Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2012, Article ID 305924, 10 pages
doi:10.1155/2012/305924

Research Article

On the Dimension of the Solution Set for Semilinear Fractional Differential Inclusions

Ravi P. Agarwal,[1,2] Bashir Ahmad,[2] Ahmed Alsaedi,[2] and Naseer Shahzad[2]

[1] *Department of Mathematics, Texas A&M University-Kingsville, 700 University Boulvard Kingsville, TX 78363-8202, USA*
[2] *Department of Mathematics, Faculty of Sciences, King Abdulaziz University, P.O. Box 80203, Jeddah 21589, Saudi Arabia*

Correspondence should be addressed to Naseer Shahzad, nshahzad@kau.edu.sa

Received 24 January 2012; Accepted 11 March 2012

Academic Editor: Ngai-Ching Wong

We investigate the existence and dimension of the solution set for a nonlocal problem of semilinear fractional differential inclusions. The main tools of our study include some well-known results on multivalued maps.

1. Introduction

The subject of fractional calculus has recently emerged as an important and popular field of research due to its numerous applications in science and engineering. Examples can be found in various disciplines such as mechanics, electricity, signal and image processing, thermodynamics, biophysics, blood flow phenomena, aerodynamics, economics, and fitting of experimental data [1–4] whereas the theoretical development of the subject can be found in [5, 6]. For some recent results on fractional differential equations and inclusions, see [7–16].

In this paper, we study the following problem for semilinear fractional differential inclusion with nonlocal condition:

$$\begin{gathered} {}^{c}D^{q}x(t) \in A(t)x(t) + F(t, x(t)), \quad t \in [0, T] \ (T > 0), \\ x(0) + g(x) = x_0, \quad x_0 \in \mathbb{R}^n, \end{gathered} \tag{1.1}$$

where ${}^cD^q$ denote the Caputo fractional derivative of order $q \in (0,1]$ [5], $A(t)$ is a bounded linear operator on $[0,T]$ (the function $t \to A(t)$ is continuous in the uniform operator topology), $g : \mathfrak{C}([0,T],\mathbb{R}^n) \to \mathbb{R}^n$, and $F : [0,T] \times \mathbb{R}^n \to P(\mathbb{R}^n)$, where $P(\mathbb{R}^n)$ is the family of all nonempty subsets of $\mathbb{R}^n$.

2. Terminology and Preliminary Results

In this section, we discuss some basic concepts of multivalued analysis and recall some results involving multivalued maps.

Let $\mathfrak{C}([0,T],\mathbb{R}^n)$ denote the Banach space of continuous functions from $[0,T]$ into $\mathbb{R}^n$ with the norm $\|x\|_\infty = \sup_{t\in[0,T]}\|x(t)\|$. Let $L^1([0,T],\mathbb{R}^n)$ be the Banach space of measureable functions $x : [0,T] \to \mathbb{R}^n$ that are Lebesgue integrable and normed by $\|x\|_{L^1} = \int_0^T \|x(t)\|dt$.

For a nonempty subset C of a complete metric space X, let $P(C) = \{Y \subseteq C : Y \neq \emptyset\}$, $P_{\rm cl}(C) = \{Y \in P(C) : Y \text{ is closed}\}$, $P_b(C) = \{Y \in P(C) : Y \text{ is bounded}\}$, $P_{b,\rm cl}(C) = \{Y \in P(C) : Y \text{ is bounded and closed}\}$, and $P_{\rm cp}(C) = \{Y \in P(C) : Y \text{ is compact}\}$. If C is a nonempty subset of a Banach space X, then we set $P_{c,\rm cl}(C) = \{Y \in P(C) : Y \text{ is closed and convex}\}$, and $P_{c,\rm cp}(C) = \{Y \in P(C) : Y \text{ is compact and convex}\}$.

A multivalued map $\mathcal{F} : C \to P(X)$ is closed (resp., compact) valued if $\mathcal{F}(x)$ is closed (resp., compact) for all $x \in C$. The map $\mathcal{F}$ is bounded on bounded sets if $\mathcal{F}(\mathbb{B}) = \cup_{x\in\mathbb{B}}\mathcal{F}(x)$ is bounded in X for all $\mathbb{B} \in P_b(C)$ (i.e., $\sup_{x\in\mathbb{B}}\{\sup\{\|y\| : y \in \mathcal{F}(x)\}\} < \infty$). The map $\mathcal{F}$ is called upper semicontinuous (u.s.c.) if $\{x \in C : \mathcal{F}(x) \subset V\}$ is open in C whenever $V \subset X$ is open. $\mathcal{F}$ is called lower semi-continuous (l.s.c.) if the set $\{y \in C : \mathcal{F}(y) \cap V \neq \emptyset\}$ is open for any open set $V \subset X$. $\mathcal{F}$ is called continuous if it is both l.s.c. and u.s.c. $\mathcal{F}$ is said to be completely continuous if $\mathcal{F}(\mathbb{B})$ is relatively compact for every $\mathbb{B} \in P_b(C)$. A mapping $f : C \to X$ is called a selection of $\mathcal{F} : C \to P(X)$ if $f(x) \in \mathcal{F}(x)$ for every $x \in C$. We say that the mapping $\mathcal{F}$ has a fixed point if there is $x \in X$ such that $x \in \mathcal{F}(x)$. The fixed points set of the multivalued operator $\mathcal{F}$ will be denoted by $\text{Fix}(\mathcal{F})$. A multivalued map $\mathcal{F} : [0,T] \to P_{\rm cl}(\mathbb{R}^n)$ is said to be measurable if, for every $y_1 \in \mathbb{R}^n$, the function

$$t \longmapsto d(y_1, \mathcal{F}(t)) = \inf\{\|y_1 - y_2\| : y_2 \in \mathcal{F}(t)\} \tag{2.1}$$

is measurable.

Definition 2.1. Let (X,d) be a metric space. Consider $H : P(X) \times P(X) \to \mathbb{R} \cup \{\infty\}$ given by

$$H(A,B) = \max\left\{\sup_{a\in A} d(a,B), \sup_{b\in B} d(b,A)\right\}, \tag{2.2}$$

where $d(a,B) = \inf_{b\in B} d(a,b)$. H is the (generalized) Pompeiu-Hausdorff functional. It is known that $(P_{b,\rm cl}(X), H)$ is a metric space and $(P_{\rm cl}(X), H)$ is a generalized metric space (see [17]).

Definition 2.2. A multivalued operator $\mathcal{F} : X \to P_{\rm cl}(X)$ is called a k-contraction if there exists $0 < k < 1$ such that

$$H(\mathcal{F}(x), \mathcal{F}(y)) \leq k d(x,y), \quad \text{for each } x, y \in X. \tag{2.3}$$

It is known that $\mathcal{F} : X \to P_{cp}(X)$ is continuous on X if and only if $\mathcal{F}$ is continuous on X with respect to the Hausdorff metric. Also, if $\mathcal{F} : X \to P_{cp}(X)$ is a k-contraction, then $\mathcal{F}$ is continuous with respect to Hausdorff metric.

Further details of multivalued maps can be found in ([18, 19]).

For the forthcoming analysis, we need the following results on multivalued maps.

Lemma 2.3 (Covitz and Nadler [20]). *Let (X, d) be a complete metric space. If $\Phi : X \to P_{cl}(X)$ is a k-contraction, then,* $\mathrm{Fix}(\Phi) \neq \emptyset$.

Lemma 2.4 (Dzedzej and Gelman [21]). *Let $F : [0, \alpha] \to P_{c,cp}(\mathbb{R}^n)$ be a measurable map such that the Lebesgue measure μ of the set $\{t : \dim F(t) < 1\}$ is zero. Then there are arbitrarily many linearly independent measurable selections $x_1(\cdot), x_2(\cdot), \ldots, x_m(\cdot)$ of F.*

Lemma 2.5 (Saint-Raymond [22]). *Let K be a compact metric space with $\dim K < n$, X a Banach space, and $\Omega : K \to P_{c,cp}(X)$ a lower semicontinuous map such that $0 \in \Omega(x)$ and $\dim \Omega(x) \geq n$ for every $x \in K$. Then, there exists a continuous selection f of Ω such that $f(x) \neq 0$ for each $x \in K$.*

Lemma 2.6 (Michael's selection theorem [23]). *Let C be a metric space, X a Banach space and $\Omega : C \to P_{c,cl}(C)$ a lower semicontinuous map. Then, there exists a continuous selection $f : C \to X$ of Ω.*

Lemma 2.7 (see Dzedzej and Gelman [21] and Petrusel [24]). *Let C be a nonempty closed convex subset of a Banach space X. Suppose that $\Omega : C \to P_{c,cp}(C)$ is a k-contraction. If $f : C \to C$ is a continuous selection of Ω, then* $\mathrm{Fix}(f)$ *is nonempty.*

3. Main Results

Definition 3.1. A function $x \in \mathfrak{C}([0, T], \mathbb{R}^n)$ is a solution of the problem (1.1) if there exists a function $f \in L^1([0, T], \mathbb{R}^n)$ such that $f(t) \in F(t, x(t))$ a.e. on $[0, T]$ and

$$x(t) = x_0 - g(x) + \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} A(s)x(s)ds + \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} f(s)ds. \tag{3.1}$$

Let $S_{x_0}([0, \alpha])$ denote the set of all solutions of (1.1) on the interval $[0, \alpha]$, where $0 < \alpha \leq T$.

Lemma 3.2. *Assume that*

(H_1) *$F : [0, T] \times \mathbb{R}^n \to P_{cp}(\mathbb{R}^n)$ is such that $F(\cdot, x) : [0, T] \to P_{cp}(\mathbb{R}^n)$ is measurable for each $x \in \mathbb{R}^n$,*

(H_2) *$H(F(t, x), F(t, \overline{x})) \leq \kappa_1(t)\|x - \overline{x}\|$ for almost all $t \in [0, T]$ and $x, \overline{x} \in \mathbb{R}^n$ with $\kappa_1 \in \mathfrak{C}([0, T], \mathbb{R}_+)$ and $\|F(t, x)\| = \sup\{\|v\| : v \in F(t, x)\} \leq \kappa_1(t)$ for almost all $t \in [0, T]$ and $x \in \mathbb{R}^n$,*

(H_3) *$g : \mathfrak{C}([0, T], \mathbb{R}^n) \to \mathbb{R}^n$ is continuous and $\|g(x) - g(y)\| \leq \kappa_2\|x - y\|_\infty$ for all $x, y \in \mathfrak{C}([0, T], \mathbb{R}^n)$ and some $\kappa_2 > 0$.*

Then, the Cauchy problem (1.1) *has at least one solution on* $[0,T]$ *if*

$$\kappa_2 + \frac{T^q}{\Gamma(q+1)}\left(A_1 + \|\kappa_1\|_\infty\right) < 1, \tag{3.2}$$

where $A_1 = \max_{t\in[0,T]}\|A(t)\|$.

Proof. For each $y \in \mathfrak{C}([0,T],\mathbb{R}^n)$, define the set of selections of F by

$$S_{F,y} := \left\{v \in L^1([0,T],\mathbb{R}^n) : v(t) \in F(t, y(t)) \text{ for a.e. } t \in [0,T]\right\}. \tag{3.3}$$

Observe that, by assumptions (H_1) and (H_2), $F(\cdot, x(\cdot))$ is measurable and has a measureable selection $v(\cdot)$ (see [25, Theorem III.6]). Also $\kappa_1 \in \mathfrak{C}([0,T],\mathbb{R}_+)$ and

$$\begin{aligned} \|v(t)\| &\le \|F(t,x(t))\| \\ &\le \kappa_1(t). \end{aligned} \tag{3.4}$$

Thus, the set $S_{F,x}$ is nonempty for each $x \in \mathfrak{C}([0,T],\mathbb{R}^n)$. Now we show that the operator Ω defined by

$$\begin{aligned} \Omega(x) = \Bigg\{h \in \mathfrak{C}([0,T],\mathbb{R}^n) : h(t) &= x_0 - g(x) + \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} A(s)x(s)ds \\ &+ \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} f(s)ds,\ f \in S_{F,x}\Bigg\} \end{aligned} \tag{3.5}$$

satisfies the assumptions of Lemma 2.3. To show that $\Omega(x) \in P_{\mathrm{cl}}(\mathfrak{C}([0,T],\mathbb{R}^n))$ for each $x \in \mathfrak{C}([0,T],\mathbb{R}^n)$, let $\{u_n\}_{n\ge 0} \in \Omega(x)$ be such that $u_n \to u(n \to \infty)$ in $\mathfrak{C}([0,T],\mathbb{R}^n)$. Then, $u \in \mathfrak{C}([0,T],\mathbb{R}^n)$ and there exists $v_n \in S_{F,x}$ such that, for each $t \in [0,T]$,

$$u_n(t) = x_0 - g(x) + \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} A(s)x(s)ds + \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} v_n(s)ds. \tag{3.6}$$

As F has compact values, we pass to a subsequence to obtain that v_n converges to v in $L^1([0,T],\mathbb{R}^n)$. Thus, $v \in S_{F,x}$ and, for each $t \in [0,T]$,

$$u_n(t) \longrightarrow u(t) = x_0 - g(x) + \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} A(s)x(s)ds + \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} v(s)ds. \tag{3.7}$$

Hence, $u \in \Omega(x)$.

Next we show that there exists $k > 0$ such that

$$H(\Omega(x), \Omega(\overline{x})) \le k\|x - \overline{x}\|_\infty \quad \text{for each } x, \overline{x} \in \mathfrak{C}([0,T],\mathbb{R}^n). \tag{3.8}$$

Let $x, \overline{x} \in \mathfrak{C}([0,T],\mathbb{R}^n)$ and $h_1 \in \Omega(x)$. Then, there exists $v_1(t) \in S_{F,x}$ such that, for each $t \in [0,T]$,

$$h_1(t) = x_0 - g(x) + \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} A(s)x(s)ds + \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} v_1(s)ds. \tag{3.9}$$

By (H_2), we have

$$H(F(t,x),F(t,\overline{x})) \le \kappa_1(t)\|x(t)-\overline{x}(t)\|. \tag{3.10}$$

So, there exists $w \in F(t,\overline{x}(t))$ such that

$$\|v_1(t) - w\| \le \kappa_1(t)\|x(t)-\overline{x}(t)\|, \quad t \in [0,T]. \tag{3.11}$$

Define $V : [0,T] \to P(\mathbb{R}^n)$ by

$$V(t) = \{w \in \mathbb{R}^n : \|v_1(t) - w\| \le \kappa_1(t)\|x(t)-\overline{x}(t)\|\}. \tag{3.12}$$

Since the nonempty closed valued operator $V(t) \cap F(t,\overline{x}(t))$ is measurable [25, Proposition III.4], there exists a function $v_2(t)$ that is a measurable selection for $V(t) \cap F(t,\overline{x}(t))$. So $v_2(t) \in F(t,\overline{x}(t))$ and, for each $t \in [0,T]$, we have $\|v_1(t) - v_2(t)\| \le \kappa_1(t)\|x(t)-\overline{x}(t)\|$.

For each $t \in [0,T]$, let us define

$$h_2(t) = x_0 - g(\overline{x}) + \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} A(s)\overline{x}(s)ds + \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} v_2(s)ds. \tag{3.13}$$

Thus,

$$\begin{aligned} \|h_1(t) - h_2(t)\| \le{}& \left\|g(x) - g(\overline{x})\right\| + \int_0^t \frac{|t-s|^{q-1}}{\Gamma(q)} \|A(s)(x-\overline{x}(s))\|ds \\ &+ \int_0^t \frac{|t-s|^{q-1}}{\Gamma(q)} \|v_1(s) - v_2(s)\|ds. \end{aligned} \tag{3.14}$$

Hence,

$$\begin{aligned} \|h_1 - h_2\|_\infty &\le \kappa_2\|x-\overline{x}\|_\infty + \frac{T^q}{\Gamma(q+1)}(A_1 + \|\kappa_1\|_\infty)\|x-\overline{x}\|_\infty \\ &= \left(\kappa_2 + \frac{T^q}{\Gamma(q+1)}(A_1 + \|\kappa_1\|_\infty)\right)\|x-\overline{x}\|_\infty. \end{aligned} \tag{3.15}$$

Analogously, interchanging the roles of x and $\overline{x}$, we obtain

$$H(\Omega(x),\Omega(\overline{x})) \le k\|x-\overline{x}\|_\infty, \quad \text{for each } x, \overline{x} \in \mathfrak{C}([0,T],\mathbb{R}^n), \tag{3.16}$$

where $k = (\kappa_2 + (T^q/\Gamma(q+1))(A_1 + \|\kappa_1\|_\infty)) < 1$. Since Ω is a contraction, it follows by Lemma 2.3 that Ω has a fixed point x that is a solution of (1.1). This completes the proof. □

Lemma 3.3. *Let $F : [0,T] \times \mathbb{R}^n \to P_{c,cp}(\mathbb{R}^n)$ satisfy (H_1), (H_2), and (H_3) and suppose that $\Omega : \mathfrak{C}([0,T],\mathbb{R}^n) \to P(\mathfrak{C}([0,T],\mathbb{R}^n))$ is defined by*

$$\begin{aligned}\Omega(x) = \Bigg\{ h \in \mathfrak{C}([0,T],\mathbb{R}^n) : h(t) &= x_0 - g(x) + \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} A(s)x(s)ds \\ &+ \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} f(s)ds,\ f \in S_{F,x} \Bigg\}.\end{aligned} \tag{3.17}$$

Then, $\Omega(x) \in P_{c,cp}(\mathfrak{C}([0,T],\mathbb{R}^n))$ for each $x \in \mathfrak{C}([0,T],\mathbb{R}^n)$.

Proof. First we show that $\Omega(x)$ is convex for each $x \in \mathfrak{C}([0,T],\mathbb{R}^n)$. For that, let $h_1, h_2 \in \Omega(x)$. Then. there exist $f_1, f_2 \in S_{F,x}$ such that, for each $t \in [0,T]$, we have

$$h_i(t) = x_0 - g(x) + \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} A(s)x(s)ds + \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} f_i(s)ds, \quad i = 1,2. \tag{3.18}$$

Let $0 \le \lambda \le 1$. Then, for each $t \in [0,T]$, we have

$$\begin{aligned}[\lambda h_1 + (1-\lambda)h_2](t) &= x_0 - g(x) + \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} A(s)x(s)ds \\ &+ \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} [\lambda f_1(s) + (1-\lambda) f_2(s)]ds.\end{aligned} \tag{3.19}$$

Since $S_{F,x}$ is convex (F has convex values), it follows that $\lambda h_1 + (1-\lambda)h_2 \in \Omega(x)$.

Next, we show that Ω maps bounded sets into bounded sets in $\mathfrak{C}([0,T],\mathbb{R}^n)$. For a positive number r, let $B_r = \{x \in \mathfrak{C}([0,T],\mathbb{R}^n) : \|x\|_\infty \le r\}$ be a bounded set in $\mathfrak{C}([0,T],\mathbb{R}^n)$. Then, for each $h \in \Omega(x)$, $x \in B_r$, there exists $f \in S_{F,x}$ such that

$$h(t) = x_0 - g(x) + \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} A(s)x(s)ds + \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} f(s)ds, \tag{3.20}$$

and, in view of (H_1), we have

$$\begin{aligned}\|h(t)\| &\leq \|x_0\| + \sup_{x\in B_r}\|g(x)\| + \int_0^t \frac{|t-s|^{q-1}}{\Gamma(q)}\|A(s)x(s)\|ds \\ &\quad + \int_0^t \frac{|t-s|^{q-1}}{\Gamma(q)}\|f(s)\|ds \\ &\leq \|x_0\| + \sup_{x\in B_r}\|g(x)\| \\ &\quad + \frac{T^q}{\Gamma(q+1)}\left(A_1 r + \|\kappa_1\|_\infty\right).\end{aligned} \tag{3.21}$$

Thus,

$$\|h\|_\infty \leq \|x_0\| + \sup_{x\in B_r}\|g(x)\| + \frac{T^q}{\Gamma(q+1)}\left(A_1 r + \|\kappa_1\|_\infty\right). \tag{3.22}$$

Now we show that Ω maps bounded sets into equicontinuous sets in $\mathfrak{C}([0,T],\mathbb{R}^n)$. Let $t', t'' \in [0,T]$ with $t' < t''$ and $x \in B_r$, where B_r is a bounded set in $\mathfrak{C}([0,T],\mathbb{R}^n)$. For each $h \in \Omega(x)$, we obtain

$$\begin{aligned}&\|h(t'') - h(t')\| \\ &\quad = \left\|\int_0^{t''} \frac{(t''-s)^{q-1}}{\Gamma(q)}\left(A(s)x(s)+f(s)\right)ds - \int_0^{t'} \frac{(t'-s)^{q-1}}{\Gamma(q)}\left(A(s)x(s)+f(s)\right)ds\right\| \\ &\quad \leq \left\|\int_0^{t'} \frac{\left[(t''-s)^{q-1} - (t'-s)^{q-1}\right]}{\Gamma(q)}\left(A(s)x(s)+f(s)\right)ds\right\| \\ &\qquad + \left\|\int_{t'}^{t''} \frac{(t''-s)^{q-1}}{\Gamma(q)}\left(A(s)x(s)+f(s)\right)ds\right\|.\end{aligned} \tag{3.23}$$

Obviously the right-hand side of the above inequality tends to zero independently of $x \in B_{r'}$ as $t''-t' \to 0$. By the Arzela-Ascoli theorem, $\Omega : \mathfrak{C}([0,T],\mathbb{R}^n) \to P(\mathfrak{C}([0,T],\mathbb{R}^n))$ is completely continuous. As in Lemma 3.2, Ω is closed valued. Consequently, $\Omega(x) \in P_{c,\mathrm{cp}}(\mathfrak{C}([0,T],\mathbb{R}^n))$ for each $x \in \mathfrak{C}([0,T],\mathbb{R}^n)$.

For $0 < \alpha \leq T$, let us consider the operator

$$\begin{aligned}\Omega(x) = \Bigg\{ h \in \mathfrak{C}([0,\alpha],\mathbb{R}^n) : h(t) &= x_0 - g(x) + \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} A(s)x(s)ds \\ &\quad + \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} f(s)ds,\ f \in S_{F,x} \Bigg\}.\end{aligned} \tag{3.24}$$

It is well known that $\mathrm{Fix}(\Omega) = S_{x_0}([0,\alpha])$ and, in view of Lemma 3.2, it is nonempty for each $0 < \alpha \le T$. □

Theorem 3.4. *Suppose that $F : [0,\alpha] \times \mathbb{R}^n \to P_{c,cp}(\mathbb{R}^n)$ satisfies (H_1), (H_2), and (H_3) and that the Lebesgue measure μ of the set$\{t : \dim F(t,x) < 1$ for some $x \in \mathbb{R}^n\}$ is zero. Then, for each α, $0 < \alpha < \min\{((1-\kappa_2)\Gamma(q+1)/(A_1 + \|\kappa_1\|_\infty))^{1/q}, T\}$, the set $S_{x_0}([0,\alpha])$ of solutions of* (1.1) *has an infinite dimension for any x_0.*

Proof. Let the operator Ω be defined by

$$\begin{aligned}\Omega(x) = \Bigg\{ & h \in \mathfrak{C}([0,\alpha],\mathbb{R}^n) : h(t) = x_0 - g(x) + \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} A(s)x(s)ds \\ & + \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} f(s)ds,\ f \in S_{F,x} \Bigg\}.\end{aligned} \tag{3.25}$$

Lemma 3.3 guarantees that $\Omega(x) \in P_{c,cp}(\mathfrak{C}([0,\alpha],\mathbb{R}^n))$ for each $x \in \mathfrak{C}([0,\alpha],\mathbb{R}^n)$ and as in the proof of Lemma 3.2, it is a contraction if $\kappa_2 + (\alpha^q/\Gamma(q+1))(A_1 + \|\kappa_1\|_\infty) < 1$ or $\alpha < \{((1-\kappa_2)\Gamma(q+1)/(A_1+\|\kappa_1\|_\infty))^{1/q}, T\}$. We shall show that $\dim \Omega(x) \ge m$ for any $x \in \mathfrak{C}([0,\alpha],\mathbb{R}^n)$ and arbitrary $m \in \mathbb{N}$. Consider $G(t) = F(t,x(t))$. By Lemma 2.4, there exist linearly independent measurable selections $x_1(\cdot), x_2(\cdot), \dots, x_m(\cdot)$ of G. Set

$$y_i(t) = x_0 - g(x) + \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} A(s)x_i(s)ds + \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} x_i(s)ds \in \Omega(x). \tag{3.26}$$

Assume that $\sum_{i=1}^m a_i y_i(t) = 0$ a.e. in $[0,\alpha]$. Taking the Caputo derivatives a.e. in $[0,\alpha]$, we have $\sum_{i=1}^m a_i x_i(t) = 0$ a.e. in $[0,\alpha]$ and hence $a_i = 0$ for all i. As a result, $y_i(\cdot)$ are linearly independent. Thus, $\Omega(x)$ contains an m-dimensional simplex. So, $\dim \Omega(x) \ge m$. As in Lemma 3.2, $\mathrm{Fix}(\Omega)$ is nonempty. It is known that every multivalued k-contraction having compact values is condensing with respect to the Hausdorff measure of noncompactness χ [26]. Since $\mathrm{Fix}(\Omega) \subset \Omega(\mathrm{Fix}(\Omega))$, we have

$$\chi(\mathrm{Fix}(\Omega)) \le \chi(\Omega(\mathrm{Fix}(\Omega))). \tag{3.27}$$

Since Ω is χ-condensing, $\mathrm{Fix}(\Omega)$ is compact. Consider a map $I - \Omega : \mathrm{Fix}(\Omega) \to P_{c,cp}(\mathbb{R}^n)$, where I is the identity operator. Assume that $\dim \mathrm{Fix}(\Omega) < n$. Then, Lemma 2.5 guarantees that there is a continuous selection g of $I - \Omega$ such that $g(x) \neq 0$ for each $x \in \mathrm{Fix}(\Omega)$. This implies that there exists a continuous selection h of $F : \mathrm{Fix(F)} \to P_{c,cp}(\mathbb{R}^n)$ without fixed points. Define $\Lambda : \mathbb{R}^n \to P_{c,cp}(\mathbb{R}^n)$ by

$$\Lambda(x) = \begin{cases} \Omega(x), & x \in \mathbb{R}^n \setminus \mathrm{Fix}(\Omega), \\ h(x), & x \in \mathrm{Fix}(\Omega). \end{cases} \tag{3.28}$$

Since Λ is lower semicontinuous, in view of Michael's selection result (Lemma 2.6), Λ admits a continuous selection $f : \mathbb{R}^n \to \mathbb{R}^n$. Thus $f : \mathbb{R}^n \to \mathbb{R}^n$ is a continuous selection of Ω with no fixed points and $f = h$ on $\mathrm{Fix}(\Omega)$ contradicting Lemma 2.7. As a result, $\mathrm{Fix}(\Omega) = S_{x_0}([0,\alpha])$ is infinite dimensional. □

Definition 3.5. A metric space X is said to be an AR-space if, whenever it is nonempty closed subset of another metric space Y, there exists a continuous retraction $r : Y \to X$, $r(x) = x$ for $x \in X$. In particular, it is contractible (and hence connected).

Theorem 3.6 (see [27]). *Let C be a nonempty closed convex subset of a Banach space X and $F : C \to P_{c,cp}(C)$ a contraction. Then* $\mathrm{Fix}(F)$ *is a nonempty AR-space.*

The following result is a consequence of Theorems 3.4 and 3.6.

Corollary 3.7. *Suppose that $F : [0,\alpha] \times \mathbb{R}^n \to P_{c,\mathrm{cp}}(\mathbb{R}^n)$ satisfies (H_1), (H_2), and (H_3) and that the Lebesgue measure μ of the set $\{t : \dim F(t,x) < 1$ for some $x \in \mathbb{R}^n\}$ is zero. Then, for each α, $0 < \alpha < \min\{((1-\kappa_2)\Gamma(q+1)/(A_1 + \|\kappa_1\|_\infty))^{1/q}, T\}$, the set $S_{x_0}([0,\alpha])$ of solutions of* (1.1) *is a compact and infinite dimensional AR-space.*

Acknowledgment

This project was funded by the Deanship of Scientifc Research (DSR), King Abdulaziz University, Jeddah, under grant no. 08/31/Gr. The authors, therefore, acknowledge with thanks DSR's technical and financial support.

References

[1] G. M. Zaslavsky, *Hamiltonian Chaos and Fractional Dynamics*, Oxford University Press, Oxford, UK, 2005.
[2] R. L. Magin, *Fractional Calculus in Bioengineering*, Begell House Publisher, Ridgefield, Conn, USA, 2006.
[3] J. Sabatier, O. P. Agrawal, and J. A. T. Machado, Eds., *Advances in Fractional Calculus: Theoretical Developments and Applications in Physics and Engineering*, Springer, Dordrecht, The Netherlands, 2007.
[4] I. Podlubny, *Fractional Differential Equations*, Academic Press, San Diego, Calif, USA, 1999.
[5] A. A. Kilbas, H. M. Srivastava, and J. J. Trujillo, *Theory and Applications of Fractional Differential Equations*, vol. 204 of *North-Holland Mathematics Studies*, Elsevier Science B.V., Amsterdam, The Netherlands, 2006.
[6] V. Lakshmikantham, S. Leela, and J. Vasundhara Devi, *Theory of Fractional Dynamic Systems*, Cambridge Academic Publishers, Cambridge, UK, 2009.
[7] R. P. Agarwal, M. Belmekki, and M. Benchohra, "A survey on semilinear differential equations and inclusions involving Riemann-Liouville fractional derivative," *Advances in Difference Equations*, vol. 2009, Article ID 981728, 47 pages, 2009.
[8] Y.-K. Chang and J. J. Nieto, "Some new existence results for fractional differential inclusions with boundary conditions," *Mathematical and Computer Modelling*, vol. 49, no. 3-4, pp. 605–609, 2009.
[9] B. Ahmad and J. J. Nieto, "Existence results for a coupled system of nonlinear fractional differential equations with three-point boundary conditions," *Computers & Mathematics with Applications*, vol. 58, no. 9, pp. 1838–1843, 2009.
[10] A. Cernea, "On the existence of solutions for nonconvex fractional hyperbolic differential inclusions," *Communications in Mathematical Analysis*, vol. 9, no. 1, pp. 109–120, 2010.
[11] B. Ahmad, "Existence results for fractional differential inclusions with separated boundary conditions," *Bulletin of the Korean Mathematical Society*, vol. 47, no. 4, pp. 805–813, 2010.
[12] B. Ahmad and S. K. Ntouyas, "Some existence results for boundary value problems of fractional differential inclusions with non-separated boundary conditions," *Electronic Journal of Qualitative Theory of Differential Equations*, no. 71, pp. 1–17, 2010.

[13] R. P. Agarwal and B. Ahmad, "Existence theory for anti-periodic boundary value problems of fractional differential equations and inclusions," *Computers & Mathematics with Applications*, vol. 62, no. 3, pp. 1200–1214, 2011.
[14] B. Ahmad and N. Shahzad, "A nonlocal boundary value problem for fractional differential inclusions of arbitrary order involving convex and non-convex valued maps," *Vietnam Journal of Mathematics*, vol. 38, no. 4, pp. 435–451, 2010.
[15] G. M. N'Guerekata, "A Cauchy problem for some fractional abstract differential equation with non local conditions," *Nonlinear Analysis*, vol. 70, no. 5, pp. 1873–1876, 2009.
[16] R. P. Agarwal and B. Ahmad, "Existence of solutions for impulsive anti-periodic boundary value problems of fractional semilinear evolution equations," *Dynamics of Continuous, Discrete and Impulsive Systems*, vol. 18, no. 4, pp. 457–470, 2011.
[17] M. Kisielewicz, *Differential Inclusions and Optimal Control*, Kluwer Academic, Dodrecht, The Netherlands, 1991.
[18] S. Hu and N. Papageorgiou, *Handbook of Multivalued Analysis: Theory*, vol. I, Kluwer Academic, Dodrecht, The Netherlands, 1997.
[19] J. Dugundji and A. Granas, *Fixed Point Theory*, Springer, New York, NY, USA, 2005.
[20] H. Covitz and S. B. Nadler, Jr., "Multi-valued contraction mappings in generalized metric spaces," *Israel Journal of Mathematics*, vol. 8, pp. 5–11, 1970.
[21] Z. Dzedzej and B. D. Gelman, "Dimension of the solution set for differential inclusions," *Demonstratio Mathematica*, vol. 26, no. 1, pp. 149–158, 1993.
[22] J. Saint-Raymond, "Points fixes des multiapplications valeurs convexes," *Comptes Rendus de l'Académie des Sciences*, vol. 298, no. 4, pp. 71–74, 1984.
[23] E. Michael, "Continuous selections. I," *The Annals of Mathematics*, vol. 63, no. 2, pp. 361–382, 1956.
[24] A. Petrusel, "Multivalued operators and continuous selections. The fixed points set," *Pure Mathematics and Applications*, vol. 9, no. 1-2, pp. 165–170, 1998.
[25] C. Castaing and M. Valadier, *Convex Analysis and Measurable Multifunctions*, vol. 580 of *Lecture Notes in Mathematics*, Springer, Berlin, Germany, 1977.
[26] R. R. Akhmerov, M. I. Kamenskii, A. S. Potapov, A. E. Rodkina, and B. N. Sadovskii, *Measures of Noncompactness and Condensing Operators*, vol. 55 of *Operator Theory: Advances and Applications*, Birkhauser, Basel, Switzerland, 1992, Translated from the 1986 Russian original by A. Iacob.
[27] B. Ricceri, "Une proprit topologique de l'ensemble des points fixes d'une contraction multivoque valeurs convexes," *Atti della Accademia Nazionale dei Lincei, Rendiconti della Classe di Scienze Fisiche, Matematiche e Naturali Serie 8*, vol. 81, no. 3, pp. 283–286, 1987.