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Research Article

Some Properties on the q-Euler Numbers and Polynomials

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We give some new identities on q-Euler numbers and polynomials by using the fermionic p-adic integral on $\mathbb{Z}_p$.

1. Introduction

Let p be a fixed odd prime number. Throughout this paper $\mathbb{Z}_p$, $\mathbb{Q}_p$, and $\mathbb{C}_p$ will denote the ring of p-adic rational integers, the field of p-adic rational numbers, and the completion of the algebraic closure of $\mathbb{Q}_p$. The p-adic absolute value $|\cdot|_p$ is defined by $|p|_p = 1/p$. In this paper, we assume that $q \in \mathbb{C}_p$ with $|1-q|_p < 1$. As is well known, the fermionic p-adic integral on $\mathbb{Z}_p$ is defined by Kim as follows:

$$I_{-1}(f) = \int_{\mathbb{Z}_p} f(x)\,d\mu_{-1}(x) = \lim_{N\to\infty} \sum_{x=0}^{p^N-1} (-1)^x f(x), \tag{1.1}$$

where $f \in C(\mathbb{Z}_p)$ = the space of continuous functions on $\mathbb{Z}_p$ (see [1]).

From (1.1), we note that

$$I_{-1}(f_1) + I_{-1}(f) = 2f(0), \quad \text{where } f_1(x) = f(x+1). \tag{1.2}$$

The q-Euler polynomials are defined by

$$\frac{2}{qe^t+1}e^{xt}=e^{E_q(x)t}=\sum_{n=0}^{\infty}E_{n,q}(x)\frac{t^n}{n!},\tag{1.3}$$

with the usual convention about replacing E_q^n by $E_{n,q}$ (see [2, 3]).

Let us take $f(y)=q^y e^{t(x+y)}$. Then, by (1.2), we get

$$\int_{\mathbb{Z}_p} e^{(x+y)t}q^y d\mu_{-1}(y)=\frac{2}{qe^t+1}e^{xt}=\sum_{n=0}^{\infty}E_{n,q}(x)\frac{t^n}{n!}.\tag{1.4}$$

By (1.3) and (1.4), we get the Witt's formula for the q-Euler polynomials as follows:

$$E_{n,q}(x)=\int_{\mathbb{Z}_p} q^y(x+y)^n d\mu_{-1}(y),\qquad n\in\mathbb{Z}_+=\mathbb{N}\cup\{0\}.\tag{1.5}$$

In the special case, $x=0$, $E_{n,q}(0)=E_{n,q}$ are called the n-th q-Euler numbers.

From (1.3), we can derive the following recurrence relation for the q-Euler numbers $E_{n,q}$:

$$E_{0,q}=\frac{2}{[2]_q},\qquad q(E_q+1)^n+E_{n,q}=2\delta_{0,n},\tag{1.6}$$

with the usual convention about replacing E_q^n by $E_{n,q}$ (see [4]).

By (1.5), we easily see that

$$E_{n,q}(x)=\sum_{\ell=0}^{n}\binom{n}{\ell}x^{n-\ell}\int_{\mathbb{Z}_p} q^y y^{\ell} d\mu_{-1}(y)=\sum_{\ell=0}^{n}\binom{n}{\ell}x^{n-\ell}E_{\ell,q},\tag{1.7}$$

where $\binom{n}{\ell}=n!/\ell!(n-\ell)!=n(n-1)\cdots(n-\ell+1)/\ell!$ (see [1, 2, 4–13]). Cohen introduced many interesting and valuable identities related to Euler and Bernoulli numbers and polynomials in his book (see [14]). In [13], Ryoo has introduced the q-Euler numbers and polynomials with weight α, and Simsek et al. have studied q-Euler numbers and polynomials, and they introduced many interesting identities and properties (see [3, 15, 16]). In this paper, we consider the q-Euler numbers and polynomials with weight $\alpha=1$. By applying the fermionic p-adic integral on $\mathbb{Z}_p$, we derive many not only new but also some interesting identities on the q-extension of Euler numbers and polynomials. In particular, we consider that Theorems 2.5, 2.6, 2.7, and 2.9 are important identities because these identities are closely related to Frobenius-Euler numbers and polynomials. As is well known, Frobenius-Euler numbers and polynomials are important to study p-adic l-functions in the number theory and mathematical physics related to fermionic distributions. In [17], Bayad and Kim have studied some interesting identities and properties on the q-Euler numbers and polynomials associated with Bernstein polynomials. Recently, several authors have studied some properties of q-Euler numbers and polynomials (see [1–19]). The purpose of this paper is to give some interesting new identities for the q-Euler numbers and polynomials by using the fermionic p-adic integral on $\mathbb{Z}_p$ and (1.7).

2. Some Identities on q-Euler Polynomials

From (1.4), we note that

$$\int_{\mathbb{Z}_p} e^{(x+y+z)t} q^z d\mu_{-1}(z) = e^{(x+y)t}\frac{2}{qe^t+1} = \sum_{n=0}^{\infty} E_{n,q}(x+y)\frac{t^n}{n!}. \tag{2.1}$$

Thus, by (1.4) and (2.1), we get

$$\begin{aligned} E_{n,q}(x+y) &= \int_{\mathbb{Z}_p} (x+y+z)^n q^z d\mu_{-1}(z) \\ &= \sum_{\ell=0}^{n}\binom{n}{\ell} y^{n-\ell}\int_{\mathbb{Z}_p}(x+z)^{\ell} q^z d\mu_{-1}(z) \\ &= \sum_{\ell=0}^{n}\binom{n}{\ell} y^{n-\ell} E_{\ell,q}(x), \quad \text{where } n \in \mathbb{Z}_+. \end{aligned} \tag{2.2}$$

By (2.2), we get

$$\begin{aligned} E_{n,q}(x+y) &= \sum_{j=0}^{n}\binom{n}{j} y^{n-j} E_{j,q}(x) \\ &= \frac{2}{[2]_q} y^n + \sum_{j=1}^{n}\frac{n}{j}\binom{n-1}{j-1} y^{n-j} E_{j,q}(x). \end{aligned} \tag{2.3}$$

From (2.3), we can derive the following equation (2.4):

$$\sum_{j=0}^{n-1}\binom{n-1}{j} y^{n-1-j}\frac{E_{j+1,q}(x)}{j+1} = \frac{E_{n,q}(x+y) - \left(2/[2]_q\right)y^n}{n}. \tag{2.4}$$

Therefore, by (2.4), we obtain the following theorem.

Theorem 2.1. *For $n \in \mathbb{Z}_+$, one has*

$$\sum_{j=0}^{n}\binom{n}{j} y^{n-j}\frac{E_{j+1,q}(x)}{j+1} = \frac{1}{n+1}\left(E_{n+1,q}(x+y) - \frac{2}{[2]_q} y^{n+1}\right). \tag{2.5}$$

Let us replace y by $-y$ in Theorem 2.1. Then we get

$$\sum_{j=0}^{n}\binom{n}{j}(-1)^{n-j}y^{n-j}\frac{E_{j+1,q}(x)}{j+1}=\frac{1}{n+1}\left(E_{n+1,q}(x-y)-\frac{(-1)^{n+1}2}{[2]_q}y^{n+1}\right). \tag{2.6}$$

Thus, we have

$$\sum_{j=0}^{n}\binom{n}{j}(-1)^{j}y^{n-j}\frac{E_{j+1,q}(x)}{j+1}=\frac{1}{n+1}\left((-1)^{n}E_{n+1,q}(x-y)+\frac{2}{[2]_q}y^{n+1}\right). \tag{2.7}$$

Therefore, by Theorem 2.1 and (2.7), we obtain the following corollary.

Corollary 2.2. *For $n\in\mathbb{Z}_+$, one has*

$$\sum_{j=1}^{[n/2]}\binom{n}{2j}y^{n-2j}\frac{E_{2j+1,q}(x)}{2j+1}=\frac{E_{n+1,q}(x+y)+(-1)^{n}E_{n+1,q}(x-y)}{2n+2}. \tag{2.8}$$

From (2.2), we have

$$\sum_{j=1}^{n}\frac{1}{j}\binom{n-1}{j-1}y^{n-j}(-1)^{j}E_{j,q}(x)=\frac{(-1)^{n}E_{n,q}(x-y)-y^{n}\left(2/[2]_q\right)}{n}. \tag{2.9}$$

Therefore, by (2.3) and (2.9), we obtain the following theorem.

Theorem 2.3. *For $n\in\mathbb{N}$, one has*

$$\sum_{j=1}^{[(n+1)/2]}\binom{n}{2j-1}y^{n+1-2j}\frac{E_{2j,q}(x)}{j}=\frac{E_{n+1,q}(x+y)+(-1)^{n+1}E_{n+1,q}(x-y)-\left(4/[2]_q\right)y^{n+1}}{4n+4}. \tag{2.10}$$

Letting $y=1$ in Theorem 2.1, we see that

$$\begin{aligned}
q\sum_{j=0}^{n}\binom{n}{j}\frac{E_{j+1,q}(x)}{j+1}&=\frac{qE_{n+1,q}(x+1)-2q/[2]_q}{n+1},\\
qE_{n+1,q}(x+1)&=\sum_{\ell=0}^{n+1}\binom{n+1}{\ell}\left(E_q+1\right)^{\ell}x^{n+1-\ell}\\
&=\left(2-E_{0,q}\right)x^{n+1}-\sum_{\ell=1}^{n+1}\binom{n+1}{\ell}E_{\ell,q}x^{n+1-\ell}
\end{aligned}$$

$$\begin{aligned}&= 2x^{n+1} - \sum_{\ell=0}^{n+1}\binom{n+1}{\ell}E_{\ell,q}x^{n+1-\ell}\\&= 2x^{n+1} - E_{n+1,q}(x).\end{aligned} \tag{2.11}$$

Therefore, by (2.11), we obtain the following theorem.

Theorem 2.4. *For $n \in \mathbb{Z}_+$, one has*

$$\sum_{j=0}^{n}\binom{n}{j}\frac{qE_{j+1,q}(x)}{j+1} = -\frac{E_{n+1,q}}{n+1} + \frac{2x^{n+1}}{n+1} - \frac{2}{[2]_q}\frac{1}{n+1}. \tag{2.12}$$

Replacing y by 1 and n by $2n$ in Corollary 2.2, we have

$$\begin{aligned}&\sum_{j=0}^{n}\binom{2n}{2j}\frac{E_{2j+1,q}(x)}{2j+1}\\&\quad= \frac{E_{2n+1,q}(x+1) + E_{2n+1,q}(x-1)}{4n+2}\\&\quad= \frac{(1/q)\left(qE_{2n+1,q}(x+1) + E_{2n+1,q}(x)\right) + \left(qE_{2n+1,q}(x) + E_{2n+1,q}(x-1)\right)}{4n+2}\\&\qquad- \frac{E_{2n+1,q}(x)}{q(4n+2)} - q\frac{E_{2n+1,q}(x)}{4n+2}\\&\quad= \frac{2x^{2n+1}}{q(4n+2)} + \frac{2(x-1)^{2n+1}}{4n+2} - \frac{E_{2n+1,q}(x)}{q(4n+2)} - q\frac{E_{2n+1,q}(x)}{4n+2}.\end{aligned} \tag{2.13}$$

Therefore, by (2.13), we obtain the following theorem.

Theorem 2.5. *For $n \in \mathbb{Z}_+$, one has*

$$\sum_{j=0}^{n}\binom{2n}{2j}\frac{E_{2j+1,q}(x)}{2j+1} = \frac{x^{2n+1}}{q(2n+1)} + \frac{(x-1)^{2n+1}}{2n+1} - \frac{E_{2n+1,q}(x)}{q(4n+2)} - q\frac{E_{2n+1,q}(x)}{4n+2}. \tag{2.14}$$

Replacing y by 1 and n by $2n$ in Theorem 2.3, we have

$$\begin{aligned}&\sum_{j=1}^{n}\binom{2n}{2j-1}\frac{E_{2j,q}(x)}{j}\\&\quad= \frac{E_{2n+1,q}(x+1) - E_{2n+1,q}(x-1)}{8n+4} - \frac{1}{(2n+1)[2]_q}\end{aligned}$$

$$
\begin{aligned}
&= \frac{2x^{2n+1}}{q(8n+4)} - \frac{2(x-1)^{2n+1}}{8n+4} - \frac{1}{(2n+1)[2]_q} - \frac{E_{2n+1,q}(x)}{q(8n+4)} + \frac{qE_{2n+1,q}(x)}{8n+4} \\
&= \frac{x^{2n+1}}{2q(2n+1)} - \frac{(x-1)^{2n+1}}{2(2n+1)} - \frac{1}{(2n+1)[2]_q} - \frac{E_{2n+1,q}(x)}{q(8n+4)} + \frac{qE_{2n+1,q}(x)}{8n+4}.
\end{aligned}
\tag{2.15}
$$

Therefore, by (2.15), we obtain the following theorem.

Theorem 2.6. *For $n \in \mathbb{Z}_+$, one has*

$$
\sum_{\ell=1}^{n} \binom{2n}{2j-1} \frac{E_{2j,q}(x)}{j} = \frac{x^{2n+1}}{2q(2n+1)} - \frac{2(x-1)^{2n+1}}{2(2n+1)} - \frac{1}{(2n+1)[2]_q} - \frac{E_{2n+1,q}(x)}{q(8n+4)} + q\frac{E_{2n+1,q}(x)}{8n+4}.
\tag{2.16}
$$

Replacing y by $1/2$ and n by $2n$ in Theorem 2.3, we get

$$
\begin{aligned}
&\sum_{j=1}^{n} \binom{2n}{2j-1} \left(\frac{1}{2}\right)^{2n+1-2j} \frac{E_{2j}(x)}{j} \\
&\quad = \frac{E_{2n+1,q}(x+1/2) - E_{2n+1}(x-1/2) - \left(4/[2]_q\right)(1/2)^{2n+1}}{8n+4}.
\end{aligned}
\tag{2.17}
$$

Thus, by (2.17), we get

$$
\sum_{j=1}^{n} \binom{2n}{2j-1} 2^{2j} \frac{E_{2j}(x)}{j} = \frac{2^{2n}\left(E_{2n+1,q}(x+1/2) - E_{2n+1}(x-1/2)\right)}{4n+2} - \frac{1}{(2n+1)[2]_q}.
\tag{2.18}
$$

Note that

$$
\begin{aligned}
qE_{2n+1,q}\left(x+\frac{1}{2}\right) &= qE_{2n+1,q}\left(x-\frac{1}{2}+1\right) \\
&= q\sum_{\ell=0}^{2n+1} \binom{2n+1}{\ell} \left(x-\frac{1}{2}\right)^{2n+1-\ell} \left(E_q+1\right)^{\ell} \\
&= \left(x-\frac{1}{2}\right)^{2n+1} \left(2-E_{0,q}\right) - \sum_{\ell=1}^{2n+1} \binom{2n+1}{\ell} \left(x-\frac{1}{2}\right)^{2n+1-\ell} E_{\ell,q} \\
&= 2\left(x-\frac{1}{2}\right)^{2n+1} - E_{2n+1,q}\left(x-\frac{1}{2}\right).
\end{aligned}
\tag{2.19}
$$

Therefore, by (2.18) and (2.19), we obtain the following theorem.

Theorem 2.7. *For $n \in \mathbb{N}$, one has*

$$\sum_{j=1}^{n}\binom{2n}{2j-1}2^{2j}\frac{E_{2j}(x)}{j}=\frac{2^{2n}(x-1/2)^{2n+1}}{2n+1}-\frac{[2]_q 2^n}{4n+2}E_{2n+1,q}\left(x-\frac{1}{2}\right)-\frac{1}{(2n+1)[2]_q}. \tag{2.20}$$

Replacing y by 1 and n by $2n+1$ in Corollary 2.2, we see that

$$\begin{aligned}
&\sum_{j=0}^{n}\binom{2n+1}{2j}\frac{E_{2j+1,q}(x)}{2j+1}\\
&\quad=\frac{E_{2n+2,q}(x+1)-E_{2n+2,q}(x-1)}{4n+4}\\
&\quad=\left(\frac{(1/q)\left(qE_{2n+2,q}(x+1)+E_{2n+2,q}(x)\right)}{4n+4}\right)-\left(\frac{qE_{2n+2,q}(x)+E_{2n+2,q}(x-1)}{4n+4}\right)\\
&\qquad-\left(\frac{E_{2n+2,q}(x)}{q(4n+4)}-q\frac{E_{2n+2,q}(x)}{4n+4}\right)\\
&\quad=\frac{2x^{2n+2}}{q(4n+4)}-\frac{2(x-1)^{2n+2}}{4n+4}-\left(\frac{E_{2n+2,q}(x)}{q(4n+4)}-q\frac{E_{2n+2,q}(x)}{4n+4}\right).
\end{aligned} \tag{2.21}$$

Therefore, by (2.21), we obtain the following theorem.

Theorem 2.8. *For $n \in \mathbb{Z}_+$, one has*

$$\sum_{j=0}^{n}\binom{2n+1}{2j}\frac{E_{2j+1,q}(x)}{2j+1}=\frac{x^{2n+2}}{q(2n+2)}-\frac{(x-1)^{2n+2}}{2n+2}-(1-q)\frac{[2]_qE_{2n+2,q}(x)}{q(4n+4)}. \tag{2.22}$$

Replacing n by $2n+1$ and y by 1 in Theorem 2.3, we get

$$\begin{aligned}
&\sum_{j=1}^{n+1}\binom{2n+1}{2j-1}\frac{E_{2j,q}(x)}{j}\\
&\quad=\frac{E_{2n+2,q}(x+1)+E_{2n+2,q}(x-1)}{8n+8}-\frac{1}{(2n+2)[2]_q}\\
&\quad=\frac{qE_{2n+2,q}(x+1)+E_{2n+2,q}(x)}{q(8n+8)}+\frac{qE_{2n+2,q}(x)+E_{2n+2,q}(x)}{8n+8}\\
&\qquad-\frac{E_{2n+2,q}(x)}{q(8n+8)}-\frac{qE_{2n+2,q}(x)}{8n+8}-\frac{1}{(2n+2)[2]_q}\\
&\quad=\frac{2x^{2n+2}}{q(8n+8)}+\frac{q2(x-1)^{2n+2}}{8n+8}-\frac{E_{2n+2,q}(x)}{q(8n+8)}-q\frac{E_{2n+2,q}(x)}{8n+8}-\frac{1}{(2n+2)[2]_q}.
\end{aligned} \tag{2.23}$$

Therefore, by (2.23), we obtain the following theorem.

Theorem 2.9. *For $n \in \mathbb{Z}_+$, one has*

$$\sum_{j=1}^{n+1}\binom{2n+1}{2j-1}\frac{E_{2j,q}(x)}{j}=\frac{x^{2n+2}}{q(4n+4)}+q\frac{(x-1)^{2n+2}}{4n+4}-\left(1+q^2\right)\frac{E_{2n+2,q}(x)}{8n+8}-\frac{1}{(2n+2)[2]_q}. \tag{2.24}$$

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