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Research Article

Boundedness of Global Solutions for a Heat Equation with Exponential Gradient Source

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We consider a one-dimensional semilinear parabolic equation with exponential gradient source and provide a complete classification of large time behavior of the classical solutions: either the space derivative of the solution blows up in finite time with the solution itself remaining bounded or the solution is global and converges in C^1 norm to the unique steady state. The main difficulty is to prove C^1 boundedness of all global solutions. To do so, we explicitly compute a nontrivial Lyapunov's functional by carrying out the method of Zelenyak.

1. Introduction and Main Results

We consider the problem:

$$\begin{aligned} &u_t = u_{xx} + e^{u_x}, \quad 0 < x < 1,\ t > 0, \\ &u(0,t) = 0, \quad u(1,t) = A, \quad t > 0, \\ &u(x,0) = u_0(x), \quad 0 < x < 1. \end{aligned} \tag{1.1}$$

Here $A > 0$ is a constant, and the initial data u_0 belongs to the space $X = \{v \in C^1([0,1]);\ v(0) = 0,\ v(1) = A\}$ with the C^1 norm. The problem (1.1) admits a unique maximum classical solution $u = u(u_0;\cdot,t)$, whose existence time will be denoted by $T = T^*(u_0) \in (0,\infty]$. Note that we make no restriction on the signs of u or u_x.

The differential equation in (1.1) possesses both mathematical and physical interest. It can serve as a typical model case in the theory of parabolic PDEs. Indeed, it is the one of the simplest examples (along with Burger's equation) of a parabolic equation with a nonlinearity depending on the first-order spatial derivatives of u. On the other hand, this equation (and its N-dimensional version) arises in the viscosity approximation of the Hamilton-Jacobi-type equations from stochastic control theory [1] and in some physical models of surface growth [2].

The aim of this paper is to provide a complete classification of large time behavior of the solutions of (1.1). A basic fact about (1.1) is that the solutions satisfy a maximum principle:

$$\min_{[0,1]} u_0 \leq u(x,t) \leq \max_{[0,1]} u_0, \quad 0 \leq x \leq 1,\ 0 \leq t < T^*. \tag{1.2}$$

Since problem (1.1) is well posed in C^1, therefore, only three possibilities can occur as follows.

(1) u exists globally and is bounded in C^1:

$$T^* = \infty \qquad \sup_{t\geq 0} \|u_x(\cdot,t)\|_\infty < \infty. \tag{1.3}$$

Moreover, due to the results in [3] (see the last part of this Introduction section for more details), u has to converge in C^1 to a steady state (which is actually unique when it exists).

(2) u blows up in finite time in C^1 norm (finite time gradient blowup):

$$T^* < \infty \qquad \lim_{t\to T^*} \|u_x(\cdot,t)\|_\infty = \infty. \tag{1.4}$$

(3) u exists globally but is unbounded in C^1 (infinite time gradient blowup):

$$T^* = \infty \qquad \limsup_{t\to \infty} \|u_x(\cdot,t)\|_\infty = \infty. \tag{1.5}$$

In [4], the first author and Hu studied the case (2) and got estimates on the gradient blowup rate under the assumptions on the initial data so that the solution is monotone in x and in t. In the present paper, our primary goal is to exclude (3), that is, infinite time gradient blowup. For the boundedness of global solutions of other problems, for example, the equation $u_t = u_{xx} + |u_x|^p$ with $p > 2$, we refer to [5] and the references therein.

For $A > 0$, the situation is slightly more involved. There exists a critical value

$$A_c = 1 \tag{1.6}$$

such that (1.1) has a unique steady-state V_A if $A < A_c$ and no steady state if $A > A_c$ (the explicit formula for V_A is recalled at the beginning of Section 2). In the critical case $A = A_c$, there still exists a steady-state V_{A_c}, but it is singular, satisfying $V_{A_c} \in C([0,1]) \cap C^2((0,1])$ with $V_{A_c,x}(0) = \infty$.

Theorem 1.1. *Assume $0 < A < A_c$. Then all global solutions of (1.1) are bounded in C^1. In other words, (3) cannot occur. Moreover, they converge in C^1 norm to V_A.*

For the case $A > A_c$, we improve the result by removing the restrictions $u_0 \geq 0$ and $(u_0)_x \geq 0$ on the initial data. Then all solutions of (1.1) blow up in finite time in C^1 norm.

Remark 1.2. In the critical case $A = A_c$, all solutions have to blow up in C^1 in either finite or infinite time. Moreover, if (3) occurs, then the solution will converge in $C([0,1])$ to the singular steady-state V_{A_c}, as $t \to \infty$. This follows from Proposition 3.2 below. However, the possibility of (3) remains an open problem in this case. We conjecture that this could occur.

As a consequence of our results, we exhibit the following interesting situation: although C^1 boundedness of global solutions is true, the global solutions of (1.1) do not satisfy a uniform a priori estimate, that is, the supremum in (1) cannot be estimated in terms of the norm of the initial data. In other words, there exists a bounded, even compact, subset $\mathcal{S} \subset X$, such that the trajectories starting from $\mathcal{S}$ describe an unbounded subset of X, although each of them is individually bounded and converges to the same limit. As a further consequence, the existence time T^*, defined as a function from X into $(0,\infty]$, is not (upper semi) continuous.

Proposition 1.3. *Assume $0 < A < A_c$. There exists $u_0 \in X$ and a sequence $\{u_{0,n}\}$ in X with the following properties:*

(a) $u_{0,n} \to u_0$ *in* C^1,

(b) $T^*(u_{0,n}) = \infty$ *for each* n, *and* $T^*(u_0) < \infty$,

(c) $\sup_{t\geq 0}\|(u_n)_x(\cdot,t)\|_\infty =: K_n \to \infty$.

To explain the ideas of our proof, let us first recall that, in a classical paper [3], Zelenyak showed that any one-dimensional quasilinear uniformly parabolic equation possesses a (strict) Lyapunov's functional, of the form:

$$\mathcal{L}(u(t)) = \int_0^1 \phi(u(x,t), u_x(x,t))dx. \tag{1.7}$$

The construction of ϕ is in principle explicit, although too complicated to be completely computed in most situations. As a consequence, for any solution u of (1.1) which is global and bounded in C^1, the (nonempty) w-limit set of u consists of equilibria. Since (1.1) admits at most one equilibrium V, such u has to converge to V. (In fact, it was also proved in [3] that whether or not equilibria are unique, any bounded solution of a one-dimensional uniformly parabolic equation converges to an equilibrium, but this need not concern us here.) For $A > 0$, our proof proceeds by contradiction and makes essential use of the Zelenyak construction. It consists of three steps as follows.

Assuming that a C^1 unbounded global solution would exist, we analyze its possible final singularities (along a sequence $t_n \to \infty$). We shall show that u_x remains bounded away from the left boundary and describe the shape of u_x near the boundary (cf. Section 2).

We shall carry out the Zelenyak construction in a sufficiently precise way to determine the density $\phi(u,v)$ of the Lyapunov functional. It will turn out that, whenever u remains in a bounded set of $\mathbb{R}$ (as it does here in view of the estimate (1.2)), $\phi(u,v)$ remains bounded from below *uniformly with respect to* v (see Proposition 3.1).

Using this property of ϕ in the classical Lyapunov's argument, together with the fact that singularities may occur only near the boundary, it will be possible to prove the following convergence result: any global solution, even unbounded in C^1, has to converge in $C([0,1])$

to a stationary solution W of (1.1) with $W(0) = 0$, $W(1) = A$ (see Proposition 3.2). On the other hand, if u were bounded, then our estimates would imply $W_x(0) = \infty$. But such a W is not available if $A \neq A_c$, leading to a contradiction.

2. Preliminary Estimates

We start with some preliminary estimates. They are collected in Lemmas 2.1–2.6.

Lemma 2.1. *Let u be a maximal solution of* (1.1). *For all $t_0 \in (0, T^*)$, there exists $C_1 > 0$ such that*

$$|u_t| \leq C_1, \quad 0 \leq x \leq 1, \ t_0 \leq t < T^*. \tag{2.1}$$

Proof. The function $h = u_t$ satisfies

$$\begin{gathered} h_t = h_{xx} + e^{u_x} h_x, \quad 0 < x < 1, \ t_0 < t < T^*, \\ h(0,t) = h(1,t) = 0, \quad t_0 < t < T^*, \\ h(x,t_0) = u_{xx}(x,t_0) + e^{u_x(t_0,x)}, \quad 0 < x < 1. \end{gathered} \tag{2.2}$$

It follows from the maximum principle that $|h| \leq \|h(t_0)\|_\infty$ in $[0,1] \times [t_0, T^*)$. □

Remark 2.2. Although the second-order compatibility condition is not assumed, the maximum principle is still valid for u_t. In fact, the system can be approximated by boundary data satisfying the second-order compatibility condition and taking the limit, or another simpler argument (without approximation procedure) is this: since $u_0 \in H^2 \cap H_0^1$, standard regularity results imply $u_t \in C([t_0, T); L^2)$, which is enough to apply the (weak) Stampacchia maximum principle to the function $h = u_t$ (which satisfies $h_t = h_{xx} + a(x,t)h_x$ with $a(x,t)$ bounded near $t = t_0$).

The following two lemmas give upper and lower bounds on u_x which show, in particular, that u_x remains bounded away from the boundary.

Lemma 2.3. *Let u be a maximal solution of* (1.1). *For all $t_0 \in (0, T^*)$, there exists $C_1 > 0$ such that, for all $0 \leq x \leq 1$ and $t_0 \leq t < T^*$,*

$$u_x(x,t) \leq C_1 x + \ln \frac{1}{x + e^{-u_x(0,t)}}, \tag{2.3}$$

$$u_x(1-x,t) \geq -C_1 x - \ln \frac{1}{x + e^{-u_x(0,t)}}. \tag{2.4}$$

Proof. Fix $t \in [t_0, T^*)$ and let $y(x) = (u_x(x,t) - C_1 x)_+$, where C_1 is given by Lemma 2.1. The function y satisfies

$$y' + e^y = (u_{xx} - C_1)|_{\{u_x > C_1 x\}} + e^{(u_x - C_1 x)_+}. \tag{2.5}$$

For each x such that $u_x(x,t) > C_1 x$, we have $y' + e^y \leq u_{xx} - C_1 + e^{u_x} \leq 0$ by Lemma 2.1. Therefore, we have $y' + e^y \leq 0$ on $(0,1)$. By integration, it follows that $y(x) \leq \ln 1/[x + e^{-u_x(0,t)}]$, hence, (2.3).

As for (2.4), it follows similarly by considering $y(x) = (-u_x(1-x,t) - C_1 x)_+$. □

Lemma 2.4. *Let u be a maximal solution of* (1.1). *There exists $C_2 > 0$ such that, for all $T \in (0,T^*)$,*

$$\max_{Q_T} u_x(x,t) \leq \max\left(C_2, \max_{0\leq t\leq T} u_x(0.t)\right), \tag{2.6}$$

where $Q_T = [0,1] \times [0,T]$ and

$$\min_{Q_T} u_x(x,t) \geq \min\left(-C_2, \min_{0\leq t\leq T} u_x(1.t)\right). \tag{2.7}$$

Proof. The function $w = u_x$ satisfies $w_t = w_{xx} + a(x,t)w_x$ in $(0,1)\times(0,T^*)$, where $a(x,t) = e^{u_x}$. Therefore, w attains its extrema in Q_T on the parabolic boundary of Q_T.

Since, by Lemma 2.3, we have $u_x(1,t) \leq C$ and $u_x(0,t) \geq -C$ for all $t \in [0,T^*)$, the conclusion follows. □

The following lemma will provide a useful lower bound on the blowup profile of u_x in case that $u_x(1,t)$ or $u_x(0,t)$ becomes unbounded.

Lemma 2.5. *Let u be a maximal solution of* (1.1). *For all $t_0 \in (0,T^*)$, there exists $C_3 > 0$ such that, for all $0 \leq x \leq 1$ and $t_0 \leq t < T^*$,*

$$e^{-[u_x^+(x,t)+C_3]} \leq e^{-[u_x^+(0,t)+C_3]} + x, \tag{2.8}$$

$$e^{-[(-u_x)^+(1-x,t)+C_3]} \leq e^{-[(-u_x)^+(1,t)+C_3]} + x. \tag{2.9}$$

Proof. Fix $t \in [t_0,T^*)$, and let $z(x) = u_x^+(x,t) + \ln(1+C_1)$, where C_1 is given by Lemma 2.1. The function z satisfies

$$\begin{aligned} z' + e^z &= u_{xx}|_{\{u_x>0\}} + e^{u_x^+(x,t)+\ln(1+C_1)} \\ &\geq (u_{xx} + e^{u_x} + C_1 e^{u_x})|_{\{u_x>0\}} \\ &\geq (u_{xx} + e^{u_x} + C_1)|_{\{u_x>0\}} \geq 0, \end{aligned} \tag{2.10}$$

on $[0,1]$ by Lemma 2.1. By integration, it follows that $e^{-z(x)} \leq e^{-z(0)} + x$, that is, (2.8) with $C_3 = \ln(1+C_1)$.

The estimate (2.9) follows similarly by considering $Z(x) = (-u_x)^+(1-x,t) + \ln(1+C_1)$. □

Lemma 2.6. *Let u be a global solution of* (1.1). *Then it holds*

$$\inf_{[0,1]\times[0,\infty)} u_x > -\infty. \tag{2.11}$$

Proof. Assume that the lemma is false. Then, by Lemma 2.4, there exists a sequence $t_n \to \infty$ such that $u_x(1,t_n) \to -\infty$.

Fix $\varepsilon > 0$. By (2.9) in Lemma 2.5, for $n > n_0(\varepsilon)$ large enough, we have

$$e^{-[(-u_x)^+(1-x,t_n)+\ln(1+C_1)]} \le e^{-[(-u_x)^+(1,t_n)+\ln(1+C_1)]} + x \le \varepsilon, \quad 0 \le x \le \varepsilon. \tag{2.12}$$

Hence,

$$(-u_x)^+(1-x,t_n) \ge -\ln\varepsilon - \ln(1+C_1), \quad 0 \le x \le \varepsilon. \tag{2.13}$$

By choosing $\varepsilon = \varepsilon(C_1)$ small, we deduce that $u_x(1-x,t_n) \le -1$ on $[0,\varepsilon]$; hence,

$$u(1-x,t_n) \ge A + x, \quad 0 \le x \le \varepsilon, \tag{2.14}$$

for all $n \ge n_0(\varepsilon)$. But this contradicts the strong maximum principle which implies that $\lim_{t\to\infty}\{\max_{x\in[0,1]} u(x,t)\} \le A$. □

3. Lyapunov's Functional and Proof of Theorem 1.1

As a main step, we now carry out the argument of Zelenyak to construct a Lyapunov's functional. The key point here is that the Lyapunov functional enjoys nice properties on any global trajectory of (1.1), even if it were unbounded in C^1.

Proposition 3.1. *Fix any $K > 0$ and let $D_K = [-K,K] \times \mathbb{R}$. There exist functions $\phi \in C^1(D_K;\mathbb{R})$ and $\psi \in C(D_K;(0,\infty))$ with the following property: for any solution u of* (1.1) *with $|u| \le K$, defining*

$$\mathcal{L}(u(t)) := \int_0^1 \phi(u(x,t),u_x(x,t))dx, \tag{3.1}$$

it holds

$$\frac{d}{dt}\mathcal{L}(u(t)) = -\int_0^1 \psi(u(x,t),u_x(x,t))u_t^2(x,t)dx, \; 0 < t < T^*. \tag{3.2}$$

Furthermore, we have

$$\phi \ge 0. \tag{3.3}$$

Proof. For a given function $\varphi(u,v)$, let us denote

$$H = \varphi_u + e^v\varphi_{vv} - v\varphi_{uv}. \tag{3.4}$$

Here we assume that φ, φ_u, φ_v, and φ_{vu} are continuous and C^1 in v in D_K and that φ_{vv} is continuous in D_K. We observe that H is continuous and differentiable in v in D_K and satisfies

$$H_v = e^v \varphi_{vvv} + e^v \varphi_{vv} - v\varphi_{uvv}. \tag{3.5}$$

Now suppose that $\psi := \varphi_{vv}$ satisfies

$$v\psi_u - e^v \psi_v - e^v \psi = 0, \quad |u| \le K. \tag{3.6}$$

It follows that $H_v = 0$; hence,

$$H = H(u) = \varphi_u(u,0). \tag{3.7}$$

Let then

$$\phi(u,v) = \varphi(u,v) - \int_0^u H(s)ds = \varphi(u,v) - \varphi(u,0) + \varphi(0,0). \tag{3.8}$$

We compute, using integration by parts and $u_t(1,t) = 0$ and $u_t(0,t) = 0$,

$$\begin{aligned} \frac{d}{dt}\mathcal{L}(u(t)) &= \int_0^1 \{(\varphi_u(u,u_x) - H(u))u_t + \varphi_v(u,\mathrm{u}_x)u_{xt}\}(x,t)dx \\ &= \int_0^1 \{(\varphi_u(u,u_x) - H(u) - \varphi_{vu}(u,u_x)u_x - \varphi_{vv}(u,u_x)u_{xx})\}u_t(x,t)dx. \end{aligned} \tag{3.9}$$

Using the definition of H and $u_{xx} = u_t - e^{u_x}$, we deduce that

$$\frac{d}{dt}\mathcal{L}(u(t)) = -\int_0^1 \psi(u(x,t),u_x(x,t))u_t^2(x,t)dx. \tag{3.10}$$

We have, thus, obtained (3.2), provided (3.6) is true.

Now, (3.6) can be solved by the method of characteristics. For each $K > 0$, one finds that the function ψ defined by

$$\psi(u,v) = e^v > 0 \tag{3.11}$$

is a solution of (3.6) on $[-K,K] \times \mathbb{R}$.

Define φ by

$$\varphi(u,v) = \int_0^v \int_0^z \psi(u,s)ds\,dz \ge 0. \tag{3.12}$$

It is easy to check that φ enjoys the regularity properties assumed at the beginning of the proof and $\phi = \varphi$; hence, $\phi \ge 0$. $\square$

As a consequence of Proposition 3.1 and of Section 2, we shall obtain the following convergence result. Of course, the main point here is that we do not assume u to be bounded, but only global.

Proposition 3.2. *Let u be a global solution of (1.1). Then, as $t \to \infty$, $u(t)$ converges in $C([0,1])$ to a stationary solution of (1.1), that is, a function $W \in C([0,1]) \cap C^2(0,1]$ of*

$$\begin{gathered} W_{xx} + e^{W_x} = 0, \quad 0 < x < 1, \\ W(0) = 0, \qquad W(1) = A. \end{gathered} \tag{3.13}$$

Moreover, the convergence also holds in $C^1([\varepsilon,1])$ for all $\varepsilon > 0$.

Proof. Fix any sequence $t_n \to \infty$, and let $u_n = u(\cdot, t_n + \cdot)$. Denote $Q := [0,1] \times [0,\infty)$ and $Q_\varepsilon := (\varepsilon, 1] \times [0, \infty\ )$, for all $\varepsilon > 0$.

From (1.2) and Lemma 2.1, we know that

$$|u| + |u_t| \le C \ \text{ in } [0,1] \times [1,\infty). \tag{3.14}$$

Also, using (2.3) and Lemma 2.6, we obtain

$$\|\partial_x u_n\|_{L^\infty(1,\infty;L^\infty(0,1))} \le C. \tag{3.15}$$

It follows from (3.14) and (3.15) that the sequence $\{(u_n)\}$ is relatively compact in $C([0,1] \times [0,T])$ for each $T > 0$.

On the other hand, using (2.3), (2.4), and (3.14), we have $|u_x| \le C(\varepsilon)$, and; hence, $|u_{xx}| \le C(\varepsilon)$ in $(\varepsilon,1] \times [1,\infty)$. Since $w := u_x$ satisfies $w_t - w_{xx} = e^{u_x} w_x$, parabolic regularity estimates then imply that

$$\|w_t(\cdot, t_n + \cdot)\|_{L^\infty((\varepsilon,1]\times(0,T))} \le C(\varepsilon, T), \quad T > 0. \tag{3.16}$$

It follows that the sequence $\{\partial_x u_n\}$ is relatively compact in $C([\varepsilon,1] \times [0,\ T])$ for each $\varepsilon,\ T > 0$. Then some subsequence $\{u_{n_k}\}$ converges to a function $W \in C(\overline{Q})$, with $w_x \in C(Q)$, which satisfies

$$\begin{gathered} W_t - W_{xx} = e^{W_x} \ \text{ in } Q, \\ W(0,t) = 0, \quad W(1,t) = A, \quad t \ge 0. \end{gathered} \tag{3.17}$$

The convergence of $\{u_{n_k}\}$ is uniform in each set $[0,1] \times [0,T]$, and the convergence of $\{\partial_x u_{n_k}\}$ is uniform in each set $[\varepsilon, 1] \times [0,T]$.

Now, by (1.2), we may find $K > 0$ such that

$$|u| \le K \quad \text{on } [0,1] \times [0,\infty). \tag{3.18}$$

Since φ, given by Proposition 3.1, is positive and continuous, we have

$$\eta(K,R) := \inf\{\varphi(u,v);\ |u| \le K,\ |v| \le R\} > 0, \qquad \forall R > 0. \tag{3.19}$$

Fix any $\varepsilon \in (0,1)$. We get, for all $T > 1$,

$$\begin{aligned}\eta(K,C(\varepsilon))\int_1^T\int_\varepsilon^1 u_t^2(x,t)dx\,dt &\le \int_1^T\int_0^1 \varphi(u,u_x)u_t^2(x,t)dx\,dt \\ &= \mathcal{L}(u(1)) - \mathcal{L}(u(T)) \le \mathcal{L}(u(1)).\end{aligned} \tag{3.20}$$

This implies that $\int_1^\infty \int_\varepsilon^1 u_t^2(x,t)dx\,dt < \infty$; hence,

$$\int_0^\infty \int_\varepsilon^1 \left(\partial_t u_{n_k}^2\right)(x,t)dx\,dt = \int_{t_{n_k}}^\infty \int_\varepsilon^1 u_t^2(x,t)dx\,dt \longrightarrow 0,\ k \longrightarrow \infty. \tag{3.21}$$

Since $\partial_t u_{n_k} \longrightarrow W_t$ in $\mathfrak{D}'((0,1)\times(0,\infty))$ and since $\varepsilon \in (0,1)$ is arbitrary, it follows that $W_t \equiv 0$. Therefore, $W = W(x) \in C([0,1]) \cap C^2(0,1]$ satisfies (3.13).

But we know (cf. the beginning of Section 2) that the solution of (3.13) is unique whenever it exists. Since the sequence $t_n \longrightarrow \infty$ was arbitrary, this readily implies that the whole solution $u(t)$ actually converges to W. The proposition is proved. □

Proof of Theorem 1.1. For $0 < A < A_c$, assume that u is a global solution of (1.1) which is unbounded in C^1. By Proposition 3.2, as $t \longrightarrow \infty$, $u(t)$ converges to $W = V_A$, with convergence in $C([0,1])$ and in $C^1([\varepsilon,1])$ for all $\varepsilon > 0$.

Since u is unbounded, by Lemmas 2.4 and 2.6, there exists a sequence $t_n \longrightarrow \infty$ such that

$$u_x(0,t_n) \longrightarrow \infty. \tag{3.22}$$

Using Lemma 2.6, (2.8), and (3.22), we deduce that $W_x(x) \ge -C$ and

$$e^{-[W_x^+(x)]+C_3} \le x \ \text{ in } (0,1]. \tag{3.23}$$

This easily implies that

$$W_x(x) \ge -\ln x - C' \text{ in } (0,1]. \tag{3.24}$$

But this is a contradiction, since $W = V_A \in C([0,1])$. We have, thus, proved that all global solutions are bounded in C^1.

Finally, once boundedness is known, the convergence of global solutions to V_A in C^1 is a standard consequence of the existence of a Lyapunov's functional, the uniqueness of the steady-state, and compactness properties of the semi-flow associated with (1.1). The proof of Theorem 1.1 is completed. □

Proof of Proposition 1.3. Let

$$D = \left\{u_0 \in X; u(u_0;\cdot,t) \text{ converges to } V_A \text{ in } C^1 \text{ as } t \to \infty\right\} \tag{3.25}$$

and fix $\overline{A} \in (A, A_c)$. We claim that,

$$\forall\ u_0 \in X, \quad u_0 \leq \min\left(A, V_{\overline{A}}\right) \quad \text{implies } u_0 \in D. \tag{3.26}$$

Indeed, by the comparison principle, as long as $u := u(u_0;\cdot,t)$ exists, we have $u \leq V_{\overline{A}}$; hence, $u_x(0,t) \leq V_{\overline{A},x}(0)$, and $u \leq A$; hence, $u_x(1,t) \geq 0$. By Lemma 2.4, we deduce that u is global and bounded in C^1. It then follows from [3] that u converges in C^1 to the unique steady-state V_A as $t \to \infty$, which proves the claim.

Let us first consider the case $A \in (0, A_c)$. By [4], there exists $\overline{u_0} \in X$ with $\overline{u}_{0,x} \geq 0$, such that $T^*(\overline{u}_0) < \infty$. For each $\lambda \in [0,1]$, denote $u_{0,\lambda} := V_A + \lambda(\overline{u}_0 - V_A) \in X$ and $u_\lambda := u(u_{0,\lambda};\cdot,t)$. For $\lambda > 0$ small, we have $u_{0,\lambda} \leq \min(A, V_{\overline{A}})$; hence, $u_{0,\lambda} \in D$. Therefore, $\lambda^* := \inf\{\lambda \in [0,1];\ u_{0,\lambda} \notin D\} \in (0,1]$. By (3.26) and a standard continuous dependence argument, we have $u_{0,\lambda^*} \notin D$. This implies that u_{λ^*} cannot be global and bounded in C^1 (since otherwise it would converge to V_A due to [3]). In view of Theorem 1.1, the only remaining possibility is that $T^*(u_{0,\lambda^*}) < \infty$. Considering u_{0,λ_n} for a sequence $\lambda_n \uparrow \lambda^*$, we obtain the conclusions (a) and (b) of Proposition 1.3. We also get (c), since otherwise u_{λ^*} would be global by continuous dependence. □

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