q$ to both sides of equation (3.95) and integrating the resultant equation with respect to x, we obtain the equality

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\|w\|_{H^q}^2 &= \int_R \left(\Lambda^q w\right)\Lambda^{q-2}\left\{\partial_x\left[-\frac{k}{m+1}wg_m - \frac{m+3}{m+2}wg_{m+1} + \frac{1}{m+2}\partial_x^2\left(wg_{m+1}\right)\right.\right. \\ &\qquad \left. -\partial_x\left(u^{m+1}\right)\partial_x w - \partial_x\left(u^{m+1} - v^{m+1}\right)\partial_x v\right] \\ &\qquad \left. + u^m u_x u_{xx} - v^m v_x v_{xx}\right\} dx. \end{aligned} \tag{3.96}$$

By the similar estimates presented in Lemma 3.14, we have

$$\frac{d}{dt}\|w\|_{H^q}^2 \leq \tilde{c}\|w\|_{H^q}^2. \tag{3.97}$$

Using the Gronwall inequality leads to the conclusion that

$$\|w\|_{H^q} \leq 0 \times e^{\tilde{c}t} = 0 \tag{3.98}$$

for $t \in [0, \widetilde{T})$. This completes the proof. □

4. Global Existence of Strong Solutions

We study the differential equation

$$p_t = u^{m+1}(t,p), \quad t \in [0,T),$$
$$p(0,x) = x. \tag{4.1}$$

Motivated by the Lagrangian viewpoint in fluid mechanics, by which one looks at the motion of individual fluid particles (see [43]), we state the following Lemma.

Lemma 4.1. *Let $u_0 \in H^s$, $s \geq 3$ and let $T > 0$ be the maximal existence time of the solution to problem* (2.2)*. Then problem* (4.1) *has a unique solution $p \in C^1([0,T)\times R)$. Moreover, the map $p(t,.)$ is an increasing diffeomorphism of R with $p_x(t,x) > 0$ for $(t,x) \in [0,T)\times R$.*

Proof. From Theorem 2.1, we have $u(t,x) \in C([0,T);H^s(R))\bigcap C^1([0,T);H^{s-1}(R))$ and $H^s(R) \in C^1(R)$, where the Sobolev imbedding theorem is used. Thus, we conclude that both functions $u(t,x)$ and $u_x(t,x)$ are bounded, Lipschitz in space and C^1 in time. Using the existence and uniqueness theorem of ordinary differential equations derives that problem (4.1) has a unique solution $p \in C^1([0,T)\times R)$.

Differentiating (4.1) with respect to x yields that

$$\frac{d}{dt}p_x = (m+1)u^m u_x(t,p)p_x, \quad t \in [0,T),\ b \neq 0,$$
$$p_x(0,x) = 1, \tag{4.2}$$

which leads to

$$p_x(t,x) = \exp\left(\int_0^t (m+1)u^m u_x(\tau,p(\tau,x))d\tau\right). \tag{4.3}$$

For every $T' < T$, using the Sobolev imbedding theorem yields that

$$\sup_{(\tau,x)\in[0,T')\times R} |u_x(\tau,x)| < \infty. \tag{4.4}$$

It is inferred that there exists a constant $K_0 > 0$ such that $p_x(t,x) \geq e^{-K_0 t}$ for $(t,x) \in [0,T)\times R$. It completes the proof. □

The next Lemma is reminiscent of a strong invariance property of the Camassa-Holm equation (the conservation of momentum [44, 45]).

Lemma 4.2. *Let $u_0 \in H^s$ with $s \geq 3$, and let $T > 0$ be the maximal existence time of the problem* (2.2)*, it holds that*

$$y(t,p(t,x))p_x^2(t,x) = y_0(x)e^{\int_0^t mu^m u_x d\tau}, \tag{4.5}$$

where $(t,x) \in [0,T)\times R$ and $y := u - u_{xx} + k/2(m+1)$.

Proof. We have

$$\begin{aligned}
\frac{d}{dt}\left[y(t,p(t,x))p_x^2(t,x)\right] &= y_t p_x^2 + 2y p_x p_{xt} + y_x p_t p_x^2 \\
&= y_t p_x^2 + 2y(m+1)u^m u_x p_x^2 + u^{m+1} y_x p_x^2 \\
&= \left[y_t + k u^m u_x + (m+2)u^m u_x y + y_x u^{m+1}\right] p_x^2 + m u^m u_x y p_x^2 \\
&= \left[u_t - u_{txx} + k u^m u_x + (m+2)u^m u_x (u - u_{xx}) + u^{m+1}(u_x - u_{xxx})\right] p_x^2 \\
&\quad + m u^m u_x y p_x^2 \\
&= m u^m u_x y p_x^2.
\end{aligned} \tag{4.6}$$

Using $p_x(0,x) = 1$ and solving the above equation, we complete the proof of this lemma. □

Lemma 4.3. *If* $u_0 \in H^s$, $s \geq 3/2$, *such that* $(1-\partial_x^2)u_0 + k/2(m+1) \geq 0$, *then the solution of problem* (2.2) *satisfies the following:*

$$\|u_x\|_{L^\infty} \leq \|u\|_{L^\infty} + \frac{k}{2(m+1)} \leq c. \tag{4.7}$$

Proof. Using $u_0 - u_{0xx} + k/2(m+1) \geq 0$, it follows from Lemma 4.2 that $u - u_{xx} + k/2(m+1) \geq 0$. Letting $Y_1 = u - u_{xx}$, we have

$$u = \frac{1}{2} e^{-x} \int_{-\infty}^{x} e^{\eta} Y_1(t,\eta)\,d\eta + \frac{1}{2} e^{x} \int_{x}^{\infty} e^{-\eta} Y_1(t,\eta)\,d\eta, \tag{4.8}$$

from which we obtain that

$$\begin{aligned}
\partial_x u(t,x) &= -\frac{1}{2}\left(e^{-x}\int_{-\infty}^{x} e^{\eta} Y_1(t,\eta)\,d\eta + e^{x}\int_{x}^{\infty} e^{-\eta} Y_1(t,\eta)\,d\eta\right) + e^{x}\int_{x}^{\infty} e^{-\eta} Y_1(t,\eta)\,d\eta \\
&= -u(t,x) + e^{x}\int_{x}^{\infty} e^{-\eta} Y_1(t,\eta)\,d\eta \\
&= -u(t,x) + e^{x}\int_{x}^{\infty} e^{-\eta}\left(Y_1(t,\eta) + \frac{k}{2(m+1)}\right)d\eta - \frac{k}{2(m+1)} e^{x}\int_{x}^{\infty} e^{-\eta}\,d\eta \\
&= -u(t,x) + e^{x}\int_{x}^{\infty} e^{-\eta}(y(t,\eta))\,d\eta - \frac{k}{2(m+1)} \\
&\geq -u(t,x) - \frac{k}{2(m+1)}.
\end{aligned} \tag{4.9}$$

On the other hand, we have

$$\begin{aligned}
\partial_x u(t,x) &= \frac{1}{2}\left(e^{-x}\int_{-\infty}^{x} e^{\eta} Y_1(t,\eta)\,d\eta + e^{x}\int_{x}^{\infty} e^{-\eta} Y_1(t,\eta)\,d\eta\right) - e^{-x}\int_{-\infty}^{x} e^{\eta} Y_1(t,\eta)\,d\eta \\
&= u(t,x) - e^{-x}\int_{-\infty}^{x} e^{\eta} Y_1(t,\eta)\,d\eta
\end{aligned}$$

$$
\begin{aligned}
&= u(t,x) - e^{-x}\int_{-\infty}^{x} e^{\eta}\left(Y_1(t,\eta) + \frac{k}{2(m+1)}\right)d\eta + \frac{k}{2(m+1)}e^{-x}\int_{-\infty}^{x} e^{\eta}d\eta \\
&= u(t,x) - e^{-x}\int_{-\infty}^{x} e^{\eta}y(t,\eta)d\eta + \frac{k}{2(m+1)} \\
&\le u(t,x) + \frac{k}{2(m+1)}.
\end{aligned}
\tag{4.10}
$$

The inequalities (3.40), (4.9), and (4.10) derive that (4.7) is valid. □

Proof of Theorem 2.2. Noting Remarks 3.6 and 3.8, $\|u\|_{H^1} \le c$ and taking $q+1=s$ in inequality (3.15), we have

$$
\|u\|_{H^s}^2 \le \|u_0\|_{H^s}^2 + c\int_0^t \|u\|_{H^s}^2\left(\|u_x\|_{L^\infty} + \|u_x\|_{L^\infty}^2\right)d\tau, \tag{4.11}
$$

from which we obtain that

$$
\|u\|_{H^s} \le \|u_0\|_{H^s} e^{c\int_0^t(\|u_x\|_{L^\infty}+\|u_x\|_{L^\infty}^2)d\tau}. \tag{4.12}
$$

Applying Lemma 4.3 derives

$$
\|u\|_{H^s} \le \|u_0\|_{H^s} e^{(c+c^2)t}. \tag{4.13}
$$

From Theorem 2.1 and (4.13), we know that the result of Theorem 2.2 holds. □

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