

Research Article

A New Approach to the Approximation of Common Fixed Points of an Infinite Family of Relatively Quasinonexpansive Mappings with Applications

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Received 23 June 2012; Revised 16 August 2012; Accepted 21 August 2012

Academic Editor: Simeon Reich

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By using a specific way of choosing the indexes, we propose an iteration algorithm generated by the monotone CQ method for approximating common fixed points of an infinite family of relatively quasinonexpansive mappings. A strong convergence theorem without the stronger assumptions of the AKTT condition and the *AKTT condition imposed on the involved mappings is established in the framework of Banach space. As application, an iterative solution to a system of equilibrium problems is studied. The result is more applicable than those of other authors with related interest.

1. Introduction

Let C be a nonempty and closed convex subset of a real Banach space E . A mapping $T : C \rightarrow E$ is said to be nonexpansive if

$$\|Tx - Ty\| \leq \|x - y\|, \quad \forall x, y \in C. \quad (1.1)$$

A mapping T is said to be quasi-nonexpansive if $F(T) := \{x \in C : x = Tx\} \neq \emptyset$ and

$$\|Tx - p\| \leq \|x - p\|, \quad \forall x \in C, p \in F(T). \quad (1.2)$$

It is easy to see that if T is nonexpansive with $F(T) \neq \emptyset$, then it is quasi-nonexpansive. There are many methods for approximating fixed points of quasi-nonexpansive mappings. In 1953, Mann [1] introduced the iteration as follows: a sequence $\{x_n\}$ is defined by

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n)Tx_n, \quad (1.3)$$

where the initial element $x_0 \in C$ is arbitrary and $\{\alpha_n\}$ is a sequence of real numbers in $[0, 1]$. Approximation of fixed points of nonexpansive mappings via Mann's algorithm has extensively been investigated. One of the fundamental convergence results was proved by Reich [2]. In infinite-dimensional Hilbert spaces, Mann iteration can yield only weak convergence (see [3, 4]).

Attempts to modify the Mann iteration method (1.3) for strong convergence have recently been made. Nakajo and Takahashi [5] proposed the following modification of Mann iteration method (1.3) for a nonexpansive mapping T from C into itself in a Hilbert space: from an arbitrary $x_0 \in C$,

$$\begin{aligned} y_n &= \alpha_n x_n + (1 - \alpha_n)Tx_n, \\ C_n &= \{z \in C : \|y_n - z\| \leq \|x_n - z\|\}, \\ Q_n &= \{z \in C : \langle x_n - z, x_0 - x_n \rangle \geq 0\}, \\ x_{n+1} &= P_{C_n \cap Q_n} x_0, \quad \forall n \geq 0, \end{aligned} \quad (1.4)$$

where P_K denotes the metric projection from a Hilbert space H onto a closed convex subset K of H . They proved that the sequence $\{x_n\}$ converges strongly to $P_{F(T)}x_0$.

Recently, Su and Qin [6] introduced a monotone CQ method for nonexpansive mapping, defined as follows: from an arbitrary $x_0 \in C$,

$$\begin{aligned} y_n &= \alpha_n x_n + (1 - \alpha_n)Tx_n, \\ C_0 &= \{z \in C : \|y_0 - z\| \leq \|x_0 - z\|\}, \quad Q_0 = C, \\ C_n &= \{z \in C_{n-1} \cap Q_{n-1} : \|y_n - z\| \leq \|x_n - z\|\}, \\ Q_n &= \{z \in C_{n-1} \cap Q_{n-1} : \langle x_n - z, x_0 - x_n \rangle \geq 0\}, \\ x_{n+1} &= P_{C_n \cap Q_n} x_0, \quad \forall n \geq 0, \end{aligned} \quad (1.5)$$

and it proved that the sequence $\{x_n\}$ converges strongly to $P_{F(T)}x_0$.

We now recall some definitions concerning relatively quasi-nonexpansive mappings. Let E be a real smooth Banach space with norm $\|\cdot\|$ and let E^* be the dual of E . The normalized duality mapping J from E to E^* is defined by

$$Jx = \left\{ f \in E^* : \langle x, f \rangle = \|x\|^2 = \|f\|^2 \right\}, \quad \forall x \in E, \quad (1.6)$$

where $\langle \cdot, \cdot \rangle$ denotes the pairing between E and E^* . Readers are directed to [7] (and its review [8]), where the properties of the duality mapping and several related topics are presented. The function $\phi : E \times E \rightarrow \mathbb{R}^+$ is defined by

$$\phi(x, y) = \|x\|^2 - 2\langle x, Jy \rangle + \|y\|^2, \quad \forall x, y \in E. \quad (1.7)$$

Let T be a mapping from C into E . A point p in C is said to be an *asymptotic fixed point* [9] of T if C contains a sequence $\{x_n\}$ which converges weakly to p and $\lim_{n \rightarrow \infty} (x_n - Tx_n) = 0$. The set of asymptotic fixed points of T is denoted by $\hat{F}(T)$.

We say that the mapping T is *relatively nonexpansive* (see [10]) if the following conditions are satisfied:

$$(R1) \ F(T) \neq \emptyset;$$

$$(R2) \ \phi(p, Tx) \leq \phi(p, x), \ \forall p \in F(T);$$

$$(R3) \ F(T) = \hat{F}(T).$$

If T satisfies (R1) and (R2), then T is called *relatively quasi-nonexpansive*.

Several articles have provided methods for approximating fixed points of relatively quasi-nonexpansive mappings [11–16]. Employing the ideas of Su and Qin [6], and of Aoyama et al. [17], in 2008, Nilsrakoo and Saejung [18] used the following iterations to obtain strong convergence theorems for common fixed points of a countable family of relatively quasi-nonexpansive mappings in a Banach space

$$\begin{aligned} x_0 &\in C, \quad C_{-1} = Q_{-1} = C; \\ y_n &= J^{-1}(\alpha_n Jx_n + (1 - \alpha_n)JT_n x_n), \\ C_n &= \{z \in C_{n-1} \cap Q_{n-1} : \phi(z, y_n) \leq \phi(z, x_n)\}, \\ Q_n &= \{z \in C_{n-1} \cap Q_{n-1} : \langle x_n - z, Jx_0 - Jx_n \rangle \geq 0\}, \\ x_{n+1} &= \Pi_n x_0, \quad \forall n \geq 0. \end{aligned} \quad (1.8)$$

However, the results were obtained under two stronger assumption conditions, namely, the *AKTT*-condition and the **AKTT*-condition imposed on the involved mappings.

Inspired and motivated by those studies mentioned above, in this paper, we use a modified type of the iteration scheme (1.8) for approximating common fixed points of an infinite family of relatively quasi-nonexpansive mappings; without stronger assumptions imposed on the involved mappings, a strong convergence theorem in Banach spaces is obtained for solving a system of equilibrium problems. The results improve those of other authors with related interest.

2. Preliminaries

Throughout the paper, let E be a real Banach space. We say that E is *strictly convex* if the following implication holds for $x, y \in E$:

$$\|x\| = \|y\| = 1, \quad x \neq y \implies \left\| \frac{x+y}{2} \right\| < 1. \quad (2.1)$$

It is also said to be *uniformly convex* if for any $\epsilon > 0$, there exists a $\delta > 0$ such that

$$\|x\| = \|y\| = 1, \quad \|x - y\| \geq \epsilon \implies \left\| \frac{x+y}{2} \right\| \leq 1 - \delta. \quad (2.2)$$

It is known that if E is uniformly convex Banach space, then E is reflexive and strictly convex. A Banach space E is said to be *smooth* if

$$\lim_{t \rightarrow 0} \frac{\|x + ty\| - \|x\|}{t} \quad (2.3)$$

exists for each $x, y \in S(E) := \{x \in E : \|x\| = 1\}$. In this case, the norm of E is said to be *Gâteaux differentiable*. The space E is said to have *uniformly Gâteaux differentiable norm* if for each $y \in S(E)$; the limit (2.3) is attained uniformly for $x \in S(E)$. The norm of E is said to be *Fréchet differentiable* if for each $x \in S(E)$; the limit (2.3) is attained uniformly for $y \in S(E)$. The norm of E is said to be *uniformly Fréchet differentiable* (and E is said to be *uniformly smooth*) if the limit (2.3) is attained uniformly for $x, y \in S(E)$.

We also know the following properties (see, e.g., [19] for details).

- (1) E (E^* , resp.) is uniformly convex $\Leftrightarrow E^*$ (E , resp.) is uniformly smooth.
- (2) $Jx \neq \emptyset$ for each $x \in E$.
- (3) If E is reflexive, then J is a mapping from E onto E^* .
- (4) If E is strictly convex, then $Jx \cap Jy = \emptyset$ as $x \neq y$.
- (5) If E is smooth, then J is single-valued.
- (6) If E has a Fréchet differentiable norm, then J is norm-to-norm continuous.
- (7) If E is uniformly smooth, then J is uniformly norm-to-norm continuous on each bounded subset of E .
- (8) If E is a Hilbert space, then J is the identity operator.

Let E be a smooth Banach space. The function $\phi : E \times E \rightarrow \mathbb{R}^+$ is defined by

$$\phi(x, y) := \|x\|^2 - 2\langle x, Jy \rangle + \|y\|^2. \quad (2.4)$$

It is obvious from the definition of the function ϕ that

$$(\|x\| - \|y\|)^2 \leq \phi(x, y) \leq (\|x\| + \|y\|)^2. \quad (2.5)$$

Moreover, we know the following results.

Lemma 2.1 (see [13]). *Let E be a strictly convex and smooth Banach space, then $\phi(x, y) = 0$ if and only if $x = y$.*

Lemma 2.2 (see [11]). *Let E be a uniformly convex and smooth Banach space and let $r > 0$. Then there exists a continuous, strictly increasing, and convex function $g : [0, 2r] \rightarrow [0, \infty)$ such that $g(0) = 0$ and*

$$g(\|x\| - \|y\|) \leq \phi(x, y) \quad (2.6)$$

for all $x, y \in B_r := \{z \in E : PzP \leq r\}$.

Let C be a nonempty and closed convex subset of E . Suppose that E is reflexive, strictly convex, and smooth. It is known in [20] that for any $x \in E$, there exists a unique point $x^* \in C$ such that

$$\phi(x^*, x) = \min_{y \in C} \phi(y, x). \quad (2.7)$$

Following Alber [21], we denote such an x^* by $\Pi_C x$. The mapping Π_C is called the *generalized projection* from E onto C . It is easy to see that in a Hilbert space, the mapping Π_C coincides with the metric projection P_C . What follows are the well-known facts concerning the generalized projection.

Lemma 2.3 (see [20]). *Let C be a nonempty closed convex subset of a smooth Banach space E and let $x \in E$. Then*

$$x^* = \Pi_C x \iff \langle x^* - y, Jx - Jx^* \rangle \geq 0, \quad \forall y \in C. \quad (2.8)$$

Lemma 2.4 (see [20]). *Let E be a reflexive, strictly convex, and smooth Banach space, let C be a nonempty closed convex subset of E , and let $x \in E$. Then*

$$\phi(y, \Pi_C x) + \phi(\Pi_C x, x) \leq \phi(y, x), \quad \forall y \in C. \quad (2.9)$$

Dealing with the generalized projection from E onto the fixed point set of a relatively quasi-nonexpansive mapping, we have the following result.

Lemma 2.5 (see [18]). *Let E be a strictly convex and smooth Banach space, let C be a nonempty and closed convex subset of E , and let T be a relatively quasi-nonexpansive mapping from C into E . Then $F(T)$ is closed and convex.*

Let C be a subset of a Banach space E and let $\{T_n\}$ be a family of mappings from C into E . For a subset B of C , we say that

(i) $(\{T_n\}, B)$ satisfies AKTT-condition if

$$\sum_{n=1}^{\infty} \sup\{\|T_{n+1}z - T_n z\| : z \in B\} < \infty; \quad (2.10)$$

(ii) $(\{T_n\}, B)$ satisfies * AKTT-condition if

$$\sum_{n=1}^{\infty} \sup\{\|JT_{n+1}z - JT_nz\| : z \in B\} < \infty. \quad (2.11)$$

3. Main Results

Recall that an operator T in a Banach space is *closed* if $x_n \rightarrow x$ and $Tx_n \rightarrow y$ as $n \rightarrow \infty$, then $Tx = y$.

Theorem 3.1. *Let E be a uniformly convex and uniformly smooth Banach space, C a nonempty and closed convex subset of E . Let $\{T_i\}_{i=1}^{\infty} : C \rightarrow E$ be a sequence of closed and relatively quasi-nonexpansive mappings with $F := \bigcap_{i=1}^{\infty} F(T_i) \neq \emptyset$. Starting from an arbitrary $x_1 \in C$, the sequence $\{x_n\}$ is define by*

$$\begin{aligned} x_1 &\in C, & C_0 &= Q_0 = C; \\ y_n &= J^{-1}(\alpha_n Jx_n + (1 - \alpha_n)JT_{i_n}x_n), \\ C_n &= \{z \in C_{n-1} \cap Q_{n-1} : \phi(z, y_n) \leq \phi(z, x_n)\}, \\ Q_n &= \{z \in C_{n-1} \cap Q_{n-1} : \langle x_n - z, Jx_1 - Jx_n \rangle \geq 0\}, \\ x_{n+1} &= \Pi_n x_1, \quad \forall n \geq 1, \end{aligned} \quad (3.1)$$

where $\Pi_n := \Pi_{C_n \cap Q_n}$ and $\{\alpha_n\}$ is a sequence in $[0, 1)$ with $\limsup_{n \rightarrow \infty} \alpha_n < 1$; i_n is the solution to the positive integer equation: $n = i + (m - 1)m/2$ ($m \geq i, n = 1, 2, \dots$), that is, for each $n \geq 1$, there exists a unique i_n such that

$$\begin{aligned} i_1 &= 1, & i_2 &= 1, & i_3 &= 2, & i_4 &= 1, & i_5 &= 2, & i_6 &= 3, & i_7 &= 1, & i_8 &= 2, \\ i_9 &= 3, & i_{10} &= 4, & i_{11} &= 1, \dots \end{aligned} \quad (3.2)$$

Then $\{x_n\}$ converges strongly to $\Pi_F x_1$.

Proof. We first claim that both C_n and Q_n are closed and convex. This follows from the fact that $\phi(z, y_n) \leq \phi(z, x_n)$ is equivalent to the following:

$$2\langle z, Jx_n - Jy_n \rangle \leq \|x_n\|^2 - \|y_n\|^2. \quad (3.3)$$

It is clear that $F \subset C = C_0 \cap Q_0$. Next, we show that

$$F \subset C_n \cap Q_n, \quad \forall n \geq 1. \quad (3.4)$$

Suppose that $F \subset C_{k-1} \cap Q_{k-1}$ for some $k \geq 2$. Letting $p \in F$, we then have

$$\begin{aligned}
 \phi(p, y_k) &= \phi\left(p, J^{-1}(\alpha_k Jx_k + (1 - \alpha_k)JT_{i_k} - x_k)\right) \\
 &= \|p\|^2 - 2\langle p, \alpha_k Jx_k + (1 - \alpha_k)JT_{i_k}x_k \rangle + \|\alpha_k Jx_k + (1 - \alpha_k)JT_{i_k}x_k\|^2 \\
 &\leq \|p\|^2 - 2\alpha_k \langle p, Jx_k \rangle - 2(1 - \alpha_k) \langle p, JT_{i_k}x_k \rangle + \alpha_k \|x_k\|^2 + (1 - \alpha_k) \|T_{i_k}x_k\|^2 \\
 &= \alpha_k \left(\|p\|^2 - 2\langle p, Jx_k \rangle + \|x_k\|^2 \right) + (1 - \alpha_k) \left(\|p\|^2 - 2\langle p, JT_{i_k}x_k \rangle + \|T_{i_k}x_k\|^2 \right) \quad (3.5) \\
 &= \alpha_k \phi(p, x_k) + (1 - \alpha_k) \phi(p, T_{i_k}x_k) \\
 &\leq \alpha_k \phi(p, x_k) + (1 - \alpha_k) \phi(p, x_k) \\
 &= \phi(p, x_k).
 \end{aligned}$$

This implies that $F \subset C_k$. It follows from $x_k = \Pi_{k-1}x_1$ and Lemma 2.3 that

$$\langle x_k - z, Jx_1 - Jx_k \rangle \geq 0, \quad \forall z \in C_{k-1} \cap Q_{k-1}. \quad (3.6)$$

Particularly,

$$\langle x_k - z, Jx_1 - Jx_k \rangle \geq 0, \quad \forall p \in F \quad (3.7)$$

and hence $F \subset Q_k$, which yields that

$$F \subset C_k \cap Q_k. \quad (3.8)$$

By induction, (3.4) holds. This implies that $\{x_n\}$ is well defined. It follows from the definition of Q_n and Lemma 2.3 that $x_n = \Pi_{Q_n}x_1$. Since $x_{n+1} = \Pi_n x_1 \in Q_n$, we have

$$\phi(x_n, x_1) \leq \phi(x_{n+1}, x_1), \quad \forall n \geq 1. \quad (3.9)$$

Therefore, $\{\phi(x_n, x_1)\}$ is nondecreasing. Using $x_n = \Pi_{Q_n}x_1$ and Lemma 2.4, we have

$$\phi(x_n, x_1) = \phi(\Pi_{Q_n}x_1, x_1) \leq \phi(p, x_1) - \phi(p, x_n) \leq \phi(p, x_1) \quad (3.10)$$

for all $p \in F$ and for all $n \geq 1$, that is, $\{\phi(x_n, x_1)\}$ is bounded. Then

$$\lim_{n \rightarrow \infty} \phi(x_n, x_1) \text{ exists.} \quad (3.11)$$

In particular, by (2.5), the sequence $\{(\|x_n\| - \|x_1\|)^2\}$ is bounded. This implies that $\{x_n\}$ is bounded. Note again that $x_n = \Pi_{Q_n}x_1$ and for any positive integer k , $x_{n+k} \in Q_{n+k-1} \subset Q_n$. By Lemma 2.4,

$$\begin{aligned}\phi(x_{n+k}, x_n) &= \phi(x_{n+k}, \Pi_{Q_n}x_1) \\ &\leq \phi(x_{n+k}, x_1) - \phi(\Pi_{Q_n}x_1, x_1) \\ &= \phi(x_{n+k}, x_1) - \phi(x_n, x_1).\end{aligned}\tag{3.12}$$

By Lemma 2.2, we have, for any positive integers m, n with $m > n$,

$$g(\|x_m - x_n\|) \leq \phi(x_m, x_n) \leq \phi(x_m, x_1) - \phi(x_n, x_1),\tag{3.13}$$

where $g : [0, \infty) \rightarrow [0, \infty)$ is a continuous, strictly increasing, and convex function with $g(0) = 0$. Then the properties of the function g yield that $\{x_n\}$ is a Cauchy sequence in C , so there exists an $x^* \in C$ such that

$$x_n \longrightarrow x^* \quad (n \longrightarrow \infty).\tag{3.14}$$

In view of $x_{n+1} = \Pi_n x_1 \in C_n$ and the definition of C_n , we also have

$$\phi(x_{n+1}, y_n) \leq \phi(x_{n+1}, x_n), \quad \forall n \geq 1.\tag{3.15}$$

This implies that

$$\lim_{n \rightarrow \infty} \phi(x_{n+1}, y_n) = \lim_{n \rightarrow \infty} \phi(x_{n+1}, x_n) = 0.\tag{3.16}$$

It follows from Lemma 2.2 that

$$\lim_{n \rightarrow \infty} \|x_{n+1} - y_n\| = \lim_{n \rightarrow \infty} \|x_{n+1} - x_n\| = 0.\tag{3.17}$$

Since J is uniformly norm-to-norm continuous on bounded sets, we have

$$\lim_{n \rightarrow \infty} \|Jx_{n+1} - Jy_n\| = \lim_{n \rightarrow \infty} \|Jx_{n+1} - Jx_n\| = 0.\tag{3.18}$$

On the other hand, we have, for each $n \geq 1$,

$$\begin{aligned}\|Jx_{n+1} - Jy_n\| &= \|Jx_{n+1} - (\alpha_n Jx_n + (1 - \alpha_n)JT_{i_n}x_n)\| \\ &= \|(1 - \alpha_n)(Jx_{n+1} - JT_{i_n}x_n) - \alpha_n(Jx_n - Jx_{n+1})\| \\ &\geq (1 - \alpha_n)\|Jx_{n+1} - JT_{i_n}x_n\| - \alpha_n\|Jx_n - Jx_{n+1}\|,\end{aligned}\tag{3.19}$$

and hence

$$\|Jx_{n+1} - JT_{i_n}x_n\| \leq \frac{1}{1 - \alpha_n} \|Jx_{n+1} - Jy_n\| + \frac{\alpha_n}{1 - \alpha_n} \|Jx_n - Jx_{n+1}\|. \quad (3.20)$$

From (3.18) and $\limsup_{n \rightarrow \infty} \alpha_n < 1$, we obtain that

$$\lim_{n \rightarrow \infty} \|Jx_{n+1} - JT_{i_n}x_n\| = 0. \quad (3.21)$$

Since J^{-1} is uniformly norm-to-norm continuous on bounded sets, we have

$$\lim_{n \rightarrow \infty} \|x_{n+1} - T_{i_n}x_n\| = \lim_{n \rightarrow \infty} \|J^{-1}(Jx_{n+1}) - J^{-1}(JT_{i_n}x_n)\| = 0. \quad (3.22)$$

It follows from (3.17) that, as $n \rightarrow \infty$,

$$\|x_n - T_{i_n}x_n\| \leq \|x_n - x_{n+1}\| + \|x_{n+1} - T_{i_n}x_n\| \rightarrow 0. \quad (3.23)$$

Now, set $\mathcal{K}_i = \{k \geq 1 : k = i + (m-1)m/2, m \geq i, m \in \mathbb{Z}^+\}$ for each $i \geq 1$. Note that $T_{i_k} = T_i$ whenever $k \in \mathcal{K}_i$. For example, by the definition of $\mathcal{K}_1$, we have $\mathcal{K}_1 = \{1, 2, 4, 7, 11, 16, \dots\}$ and $i_1 = i_2 = i_4 = i_7 = i_{11} = i_{16} = \dots = 1$. Then it follows from (3.23) that

$$\lim_{\mathcal{K}_i \ni k \rightarrow \infty} \|T_i x_k - x_k\| = 0, \quad \forall i \geq 1. \quad (3.24)$$

Since $\{x_k\}_{k \in \mathcal{K}_i}$ is a subsequence of $\{x_n\}$, (3.14) implies that $x_k \rightarrow x^*$ as $\mathcal{K}_i \ni k \rightarrow \infty$. It immediately follows from (3.24) and the closedness of T_i that $x^* \in F(T_i)$ for each $i \geq 1$, and hence $x^* \in F$. Furthermore, by (3.10),

$$\phi(x_*, x_1) = \lim_{n \rightarrow \infty} \phi(x_n, x_1) \leq \phi(p, x_1), \quad \forall p \in F. \quad (3.25)$$

This implies that $x^* = \Pi_F x_1$. The proof is completed. $\square$

Remark 3.2. Note that the algorithm (3.1) is based on the projection onto an intersection of two closed and convex sets. An example [22] of how to compute such a projection is given as follows.

Dijkstra's Algorithm

Let $\Omega_1, \Omega_2, \dots, \Omega_p$ be closed and convex subsets of $\mathbb{R}^n$. For any $i = 1, 2, \dots, p$ and $x^0 \in \mathbb{R}^n$, the sequences $\{x_i^k\}$ are defined by the following recursive formulae:

$$\begin{aligned} x_0^k &= x_p^{k-1}, \\ x_i^k &= P_{\Omega_i} \left(x_{i-1}^k - y_i^{k-1} \right), \quad i = 1, 2, \dots, p, \\ y_i^k &= x_i^k - \left(x_{i-1}^k - y_i^{k-1} \right), \quad i = 1, 2, \dots, p, \end{aligned} \quad (3.26)$$

for $k = 1, 2, \dots$ with initial values $x_p^0 = x^0$ and $y_i^0 = 0$ for $i = 1, 2, \dots, p$. If $\Omega := \bigcap_{i=1}^p \Omega_i \neq \emptyset$, then $\{x_i^k\}$ converges to $x^* = P_{\Omega}(x^0)$, where $P_{\Omega}(x) := \arg \inf_{y \in \Omega} \|y - x\|^2$, for all $x \in \mathbb{R}^n$.

4. Applications

The so-called *convex feasibility problem* for a family of mappings $\{T_i\}_{i=1}^{\infty}$ is to find a point in the nonempty intersection $\bigcap_{i=1}^{\infty} F(T_i)$, which exactly illustrates the importance of finding fixed points of infinite families. The following example also clarifies the same thing.

Example 4.1. Let E be a smooth, strictly convex, and reflexive Banach space, C a nonempty and closed convex subset of E , and $\{f_i\}_{i=1}^{\infty} : C \rightarrow C$ a countable family of bifunctions satisfying the conditions: for each $i \geq 1$,

- (A₁) $f_i(x, x) = 0$;
- (A₂) f_i is monotone, that is, $f_i(x, y) + f_i(y, x) \leq 0$;
- (A₃) $\limsup_{t \downarrow 0} f_i(x + t(z - x), y) \leq f_i(x, y)$;
- (A₄) the mapping $y \mapsto f_i(x, y)$ is convex and lower semicontinuous.

A system of equilibrium problems for $\{f_i\}_{i=1}^{\infty}$ is to find an $x^* \in C$ such that

$$f_i(x^*, y) \geq 0, \quad \forall y \in C, \quad i \geq 1, \quad (4.1)$$

whose set of common solutions is denoted by $EP := \bigcap_{i=1}^{\infty} EP(f_i)$, where $EP(f_i)$ denotes the set of solutions to the equilibrium problem for f_i ($i = 1, 2, \dots$). It will be shown in Theorem 4.3 that such a system of problems can be reduced to approximation of some fixed points of a countable family of nonexpansive mappings.

Example 4.2 (see [23]). Let $r > 0$. Define a countable family of mappings $\{T_{r,i}\}_{i=1}^{\infty} : E \rightarrow C$ as follows:

$$T_{r,i}(x) = \left\{ z \in C : f_i(z, y) + \frac{1}{r} \langle y - z, Jz - Jx \rangle \geq 0, \forall y \in C \right\}, \quad \forall i \geq 1. \quad (4.2)$$

Then we have that

- (1) $\{T_{r,i}\}_{i=1}^{\infty}$ is a sequence of single-valued mappings;
- (2) $\{T_{r,i}\}_{i=1}^{\infty}$ is a sequence of closed relatively quasi-nonexpansive mappings;
- (3) $F := \bigcap_{i=1}^{\infty} F(T_{r,i}) = EP$.

Now, we have the following result.

Theorem 4.3. *Let C, E , and $\{\alpha_n\}$ be the same as those in Theorem 3.1. Let $\{f_i\}_{i=1}^{\infty} : C \rightarrow C$ be a countable family of bifunctions satisfying the conditions (A_1) – (A_4) . Let $\{T_{r,i}\}_{i=1}^{\infty} : E \rightarrow C$ be a countable family of mappings defined by (4.2). Let $\{x_n\}$ be the sequence generated by*

$$\begin{aligned}
 x_1 &\in C, \quad C_0 = Q_0 = C; \\
 f_{i_n}(u_n, y) + \frac{1}{r} \langle y - u_n, Ju_n - Jx_n \rangle &\geq 0, \quad \forall y \in C, \\
 y_n &= J^{-1}(\alpha_n Jx_n + (1 - \alpha_n)Ju_n), \\
 C_n &= \{z \in C_{n-1} \cap Q_{n-1} : \phi(z, y_n) \leq \phi(z, x_n)\}, \\
 Q_n &= \{z \in C_{n-1} \cap Q_{n-1} : \langle x_n - z, Jx_1 - Jx_n \rangle \geq 0\}, \\
 x_{n+1} &= \Pi_n x_1, \quad \forall n \geq 1,
 \end{aligned} \tag{4.3}$$

where i_n satisfies the positive integer equation: $n = i + (m - 1)m/2$ ($m \geq i, n = 1, 2, \dots$). If $F := \bigcap_{i=1}^{\infty} F(T_{r,i}) \neq \emptyset$, then $\{x_n\}$ strongly converges to $\Pi_F x_1$ which is a common solution of the system of equilibrium problems for $\{f_i\}_{i=1}^{\infty}$.

Proof. Since each $T_{r,i}$ is single-valued, $u_n = T_{r,i_n}x_n$ for all $n \geq 1$. In addition, we have pointed out in Example 4.2 that $F = EP$ and $\{T_{r,i}\}_{i=1}^{\infty}$ is a sequence of closed relatively quasi-nonexpansive mappings. Hence, (4.3) can be rewritten as follows:

$$\begin{aligned}
 x_1 &\in C, \quad C_0 = Q_0 = C; \\
 y_n &= J^{-1}(\alpha_n Jx_n + (1 - \alpha_n)JT_{r,i_n}x_n), \\
 C_n &= \{z \in C_{n-1} \cap Q_{n-1} : \phi(z, y_n) \leq \phi(z, x_n)\}, \\
 Q_n &= \{z \in C_{n-1} \cap Q_{n-1} : \langle x_n - z, Jx_1 - Jx_n \rangle \geq 0\}, \\
 x_{n+1} &= \Pi_n x_1, \quad \forall n \geq 1.
 \end{aligned} \tag{4.4}$$

Therefore, this conclusion can be obtained immediately from Theorem 3.1. $\square$

Acknowledgments

The author is greatly grateful to the referees for their useful suggestions by which the contents of this article are improved. This study was supported by the General Project of Scientific Research Foundation of Yunnan University of Finance and Economics (YC2012A02).

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