

## Research Article

# A Hamilton-Poisson Model of the Chen-Lee System

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We will present some dynamical and geometrical properties of Chen-Lee system from the Poisson geometry point of view.

## 1. Introduction

Let us consider the Chen-Lee's system (see [1, 2]) given by the following differential equations system on  $\mathbb{R}^3$ :

$$\begin{aligned}\dot{x} &= ax - yz, \\ \dot{y} &= -by + xz, \\ \dot{z} &= -cz + \frac{xy}{3},\end{aligned}\tag{1.1}$$

where  $a, b, c$  are real parameters.

In this paper we consider a special case of the Chen-Lee system, realizing this system as a Hamiltonian system and then study it from the mechanical geometry point of view. This means the study of the nonlinear stability, the existence of periodic solutions, and numerical integration. The paper is structured as follows: Section 2 presents the special case of Chen-Lee system for which it admits a Hamilton-Poisson structure; in Section 3 we study the nonlinear stability of the equilibrium states of our dynamics using energy-Casimir method. Periodical orbits are the subject of Section 4. In Section 5 of the paper we give a Lax formulation of the system; Section 6 discusses numerical integration of the system using Poisson and non-Poisson integrators. Numerical simulations using MATHEMATICA 8.0 are presented, too.

For details on Poisson geometry and Hamiltonian dynamics, (see [3–5]).

## 2. The Poisson Geometry Associated to the Chen-Lee's System

In this section we will find the parameters values for which Chen-Lee's system admits a Poisson structure. In order to do this, we need to find the system's Hamiltonians. Due to the existence of a numerous parameters, we are looking for polynomial Hamiltonians.

**Proposition 2.1.** *The following smooth real functions  $H$  are two-degree polynomial constants of the motion defined by the system (1.1).*

(i) *If  $a \in \mathbb{R}^*$ ,  $b = c = 0$ , then the function*

$$H(x, y, z) = \alpha(y^2 - 3z^2) + \beta, \quad \alpha, \beta \in \mathbb{R} \quad (2.1)$$

*is the Hamiltonian of the system (1.1).*

(ii) *If  $b \in \mathbb{R}^*$ ,  $a = c = 0$ , then the function*

$$H(x, y, z) = \alpha(x^2 + 3z^2) + \beta, \quad \alpha, \beta \in \mathbb{R} \quad (2.2)$$

*is the Hamiltonian of the system (1.1).*

(iii) *If  $c \in \mathbb{R}^*$ ,  $a = b = 0$ , then the function*

$$H(x, y, z) = \alpha(x^2 + y^2) + \beta, \quad \alpha, \beta \in \mathbb{R} \quad (2.3)$$

*is the Hamiltonian of the system (1.1).*

Let us focus now on the first case; if  $a \in \mathbb{R}^*$ ,  $b = c = 0$ , the system (1.1) becomes

$$\begin{aligned} \dot{x} &= ax - yz, \\ \dot{y} &= xz, \\ \dot{z} &= \frac{xy}{3}, \end{aligned} \quad (2.4)$$

and we will consider the Hamiltonian given by

$$H(x, y, z) = \frac{1}{2}(y^2 - 3z^2). \quad (2.5)$$

In order to find the Poisson structure in this case we will use a method described by Haas and Goedert (see [6] for details). Let us consider the skew-symmetric matrix given by

$$\Pi := \begin{bmatrix} 0 & p_1(x, y, z) & p_2(x, y, z) \\ -p_1(x, y, z) & 0 & p_3(x, y, z) \\ -p_2(x, y, z) & -p_3(x, y, z) & 0 \end{bmatrix}. \quad (2.6)$$

In the beginning, let us denote

$$\begin{aligned}v_1 &:= ax - yz, \\v_2 &:= xz, \\v_3 &:= \frac{xy}{3}.\end{aligned}\tag{2.7}$$

The function  $p$  is the solution of the following first order ODE:

$$v_1 \frac{\partial p}{\partial x} + v_2 \frac{\partial p}{\partial y} + v_3 \frac{\partial p}{\partial z} = A \cdot p + B,\tag{2.8}$$

where

$$\begin{aligned}A &= \frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y} + \frac{\partial v_3}{\partial z} - \frac{(\partial v_1 / \partial z)(\partial H / \partial x) + (\partial v_2 / \partial z)(\partial H / \partial y) + (\partial v_3 / \partial z)(\partial H / \partial z)}{\partial H / \partial z}, \\B &= \frac{v_1(\partial v_2 / \partial z) - v_2(\partial v_1 / \partial z)}{\partial H / \partial z}.\end{aligned}\tag{2.9}$$

the Equation (2.8) becomes

$$(ax - yz) \frac{\partial p}{\partial x} + xz \frac{\partial p}{\partial y} + \frac{xy}{3} \frac{\partial p}{\partial z} = \left(-a + \frac{xy}{3z}\right)p - \frac{ax^2}{3z}.\tag{2.10}$$

If  $a = 0$  then (2.10) has the solution  $p(x, y, z) = -z$ .

If  $a \neq 0$  then finding the solution of (2.10) remains an open problem.

If  $a = 0$  then the system (2.4) becomes

$$\begin{aligned}\dot{x} &= -yz, \\ \dot{y} &= xz, \\ \dot{z} &= \frac{xy}{3}.\end{aligned}\tag{2.11}$$

Now, one can reach the following result.

**Proposition 2.2.** *The system (2.11) has the Hamilton-Poisson realization:*

$$\left(\mathbb{R}^3, \Pi := [\Pi^{ij}], H\right),\tag{2.12}$$

where

$$\Pi = \begin{bmatrix} 0 & -z & 0 \\ z & 0 & -\frac{x}{3} \\ 0 & \frac{x}{3} & 0 \end{bmatrix}, \quad (2.13)$$

$$H(x, y, z) = \frac{1}{2}(y^2 - 3z^2).$$

*Proof.* Indeed, we have

$$\Pi \cdot \nabla H = \begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{bmatrix}, \quad (2.14)$$

and the matrix  $\Pi$  is a Poisson matrix, see [7].  $\square$

It is easy to see that the Poisson structure is degenerate, so we can proceed to find the Casimir functions of our configuration.

**Proposition 2.3.** *The real smooth function  $C : \mathbb{R}^3 \rightarrow \mathbb{R}$ ,*

$$C(x, y, z) = \frac{1}{2}(x^2 + 3z^2), \quad (2.15)$$

*is the only one functionally independent Casimir of the Hamilton-Poisson realization given by Proposition 2.2.*

*Proof.* Indeed, we have  $(\nabla C)^t \Pi = 0$  and  $\text{rank } \Pi = 2$ , as required.  $\square$

The phase curves of the dynamics (2.11) are the intersections of the surfaces:

$$\begin{aligned} H(x, y, z) &= \text{const.}, \\ C(x, y, z) &= \text{const.}, \end{aligned} \quad (2.16)$$

see the Figure 1.

The next proposition gives other Hamilton-Poisson realizations of the system (2.11).

**Proposition 2.4.** *The system (2.11) may be modeled as a Hamilton-Poisson system in an infinite number of different ways, that is, there exist infinitely more different (in general nonisomorphic) Poisson structures on  $\mathbb{R}^3$  such that the system (2.11) is induced by an appropriate Hamiltonian.*

*Proof.* The triplets

$$\left( \mathbb{R}^3 \{ \cdot, \cdot \}_{\alpha\beta}, H_{\gamma\delta} \right), \quad (2.17)$$

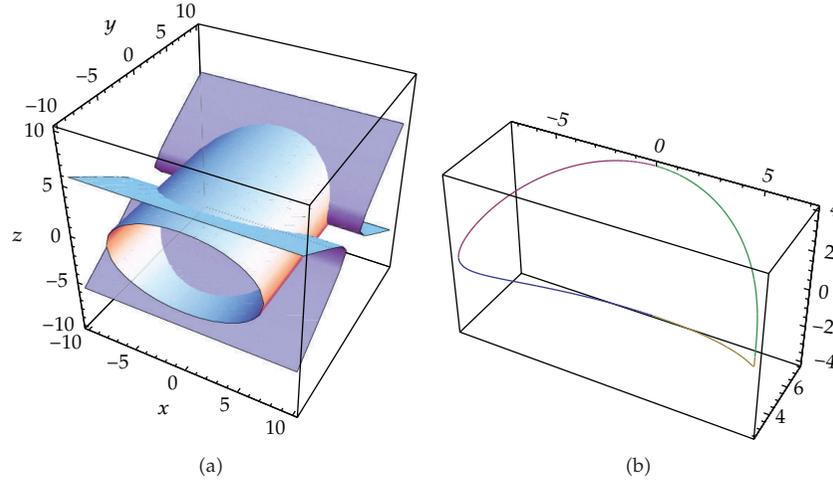


Figure 1: The phase curves of the system (2.11).

where

$$\begin{aligned} \{f, g\}_{\alpha\beta} &= -\nabla C_{\alpha\beta} \cdot (\nabla f \times \nabla g), \quad \forall f, g \in C^\infty(\mathbb{R}^3, \mathbb{R}), \\ C_{\alpha\beta} &= \alpha C + \beta H, \quad H_{\gamma\delta} = \gamma C + \delta H, \quad \alpha, \beta, \gamma, \delta \in \mathbb{R}, \alpha\delta - \beta\gamma = \frac{1}{3}, \\ H &= \frac{1}{2}(y^2 - 3z^2), \quad C = \frac{1}{2}(x^2 + 3z^2), \end{aligned} \quad (2.18)$$

define Hamilton-Poisson realizations of the dynamics (2.11).  $\square$

### 3. The Stability Problem

Let us pass now to discuss the stability problem of the system (2.11). It is not difficult to see that the equilibrium states of our dynamics are

$$e_1^M = (M, 0, 0), \quad e_2^M = (0, M, 0), \quad e_3^M = (0, 0, M), \quad M \in \mathbb{R}. \quad (3.1)$$

Let  $A$  be the matrix of the linear part of our system, that is,

$$A = \begin{bmatrix} 0 & -z & -y \\ z & 0 & x \\ \frac{y}{3} & \frac{x}{3} & 0 \end{bmatrix}. \quad (3.2)$$

Then the characteristic roots of  $A(e_1^M)$  (resp.,  $A(e_2^M)$ , resp.  $A(e_3^M)$ ) are given by,

$$\lambda_1 = 0, \quad \lambda_{2,3} = \pm \frac{M}{\sqrt{3}}, \quad (3.3)$$

(resp.,  $\lambda_1 = 0, \lambda_{2,3} = (\pm iM/\sqrt{3})$ , resp.,  $\lambda_1 = 0, \lambda_{2,3} = \pm iM$ ), so one gets that the following.

**Proposition 3.1.** *The equilibrium states  $e_1^M, e_2^M, e_3^M, M \in \mathbb{R}$ , have the following behavior:*

- (i)  $e_1^M$  are unstable for any  $M \in \mathbb{R}$ ;
- (ii)  $e_2^M$  are spectrally stable for any  $M \in \mathbb{R}$ ;
- (iii)  $e_3^M$  are spectrally stable for any  $M \in \mathbb{R}$ .

Let us begin the nonlinear stability analysis using the energy-Casimir method for the equilibrium state  $e_2^M$  and  $e_3^M$ .

**Proposition 3.2.** *The equilibrium states  $e_2^M, M \in \mathbb{R}$ , are nonlinearly stable for any  $M \in \mathbb{R}$ .*

*Proof.* To study the nonlinear stability of the equilibrium state  $e^M$  we are using energy-Casimir method ([8]). To do that, let  $H_\varphi \in C^\infty(\mathbb{R}^3, \mathbb{R})$  be defined by

$$H_\varphi := C + \varphi(H), \quad (3.4)$$

where  $\varphi : \mathbb{R} \rightarrow \mathbb{R}$  is a smooth real valued function defined on  $\mathbb{R}$ .

Now, the first variation of  $H_\varphi$  at the equilibrium of interest equals zero if and only if

$$\dot{\varphi}\left(\frac{M^2}{2}\right) = 0. \quad (3.5)$$

Using (3.5), the second variation of  $H_\varphi$  at the equilibrium of interest is given by

$$\delta^2 H_\varphi(e_2^M) = (\delta x)^2 + 3(\delta z)^2 + M^2 \ddot{\varphi}\left(\frac{M^2}{2}\right) (\delta y)^2. \quad (3.6)$$

If we choose now the function  $\varphi$  such that

$$\dot{\varphi}\left(\frac{M^2}{2}\right) = 0, \quad \ddot{\varphi}\left(\frac{M^2}{2}\right) > 0 \quad (3.7)$$

we can conclude that the second variation of  $H_\varphi$  at the equilibrium of interest is positively defined for any  $M \in \mathbb{R}$  and thus  $e_2^M$  are nonlinear stable for any  $M \in \mathbb{R}$ .  $\square$

**Proposition 3.3.** *The equilibrium states  $e_3^M, M \in \mathbb{R}$ , are nonlinearly stable for any  $M \in \mathbb{R}$ .*

*Proof.* To study the nonlinear stability of the equilibrium state  $e^M$  we are using energy-Casimir method ([8]). To do that, let  $H_\varphi \in C^\infty(\mathbb{R}^3, \mathbb{R})$  be defined by

$$H_\varphi := C + \varphi(H), \quad (3.8)$$

where  $\varphi : \mathbb{R} \rightarrow \mathbb{R}$  is a smooth real valued function defined on  $\mathbb{R}$ .

Now, the first variation of  $H_\varphi$  at the equilibrium of interest equals zero if and only if

$$\dot{\varphi}\left(-\frac{3M^2}{2}\right) = 1. \quad (3.9)$$

Using (3.9), the second variation of  $H_\varphi$  at the equilibrium of interest is given by

$$\delta^2 H_\varphi(e_2^M) = (\delta x)^2 + (\delta y)^2 + 9M^2 \ddot{\varphi}\left(-\frac{3M^2}{2}\right) (\delta z)^2. \quad (3.10)$$

If we choose now the function  $\varphi$  such that

$$\dot{\varphi}\left(-\frac{3M^2}{2}\right) = 1, \quad \ddot{\varphi}\left(-\frac{3M^2}{2}\right) > 0 \quad (3.11)$$

we can conclude that the second variation of  $H_\varphi$  at the equilibrium of interest is positively defined for any  $M \in \mathbb{R}$  and thus  $e_2^M$  are nonlinear stable for any  $M \in \mathbb{R}$ .  $\square$

#### 4. Periodical Orbits

As we have proved in the previous section, the equilibrium states  $e_2^M$  are nonlinear stable so we can try to find the periodic orbits around them. As we know, the dynamics described by a Hamilton-Poisson system take place on the symplectic leaves of the Poisson configuration. In order to do this we consider the system restricted to a regular coadjoint orbit of  $(\mathbb{R}^3)^*$  that contains  $e_2^M$ ; we will obtain the existence of periodic solutions for the restricted system; these are, also, the periodic solutions for the unrestricted system.

**Proposition 4.1.** *Near to  $e_2^M$  the reduced dynamics has for each sufficiently small value of the reduced energy at least 1-periodic solution whose period is close to  $(2\pi\sqrt{3}/M)$ .*

*Proof.* Indeed, we have successively the following.

- (i) The reduction of the system (2.11) to the regular coadjoint orbit:

$$\Omega_{e_2^M} = \left\{ (x, y, z) \in \mathbb{R}^3, y^2 - 3z^2 = M^2 \right\} \quad (4.1)$$

gives rise to a classical Hamiltonian system.

- (ii) The matrix of the linear part of the reduced dynamics has purely imaginary roots. More exactly,

$$\lambda_{2,3} = \pm i \frac{M}{\sqrt{3}}. \quad (4.2)$$

- (iii) One has  $\text{span}(\nabla C(e_2^M)) = V_0$ , where

$$V_0 = \ker\left(A\left(e_2^M\right)\right). \quad (4.3)$$

- (iv) The equilibrium state  $e_2^M$  are nonlinear stable for any  $M \in \mathbb{R}$ .

Then our assertion follows via the Moser-Weinstein theorem with zero eigenvalue, see [9] for details.  $\square$

Similar arguments lead us to the following result.

**Proposition 4.2.** *Near to  $e_3^M$  the reduced dynamics has for each sufficiently small value of the reduced energy at least 1-periodic solution whose period is close to  $2\pi/M$ .*

## 5. Lax Formulation of the Dynamics (2.11)

**Proposition 5.1.** *The dynamics (2.11) allows a formulation in terms of Lax pairs.*

*Proof.* Let us take the following.

$$L = \begin{bmatrix} 0 & 2ix - \frac{i}{\sqrt{3}}y - 3z & \frac{2}{\sqrt{3}}x - y + i\sqrt{3}z \\ -2ix + \frac{i}{\sqrt{3}}y + 3z & 0 & -2x - 2iz \\ -\frac{2}{\sqrt{3}}x + y - i\sqrt{3}z & 2x + 2iz & \\ 0 & & \end{bmatrix}, \quad (5.1)$$

$$B = \begin{bmatrix} 0 & -ix + \frac{i}{\sqrt{3}}y & -\frac{1}{\sqrt{3}}x + y \\ ix - \frac{i}{\sqrt{3}}y & 0 & x \\ \frac{1}{\sqrt{3}}x - y & -x & 0 \end{bmatrix}.$$

Then, using MATHEMATICA 8.0, we can put the system (2.11) in the equivalent form:

$$\dot{L} = [L, B], \quad (5.2)$$

as desired.  $\square$

## 6. Numerical Integration of the Dynamics (2.11)

We will discuss now the numerical integration of the dynamics (2.11) via the Lie-Trotter integrator [7]. For the beginning, let us observe that the Hamiltonian vector field  $X_H$  splits in  $X_H = X_{H_1} + X_{H_2}$ , where

$$H_1 = \frac{1}{2}y^2, \quad H_2 = -\frac{1}{2}z^2. \quad (6.1)$$

Their corresponding integral curves are, respectively, given by

$$[x_1(t) \ x_2(t) \ x_3(t)] = A_i[x_1(0) \ x_2(0) \ x_3(0)], \quad i = 1, 2, \text{ where}$$

$$A_1 = \begin{bmatrix} \cos \frac{at}{\sqrt{3}} & 0 & -\sqrt{3} \sin \frac{at}{\sqrt{3}} \\ 0 & 1 & 0 \\ \frac{1}{\sqrt{3}} \sin \frac{at}{\sqrt{3}} & 0 & \cos \frac{at}{\sqrt{3}} \end{bmatrix}, \quad A_2 = \begin{bmatrix} 1 & 0 & 0 \\ bt & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad (6.2)$$

and  $a = x(0), b = z(0)$ .

Then the Lie-Trotter integrator is given by

$$[x_1^{n+1} \ x_2^{n+1} \ x_3^{n+1}]^t = A_1 A_2 [x_1^n \ x_2^n \ x_3^n]^t, \quad (6.3)$$

that is,

$$\begin{aligned} x^{n+1} &= \cos \frac{at}{\sqrt{3}} x^n - \sqrt{3} \sin \frac{at}{\sqrt{3}} z^n, \\ y^{n+1} &= b t x^n + y^n, \\ z^{n+1} &= \frac{1}{\sqrt{3}} \sin \frac{at}{\sqrt{3}} x^n + \cos \frac{at}{\sqrt{3}} z^n, \end{aligned} \quad (6.4)$$

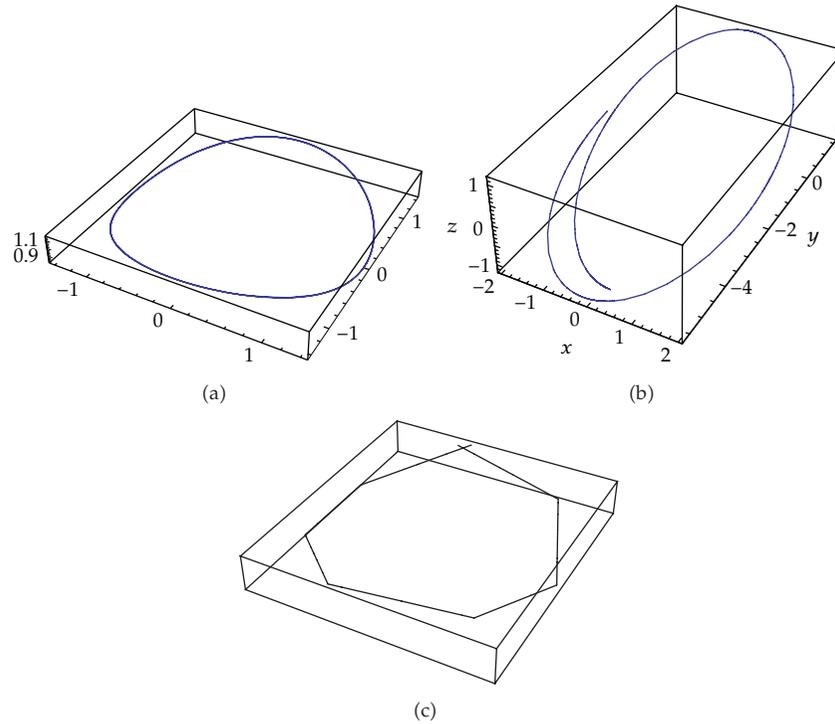
The following proposition sketches the Lie-Trotter integrator properties.

**Proposition 6.1.** *The Lie-Trotter integrator (6.4) has the following properties.*

- (i) *It preserves the Poisson structure  $\Pi$ .*
- (ii) *It preserves the Casimir  $C$  of our Poisson configuration  $(\mathbb{R}^3, \Pi)$ .*
- (iii) *It does not preserve the Hamiltonian  $H$  of our system (2.11).*
- (iv) *Its restriction to the coadjoint orbit  $(\mathcal{O}_k, \omega_k)$ , where*

$$\mathcal{O}_k = \left\{ (x, y, z) \in \mathbb{R}^3, y^2 - 3z^2 = k \right\} \quad (6.5)$$

*and  $\omega_k$  is the Kirilov-Kostant-Souriau symplectic structure on  $\mathcal{O}_k$ , gives rise to a symplectic integrator.*



**Figure 2:** Runge-Kutta 4 steps, Lie-Trotter, and Kahan integrator, respectively ( $x(0) = y(0) = z(0) = 1$ ).

We will discuss now the numerical integration of the dynamics (2.11) via the Kahan integrator and also via Runge-Kutta 4th steps integrator and we will point out some properties of Kahan integrator. The Kahan integrator [10] of the system (2.11) is given by

$$\begin{aligned}
 x^{n+1} - x^n &= -\frac{h}{2} \left( y^{n+1} z^n + z^{n+1} y^n \right), \\
 y^{n+1} - y^n &= \frac{h}{2} \left( x^{n+1} z^n + z^{n+1} x^n \right), \\
 z^{n+1} - z^n &= \frac{h}{6} \left( x^{n+1} y^n + y^{n+1} x^n \right).
 \end{aligned} \tag{6.6}$$

Using MATHEMATICA 8.0, we can prove the following proposition which shows the incompatibility of the Kahan integrator with the Poisson structure of the system (2.11).

**Proposition 6.2.** *The Kahan integrator (6.6) does not preserve the Poisson structure and does not preserve the Hamiltonian and the Casimir of our configuration.*

*Remark 6.3.* As we can see from Figure 2 the three integrators give us almost the same results. However, Lie-Trotter and Kahan integrators have the advantage of being easily implemented.

## 7. Conclusion

The Chen-Lee system is a system arisen from engineering field. Its chaotic behavior makes it good to applied in secure communications, complete synchronization, or optimization of nonlinear system performance. The geometric overview gives it a different perspective and points out new properties. It is easy to see that, like other chaotic systems studied before—the Rikitake system [11], the Lü system [12], the Lorenz system [13]—finding the corresponding Poisson structure implies the study of particular values for its parameters. Unlike the other studied systems, the Chen-Lee one needs to vanish all its parameters to admit a Hamilton-Poisson realization. The connexion between the existence of a Hamilton-Poisson realization and the number of the parameters which should be vanished of a chaotic system remains an open problem.

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