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Research Article

Viscosity Methods of Asymptotically Pseudocontractive and Asymptotically Nonexpansive Mappings for Variational Inequalities

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Let $\{t_n\} \subset (0,1)$ be such that $t_n \to 1$ as $n \to \infty$, let α and β be two positive numbers such that $\alpha + \beta = 1$, and let f be a contraction. If T be a continuous asymptotically pseudocontractive self-mapping of a nonempty bounded closed convex subset K of a real reflexive Banach space with a uniformly Gateaux differentiable norm, under suitable conditions on the sequence $\{t_n\}$, we show the existence of a sequence $\{x_n\}_n$ satisfying the relation $x_n = (1 - t_n/k_n)f(x_n) + (t_n/k_n)T^n x_n$ and prove that $\{x_n\}$ converges strongly to the fixed point of T, which solves some variational inequality provided T is uniformly asymptotically regular. As an application, if T be an asymptotically nonexpansive self-mapping of a nonempty bounded closed convex subset K of a real Banach space with a uniformly Gateaux differentiable norm and which possesses uniform normal structure, we prove that the iterative process defined by $z_0 \in K$, $z_{n+1} = (1 - t_n/k_n)f(z_n) + (\alpha t_n/k_n)T^n z_n + (\beta t_n/k_n)z_n$ converges strongly to the fixed point of T.

1. Introduction

Let E be a real Banach space with dual E^* and K a nonempty closed convex subset of E. Recall that a mapping $T : K \to K$ is said to be asymptotically pseudocontractive if, for each $n \in N$ and $x, y \in K$, there exist $j \in J(x - y)$ and a constant $k_n \geq 1$ with $\lim_{n\to\infty} k_n = 1$ such that

$$\langle T^n x - T^n y \rangle \leq k_n \|x - y\|^2, \tag{1.1}$$

where $J : E \to 2^{E^*}$ denote the normalized duality mapping defined by

$$J(x) = \left\{x^* \in E^* : \langle x, x^* \rangle = \|x\|^2, \|x^*\| = \|x\|, x \in E\right\}. \tag{1.2}$$

The class of asymptotically pseudocontractive mappings is essentially wider than the class of asymptotically nonexpansive mappings. A mapping T is called asymptotically nonexpansive if there exists a sequence $\{k_n\} \subset [1,\infty)$ with $\lim_{n\to\infty} k_n = 1$ such that

$$\|T^n x - T^n y\| \le k_n \|x - y\| \tag{1.3}$$

for all integers $n \ge 0$ and all $x, y \in K$. A mapping $f : K \to K$ is called a contraction if there exists a constant $\gamma \in [0,1)$ such that

$$\|f(x) - f(y)\| \le \gamma \|x - y\|, \quad \forall x, y \in K. \tag{1.4}$$

It is clear that every contraction is nonexpansive, every nonexpansive mapping is asymptotically nonexpansive, and every asymptotically nonexpansive mapping is asymptotically pseudocontractive. The converses do not hold. The asymptotically nonexpansive mappings are important generalizations of nonexpansive mappings. For details, you may see [1].

The mapping T is called uniformly asymptotically regular (in short u.a.r.) if for each $\epsilon > 0$ there exists $n_0 \in N$ such that

$$\left\|T^{n+1} x - T^n x\right\| \le \epsilon, \tag{1.5}$$

for all $n \ge n_0$ and $x \in K$ and it is called uniformly asymptotically regular with sequence $\{\epsilon_n\}$ (in short u.a.r.s.) if

$$\left\|T^{n+1} x - T^n x\right\| \le \epsilon_n, \tag{1.6}$$

for all integers $n \ge 1$ and all $x \in K$, where $\epsilon_n \to 0$ as $n \to \infty$.

The viscosity approximation method of selecting a particular fixed point of a given nonexpansive mapping was proposed by Moudafi [2] who proved the strong convergence of both the implicit and explicit methods in Hilbert spaces, see [2, Theorems 2.1 and 2.2]. Xu [3] studied the viscosity approximation methods proposed by Moudafi [2] for a nonexpansive mapping in a uniformly smooth Banach space.

Very recently, Shahzad and Udomene [4] obtained fixed point solutions of variational inequalities for an asymptotically nonexpansive mapping defined on a real Banach space with uniformly Gateaux differentiable norm possessing uniform normal structure. They proved the following theorem.

Theorem 1.1. *Let E be a real Banach space with a uniformly Gateaux differentiable norm possessing uniform normal structure, let K be a nonempty closed convex and bounded subset of E, let $T : K \to K$ be an asymptotically nonexpansive mapping with sequence $\{k_n\}_n \subset [1,\infty)$, and let $f : K \to K$ be a contraction with constant $\alpha \in [0,1)$. Let $\{t_n\}_n \subset (0, \xi_n)$ be such that*

$\lim_{n\to\infty} t_n = 1$, $\sum_{n=1}^{\infty} t_n(1-t_n) = \infty$, *and* $\lim_{n\to\infty}((k_n-1)/(k_n-t_n)) = 0$, *where* $\xi_n = \min\{(1-\alpha)k_n/(k_n-\alpha), 1/k_n\}$. *For an arbitrary* $z_0 \in K$ *let the sequence* $\{z_n\}$ *be iteratively defined by*

$$z_{n+1} = \left(1 - \frac{t_n}{k_n}\right) f(z_n) + \frac{t_n}{k_n} T^n z_n, \quad n \in N. \tag{1.7}$$

Then

(i) *for each integer* $n \geq 0$, *there is a unique* $x_n \in K$ *such that*

$$x_n = \left(1 - \frac{t_n}{k_n}\right) f(x_n) + \frac{t_n}{k_n} T^n x_n; \tag{1.8}$$

if in addition

$$\lim_{n\to\infty} \|x_n - Tx_n\| = 0, \qquad \lim_{n\to\infty} \|z_n - Tz_n\| = 0, \tag{1.9}$$

then

(ii) *the sequence* $\{z_n\}_n$ *converges strongly to the unique solution of the variational inequality:*

$$p \in F(T) \text{ such that } \langle (I-f)p, j(p-x^*) \rangle \leq 0, \quad \forall x^* \in F(T). \tag{1.10}$$

Remark 1.2. We note that $\|T^{n+1}x - T^n x\| \leq k_n \|Tx - x\|$, then the condition (1.9) $\lim_{n\to\infty}\|x_n - Tx_n\| = 0$ and $\lim_{n\to\infty}\|z_n - Tz_n\| = 0$ imply that

$$\begin{aligned} &\lim_{n\to\infty} \left\| T^{n+1}x_n - T^n x_n \right\| = 0, \\ &\lim_{n\to\infty} \left\| T^{n+1}z_n - T^n z_n \right\| = 0, \end{aligned} \tag{1.11}$$

respectively. In other words, if an asymptotically nonexpansive mapping T satisfies the condition (1.9) then T must be u.a.r.s.

Inspired by the works in [4–8], in this paper, we suggest and analyze a modification of the iterative algorithm.

Let $\{t_n\} \subset (0,1)$, let α and β be two positive numbers such that $\alpha + \beta = 1$, and let f be a contraction on K, a sequence $\{z_n\}$ iteratively defined by: $z_0 \in K$,

$$z_{n+1} = \left(1 - \frac{t_n}{k_n}\right) f(z_n) + \frac{\alpha t_n}{k_n} T^n z_n + \frac{\beta t_n}{k_n} z_n. \tag{1.12}$$

Remark 1.3. The algorithm (1.12) includes the algorithm (1.7) of Chidume et al. [5] and Shahzad and Udomene [4] as a special case.

We show the convergence of the proposed algorithm (1.12) to the unique solution of some variational inequality (some related works on VI, please see [9–12]). In this respect, our results can be considered as a refinement and improvement of the known results of Chidume et al. [5], Shahzad and Udomene [4], and Lim and Xu [13].

2. Preliminaries

Let $S = \{x \in E : \|x\| = 1\}$ denote the unit sphere of the Banach space E. The space E is said to have a Gateaux differentiable norm if the limit

$$\lim_{n \to \infty} \frac{\|x + ty\| - \|x\|}{t} \tag{2.1}$$

exists for each $x, y \in S$, and we call E smooth; E is said to have a uniformly Gateaux differentiable norm if for each $y \in S$ the limit (2.1) is attained uniformly for $x \in S$. Further, E is said to be uniformly smooth if the limit (2.1) exists uniformly for $(x, y) \in S \times S$. It is well known [14] that if E is smooth then any duality mapping on E is single-valued, and if E has a uniformly Gateaux differentiable norm then the duality mapping is norm-to-weak* uniformly continuous on bounded sets.

Let K be a nonempty closed convex and bounded subset of the Banach space E and let the diameter of K be defined by $d(K) = \sup\{\|x - y\| : x, y \in K\}$. For each $x \in K$, let $r(x, K) = \sup\{\|x - y\| : y \in K\}$ and let $r(K) = \inf\{r(x, K) : x \in K\}$ denote the Chebyshev radius of K relative to itself. The normal structure coefficient $N(E)$ of E is defined by

$$N(E) = \inf\left\{ \frac{d(K)}{r(K)} : K \text{ is a closed convex and bounded subset of } E \text{ with } d(K) > 0 \right\}. \tag{2.2}$$

A space E such that $N(E) > 1$ is said to have uniform normal structure. It is known that every space with a uniform normal structure is reflexive, and that all uniformly convex and uniformly smooth Banach spaces have uniform normal structure (see [13]).

We will let LIM be a Banach limit. Recall that $\mathrm{LIM} \in (l^\infty)^*$ such that $\|\mathrm{LIM}\| = 1$, $\liminf_{n \to \infty} a_n \le \mathrm{LIM}_n a_n \le \limsup_{n \to \infty} a_n$, and $\mathrm{LIM}_n a_n = \mathrm{LIM}_n a_{n+1}$ for all $\{a_n\}_n \in l^\infty$. Let $\{x_n\}$ be a bounded sequence of E. Then we can define the real-valued continuous convex function g on E by $g(z) = \mathrm{LIM}_n \|x_n - z\|^2$ for all $z \in K$.

Let $T : K \to K$ be a nonlinear mapping and $M = \{x \in K : g(x) = \min_{z \in K} g(z)\}$. T is said to satisfy the property (S) if for any bounded sequence $\{x_n\}$ in K, $\lim_{n \to \infty} \|x_n - Tx_n\| = 0$ implies $M \cap F(T) \ne \emptyset$.

Lemma 2.1 (see [15]). *Let E be a Banach space with the uniformly Gateaux differentiable norm and $u \in E$. Then*

$$g(u) = \inf_{z \in E} g(z) \tag{2.3}$$

if and only if

$$LIM\langle z, J(x_n - u) \rangle \le 0 \tag{2.4}$$

for all $z \in E$.

Lemma 2.2 (see [16]). *Assume $\{a_n\}$ is a sequence of nonnegative real numbers such that*

$$a_{n+1} \leq (1-\gamma_n)a_n + \delta_n\gamma_n, \tag{2.5}$$

where $\{\gamma_n\}$ is a sequence in $(0,1)$ and $\{\delta_n\}$ is a sequence such that

(1) $\sum_{n=1}^{\infty}\gamma_n = \infty$;

(2) $\limsup_{n\to\infty}\delta_n \leq 0$ *or* $\sum_{n=1}^{\infty}|\delta_n\gamma_n| < \infty$.

Then $\lim_{n\to\infty}a_n = 0$.

Lemma 2.3 (see [17]). *Let $\{x_n\}$ and $\{y_n\}$ be bounded sequences in a Banach space X and let $\{\beta_n\}$ be a sequence in* $[0,1]$ *with* $0 < \liminf_{n\to\infty}\beta_n \leq \limsup_{n\to\infty}\beta_n < 1$. *Suppose that*

$$x_{n+1} = (1-\beta_n)y_n + \beta_n x_n \tag{2.6}$$

for all $n \geq 0$ and

$$\limsup_{n\to\infty}\left(\|y_{n+1} - y_n\| - \|x_{n+1} - x_n\|\right) \leq 0. \tag{2.7}$$

Then $\lim_{n\to\infty}\|y_n - x_n\| = 0$.

Lemma 2.4. *Let E be an arbitrary real Banach space. Then*

$$\|x+y\|^2 \leq \|x\|^2 + 2\langle y, j(x+y)\rangle, \tag{2.8}$$

for all $x, y \in E$ and for all $j(x+y) \in J(x+y)$.

Lemma 2.5 (see [5]). *Let E be a Banach space with uniform normal structure, K a nonempty closed convex and bounded subset of E, and $T : K \to K$ an asymptotically nonexpansive mapping. Then T has a fixed point.*

3. Main Results

Theorem 3.1. *Let E be a real reflexive Banach space with a uniformly Gateaux differentiable norm, K a nonempty closed convex and bounded subset of E, $T : K \to K$ a continuous asymptotically pseudocontractive mapping with sequence $\{k_n\}_n \subset [1,\infty)$, and $f : K \to K$ a contraction with constant $\gamma \in [0,1)$. Let $\{t_n\} \subset (0,(1-\gamma)k_n/(k_n-\gamma))$ be such that* $\lim_{n\to\infty}t_n = 1$ *and* $\lim_{n\to\infty}((k_n-1)/(k_n-t_n)) = 0$. *Suppose T satisfies the property (S). Then*

(i) *for each integer $n \geq 0$, there is a unique $x_n \in K$ such that*

$$x_n = \left(1 - \frac{t_n}{k_n}\right)f(x_n) + \frac{t_n}{k_n}T^n x_n; \tag{3.1}$$

if T is u.a.r.s., then

(ii) *the sequence $\{x_n\}_n$ converges strongly to the unique solution of the variational inequality:*

$$p \in F(T) \text{ such that } \langle (I-f)p, j(p-x^*)\rangle \leq 0, \quad \forall x^* \in F(T). \tag{3.2}$$

Proof. By the conditions on $\{t_n\}$, $t_n < (1-\gamma)k_n/(k_n-\gamma)$ implies $(1-t_n/k_n)\gamma + t_n < 1$ for each integer $n \geq 0$, then the mapping $S_n : K \to K$ defined for each $x \in K$ by $S_n x = (1 - t_n/k_n)f(x) + (t_n/k_n)T^n x$ is a strictly pseudocontractive mapping.

Indeed, for all $x, y \in K$, we have

$$\begin{aligned}\langle S_n x - S_n y, j(x-y)\rangle &= \left(1-\frac{t_n}{k_n}\right)\langle f(x)-f(y), j(x-y)\rangle \\ &\quad + \frac{t_n}{k_n}\langle T^n x - T^n y, j(x-y)\rangle \\ &\leq \left(1-\frac{t_n}{k_n}\right)\|f(x)-f(y)\|\|x-y\| + t_n\|x-y\|^2 \\ &\leq \left[\left(1-\frac{t_n}{k_n}\right)\gamma + t_n\right]\|x-y\|^2.\end{aligned} \tag{3.3}$$

It follows [18, Corollary 1] that S_n possesses exactly one fixed point x_n in K such that $S_n x_n = x_n$.

From (3.1), we have

$$\begin{aligned}\|x_n - T^n x_n\| &= \left\|\left(1-\frac{t_n}{k_n}\right)f(x_n) + \left(\frac{t_n}{k_n}-1\right)T^n x_n\right\| \\ &= \left(1-\frac{t_n}{k_n}\right)\|f(x_n) - T^n x_n\| \longrightarrow 0 \quad \text{as } n \longrightarrow \infty.\end{aligned} \tag{3.4}$$

Notice that

$$\begin{aligned}\|x_n - Tx_n\| &= \left\|\left(1-\frac{t_n}{k_n}\right)(f(x_n)-Tx_n) + \frac{t_n}{k_n}(T^n x_n - Tx_n)\right\| \\ &\leq \left(1-\frac{t_n}{k_n}\right)\|f(x_n)-Tx_n\| + \frac{t_n}{k_n}\left\|T^n x_n - T^{n+1}x_n\right\| \\ &\quad + \frac{t_n}{k_n}\left\|T^{n+1}x_n - Tx_n\right\| \\ &\leq \left(1-\frac{t_n}{k_n}\right)\|f(x_n)-Tx_n\| + \frac{t_n}{k_n}\left\|T^n x_n - T^{n+1}x_n\right\| \\ &\quad + \frac{t_n}{k_n}k_1\|x_n - T^n x_n\|.\end{aligned} \tag{3.5}$$

Therefore, from (3.4), (3.5), and T which is u.a.r.s., we obtain $\|x_n - Tx_n\| \to 0$ as $n \to \infty$.

Define a function $g : K \to R^+$ by

$$g(z) = \text{LIM}_n\|x_n - z\|^2 \tag{3.6}$$

for all $z \in K$. Since g is continuous and convex, $g(z) \to \infty$ as $\|z\| \to \infty$, and E is reflexive, g attains it infimum over K. Let $z_0 \in K$ such that $g(z_0) = \min_{z\in K} g(z)$ and let

$M = \{x \in K : g(x) = \min_{z \in K} g(z)\}$. Then M is nonempty because $z_0 \in M$. Since T satisfies the property (S), it follows that $M \cap F(T) \neq \emptyset$. Suppose that $p \in M \cap F(T)$. Then, by Lemma 2.1, we have

$$\mathrm{LIM}_n \langle x - p, j(x_n - p) \rangle \leq 0 \tag{3.7}$$

for all $x \in K$. In particular, we have

$$\mathrm{LIM}_n \langle f(p) - p, j(x_n - p) \rangle \leq 0. \tag{3.8}$$

On the other hand, from (3.1), we have

$$x_n - T^n x_n = \left(1 - \frac{t_n}{k_n}\right)(f(x_n) - T^n x_n) = \frac{1 - t_n/k_n}{t_n/k_n}(f(x_n) - x_n). \tag{3.9}$$

Now, for any $p \in F(T)$, we have

$$\begin{aligned} \langle x_n - T^n x_n, J(x_n - p) \rangle &= \langle x_n - p + T^n p - T^n x_n, J(x_n - p) \rangle \\ &\geq -(k_n - 1)\|x_n - p\|^2 \\ &\geq -(k_n - 1)B^2 \end{aligned} \tag{3.10}$$

for some $B > 0$ and it follows from (3.9) that

$$\langle x_n - f(x_n), j(x_n - p) \rangle \leq \frac{t_n(k_n - 1)}{k_n - t_n} B^2, \tag{3.11}$$

which implies that

$$\limsup_{n \to \infty} \langle x_n - f(x_n), j(x_n - p) \rangle \leq 0. \tag{3.12}$$

Consequently, similar to the lines of the proof of [4, Theorem 3.1], Theorem 3.1 is easily proved. This completes the proof. □

Corollary 3.2. *Let E be a real Banach space with a uniformly Gateaux differentiable norm possessing uniform normal structure, K a nonempty closed convex and bounded subset of E, $T : K \to K$ be an asymptotically nonexpansive mapping with sequence $\{k_n\}_n \subset [1, \infty)$, and $f : K \to K$ a contraction with constant $\gamma \in [0, 1)$. Let $\{t_n\} \subset (0, (1-\gamma)k_n/(k_n - \gamma))$ be such that $\lim_{n\to\infty} t_n = 1$ and $\lim_{n\to\infty}((k_n - 1)/(k_n - t_n)) = 0$. Then*

(i) *for each integer $n \geq 0$, there is a unique $x_n \in K$ such that*

$$x_n = \left(1 - \frac{t_n}{k_n}\right) f(x_n) + \frac{t_n}{k_n} T^n x_n; \tag{3.13}$$

if T is u.a.r.s., then

(ii) *the sequence $\{x_n\}_n$ converges strongly to the unique solution of the variational inequality:*

$$p \in F(T) \text{ such that } \langle (I - f)p, j(p - x^*) \rangle \leq 0, \quad \forall x^* \in F(T). \tag{3.14}$$

Theorem 3.3. *Let E be a real Banach space with a uniformly Gateaux differentiable norm possessing uniform normal structure, K a nonempty closed convex and bounded subset of E, $T : K \to K$ an asymptotically nonexpansive mapping with sequence $\{k_n\}_n \subset [1,\infty)$, and $f : K \to K$ a contraction with constant $\gamma \in [0,1)$. Let $\{t_n\} \subset (0,\xi_n)$ be such that $\lim_{n\to\infty} t_n = 1$, $\sum_{n=1}^{\infty} t_n(1-t_n) = \infty$, and $\lim_{n\to\infty}((k_n-1)/(k_n-t_n)) = 0$, where $\xi_n = \min\{(1-\gamma)k_n/(k_n-\gamma), 1/k_n\}$. For an arbitrary $z_0 \in K$, let the sequence $\{z_n\}_n$ be iteratively defined by* (1.12). *Then*

(i) *for each integer $n \geq 0$, there is a unique $x_n \in K$ such that*

$$x_n = \left(1 - \frac{t_n}{k_n}\right) f(x_n) + \frac{t_n}{k_n} T^n x_n; \tag{3.15}$$

if T is u.a.r.s., then

(ii) *the sequence $\{z_n\}_n$ converges strongly to the unique solution of the variational inequality:*

$$p \in F(T) \text{ such that } \langle (I-f)p, j(p-x^*) \rangle \leq 0, \quad \forall x^* \in F(T). \tag{3.16}$$

Proof. Set $\alpha_n = t_n/k_n$, then $\alpha_n \to 1$ as $n \to \infty$. Define

$$z_{n+1} = \beta\alpha_n z_n + (1-\beta\alpha_n) y_n. \tag{3.17}$$

Observe that

$$\begin{aligned} y_{n+1} - y_n &= \frac{z_{n+2} - \beta\alpha_{n+1} z_{n+1}}{1-\beta\alpha_{n+1}} - \frac{z_{n+1} - \beta\alpha_n z_n}{1-\beta\alpha_n} \\ &= \frac{(1-\alpha_{n+1}) f(z_{n+1}) + \alpha\alpha_{n+1} T^{n+1} z_{n+1}}{1-\beta\alpha_{n+1}} \\ &\quad - \frac{(1-\alpha_n) f(z_n) + \alpha\alpha_n T^n z_n}{1-\beta\alpha_n} \\ &= \frac{1-\alpha_{n+1}}{1-\beta\alpha_{n+1}} [f(z_{n+1}) - f(z_n)] + \left(\frac{1-\alpha_{n+1}}{1-\beta\alpha_{n+1}} - \frac{1-\alpha_n}{1-\beta\alpha_n}\right) f(z_n) \\ &\quad + \frac{\alpha\alpha_{n+1}}{1-\beta\alpha_{n+1}} \left(T^{n+1} z_{n+1} - T^{n+1} z_n\right) \\ &\quad + \frac{\alpha\alpha_{n+1}}{1-\beta\alpha_{n+1}} \left(T^{n+1} z_n - T^n z_n\right) \\ &\quad + \left(\frac{\alpha\alpha_{n+1}}{1-\beta\alpha_{n+1}} - \frac{\alpha\alpha_n}{1-\beta\alpha_n}\right) T^n z_n. \end{aligned} \tag{3.18}$$

It follows that

$$\begin{aligned}
&\|y_{n+1}-y_n\|-\|z_{n+1}-z_n\| \\
&\quad\le \frac{1-\alpha_{n+1}}{1-\beta\alpha_{n+1}}\gamma\|z_{n+1}-z_n\|+\left|\frac{1-\alpha_{n+1}}{1-\beta\alpha_{n+1}}-\frac{1-\alpha_n}{1-\beta\alpha_n}\right|\|f(z_n)\| \\
&\qquad+\frac{\alpha\alpha_{n+1}}{1-\beta\alpha_{n+1}}\left\|T^{n+1}z_{n+1}-T^{n+1}z_n\right\| \\
&\qquad+\frac{\alpha\alpha_{n+1}}{1-\beta\alpha_{n+1}}\left\|T^{n+1}z_n-T^nz_n\right\| \\
&\qquad+\left|\frac{\alpha\alpha_{n+1}}{1-\beta\alpha_{n+1}}-\frac{\alpha\alpha_n}{1-\beta\alpha_n}\right|\|T^nz_n\|-\|z_{n+1}-z_n\| \\
&\quad\le\left|\frac{1-\alpha_{n+1}}{1-\beta\alpha_{n+1}}-\frac{1-\alpha_n}{1-\beta\alpha_n}\right|\|f(z_n)\|+\frac{\alpha\alpha_{n+1}}{1-\beta\alpha_{n+1}}\left\|T^{n+1}z_n-T^nz_n\right\| \\
&\qquad+\left|\frac{\alpha\alpha_{n+1}}{1-\beta\alpha_{n+1}}-\frac{\alpha\alpha_n}{1-\beta\alpha_n}\right|\|T^nz_n\| \\
&\qquad+\left(\frac{1-\alpha_{n+1}}{1-\beta\alpha_{n+1}}\gamma+\frac{\alpha\alpha_{n+1}}{1-\beta\alpha_{n+1}}k_{n+1}-1\right)\|z_{n+1}-z_n\|.
\end{aligned}\tag{3.19}$$

We note that

$$\begin{aligned}
k_{n+1}-\gamma-(\alpha k_{n+1}+\beta-\gamma)&=(1-\alpha)k_{n+1}-\beta \\
&\ge 1-\alpha-\beta=0.
\end{aligned}\tag{3.20}$$

It follows that

$$t_{n+1}\le\frac{(1-\gamma)k_{n+1}}{k_{n+1}-\gamma}\le\frac{(1-\gamma)k_{n+1}}{\alpha k_{n+1}+\beta-\gamma},\tag{3.21}$$

which implies that

$$\begin{aligned}
&k_{n+1}t_{n+1}\alpha+t_{n+1}\beta-t_{n+1}\gamma\le(1-\gamma)k_{n+1} \\
&\Longrightarrow \alpha k_{n+1}\alpha_{n+1}+\alpha_{n+1}\beta-\alpha_{n+1}\gamma\le 1-\gamma \\
&\Longrightarrow \alpha k_{n+1}\alpha_{n+1}+(1-\alpha_{n+1})\gamma\le 1-\alpha_{n+1}\beta \\
&\Longrightarrow \frac{\alpha k_{n+1}\alpha_{n+1}+(1-\alpha_{n+1})\gamma}{1-\alpha_{n+1}\beta}\le 1.
\end{aligned}\tag{3.22}$$

From (3.19) and (3.22), we obtain

$$\limsup_{n\to\infty}\left(\|y_{n+1}-y_n\|-\|z_{n+1}-z_n\|\right)\le 0.\tag{3.23}$$

Hence, by Lemma 2.3 we know

$$\lim_{n\to\infty}\left\|y_n - z_n\right\| = 0, \tag{3.24}$$

consequently

$$\lim_{n\to\infty}\|z_{n+1} - z_n\| = 0. \tag{3.25}$$

On the other hand,

$$\begin{aligned}\|z_n - T^n z_n\| &\le \|z_{n+1} - z_n\| + \|z_{n+1} - T^n z_n\| \\ &\le \|z_{n+1} - z_n\| + (1-\alpha_n)\left\|f(z_n) - T^n z_n\right\| \\ &\quad + \beta\alpha_n\|z_n - T^n z_n\|,\end{aligned} \tag{3.26}$$

which implies that

$$\lim_{n\to\infty}\|z_n - T^n z_n\| = 0. \tag{3.27}$$

Hence, we have

$$\begin{aligned}\|z_n - Tz_n\| &\le \|z_n - T^n z_n\| + \left\|T^n z_n - T^{n+1} z_n\right\| + \left\|T^{n+1} z_n - Tz_n\right\| \\ &\le \|z_n - T^n z_n\| + \left\|T^n z_n - T^{n+1} z_n\right\| + k_1\|z_n - T^n z_n\| \\ &= (1+k_1)\|z_n - T^n z_n\| + \left\|T^n z_n - T^{n+1} z_n\right\| \longrightarrow 0 \quad (n \longrightarrow \infty).\end{aligned} \tag{3.28}$$

From (3.15), $x_m - z_n = (1-\alpha_m)(f(x_m) - z_n) + \alpha_m(T^m x_m - z_n)$. Applying Lemma 2.4, we estimate as follows:

$$\begin{aligned}\|x_m - z_n\|^2 &\le \alpha_m^2\|T^m x_m - z_n\|^2 + 2(1-\alpha_m)\langle f(x_m) - z_n, j(x_m - z_n)\rangle \\ &\le \alpha_m^2(\|T^m x_m - T^m z_n\| + \|T^m z_n - z_n\|)^2 \\ &\quad + 2(1-\alpha_m)\left[\langle f(x_m) - x_m, j(x_m - z_n)\rangle + \|x_m - z_n\|^2\right] \\ &\le \alpha_m^2(k_m\|x_m - z_n\| + \|T^m z_n - z_n\|)^2 \\ &\quad + 2(1-\alpha_m)\left(\langle f(x_m) - x_m, j(x_m - z_n)\rangle + k_m^2\|x_m - z_n\|^2\right) \\ &= \alpha_m^2\left(k_m^2\|x_m - z_n\|^2 + 2k_m\|x_m - z_n\|\|T^m z_n - z_n\|\right. \\ &\qquad \left. + \|T^m z_n - z_n\|^2\right) \\ &\quad + 2(1-\alpha_m)\left(\left\langle f(x_m) - x_m, j(x_m - z_n) + k_m^2\|x_m - z_n\|^2\right\rangle\right)\end{aligned}$$

$$\begin{aligned}
&= (1-(1-\alpha_m))^2 k_m^2 \|x_m - z_n\|^2 \\
&\quad + \|T^m z_n - z_n\| (2k_m \|x_m - z_n\| + \|T^m z_n - z_n\|) \\
&\quad + 2(1-\alpha_m)\left(\langle f(x_m) - x_m, j(x_m - z_n)\rangle + k_m^2 \|x_m - z_n\|^2\right) \\
&\le \left(1 + (1-\alpha_m)^2\right) k_m^2 \|x_m - z_n\|^2 \\
&\quad + \|T^m z_n - z_n\| (2k_m \|x_m - z_n\| + \|T^m z_n - z_n\|) \\
&\quad + 2(1-\alpha_m)\langle f(x_m) - x_m, j(x_m - z_n)\rangle.
\end{aligned} \tag{3.29}$$

Since K is bounded, for some constant $M > 0$, it follows that

$$\begin{aligned}
\limsup_{n\to\infty} \langle f(x_m) - x_m, j(z_n - x_m)\rangle \le{}& \frac{\left[k_m^2 - 1 + k_m^2 (1-\alpha_m)^2\right]}{1-\alpha_m} M \\
&+ \limsup_{n\to\infty} \frac{M\|z_n - T^m z_n\|}{1-\alpha_m},
\end{aligned} \tag{3.30}$$

so that

$$\limsup_{n\to\infty} \langle f(x_m) - x_m, j(z_n - x_m)\rangle \le \frac{\left[k_m^2 - 1 + k_m^2 (1-\alpha_m)^2\right]}{1-\alpha_m} M. \tag{3.31}$$

By Corollary 3.2, $x_m \to p \in F(T)$, which solve the variational inequality (3.16). Since j is norm to weak* continuous on bounded sets, in the limit as $m \to \infty$, we obtain that

$$\limsup_{n\to\infty} \langle f(p) - p, j(z_n - p)\rangle \le 0. \tag{3.32}$$

From Lemma 2.4, we estimate as follows:

$$\begin{aligned}
\|z_{n+1} - p\|^2 &= \|(1-\alpha_n)(f(z_n) - p) + \alpha\alpha_n (T^n z_n - p) + \beta\alpha_n (z_n - p)\|^2 \\
&\le \|\alpha\alpha_n (T^n z_n - p) + \beta\alpha_n (z_n - p)\|^2 \\
&\quad + 2(1-\alpha_n)\langle f(z_n) - p, j(z_{n+1} - p)\rangle \\
&\le \alpha^2\alpha_n^2 \|T^n z_n - p\|^2 + 2\alpha\beta\alpha_n^2 \|T^n z_n - p\| \|z_n - p\| \\
&\quad + \beta^2\alpha_n^2 \|z_n - p\|^2 + 2(1-\alpha_n)\langle f(z_n) - f(p), j(z_{n+1} - p)\rangle \\
&\quad + 2(1-\alpha_n)\langle f(p) - p, j(z_{n+1} - p)\rangle
\end{aligned}$$

$$
\begin{aligned}
&\leq \left(\alpha^2 k_n^2 + 2\alpha\beta k_n + \beta^2\right)\alpha_n^2 \|z_n - p\|^2 \\
&\quad + 2(1-\alpha_n)\gamma \|z_n - p\| \|z_{n+1} - p\| \\
&\quad + 2(1-\alpha_n)\langle f(p) - p, j(z_{n+1} - p)\rangle \\
&\leq \alpha_n^2 k_n^2 \|z_n - p\|^2 + \gamma(1-\alpha_n)\left(\|z_n - p\|^2 + \|z_{n+1} - p\|^2\right) \\
&\quad + 2(1-\alpha_n)\langle f(p) - p, j(z_{n+1} - p)\rangle,
\end{aligned}
\tag{3.33}
$$

so that

$$
\begin{aligned}
\|z_{n+1} - p\|^2 &\leq \frac{\left[t_n^2 + (1-\alpha_n)\gamma\right]}{1-(1-\alpha_n)\gamma}\|z_n - p\|^2 \\
&\quad + 2\frac{(1-\alpha_n)}{1-(1-\alpha_n)\gamma}\langle f(p) - p, j(z_{n+1} - p)\rangle \\
&= \left(1 - \frac{\left[1 - 2(1-\alpha_n)\gamma - t_n^2\right]}{1-(1-\alpha_n)\gamma}\right)\|z_n - p\|^2 \\
&\quad + 2\frac{(1-\alpha_n)}{1-(1-\alpha_n)\gamma}\langle f(p) - p, j(z_{n+1} - p)\rangle.
\end{aligned}
\tag{3.34}
$$

Let

$$
\lambda_n = \frac{\left[1 - 2(1-\alpha_n)\gamma - t_n^2\right]}{1-(1-\alpha_n)\gamma}. \tag{3.35}
$$

Consequently, following the lines of the proof of [4, Theorem 3.3], Theorem 3.3 is easily proved. □

From the lines of the proof of Theorem 3.3, we can obtain the following corollary.

Corollary 3.4. *Let E be a real Banach space with a uniformly Gateaux differentiable norm possessing uniform normal structure, K a nonempty closed convex and bounded subset of E, $T : K \to K$ an asymptotically nonexpansive mapping with sequence $\{k_n\}_n \subset [1,\infty)$, and $f : K \to K$ a contraction with constant $\gamma \in [0,1)$. Let $\{t_n\} \subset (0,\xi_n)$ be such that $\lim_{n\to\infty} t_n = 1$, $\sum_{n=1}^{\infty} t_n(1-t_n) = \infty$, and $\lim_{n\to\infty}((k_n - 1)/(k_n - t_n)) = 0$, where $\xi_n = \min\{(1-\gamma)k_n/(k_n-\gamma), 1/k_n\}$. For an arbitrary $z_0 \in K$, let the sequence $\{z_n\}_n$ be iteratively defined by* (1.12). *Then*

(i) *for each integer $n \geq 0$, there is a unique $x_n \in K$ such that*

$$
x_n = \left(1 - \frac{t_n}{k_n}\right) f(x_n) + \frac{t_n}{k_n} T^n x_n; \tag{3.36}
$$

if T satisfies $\lim_{n\to\infty}\|T^{n+1}x_n - T^n x_n\| = 0$ and $\lim_{n\to\infty}\|T^{n+1}z_n - T^n z_n\| = 0$ then

(ii) *the sequence $\{z_n\}_n$ converges strongly to the unique solution of the variational inequality:*

$$
p \in F(T) \text{ such that } \langle (I - f)p, j(p - x^*)\rangle \leq 0, \quad \forall x^* \in F(T). \tag{3.37}
$$

Remark 3.5. Since every nonexpansive mapping is asymptotically nonexpansive, our theorems hold for the case when T is simply nonexpansive. In this case, the boundedness requirement on K can be removed from the above results.

Remark 3.6. Our results can be viewed as a refinement and improvement of the corresponding results by Shahzad and Udomene [4], Chidume et al. [5], and Lim and Xu [13].

Example 3.7. Let $T : C \to C$ be a nonexpansive mapping. Let the iterative sequence $\{x_n\}$ be defined by

$$x_{n+1} = \frac{1}{n}u + \left(1 - \frac{1}{n}\right)Tx_n. \tag{3.38}$$

It is easy to see that $\{x_n\}$ converges strongly to some fixed point of T.

In particular, let $H = R^2$ and define $T : R^2 \to R^2$ by

$$T\left(re^{i\theta}\right) = re^{i(\theta+\pi/2)}, \tag{3.39}$$

and take that $u = e^{i\pi}$ is a fix element in C. It is obvious that T is a nonexpansive mapping with a unique fixed point $x^* = 0$. In this case, (3.38) becomes

$$x_{n+1} = \frac{1}{n}e^{i\pi} + \left(1 - \frac{1}{n}\right)r_n e^{i(\theta_n+\pi/2)}. \tag{3.40}$$

It is clear that the complex number sequence $\{x_n\}$ converges strongly to a fixed point $x^* = 0$.

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