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Research Article

Common Fixed Point for Two Pairs of Mappings Satisfying (E.A) Property in Generalized Metric Spaces

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Recently, Abbas et al. (2012) obtained some unique common fixed-point results for a pair of mappings satisfying (E.A) property under certain generalized strict contractive conditions in the framework of a generalized metric space. In this paper, we present common coincidence and common fixed points of two pairs of mappings when only one pair satisfies (E.A) property in the setup of generalized metric spaces. We present some examples to support our results. We also study well-posedness of common fixed-point problem.

1. Introduction and Preliminaries

Mustafa and Sims [1] generalized the concept of a metric in which the real number is assigned to every triplet of an arbitrary set. Based on the notion of generalized metric spaces, Mustafa et al. [2–6] obtained some fixed-point theorems for mappings satisfying different contractive conditions. Chugh et al. [7] obtained some fixed-point results for maps satisfying property p in G-metric spaces. Saadati et al. [8] studied fixed-point of contractive mappings in partially ordered G-metric spaces. Shatanawi [9] obtained fixed-points of Φ-maps in G-metric spaces. Study of common fixed-point theorems in generalized metric spaces was initiated by Abbas and Rhoades [10] (see also, [11–14]). Recently, Abbas et al. [15] obtained some unique common fixed-point results for a pair of mappings satisfying (E.A) property under certain generalized strict contractive conditions in the framework of a generalized metric space.

The aim of this paper is to study common fixed-point of two pairs of mappings for which only one pair needs to satisfy (E.A) property in the framework of G-metric spaces. Our results do not rely on any commuting or continuity condition of mappings.

Consistent with Mustafa and Sims [2], the following definitions and results will be needed in the sequel.

Definition 1.1. Let X be a nonempty set. Suppose that a mapping $G : X \times X \times X \to R^+$ satisfies the following:

G$_1$: $G(x, y, z) = 0$ if $x = y = z$;

G$_2$: $0 < G(x, y, z)$ for all $x, y, z \in X$, with $x \neq y$;

G$_3$: $G(x, x, y) \leq G(x, y, z)$ for all $x, y, z \in X$, with $y \neq z$;

G$_4$: $G(x, y, z) = G(x, z, y) = G(y, z, x) = \ldots$ (symmetry in all three variables);

G$_5$: $G(x, y, z) \leq G(x, a, a) + G(a, y, z)$ for all $x, y, z, a \in X$.

Then G is called a G-metric on X and (X, G) is called a G-metric.

Definition 1.2. A sequence $\{x_n\}$ in a G-metric space X is:

(i) a G-*Cauchy* sequence if, for any $\varepsilon > 0$, there is an $n_0 \in N$ (the set of natural numbers) such that for all $n, m, l \geq n_0, G(x_n, x_m, x_l) < \varepsilon$,

(ii) a G-*convergent* sequence if, for any $\varepsilon > 0$, there is an $x \in X$ and an $n_0 \in N$, such that for all $n,\ m \geq n_0,\ G(x, x_n, x_m) < \varepsilon$.

A G-metric space on X is said to be G-complete if every G-Cauchy sequence in X is G-convergent in X. It is known that $\{x_n\}$ G-converges to $x \in X$ if and only if $G(x_m, x_n, x) \to 0$ as $n, m \to \infty$.

Proposition 1.3. *Let X be a G-metric space. Then the following are equivalent:*

(1) *$\{x_n\}$ is G-convergent to x.*

(2) *$G(x_n, x_m, x) \to 0$ as $n, m \to \infty$.*

(3) *$G(x_n, x_n, x) \to 0$ as $n \to \infty$.*

(4) *$G(x_n, x, x) \to 0$ as $n \to \infty$.*

Definition 1.4. A G-metric on X is said to be symmetric if $G(x, y, y) = G(y, x, x)$ for all $x, y \in X$.

Proposition 1.5. *Every G-metric on X will define a metric d_G on X by*

$$d_G(x, y) = G(x, y, y) + G(y, x, x), \quad \forall x, y \in X. \tag{1.1}$$

For a symmetric G-metric

$$d_G(x, y) = 2G(x, y, y), \quad \forall x, y \in X. \tag{1.2}$$

However, if G is nonsymmetric, then the following inequality holds:

$$\frac{3}{2}G(x, y, y) \leq d_G(x, y) \leq 3G(x, y, y), \quad \forall x, y \in X. \tag{1.3}$$

It is also obvious that

$$G(x,x,y) \le 2G(x,y,y). \tag{1.4}$$

Now, we give an example of a nonsymmetric G-metric.

Example 1.6. Let $X = \{1,2\}$ and a mapping $G : X \times X \times X \to \mathbb{R}^+$ be defined as shown in Table 1.

Note that G satisfies all the axioms of a generalized metric but $G(x,x,y) \neq G(x,y,y)$ for distinct x, y in X. Therefore, G is a nonsymmetric G-metric on X.

Sessa [16] introduced the notion of the weak commutativity of mappings in metric spaces.

Definition 1.7 (see [13]). Let X be a G-metric space. Mappings $f, g : X \to X$ are called (i) compatible if, whenever a sequence $\{x_n\}$ in X is such that $\{fx_n\}$ and $\{gx_n\}$ are G-convergent to some $t \in X$, then $\lim_{n\to\infty} G(fgx_n, fgx_n, gfx_n) = 0$, (ii) noncompatible if there exists at least one sequence $\{x_n\}$ in X such that $\{fx_n\}$ and $\{gx_n\}$ are G-convergent to some $t \in X$, but $\lim_{n\to\infty} G(fgx_n, fgx_n, gfx_n)$ is either nonzero or does not exist.

Jungck [17] defined f and g to be weakly compatible if $fx = gx$ implies $fgx = gfx$.

In 2002, Aamri and Moutaawakil [18] introduced (E.A) property to obtain common fixed-point of two mappings. Recently, Babu and Negash [19] employed this concept to obtain some new common fixed-point results (see also [20–22]).

Recently, Abbas et al. [15] studied (E.A) property in the frame work of G-metric space.

Definition 1.8 (see [15]). Let X be a G-metric space. Self-maps f and g on X are said to satisfy the (E.A) property if there exists a sequence $\{x_n\}$ in X such that $\{fx_n\}$ and $\{gx_n\}$ are G-convergent to some $t \in X$.

Example 1.9 (see [15]). Let $X = [0,2]$ be a G-metric space with

$$G(x,y,z) = \max\{|x-y|, |x-z|, |y-z|\}. \tag{1.5}$$

Let $f, g : X \to X$ be defined by

$$\begin{gathered} fx = \begin{cases} 2-x, & x \in [0,1], \\ \dfrac{2-x}{2}, & x \in (1,2], \end{cases} \\ gx = \begin{cases} \dfrac{3-x}{2}, & x \in [0,1], \\ \dfrac{x}{2}, & x \in (1,2]. \end{cases} \end{gathered} \tag{1.6}$$

For a decreasing sequence $\{x_n\}$ in X such that $x_n \to 1$, $gx_n \to 1/2$, $fx_n \to 1/2$, $gfx_n = (4+x_n)/4 \to 5/4$ and $fgx_n = (4-x_n)2 \to 3/2$. So, f and g are noncompatible. Note that, there exists a sequence $\{x_n\}$ in X such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = 1 \in X$, take $x_n = 1$ for each $n \in N$. Hence f and g satisfy (E.A) property.

Table 1

(x, y, z)	$G(x, y, z)$
(1,1,1),(2,2,2)	0
(1,1,2),(1,2,1),(2,1,1)	0.5
(1,2,2),(2,1,2),(2,2,1)	1

Definition 1.10 (see [23]). The control functions ψ and ϕ are defined as follows:

(a) $\psi : [0, \infty) \to [0, \infty)$ is a continuous nondecreasing function with $\psi(t) = 0$ if and only if $t = 0$,

(b) $\phi : [0, \infty) \to [0, \infty)$ is a lower semicontinuous function with $\phi(t) = 0$ if and only if $t = 0$.

2. Common Fixed-Point Theorems

In this section, we obtain some common fixed-point results for two pairs of mappings satisfying certain contractive conditions in the frame work of a generalized metric space. It is worth mentioning to note that, one needs (E.A)property of only one pair to prove the existence of coincidence point of mappings involved therein. We start with the following result.

Theorem 2.1. *Let X be a G-metric space and $f, g, S, T : X \to X$ be mappings with $fX \subseteq TX$ and $gX \subseteq SX$ such that*

$$\begin{aligned}&\psi(G(fx, gy, gy)) \leq \psi(M(x, y, y)) - \phi(M(x, y, y)),\\ &\text{where } M(x, y, y) = \max\left\{G(Sx, Ty, Ty), G(fx, Sx, Sx), G(Ty, gy, gy), \right.\\ &\left.\qquad \frac{[G(fx, Ty, Ty) + G(Sx, gy, gy)]}{2}\right\}\end{aligned} \tag{2.1}$$

or

$$\begin{aligned}&\psi(G(fx, fx, gy)) \leq \psi(M(x, x, y)) - \phi(M(x, x, y))\\ &\text{where } M(x, x, y) = \max\left\{G(Sx, Sx, Ty), G(fx, fx, Sx), G(Ty, Ty, gy), \right.\\ &\left.\qquad \frac{[G(fx, fx, Ty) + G(Sx, Sx, gy)]}{2}\right\}\end{aligned} \tag{2.2}$$

hold for all $x, y \in X$, where ψ and ϕ are control functions. Suppose that one of the pairs (f, S) and (g, T) satisfies (E.A) property and one of the subspace $f(X)$, $g(X)$, $S(X)$, $T(X)$ is closed in X. If for every sequence $\{y_n\}$ in X, one of the following conditions hold:

(a) *$\{gy_n\}$ is bounded in case (f, S) satisfies (E.A) property,*

(b) *$\{fy_n\}$ is bounded in case (g, T) satisfies (E.A) property.*

Then, the pairs (f, S) and (g, T) have a common point of coincidence in X. Moreover, if the pairs (f, S) and (g, T) are weakly compatible, then f, g, S, and T have a unique common fixed-point.

Proof. Suppose that the pair (f, S) satisfies (E.A) property, there exists a sequence $\{x_n\}$ in X satisfying $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} Sx_n = q$ for some $q \in X$. As $fX \subseteq TX$, there exists a sequence $\{y_n\}$ in X such that $fx_n = Ty_n$. As $\{gy_n\}$ is bounded, $\overline{\lim}_{n\to\infty} G(fx_n, gy_n, gy_n)$ and $\overline{\lim}_{n\to\infty} G(Sx_n, gy_n, gy_n)$ are finite numbers. Note that

$$\left|G(fx_n, gy_n, gy_n) - G(Sx_n, gy_n, gy_n)\right| \leq 2G(fx_n, Sx_n, Sx_n). \tag{2.3}$$

Since $G(fx_n, Sx_n, Sx_n) \to 0$ as $n \to \infty$, therefore $\overline{\lim}_{n\to\infty} G(fx_n, gy_n, gy_n) = \overline{\lim}_{n\to\infty} G(Sx_n, gy_n, gy_n)$. Indeed, using that $\overline{\lim}_{n\to\infty} G(fx_n, gy_n, gy_n) = \overline{\lim}_{n\to\infty} G(Sx_n, gy_n, gy_n) = l$, we obtain subsequences $\{x_{n_k}\}$ and $\{y_{n_k}\}$ such that $G(Sx_{n_k}, gy_{n_k}, gy_{n_k})$ and $G(fx_{n_k}, gy_{n_k}, gy_{n_k})$ are G-convergent to l. Replacing x by x_{n_k} and y by y_{n_k} in (2.1), we have

$$\begin{aligned} M(x_{n_k}, y_{n_k}, y_{n_k}) = \max\Bigg\{ & G(Sx_{n_k}, fx_{n_k}, fx_{n_k}), G(fx_{n_k}, Sx_{n_k}, Sx_{n_k}), G(fx_{n_k}, gy_{n_k}, gy_{n_k}), \\ & \frac{[G(fx_{n_k}, fx_{n_k}, fx_{n_k}) + G(Sx_{n_k}, gy_{n_k}, gy_{n_k})]}{2} \Bigg\} \end{aligned} \tag{2.4}$$

which on taking limit as $k \to \infty$ implies that

$$\lim_{n\to\infty} M(x_{n_k}, y_{n_k}, y_{n_k}) = \max\left\{G(q,q,q), G(q,q,q), l, \frac{l}{2}\right\} = l. \tag{2.5}$$

Now

$$\psi(G(fx_{n_k}, gy_{n_k}, gy_{n_k})) \leq \psi(M(x_{n_k}, y_{n_k}, y_{n_k})) - \phi(M(x_{n_k}, y_{n_k}, y_{n_k})) \tag{2.6}$$

which on taking upper limit gives

$$\psi(l) \leq \psi(l) - \phi(l), \tag{2.7}$$

and so $l = 0$. Hence, $\lim_{n\to\infty} G(fx_n, gy_n, gy_n) = \lim_{n\to\infty} G(Sx_n, gy_n, gy_n) = 0$, and so, $\lim_{n\to\infty} gy_n = q$.

If $T(X)$ is a closed subspace of X. Then, there exist a p in X such that $q = Tp$. From (2.1), we have

$$\psi(G(fx_n, gp, gp)) \leq \psi(M(x_n, p, p)) - \phi(M(x_n, p, p)), \tag{2.8}$$

where

$$\begin{aligned}
M(x_n,p,p) &= \max\left\{G(Sx_n,Tp,Tp),G(fx_n,Sx_n,Sx_n),G(Tp,gp,gp),\right.\\
&\qquad \left.\frac{[G(fx_n,Tp,Tp)+G(Sx_n,gp,gp)]}{2}\right\}\\
&= \max\left\{G(Sx_n,q,q),G(fx_n,Sx_n,Sx_n),G(q,gp,gp),\right.\\
&\qquad \left.\frac{[G(fx_n,q,q)+G(Sx_n,gp,gp)]}{2}\right\},\\
\lim_{n\to\infty} M(x_n,p,p) &= \max\left\{G(q,q,q),G(q,q,q),G(q,gp,gp),\frac{[G(q,q,q)+G(q,gp,gp)]}{2}\right\}\\
&= G(q,gp,gp).
\end{aligned} \tag{2.9}$$

Hence, we have

$$\psi(G(q,gp,gp)) \le \psi(G(q,gp,gp)) - \phi(G(q,gp,gp)) \tag{2.10}$$

and $\phi(G(q,gp,gp)) \le 0$. Hence $gp = q$, p is the coincidence point of pair (g,T). As $g(X) \subseteq S(X)$, there exist a point u in X such that $q = Su$. We claim that $Su = fu$. From (2.1), we get

$$\psi(G(fu,gp,gp)) \le \psi(M(u,p,p)) - \phi(M(u,p,p)), \tag{2.11}$$

where

$$\begin{aligned}
M(u,p,p) &= \max\left\{G(Su,Tp,Tp),G(fu,Su,Su),G(Tp,gp,gp),\right.\\
&\qquad \left.\frac{[G(fu,Tp,Tp)+G(Su,gp,gp)]}{2}\right\}\\
&= \max\left\{G(Su,Su,Su),G(fu,Su,Su),G(Su,Su,Su),\right.\\
&\qquad \left.\frac{[G(fu,Su,Su)+G(Su,Su,Su)]}{2}\right\}\\
&= G(fu,Su,Su).
\end{aligned} \tag{2.12}$$

Hence, we have

$$\psi(G(fu,Su,Su)) \leq \psi(G(fu,Su,Su)) - \phi(G(fu,Su,Su)) \tag{2.13}$$

which implies $\phi(G(fu,Su,Su)) \leq 0$. Hence $fu = Su$, so u is the coincidence point of pair (f,S). Thus $fu = Su = Tp = gp = q$. Now, weakly compatibility of pairs (f,S) and (g,T) give that $fq = Sq$ and $Tq = gq$. From (2.1), we have

$$\psi(G(fq,q,q)) = \psi(G(fq,gp,gp)) \leq \psi(M(q,p,p)) - \phi(M(q,p,p)), \tag{2.14}$$

where

$$\begin{aligned} M(q,p,p) &= \max\left\{G(Sq,Tp,Tp), G(fq,Sq,Sq), G(Tp,gp,gp), \right. \\ &\qquad \left. \frac{[G(fq,Tp,Tp) + G(Sq,gp,gp)]}{2}\right\} \\ &= \max\left\{G(fq,q,q), G(fq,fq,fq), G(q,q,q), \right. \\ &\qquad \left. \frac{[G(fq,q,q) + G(fq,q,q)]}{2}\right\} \\ &= G(fq,q,q). \end{aligned} \tag{2.15}$$

From (2.14), we obtain

$$\psi(G(fq,q,q)) \leq \psi(G(fq,q,q)) - \phi(G(fq,q,q)), \tag{2.16}$$

and so $\phi(G(fq,q,q)) \leq 0$. Therefore $fq = Sq = q$. Similarly, it can be shown that $gq = q$. Therefore $gq = Tq = q$. To prove uniqueness of q, suppose that $fp = gp = Sp = Tp = p$. From (2.1) we have the following:

$$\psi(G(q,p,p)) = \psi(G(fq,gp,gp)) \leq \psi(M(q,p,p)) - \phi(M(q,p,p)), \tag{2.17}$$

where

$$
\begin{aligned}
M(q,p,p) &= \max\Big\{G(Sq,Tp,Tp), G(fq,Sq,Sq), G(Tp,gp,gp),\\
&\qquad \frac{[G(fq,Tp,Tp)+G(Sq,gp,gp)]}{2}\Big\}\\
&= \max\Big\{G(q,p,p), G(q,q,q), G(q,q,q),\\
&\qquad \frac{[G(q,p,p)+G(q,p,p)]}{2}\Big\}\\
&= G(q,p,p).
\end{aligned}
\tag{2.18}
$$

Thus from (2.17), we obtain

$$
\psi(G(q,p,p)) \le \psi(G(q,p,p)) - \phi(G(q,p,p)), \tag{2.19}
$$

which implies that $G(q,p,p) \le 0$, and so $q = p$. The proof using (2.2) is similar. □

Example 2.2. Let $X = \{0,1,2\}$ be a set with G-metric defined by Table 2.

Note that G is a nonsymmetric as $G(1,2,2) \ne G(1,1,2)$. Let $f,g,S,T : X \to X$ be defined by Table 3.

Clearly, $f(X) \subseteq T(X)$ and $g(X) \subseteq S(X)$ with the pairs (f,S) and (g,T) being weakly compatible. Also a pair (f,S) satisfy (E.A)property, indeed, $x_n = 0$ for each $n \in N$ is the required sequence. Note that pair (g,T) is not commuting at 2. The control functions $\psi, \phi : [0,\infty) \to [0,\infty)$ are defined by

$$
\psi(t) = 3t \text{ and } \phi(t) = \begin{cases} \dfrac{t}{4}, & \text{if } t \in [0,4],\\[2mm] \dfrac{e^{4-t}}{2}, & \text{if } t > 4. \end{cases} \tag{2.20}
$$

To check contractive conditions (2.1) and (2.2) for all $x,y \in X$, we consider the following cases:

Note that for cases (I) $x = y = 0$, (II) $x = 0,\ y = 1$, (III) $x = 1,\ y = 0$, (IV) $x = 1,\ y = 1$, (V) $x = 2,\ y = 0$, and (VI) $x = 2,\ y = 1$,

We have $G(fx,gy,gy) = 0,\ G(fx,fx,gy) = 0$, and hence (2.1) and (2.2) are obviously satisfied now.

Table 2

(x, y, z)	$G(x, y, z)$
(0, 0, 0), (1, 1, 1), (2, 2, 2),	0
(0, 0, 1), (0, 1, 0), (1, 0, 0), (0, 1, 1), (1, 0, 1), (1, 1, 0),	1
(1, 2, 2), (2, 1, 2), (2, 2, 1),	2
(0, 0, 2), (0, 2, 0), (2, 0, 0), (0, 2, 2), (2, 0, 2), (2, 2, 0),	3
(1, 1, 2), (1, 2, 1), (2, 1, 1), (0, 1, 2), (0, 2, 1), (1, 0, 2), (1, 2, 0), (2, 0, 1), (2, 1, 0)	4.

Table 3

x	$f(x)$	$g(x)$	$S(x)$	$T(x)$
0	0	0	0	0
1	0	0	2	2
2	0	1	1	2

(VII) If $x = 0,\ y = 2$, then $fx = 0,\ gy = 1, Sx = 0,\ Ty = 2$.

$$\begin{aligned}\psi(G(fx, gy, gy)) &= 3G(0,1,1) = 3\\ &< \frac{11}{4}(3) = \frac{11}{4}G(0,2,2)\\ &= \frac{11}{4}G(Sx, Ty, Ty) \leq \frac{11}{4}M(x,y,y)\\ &= \psi(M(x,y,y)) - \phi(M(x,y,y)).\end{aligned} \tag{2.21}$$

Also

$$\begin{aligned}\psi(G(fx, fx, gy)) &= 3G(0,0,1) = 3\\ &< \frac{11}{4}(3) = \frac{11}{4}G(0,0,2)\\ &= \frac{11}{4}G(Sx, Sx, Ty) \leq \frac{11}{4}M(x,x,y)\\ &= \psi(M(x,x,y)) - \phi(M(x,x,y)).\end{aligned} \tag{2.22}$$

(VIII) For $x = 1,\ y = 2$, then $fx = 0,\ gy = 1,\ Sx = 2,\ Ty = 2$.

$$\begin{aligned}\psi(G(fx, gy, gy)) &= 3G(0,1,1) = 3\\ &< \frac{11}{8}(3+4) = \frac{11}{4}\frac{[G(0,2,2) + G(2,1,1)]}{2}\\ &= \frac{11}{4}\frac{[G(fx, Ty, Ty) + G(Sx, gy, gy)]}{2} \leq \frac{11}{4}M(x,y,y)\\ &= \psi(M(x,y,y)) - \phi(M(x,y,y)),\\ \psi(G(fx, fx, gy)) &= 3G(0,0,1) = 3\end{aligned}$$

$$
\begin{aligned}
&< \frac{11}{8}(3+2) = \frac{11}{4}\frac{[G(0,0,2)+G(2,2,1)]}{2} \\
&= \frac{11}{4}\frac{[G(fx,fx,Ty)+G(Sx,Sx,gy)]}{2} \le \frac{11}{4}M(x,x,y) \\
&= \psi(M(x,x,y)) - \phi(M(x,x,y)).
\end{aligned}
\tag{2.23}
$$

(IX) Now, when $x = 2, y = 2$, then $fx = 0, gy = 1, Sx = 1, Ty = 2$.

$$
\begin{aligned}
\psi(G(fx,gy,gy)) &= 3G(0,1,1) = 3 \\
&< \frac{11}{4}(2) = \frac{11}{4}G(1,2,2) \\
&= \frac{11}{4}G(Sx,Ty,Ty) \le \frac{11}{4}M(x,y,y) \\
&= \psi(M(x,y,y)) - \phi(M(x,y,y)).
\end{aligned}
\tag{2.24}
$$

Also

$$
\begin{aligned}
\psi(G(fx,fx,gy)) &= 3G(0,0,1) = 3 \\
&< \frac{11}{4}(4) = \frac{11}{4}G(1,1,2) \\
&= \frac{11}{4}G(Sx,Sx,Ty) \le \frac{11}{4}M(x,x,y) \\
&= \psi(M(x,x,y)) - \phi(M(x,x,y)).
\end{aligned}
\tag{2.25}
$$

Hence, all of the conditions of Theorem 2.1 are satisfied. Moreover, 0 is the unique common fixed-point of f, g, S, and T.

As two noncompatible selfmappings on G-metric space X satisfy the (E.A) property, so above result remains true if any one of the pair of mapping is noncompatible.

Above theorem is true for any choice of control functions, for example if we take $\psi(t) = t$ and $\phi(t) = (1-\gamma)t$ for $\gamma \in [0,1)$ in Theorem 2.1, we have the following corollary.

Corollary 2.3. *Let X be a G-metric space and $f, g, S, T : X \to X$ be mappings with $f(X) \subseteq T(X)$ and $g(X) \subseteq S(X)$ such that*

$$
\begin{aligned}
G(fx,gy,gy) \le \gamma \max\Bigg\{ & G(Sx,Ty,Ty), G(fx,Sx,Sx), G(Ty,gy,gy), \\
& \frac{[G(fx,Ty,Ty)+G(Sx,gy,gy)]}{2} \Bigg\}
\end{aligned}
\tag{2.26}
$$

or

$$G(fx,fx,gy) \leq \gamma \max\left\{G(Sx,Sx,Ty), G(fx,fx,Sx), G(Ty,Ty,gy), \frac{[G(fx,fx,Ty)+G(Sx,Sx,gy)]}{2}\right\} \tag{2.27}$$

hold for all $x,y \in X$*, where* $\gamma \in [0,1)$ *hold for all* $x,y \in X$*, where* ψ *and* ϕ *are control functions. Suppose that one of the pairs* (f,S) *and* (g,T) *satisfies (E.A) property and one of the subspace* $f(X), g(X), S(X)$*, and* $T(X)$ *is closed in* X*. If for every sequence* $\{y_n\}$ *in* X*, one of the following conditions hold:*

(a) $\{gy_n\}$ *is bounded in case* (f,S) *satisfies (E.A) property*

(b) $\{fy_n\}$ *is bounded in case* (g,T) *satisfies (E.A) property.*

Then, the pairs (f,S) *and* (g,T) *have a point of coincidence in* X*. Moreover, if the pairs* (f,S) *and* (g,T) *are weakly compatible, then* f, g, S*, and* T *have a unique common fixed-point.*

If we take $f = g$ *and* $S = T$ *with* $\psi(t) = t$ *for all* $t \in 211d$ *in Theorem 2.1, we obtain the following corollary which extends Theorem 3.1 of [19] to generalized metric space.*

Corollary 2.4. *Let* X *be a G-metric space and* $f,S : X \to X$ *be mappings with* $fX \subseteq SX$ *such that*

$$\begin{aligned} &G(fx,fy,fy) \leq M(x,y,y) - \phi(M(x,y,y)), \\ &\text{where } M(x,y,y) = \max\left\{G(Sx,Sy,Sy), G(fx,Sx,Sx), G(Sy,fy,fy), \frac{[G(fx,Sy,Sy)+G(Sx,fy,fy)]}{2}\right\} \end{aligned} \tag{2.28}$$

or

$$\begin{aligned} &G(fx,fx,fy) \leq M(x,x,y) - \phi(M(x,x,y)), \\ &\text{where } M(x,x,y) = \max\left\{G(Sx,Sx,Sy), G(fx,fx,Sx), G(Sy,Sy,fy), \frac{[G(fx,fx,Sy)+G(Sx,Sx,fy)]}{2}\right\} \end{aligned} \tag{2.29}$$

hold for all $x,y \in X$*, where* ϕ *are control functions. Suppose that the pair* (f,S) *satisfy (E.A) property and one of the subspaces* $f(X)$*,* $S(X)$ *is closed in* X*. Then, the pair* (f,S) *has a common point of coincidence in* X*. Moreover, if the pair* (f,S) *is weakly compatible, then* f *and* S *have a unique common fixed-point.*

3. Well-Posedness

The notion of well-posedness of a fixed-point problem has evoked much interest of several mathematicians, (see [24–27]).

Definition 3.1. Let X be a G-metric space and $f : X \to X$ be a mapping. The fixed-point problem of f is said to be well-posed if:

(a) f has a unique fixed-point z in X;

(b) for any sequence $\{x_n\}$ of points in X such that $\lim_{n\to\infty} G(fx_n, x_n, x_n) = 0$, we have $\lim_{n\to\infty} G(x_n, z, z) = 0$.

Definition 3.2. Let X be a G-metric space and Σ be a set of mappings on X. Common fixed-point problem $CF(\Sigma)$ is said to be well-posed if:

(a) $z \in X$ is a unique common fixed-point of all mappings in Σ;

(b) for any sequence $\{x_n\}$ of points in X such that $\lim_{n\to\infty} G(fx_n, x_n, x_n) = 0$ for each $f \in \Sigma$, we have $\lim_{n\to\infty} G(x_n, z, z) = 0$.

Theorem 3.3. *Let X be a G-metric space and $f, g, S, T : X \to X$ be mappings such that*

$$\begin{aligned} & G(fx, gy, gy) \le G(Sx, Ty, Ty) - \psi(M(x, y, y)), \\ \text{where } \ M(x, y, y) = \max\Bigg\{ & G(Sx, Ty, Ty), G(fx, Sx, Sx), G(Ty, gy, gy), \\ & \frac{[G(fx, Ty, Ty) + G(Sx, gy, gy)]}{2} \Bigg\} \end{aligned} \tag{3.1}$$

or

$$\begin{aligned} & G(fx, fx, gy) \le G(Sx, Sx, Ty)) - \psi(M(x, x, y)), \\ \text{where } \ M(x, x, y) = \max\Bigg\{ & G(Sx, Sx, Ty), G(fx, fx, Sx), G(Ty, Ty, gy), \\ & \frac{[G(fx, fx, Ty) + G(Sx, Sx, gy)]}{2} \Bigg\} \end{aligned} \tag{3.2}$$

hold for all $x, y \in X$, where ψ is a control function. Suppose that one of the pairs (f, S) and (g, T) satisfies (E.A) property and one of the subspace $f(X)$, $g(X)$, $S(X)$, $T(X)$ is closed in X. If for every sequence $\{y_n\}$ in X, one of the following conditions hold:

(a) *$\{gy_n\}$ is bounded in case (f, S) satisfies (E.A) property;*

(b) *$\{fy_n\}$ is bounded in case (g, T) satisfies (E.A) property.*

If pairs (f, S) and (g, T) are weakly compatible, then $CF(\{f, g, S, T\})$ is well-posed.

Proof. From Theorem 2.1, the mappings $f, g, S, T : X \to X$ have a unique common fixed-point (say) z in X. Let $\{x_n\}$ be a sequence in X such that

$$\begin{aligned}\lim_{n\to\infty} G(fx_n, x_n, x_n) &= \lim_{n\to\infty} G(gx_n, x_n, x_n) \\ &= \lim_{n\to\infty} G(Sx_n, x_n, x_n) = \lim_{n\to\infty} G(Tx_n, x_n, x_n) = 0.\end{aligned} \tag{3.3}$$

Now by using (3.1), we have

$$\begin{aligned}G(z, x_n, x_n) &\le G(fz, gx_n, gx_n) + G(gx_n, x_n, x_n) \\ &\le G(Sz, Tx_n, Tx_n) - \varphi(M(z, x_n, x_n)) + G(gx_n, x_n, x_n) \\ &\le G(z, x_n, x_n) + G(x_n, Tx_n, Tx_n) \\ &\quad - \varphi\left(\frac{[G(fz, Tx_n, Tx_n) + G(gx_n, Sz, Sz)]}{2}\right) + G(gx_n, x_n, x_n)\end{aligned} \tag{3.4}$$

which further implies

$$\begin{aligned}\varphi\left(\frac{[G(z, Tx_n, Tx_n) + G(x_n, z, z)]}{2}\right) &\le G(x_n, Tx_n, Tx_n) + G(gx_n, x_n, x_n) \\ &\le 2G(Tx_n, x_n, x_n) + G(gx_n, x_n, x_n).\end{aligned} \tag{3.5}$$

On taking limit as $n \to \infty$ implies that

$$\lim_{n\to\infty} \varphi\left(\frac{[G(z, Tx_n, Tx_n) + G(x_n, z, z)]}{2}\right) = 0, \tag{3.6}$$

and by the property of φ, we have

$$\lim_{n\to\infty} G(z, Tx_n, Tx_n) = \lim_{n\to\infty} G(x_n, z, z) = 0. \tag{3.7}$$

Hence the result follows. □

Remark 3.4. A G-metric naturally induces a metric d_G given by $d_G(x, y) = G(x, y, y) + G(x, x, y)$. If G-metric is not symmetric, either of the inequalities (2.1) or (2.2) does not reduce to any metric inequality with the metric d_G. Hence our theorems do not reduce to fixed-point problems in the corresponding metric space (X, d_G).

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