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*Research Article*

# Krasnosel'skii Type Fixed Point Theorems for Mappings on Nonconvex Sets

**Maryam A. Alghamdi,[1] Donal O'Regan,[2] and Naseer Shahzad[3]**

[1] *Department of Mathematics, Faculty of Science for Girls, King Abdulaziz University, P.O. Box 4087, Jeddah 21491, Saudi Arabia*
[2] *Department of Mathematics, School of Mathematics, Statistics and Applied Mathematics, National University of Ireland, Galway, Ireland*
[3] *Department of Mathematics, King Abdulaziz University, P.O. Box 80203, Jeddah 21859, Saudi Arabia*

Correspondence should be addressed to Naseer Shahzad, nshahzad@kau.edu.sa

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We prove Krasnosel'skii type fixed point theorems in situations where the domain is not necessarily convex. As an application, the existence of solutions for perturbed integral equation is considered in $p$-normed spaces.

## 1. Introduction

Let $X$ be a linear space over $K$ ($K = \mathbb{R}$ or $K = \mathbb{C}$) with the origin $\theta$. A functional $\|\cdot\|_p : X \to [0, \infty)$ with $0 < p \leq 1$ is called a $p$-norm on $X$ if the following conditions hold

(a) $\|x\|_p = 0$ if and only if $x = \theta$;

(b) $\|\lambda x\|_p = |\lambda|^p \|x\|_p$, for all $x \in X$, $\lambda \in K$;

(c) $\|x + y\|_p \leq \|x\|_p + \|y\|_p$, for all $x, y \in X$.

The pair $(X, \|\cdot\|_p)$ is called a $p$-normed space. If $p = 1$, then $X$ is a usual normed space. A $p$-normed space is a metric linear space with a translation invariant metric $d_p$ given by $d_p(x, y) = \|x - y\|_p$ for all $x, y \in X$.

Let $\Omega$ be a nonempty set, let $M$ be a $\sigma$-algebra in $\Omega$, and let $\mu : M \to [0, +\infty)$ be a positive measure. The space $L^p(\mu)$ based on the complete measure space $(\Omega, M, \mu)$ is an example of a $p$-normed space with the $p$-norm defined by

$$\begin{gathered}
\|f(t)\|_p = \int_\Omega |f(t)|^p d\mu, \quad \text{for } f \in L^p(\mu), \\
L^p(\mu) = \left\{ f : f : \Omega \longrightarrow K \text{ is measurable, } \int_\Omega |f(t)|^p d\mu < +\infty \right\}.
\end{gathered} \tag{1.1}$$

Another example of a $p$-normed space is $C_p[0,1]$, the space of all continuous functions defined on the unit interval $[0,1]$ with the sup $p$-norm given by

$$\|x\|_p = \sup_{0 \le t \le 1} |x(t)|^p, \quad \text{for } x \in C_p[0,1]. \tag{1.2}$$

The class of $p$-normed spaces $(0 < p \le 1)$ is a significant generalization of the class of usual normed spaces. For more details about $p$-normed spaces, we refer the reader to [1, 2].

It is noted that most fixed point theorems are concerned with convex sets. As we know, there exists nonconvex sets also, for example, the unit ball with center $\theta$ in a $p$-normed space $(0 < p < 1)$ is not a convex set. It is a natural question whether the well-known fixed point theorems could be extended to nonconvex sets. Xiao and Zhu [3] established the existence of fixed points of mappings on $s$-convex sets in $p$-normed spaces, where $0 < p \le 1$, $0 < s \le p$.

**Theorem 1.1** (see [3] (Krasnosel'skii-type)). *Let $(X, \|\cdot\|_p)$ be a complete $p$-normed space and $C$ a bounded closed $s$-convex subset of $X$, where $0 < p \le 1$, $0 < s \le p$. Let $T : C \to X$ be a contraction mapping and $S : C \to X$ a completely continuous mapping. If $Tx + Sy \in C$ for all $x, y \in C$, then there exists $x^* \in C$ such that $Sx^* + Tx^* = x^*$.*

In this paper, we investigate the fixed point problem of the sum of an expansive mapping and a compact mapping. Our results extend and complement the classical Krasnosel'skii fixed point theorem. We also prove the Sadovskii theorem for $s$-convex sets in $p$-normed spaces, where $0 < p \le 1$, $0 < s \le p$, and from it we obtain some fixed point theorems for the sum of two mappings. In the last section, as an application of a Krasnosel'skii-type theorem, the existence of solutions for perturbed integral equation is considered in $p$-normed spaces.

## 2. Preliminaries

Throughout this paper, we denote the closure and the boundary of a subset $A$ of $X$ by $\overline{A}$ and $\partial A$, respectively. $B(x, r)$ will be the open ball of $X$ with center $x \in X$ and radius $r > 0$.

*Definition 2.1.* Let $(X, d_X)$ and $(Y, d_Y)$ be two metric spaces and $T : X \to Y$. The mapping $T$ is said to be $k$-Lipschitz, where $k$ is a positive constant, if

$$d_Y(Tx, Ty) \le k d_X(x, y), \quad \forall x, y \in X. \tag{2.1}$$

$T$ is said to be nonexpansive if $k = 1$, and to be a contraction if $k < 1$.

It is clear that every $k$-Lipschitz mapping is continuous. Moreover, the Banach contraction principle holds for a closed subset in a complete $p$-normed space.

*Definition 2.2* (see [3]). Let $(X, \|\cdot\|_p)$ $(0 < p \leq 1)$ be a $p$-normed space and $0 < s \leq p$. A set $C \subset X$ is said to be $s$-convex if the following condition is satisfied

$$(1-t)^{1/s}x + t^{1/s}y \in C, \quad \text{whenever } x, y \in C,\ t \in [0,1]. \tag{2.2}$$

Let $A \subset X$. The $s$-convex hull of $A$ denoted by $co_s A$ is the smallest $s$-convex set containing $A$ and the closed $s$-convex hull of $A$ denoted by $\overline{co_s}A$ is the smallest closed $s$-convex set containing $A$.

In other words, the $s$-convexity of the set $C$ is equivalent to that

$$t_1 x + t_2 y \in C, \quad \text{whenever } x, y \in C,\ t_1, t_2 \geq 0,\ t_1^s + t_2^s = 1. \tag{2.3}$$

For $s = 1$, we obtain the usual definition of convex sets. For a subset $A$ of $X$, the $s$-convex hull of $A$ is given by

$$co_s A = \left\{ \sum_{i=1}^{n} t_i x_i : t_i \geq 0,\ \sum_{i=1}^{n} t_i^s = 1,\ x_i \in A,\ n \geq 2 \right\}. \tag{2.4}$$

It is easy to see that if $C$ is a closed $s$-convex set, then $\theta \in C$.

**Lemma 2.3** (see [3]). *Let $(X, \|\cdot\|_p)$ $(0 < p \leq 1)$ be a $p$-normed space and $0 < s \leq p$.*

(a) *The ball $B(\theta, r)$ is $s$-convex, where $r > 0$.*

(b) *If $C \subset X$ is $s$-convex and $\alpha \in K$, then $\alpha C$ is $s$-convex.*

(c) *If $C_1, C_2 \subset X$ are $s$-convex, then $C_1 + C_2$ is $s$-convex.*

(d) *If $C_i \subset X$, $i = 1, 2, \ldots$ are all $s$-convex, then $\bigcap_{i=1}^{\infty} C_i$ is $s$-convex.*

(e) *If $A \subset X$ and $\theta \in A$, then $co_s A \subset coA$, where $coA$ is the convex hull of $A$.*

(f) *If $C$ is a closed $s$-convex set and $0 < k < s$, then $C$ is a closed $k$-convex set.*

(g) *If $X$ is complete and $A$ is a totally bounded subset of $X$, then $\overline{co_s}A(0 < s \leq p)$ is compact.*

**Theorem 2.4** (see [3] (Schauder-type)). *Let $(X, \|\cdot\|_p)$ be a complete $p$-normed space and $C$ a compact $s$-convex subset of $X$, where $0 < p \leq 1, 0 < s \leq p$. If $S : C \to C$ is continuous, then $S$ has a fixed point (i.e., there exists $x^* \in C$ such that $Sx^* = x^*$).*

**Theorem 2.5.** *Let $(X, \|\cdot\|_p)$ be a complete $p$-normed space and $C$ a closed $s$-convex subset of $X$, where $0 < p \leq 1, 0 < s \leq p$. If $S : C \to C$ is a continuous compact map (i.e., the image of $C$ under $S$ is compact), then $S$ has a fixed point.*

*Proof.* Let $Q = \overline{co_s}(\overline{S(C)})$. Note $Q$ is a closed compact $s$-convex subset of $X$ and $S(Q) \subseteq S(C) \subseteq Q$. The result follows from Theorem 2.4. □

We will need the following definition.

*Definition 2.6* (see [4]). Let $(X, d)$ be a metric space and $C$ a subset of $X$. The mapping $T : C \to X$ is said to be expansive, if there exists a constant $h > 1$ such that

$$d(Tx, Ty) \geq hd(x, y), \quad \forall x, y \in C. \tag{2.5}$$

**Theorem 2.7** (see [4]). *Let $C$ be a closed subset of a complete metric space $(X, d)$. Assume that the mapping $T : C \to X$ is expansive and $T(C) \supset C$, then there exists a unique point $x^* \in C$ such that $Tx^* = x^*$.*

Recently, Xiang and Yuan [4] established a Krasnosel'skii type fixed point theorem when the mapping $T$ is expansive. For other related results, see also [5, 6].

## 3. Main Results

**Theorem 3.1.** *Let $(X, \|\cdot\|_p)$ be a complete $p$-normed space and $C$ a closed $s$-convex subset of $X$, where $0 < p \leq 1$, $0 < s \leq p$. Suppose that*

(i) *$S : C \to X$ is a continuous compact mapping (i.e., the image of $C$ under $S$ is compact);*

(ii) *$T : C \to X$ is an expansive mapping;*

(iii) *$z \in S(C)$ implies $T(C) + z \supset C$ where $T(C) + z = \{y + z : y \in T(C)\}$.*

*Then there exists a point $x^* \in C$ such that $Sx^* + Tx^* = x^*$.*

*Proof.* Let $z \in S(C)$. Then the mapping $T+z : C \to X$ satisfies the assumptions of Theorem 2.7 by virtue of (ii) and (iii), which guarantees that the equation

$$Tx + z = x \tag{3.1}$$

has a unique solution $x = \tau(z) \in C$. For any $z_1, z_2 \in S(C)$, we have

$$T(\tau(z_1)) + z_1 = \tau(z_1), \qquad T(\tau(z_2)) + z_2 = \tau(z_2), \tag{3.2}$$

and so

$$\begin{aligned} \|T(\tau(z_1)) - T(\tau(z_2))\|_p &= \|\tau(z_1) - z_1 - \tau(z_2) + z_2\|_p \\ &\leq \|z_1 - z_2\|_p + \|\tau(z_1) - \tau(z_2)\|_p. \end{aligned} \tag{3.3}$$

Since $T$ is expansive, there exists a constant $h > 1$ such that

$$\|T(\tau(z_1)) - T(\tau(z_2))\|_p \geq h\|\tau(z_1) - \tau(z_2)\|_p. \tag{3.4}$$

As a result

$$\|\tau(z_1) - \tau(z_2)\|_p \leq \frac{1}{h-1}\|z_1 - z_2\|_p. \tag{3.5}$$

This implies that $\tau : S(C) \to C$ is continuous. Since $S$ is continuous on $C$, it follows that $\tau S : C \to C$ is also continuous. Since $S$ is compact, so is $\tau S$. By Theorem 2.5, there exists $x^* \in C$, such that $\tau(S(x^*)) = x^*$. From (3.1), we have

$$T(\tau(S(x^*))) + S(x^*) = \tau(S(x^*)), \tag{3.6}$$

that is,

$$Tx^* + Sx^* = x^*. \tag{3.7}$$

This completes the proof. □

**Corollary 3.2.** *Let $(X, \|\cdot\|_p)$ be a complete $p$-normed space and $C$ a closed $s$-convex subset of $X$, where $0 < p \leq 1$, $0 < s \leq p$. Suppose that*

(i) *$S : C \to X$ is a continuous compact mapping;*

(ii) *$T : C \to X$ is an expansive and onto mapping.*

*Then there exists a point $x^* \in C$ such that $Sx^* + Tx^* = x^*$.*

The following example shows that there are mappings which are expansive and satisfy $T(C) \subset C$.

*Example 3.3.* Let $X = C = \mathbb{R}$ with the usual metric and consider $Tx = x^3 + 2x + 1$ for $x \in C$. Then for all $x, y \in C$, we have

$$\begin{aligned} |Tx - Ty| &= \left|\left(x^3 - y^3\right) + 2(x - y)\right| \\ &= \left|(x - y)\left(x^2 + xy + y^2\right) + 2(x - y)\right| \\ &= \left|\left(x^2 + xy + y^2 + 2\right)\right||x - y| \\ &\geq 2|x - y|. \end{aligned} \tag{3.8}$$

Thus $T$ is expansive with $T(C) \subset C$.

**Theorem 3.4.** *Let $(X, \|\cdot\|_p)$ be a complete $p$-normed space and $C$ a closed $s$-convex subset of $X$, where $0 < p \leq 1, 0 < s \leq p$. Suppose that*

(i) *$S : C \to X$ is a continuous compact mapping;*

(ii) *$T : C \to X$ is an expansive mapping;*

(iii) *$z \in S(C)$ implies $C + z \subset T(C) \subset C$ where $C + z = \{y + z : y \in C\}$.*

*Then there exists a point $x^* \in C$ such that $T \circ (I - S)x^* = x^*$.*

*Proof.* Since $T$ is expansive, it follows that the inverse of $T : C \to T(C)$ exists, $T^{-1} : T(C) \to C$ is a contraction and hence continuous. Thus $T(C)$ is a closed set. Then, for each fixed $z \in S(C)$, the equation

$$T^{-1}x + z = x \tag{3.9}$$

has a unique solution $x = \tau(z) \in T(C)$. For any $z_1, z_2 \in S(C)$, we have

$$T^{-1}(\tau(z_1)) + z_1 = \tau(z_1), \qquad T^{-1}(\tau(z_2)) + z_2 = \tau(z_2), \tag{3.10}$$

so that

$$\begin{aligned} \|\tau(z_1) - \tau(z_2)\|_p &\le \|z_1 - z_2\|_p + \left\|T^{-1}(\tau(z_1)) - T^{-1}(\tau(z_2))\right\|_p \\ &\le \|z_1 - z_2\|_p + \frac{1}{h}\|\tau(z_1) - \tau(z_2)\|_p. \end{aligned} \tag{3.11}$$

Thus,

$$\|\tau(z_1) - \tau(z_2)\|_p \le \frac{h}{h-1}\|z_1 - z_2\|_p. \tag{3.12}$$

This shows that $\tau : S(C) \to T(C)$ is continuous. Since $S$ is continuous on $C$, it follows that $\tau S : C \to T(C) \subset C$ is also continuous and since $S$ is compact, so is $\tau S$. Then, by Theorem 2.5, there exists $x^* \in C$, such that $\tau(S(x^*)) = x^*$. From (3.9) we have

$$T^{-1}(\tau(S(x^*))) + S(x^*) = \tau(S(x^*)), \tag{3.13}$$

that is,

$$T^{-1}x^* + Sx^* = x^*, \tag{3.14}$$

that is,

$$T \circ (I - S)x^* = x^*. \tag{3.15}$$

This completes the proof. □

**Lemma 3.5.** *Let $(X, \|\cdot\|_p)$ $(0 < p \le 1)$ be a $p$-normed space and $C \subset X$. Suppose that the mapping $T : C \to X$ is expansive with constant $h > 1$. Then the inverse of $F = I - T : C \to (I - T)(C)$ exists and*

$$\left\|F^{-1}x - F^{-1}y\right\|_p \le \frac{1}{h-1}\|x - y\|_p, \quad x, y \in F(C). \tag{3.16}$$

*Proof.* Let $x, y \in C$, we have

$$\begin{aligned}\|Fx - Fy\|_p &= \|(Tx - Ty) - (x - y)\|_p \\ &\geq \|Tx - Ty\|_p - \|x - y\|_p \\ &\geq (h-1)\|x - y\|_p.\end{aligned} \tag{3.17}$$

From (3.17) we see that $F$ is one-to-one. Therefore, the inverse of $F : C \to F(C)$ exists. Thus, for $x, y \in F(C)$, we have $F^{-1}x, F^{-1}y \in C$. Now, using $F^{-1}x, F^{-1}y$ and substituting for $x, y$ in (3.17), respectively, we obtain

$$\left\|F^{-1}x - F^{-1}y\right\|_p \leq \frac{1}{h-1}\|x - y\|_p. \tag{3.18}$$

□

**Theorem 3.6.** *Let $(X, \|\cdot\|_p)$ be a complete $p$-normed space and $C$ a closed $s$-convex subset of $X$, where $0 < p \leq 1, 0 < s \leq p$. Suppose that*

(i) $S : C \to X$ *is a continuous compact mapping;*

(ii) $T : X \to X$ *(or $T : C \to X$) is an expansive mapping with constant $h > 1$;*

(iii) $S(C) \subset (I - T)(X)$ *and [$x = Tx + Sy, y \in C$ implies $x \in C$] or $S(C) \subset (I - T)(C)$.*

*Then there exists a point $x^* \in C$ such that $Sx^* + Tx^* = x^*$.*

*Proof.* From (iii), for each $y \in C$ since $S(C) \subset (I-T)(X)$ or $S(C) \subset (I-T)(C)$ there is an $x \in X$ such that

$$x - Tx = Sy. \tag{3.19}$$

If $S(C) \subset (I - T)(C)$ then $x \in C$ whereas if $S(C) \subset (I - T)(X)$ then Lemma 3.5 and (iii) imply $x = (I - T)^{-1}Sy \in C$. Now $(I - T)^{-1}$ is continuous, and so $(I - T)^{-1}S$ is a continuous mapping of $C$ into $C$. Since $S$ is compact, so is $(I - T)^{-1}S$. By Theorem 2.5, $(I - T)^{-1}S$ has a fixed point $x^* \in C$ with $x^* = (I - T)^{-1}Sx^*$, that is, $Sx^* + Tx^* = x^*$. This completes the proof. □

**Theorem 3.7** (Petryshyn-type). *Let $(X, \|\cdot\|_p)$ be a complete $p$-normed space and $D$ an open $s$-convex subset of $X$ with $\theta \in D$, where $0 < p \leq 1$, $0 < s \leq p$. Let $S : \overline{D} \to X$ be a continuous compact mapping and*

$$\|x - Sx\|_p \geq \|Sx\|_p, \quad \forall x \in \partial D. \tag{3.20}$$

*Then there exists $z \in \overline{D}$ such that $Sz = z$.*

*Proof.* The proof is exactly the same as the proof of Theorem 2.20 (b) [3]. Here we use Theorem 2.5 instead of Theorem 2.14 [3]. □

**Theorem 3.8.** *Let $(X, \|\cdot\|_p)$ be a complete p-normed space and $D$ an open s-convex subset of $X$, where $0 < p \leq 1$, $0 < s \leq p$. Suppose that*

(i) $S : \overline{D} \to X$ *is a continuous compact mapping;*

(ii) $T : X \to X$ *is an expansive map with constant* $h > 1$*;*

(iii) $S(\overline{D}) \subset (I - T)(X)$*;*

(iv) $\|Sx + T\theta\|_p \leq ((h-1)/2)\|x\|_p$ *for each* $x \in \partial D$.

*Then there exists a point* $x^* \in \overline{D}$ *such that* $Sx^* + Tx^* = x^*$.

*Proof.* From (iii), for each $x \in \overline{D}$, there is a $y \in X$ such that

$$y - Ty = Sx. \tag{3.21}$$

Thus by Lemma 3.5, we have $y = (I - T)^{-1}Sx := GSx \in X$. Again by Lemma 3.5 and (i), we see that $GS : \overline{D} \to X$ is compact. We now prove that (3.20) holds with $S$ replaced by $GS$. In fact, for each $x \in \overline{D}$, from (3.21), we have

$$T(GSx) + Sx = GSx, \tag{3.22}$$

so

$$\|T(GSx) - T\theta\|_p \leq \|GSx\|_p + \|Sx + T\theta\|_p. \tag{3.23}$$

Since $T$ is expansive, we have

$$\|T(GSx) - T\theta\|_p \geq h\|GSx\|_p. \tag{3.24}$$

It follows from (3.23) and (3.24) that

$$\|GSx\|_p \leq \frac{1}{h-1}\|Sx + T\theta\|_p. \tag{3.25}$$

Thus, by (3.25) and (iv), for each $x \in \partial D$, we have

$$\begin{aligned}\|GSx\|_p^2 - \left(\|GSx\|_p - \|x\|_p\right)^2 &= \|x\|_p\left(2\|GSx\|_p - \|x\|_p\right)\\ &\leq \|x\|_p\left(\frac{2}{h-1}\|Sx + T\theta\|_p - \|x\|_p\right) \leq 0,\end{aligned} \tag{3.26}$$

which implies (3.20). This completes the proof. □

**Corollary 3.9.** *Let $(X, \|\cdot\|_p)$ be a complete $p$-normed space and $D$ an open $s$-convex subset of $X$, where $0 < p \leq 1, 0 < s \leq p$. Suppose that*

(i) *$S : \overline{D} \to X$ is a continuous compact mapping;*

(ii) *$T : X \to X$ is an expansive map with constant $h > 1$;*

(iii) *$z \in S(\overline{D})$ implies $T(X) + z = X$ where $T(X) + z = \{y + z : y \in T(X)\}$;*

(iv) *$\|Sx + T\theta\|_p \leq ((h-1)/2)\|x\|_p$ for each $x \in \partial D$.*

*Then there exists a point $x^* \in \overline{D}$ such that $Sx^* + Tx^* = x^*$.*

**Theorem 3.10.** *Let $(X, \|\cdot\|_p)$ be a complete $p$-normed space and $D$ an open $s$-convex subset of $X$, where $0 < p \leq 1, 0 < s \leq p$. Suppose that*

(i) *$S : \overline{D} \to X$ is a continuous compact mapping;*

(ii) *$T : X \to X$ is a contraction with contractive constant $\alpha < 1$;*

(iii) *$\|Sx + T\theta\|_p \leq ((1-\alpha)/2)\|x\|_p$ for each $x \in \partial D$.*

*Then there exists a point $x^* \in \overline{D}$ such that $Sx^* + Tx^* = x^*$.*

*Proof.* For each $z \in S(\overline{D})$, the mapping $T + z : X \to X$ is a contraction. Thus, the equation

$$Tx + z = x \tag{3.27}$$

has a unique solution $x = \sigma(z) \in X$. For any $z_1, z_2 \in S(\overline{D})$, from

$$T(\sigma(z_1)) + z_1 = \sigma(z_1), \qquad T(\sigma(z_2)) + z_2 = \sigma(z_2), \tag{3.28}$$

it follows that

$$\begin{aligned} \|\sigma(z_1) - \sigma(z_2)\|_p &= \|T(\sigma(z_1)) + z_1 - T(\sigma(z_2)) - z_2\|_p \\ &\leq \|T(\sigma(z_1)) - T(\sigma(z_2))\|_p + \|z_1 - z_2\|_p. \end{aligned} \tag{3.29}$$

Since $T$ is a contraction with contractive constant $\alpha < 1$, we have

$$\|T(\sigma(z_1)) - T(\sigma(z_2))\|_p \leq \alpha\|\sigma(z_1) - \sigma(z_2)\|_p. \tag{3.30}$$

Thus, we have

$$\|\sigma(z_1) - \sigma(z_2)\|_p \leq \frac{1}{1-\alpha}\|z_1 - z_2\|_p, \quad z_1, z_2 \in S\left(\overline{D}\right). \tag{3.31}$$

It follows from (3.31) and (i) that $\sigma S : \overline{D} \to X$ is compact. From (3.27), we have

$$\|\sigma Sx\|_p \leq \frac{1}{1-\alpha}\|Sx + T\theta\|_p, \quad x \in \overline{D}. \tag{3.32}$$

For every $x \in \partial D$, from (3.32) and (iii), we deduce that

$$\begin{aligned}\|\sigma Sx\|_p^2 - \left(\|\sigma Sx\|_p - \|x\|_p\right)^2 &= \|x\|_p\left(2\|\sigma Sx\|_p - \|x\|_p\right) \\ &\leq \|x\|_p\left(\frac{2}{1-\alpha}\|Sx + T\theta\|_p - \|x\|_p\right) \leq 0,\end{aligned} \tag{3.33}$$

which implies (3.20). This completes the proof. □

## 4. Condensing Mappings

Now, we extend the above results to a class of condensing mappings. For convenience, we recall some definitions, see [4, 7].

*Definition 4.1.* Let $A$ be a bounded subset of a metric space $(X, d)$. The Kuratowski measure of noncompactness $\chi(A)$ of $A$ is defined as follows:

$$\chi(A) = \inf\left\{\begin{matrix}\delta > 0 : \text{there is a finite number of subsets } A_i \subset A \text{ such that} \\ A \subseteq \bigcup_i A_i \text{ and } \operatorname{diam}(A_i) \leq \delta\end{matrix}\right\}, \tag{4.1}$$

where $\operatorname{diam}(A_i)$ denotes the diameter of set $A_i$.

It is easy to prove the following fundamental properties of $\chi$, see [7]

(i) $\chi(A) = \chi(\overline{A})$.

(ii) $\chi(A) = 0$ if and only if $\overline{A}$ is compact.

(iii) $A \subseteq B \Rightarrow \chi(A) \leq \chi(B)$.

(iv) $\chi(A \cup B) = \max\{\chi(A), \chi(B)\}$.

(v) If $A$, $B$ are bounded, then $\chi(A + B) \leq \chi(A) + \chi(B)$.

(vi) If $A$ is bounded and $\lambda \in \mathbb{R}$, then $\chi(\lambda A) = |\lambda|\chi(A)$.

For (ii) one should remember that a set is compact if and only if it is closed and totally bounded.

**Proposition 4.2.** *Let* $(X.\|\cdot\|_p)$ $(0 < p \leq 1)$ *be a complete* $p$*-normed space and let* $A$, $B$ *be two bounded subsets of* X*. Then*

$$\chi(co_s(A)) = \chi(A), \tag{4.2}$$

*where* $0 < s \leq p$.

*Proof.* Let $\varepsilon > 0$ and $U(x; \varepsilon)$ the open ball with center $x$ and radius $\varepsilon$. Note

$$\operatorname{diam}(N_\varepsilon(S)) \leq \operatorname{diam}(S) + 2\varepsilon, \tag{4.3}$$

holds for any bounded set $S$ in a metric space; here $N_\varepsilon(A) = \bigcup_{x\in A} U(x;\varepsilon)$. Then, if $A$ is a bounded set, we have that

$$\chi(N_\varepsilon(A)) \leq \chi(A) + 2\varepsilon. \tag{4.4}$$

Assume first that $A$ and $B$ are bounded $s$-convex sets. Let $C = A \cup B$. We now prove that

$$\chi(co_s(C)) \leq \max\{\chi(A), \chi(B)\}. \tag{4.5}$$

To do it, suppose that $x \in co_s(C)$ and $x \notin A \cup B$. Then there exist $x_i \in C$ and $t_i \geq 0$, $i = 1, \ldots, n$, $\sum_{i=1}^{n} t_i^s = 1$, such that $x = \sum_{i=1}^{n} t_i x_i$. Let $x_i \in A$ for $1 \leq i \leq m$ and $x_i \in B$ for $m+1 \leq i \leq n$. Then, we have

$$x = \left(\sum_{i=1}^{m} t_i^s\right)^{1/s} \frac{\sum_{i=1}^{m} t_i x_i}{\left(\sum_{i=1}^{m} t_i^s\right)^{1/s}} + \left(\sum_{i=m+1}^{n} t_i^s\right)^{1/s} \frac{\sum_{i=m+1}^{n} t_i x_i}{\left(\sum_{i=m+1}^{n} t_i^s\right)^{1/s}}. \tag{4.6}$$

Consequently

$$x = \left(\sum_{i=1}^{m} t_i^s\right)^{1/s} u + \left(\sum_{i=m+1}^{n} t_i^s\right)^{1/s} v, \quad \text{where } u \in A,\ v \in B, \tag{4.7}$$

and then (4.7) leads to

$$x = t^{1/s} u + (1-t)^{1/s} v, \quad \text{where } u \in A,\ v \in B,\ t \in [0,1]. \tag{4.8}$$

Assume that for each $\varepsilon > 0$, there is a positive integer $N$ such that $1/N < \varepsilon$. For each $i = 0, 1, \ldots, N$, let

$$C_i = \left\{x \in co_s(C) : x = \left(\frac{i}{N}\right)^{1/s} u + \left(1 - \frac{i}{N}\right)^{1/s} v \text{ for some } u \in A,\ v \in B\right\}. \tag{4.9}$$

Note if $t \in (0,1)$, then $t \in [i/N, (i+1)/N]$ for some $0 \leq i \leq N-1$. This implies that if $x \in co_s(C)$ and $x \notin A \cup B$, then $x \in N_\varepsilon(C_i)$ for some $i = 0, \ldots, N$. Thus,

$$co_s(C) \subseteq \left(\bigcup_{i=1}^{n} N_\varepsilon(C_i)\right) \cup A \cup B. \tag{4.10}$$

By (iv) and (4.4), we have

$$\begin{aligned}\chi(co_s(C)) &\leq \max\left\{\chi(A), \chi(B), \max_{1\leq i\leq N}\chi(N_\varepsilon(C_i))\right\} \\ &\leq \max\left\{\chi(A), \chi(B), \max_{1\leq i\leq N}\left[\chi(C_i)+2\varepsilon\right]\right\}.\end{aligned} \tag{4.11}$$

For each $i$, by (v) and (vi), we deduce that

$$\begin{aligned}\chi(C_i) &= \chi\left(\left(\frac{i}{N}\right)^{1/s}A + \left(1-\frac{i}{N}\right)^{1/s}B\right) \\ &\leq \left(\frac{i}{N}\right)^{1/s}\chi(A) + \left(1-\frac{i}{N}\right)^{1/s}\chi(B) \\ &\leq \left[\left(\frac{i}{N}\right)^{1/s} + \left(1-\frac{i}{N}\right)^{1/s}\right]\max\{\chi(A), \chi(B)\}.\end{aligned} \tag{4.12}$$

Since

$$\left(\frac{i}{N}\right)^{1/s} + \left(1-\frac{i}{N}\right)^{1/s} \leq 1, \tag{4.13}$$

we have

$$\chi(C_i) \leq \max\{\chi(A), \chi(B)\}. \tag{4.14}$$

Hence,

$$\chi(co_s(C)) \leq \max\{\chi(A), \chi(B)\} + 2\varepsilon. \tag{4.15}$$

Since $\varepsilon > 0$ is arbitrary we obtain (4.5).

Consequently, if $C = \bigcup_{i=1}^{n} A_i$, where each $A_i$ is bounded $s$-convex, we have that

$$\chi(co_s(C)) \leq \max_{1\leq i\leq N}\chi(A_i). \tag{4.16}$$

Now, to prove (4.2), let $q > \chi(A)$. Then $A \subseteq \bigcup_{i=1}^{n} A_i$, where $\operatorname{diam}(A_i) \leq q$. We now claim that $\operatorname{diam}(co_s(A_i)) = \operatorname{diam}(A_i)$. In fact, since $A_i \subseteq co_s(A_i)$, we have

$$\operatorname{diam}(A_i) \leq \operatorname{diam}(co_s(A_i)), \tag{4.17}$$

for each $i$. Also, let $x, y \in co_s(A_i)$, such that $x = \sum_{i=1}^n t_i x_i$, $y = \sum_{i=1}^n t_i y_i$, $x_i, y_i \in A_i$ and $\sum_{i=1}^n t_i^s = 1$. Then

$$\begin{aligned}\sup_{x,y\in co_s(A_i)} \|x-y\|_p &\le \sum_{i=1}^n t_i^p \|x_i - y_i\|_p \\ &\le \sum_{i=1}^n t_i^s \|x_i - y_i\|_p \\ &\le \sum_{i=1}^n t_i^s \operatorname{diam}(A_i),\end{aligned} \tag{4.18}$$

that is,

$$\operatorname{diam}(co_s(A_i)) \le \operatorname{diam}(A_i). \tag{4.19}$$

Hence, $\operatorname{diam}(co_s(A_i)) = \operatorname{diam}(A_i)$.

Now we may assume that each $A_i$ is $s$-convex for each $i$. By (4.16), we have

$$\chi(co_s(A)) \le \max_{1\le i\le N} \chi(A_i) \le \max_{1\le i\le N} \operatorname{diam}(A_i) \le q. \tag{4.20}$$

Since this is true for all $q > \chi(A)$ then $\chi(co_s(A)) \le \chi(A)$ so (4.2) is proved. □

*Definition 4.3.* Let X, $Y$ be two metric spaces and $\Omega$ a subset of X. A bounded continuous map $T : \Omega \to Y$ is $k$-set contractive if for any bounded set $A \subset \Omega$, we have

$$\chi(T(A)) \le k\chi(A). \tag{4.21}$$

$T$ is strictly $k$-set contractive if $T$ is $k$-set contractive and

$$\chi(T(A)) < k\chi(A) \tag{4.22}$$

for all bounded sets $A \subset \Omega$ with $\chi(A) \neq 0$. We say $T$ is a condensing map if $T$ is a bounded continuous 1-set contractive map and

$$\chi(T(A)) < \chi(A) \tag{4.23}$$

for all bounded sets $A \subset \Omega$ with $\chi(A) \neq 0$.

Notice that $T$ is a compact map if and only if $T$ is a 0-set contractive map.

Now, we extend Sadovskii theorem in [7] to a map, that is, defined on a $p$-normed space.

**Theorem 4.4** (Sadovskii type). *Let $(X.\|\cdot\|_p)$ be a complete $p$-normed space and C a closed $s$-convex subset of X, where $0 < p \le 1$, $0 < s \le p$. If $S : C \to C$ is condensing and $S(C)$ is bounded, then $S$ has a fixed point.*

*Proof.* For each fixed $x \in C$, let $\mathfrak{B}$ be the family of all closed $s$-convex subsets $A$ of $C$, such that $x \in A$ and $S : A \to A$. Suppose that

$$B = \bigcap_{A \in \mathfrak{B}} A,$$
$$D = \overline{co_s}\{S(B) \cup \{x\}\}. \tag{4.24}$$

Since $x \in B$ and $S : B \to B$, it follows that $D \subseteq B$. This implies that $S(D) \subseteq S(B) \subseteq D$. On the other hand, since $x \in D$, we have $D \in \mathfrak{B}$ and $B \subseteq D$. Therefore, $B = D$.

As a result, we have $S(D) = S(B) \subseteq D$ and

$$\chi(D) = \chi(\overline{co_s}\{S(B) \cup \{x\}\}) = \chi(\{S(B) \cup \{x\}\}) = \chi(S(B)) = \chi(S(D)). \tag{4.25}$$

Since $S$ is condensing, it follows that $\chi(D) = 0$, that is, $D$ is compact. Therefore $S$ is a compact mapping of $s$-convex set $D$ into itself. By Theorem 2.5, $S$ has a fixed point. □

We next make use of the main ideas established in [5, 6, 8] to obtain a Krasnosel'skii fixed point theorem in a $p$-normed space.

**Theorem 4.5.** *Let $(X, \|\cdot\|_p)$ be a complete $p$-normed space and $C$ a closed $s$-convex subset of $X$, where $0 < p \leq 1, 0 < s \leq p$. Suppose that $S : C \to X$ and*

(i) *$T : C \to X$ is such that the inverse of $(I - T) : C \to (I - T)(C)$ exists;*

(ii) *$S(C) \subseteq (I - T)(C)$;*

(iii) *$(I - T)^{-1}S : C \to X$ is condensing and $(I - T)^{-1}S(C)$ is bounded.*

*Then there exists a point $x^* \in C$ such that $Sx^* + Tx^* = x^*$.*

*Proof.* From (ii), we have $(I - T)^{-1}S : C \to C$. Thus, $(I - T)^{-1}S$ is a condensing map of $C$ into itself. By Theorem 4.4, $(I - T)^{-1}S$ has a fixed point. This completes the proof. □

The following lemma is easy to prove.

**Lemma 4.6.** *Let $(X, \|\cdot\|_p)$ $(0 < p \leq 1)$ be a complete $p$-normed space and $C \subset X$. Assume that $T : C \to X$ is a $k$-Lipschitizian map, that is,*

$$\|Tx - Ty\|_p \leq k\|x - y\|_p, \quad x, y \in C. \tag{4.26}$$

*Then for each bounded subset $\Omega$ of $C$, we have $\chi(T(\Omega)) \leq k\chi(\Omega)$.*

**Theorem 4.7.** *Let $(X, \|\cdot\|_p)$ be a complete $p$-normed space and $C$ a closed $s$-convex subset of $X$, where $0 < p \leq 1,\ 0 < s \leq p$. Suppose that*

(i) *$S : C \to X$ is a 1-set contractive map (condensing) and $S(C)$ is bounded;*

(ii) *$T : C \to X$ is an expansive map with constant $h > 2$ $(h \geq 2)$;*

(iii) *$z \in S(C)$ implies $T(C) + z \supset C$ where $T(C) + z = \{y + z : y \in T(C)\}$.*

*Then there exists a point $x^* \in C$ such that $Sx^* + Tx^* = x^*$.*

*Proof.* Let $\tau$ be the function defined as in Theorem 3.1. We will show that $\tau S : C \to C$ is a condensing map. Let $\Omega$ be bounded in $C$. From (3.5) and Lemma 4.6, it follows that

$$\chi(\tau(S(\Omega))) \leq \frac{1}{h-1}\chi(S(\Omega)). \tag{4.27}$$

Suppose first that $S$ is 1-set contractive. Then

$$\chi((\tau S)(\Omega)) \leq \frac{1}{h-1}\chi(\Omega), \tag{4.28}$$

which implies that $\tau S : C \to C$ is a condensing map. The other case when $S$ is condensing and $h \geq 2$ also guarantees that $\tau S : C \to C$ is a condensing map. The result follows from Theorem 4.4. This completes the proof. □

**Theorem 4.8.** *Let $(X, \|\cdot\|_p)$ be a complete $p$-normed space and $C$ a closed $s$-convex subset of $X$, where $0 < p \leq 1$, $0 < s \leq p$. Suppose that*

(i) *$S : C \to X$ is a 1-set contractive map (condensing) and $S(C)$ is bounded;*

(ii) *$T : X \to X$ (or $T : C \to X$) is an expansive map with constant $h > 2$ ($h \geq 2$);*

(iii) *$S(C) \subset (I-T)(X)$ and [$x = Tx + Sy, y \in C$ implies $x \in C$] or $S(C) \subset (I-T)(C)$.*

*Then there exists a point $x^* \in C$ such that $Sx^* + Tx^* = x^*$.*

*Proof.* For each $x \in C$, by (iii), there exists a $y \in X$ such that

$$y - Ty = Sx. \tag{4.29}$$

If $S(C) \subset (I-T)(C)$ then $y \in C$ whereas if $S(C) \subset (I-T)(X)$ then it follows from Lemma 3.5 and (iii), that $y = (I-T)^{-1}Sx \in C$. Now, if $A \subset C$ is bounded, then by Lemma 4.6 and (3.16), we have

$$\chi\Big(\big((I-T)^{-1}S\big)(A)\Big) \leq \frac{1}{h-1}\chi(S(A)). \tag{4.30}$$

Suppose first that $S$ is 1-set contractive. Then

$$\chi\Big(\big((I-T)^{-1}S\big)(A)\Big) \leq \frac{1}{h-1}\chi(A), \tag{4.31}$$

which implies since $h > 2$ that $(I-T)^{-1}S : C \to C$ is a condensing map. The other case when $S$ is condensing and $h \geq 2$ also guarantees that $(I-T)^{-1}S : C \to C$ is a condensing map.

The result follows from Theorem 4.4. This completes the proof. □

**Lemma 4.9.** *Let* $(X, \|\cdot\|_p)$ $(0 < p \le 1)$ *be a* $p$*-normed space and* $C \subset X$*. Suppose that the mapping* $T : C \to X$ *is a contraction with contractive constant* $\alpha < 1$*. Then the inverse of* $F = I - T : C \to (I-T)(C)$ *exists and*

$$\left\|F^{-1}x - F^{-1}y\right\|_p \le \frac{1}{1-\alpha}\|x-y\|_p, \quad x, y \in F(C). \tag{4.32}$$

*Proof.* Since, for each $x, y \in C$, we have

$$\begin{aligned}\|Fx - Fy\|_p &= \|(x-y) - (Tx - Ty)\|_p \\ &\ge \|x-y\|_p - \|Tx - Ty\|_p \\ &\ge (1-\alpha)\|x-y\|_p.\end{aligned} \tag{4.33}$$

Then $F$ is one-to-one and the inverse of $F : C \to F(C)$ exists. Suppose that

$$G := F^{-1} - I : F(C) \longrightarrow X. \tag{4.34}$$

From the identity

$$I = F \circ F^{-1} = (I - T) \circ (I + G) = I + G - T \circ (I + G), \tag{4.35}$$

we have that

$$G = T \circ (I + G). \tag{4.36}$$

Thus,

$$\begin{aligned}\|Gx - Gy\|_p &\le \alpha\|(I+G)x - (I+G)y\|_p \\ &\le \alpha\Big(\|x-y\|_p + \|Gx - Gy\|_p\Big),\end{aligned} \tag{4.37}$$

and so

$$\|Gx - Gy\|_p \le \frac{\alpha}{1-\alpha}\|x-y\|_p, \quad x, y \in F(C). \tag{4.38}$$

Therefore,

$$\left\|F^{-1}x - F^{-1}y\right\|_p \le \|Gx - Gy\|_p + \|x-y\|_p \le \frac{1}{1-\alpha}\|x-y\|_p. \tag{4.39}$$

□

*Remark 4.10.* If $T : C \to X$ is a contraction with contractive constant $\alpha < 1$, by Lemma 4.9, it follows that $(I - T)^{-1}$ exists and is continuous. If in addition

$$Tx + Sy \in C, \quad \text{for any } x, y \in C \tag{4.40}$$

holds, then $(I - T)^{-1}S : C \to C$. To see this first note, since $S(C) + T(C) \subseteq C$, for each $z \in S(C)$, the mapping $T + z : C \to C$ is a contraction. Thus,

$$Tx + z = x \tag{4.41}$$

has a unique solution $x \in C$. Hence, $z \in (I - T)(C)$ and we have $S(C) \subseteq (I - T)(C)$. This shows that $(I - T)^{-1}S : C \to C$.

From Theorem 4.4, we generalize a Krasnosel'skii type theorem (Theorem 1.1) as follows.

**Theorem 4.11.** *Let $(X, \|\cdot\|_p)$ be a complete $p$-normed space and $C$ a closed $s$-convex subset of $X$, where $0 < p \leq 1$, $0 < s \leq p$. Suppose that*

(i) *$S : C \to X$ is a strictly $(1 - \alpha)$-set contractive map (or a $\beta$-set contractive map with $\beta < 1 - \alpha$) and $S(C)$ is bounded;*

(ii) *$T : C \to X$ is a contraction with contractive constant $\alpha < 1$;*

(iii) *any $x, y \in C$ imply $Tx + Sy \in C$.*

*Then there exists a point $x^* \in C$ such that $Sx^* + Tx^* = x^*$.*

*Proof.* Let $A \subset C$ be bounded. By Remark 4.10, Lemma 4.6 and (4.32), we have

$$\chi\Big(\big((I - T)^{-1}S\big)(A)\Big) \leq \frac{1}{1 - \alpha}\chi(S(A)), \tag{4.42}$$

and so $(I - T)^{-1}S : C \to C$ is a condensing map. Hence, from Theorem 4.4, there exists a point $x^* \in C$ such that $Sx^* + Tx^* = x^*$. This completes the proof. □

*Remark 4.12.* If $(I - T)^{-1}$ exists and is continuous on $X$ and if in addition

$$x = Tx + Sy, \quad y \in C \Longrightarrow x \in C \tag{4.43}$$

holds, then $(I - T)^{-1}S : C \to C$. To see this first note if $z \in S(C)$, then there exists $y \in C$ such that $z = S(y)$. Let $x = (I - T)^{-1}z$, so $x - Tx = z$. This implies $x = Tx + Sy$, and then $x \in C$. Hence, $z \in (I - T)(C)$ and we have $S(C) \subseteq (I - T)(C)$. This shows that $(I - T)^{-1}S : C \to C$.

**Theorem 4.13.** *Let $(X, \|\cdot\|_p)$ be a complete $p$-normed space and $C$ a closed $s$-convex subset of $X$, where $0 < p \leq 1, 0 < s \leq p$. Suppose that*

(i) *$S : C \to X$ is a strictly $(1 - \alpha)$-set contractive map (or a $\beta$-set contractive map with $\beta < 1 - \alpha$) and $S(C)$ is bounded;*

(ii) $T : X \to X$ *is a contraction with contractive constant* $\alpha < 1$;

(iii) $[x = Tx + Sy, y \in C] \Rightarrow x \in C$.

*Then there exists a point* $x^* \in C$ *such that* $Sx^* + Tx^* = x^*$.

*Proof.* Let $A \subset C$ be bounded. By Remark 4.12, Lemma 4.6 and (4.32), we have

$$\chi\Big(\Big((I-T)^{-1}S\Big)(A)\Big) \le \frac{1}{1-\alpha}\chi(S(A)), \tag{4.44}$$

and so $(I-T)^{-1}S : C \to C$ is a condensing map. Hence, from Theorem 4.4, there exists a point $x^* \in C$ such that $Sx^* + Tx^* = x^*$. This completes the proof. □

## 5. Application

In [9], Hajji and Hanebaly presented a modular version of Krasnosel'skii fixed point theorem and applied their result to the existence of solutions to perturbed integral equations in modular spaces. In this section, we use the same argument as in [9] to give an application of Krasnosel'skii fixed point theorem to a $p$-normed space. For more details about modular spaces, we refer the reader to [10, 11]. Now, we recall some definitions.

*Definition 5.1* (see [12]). Let $X$ be an arbitrary linear space over $K$, where $K = \mathbb{R}$ or $K = \mathbb{C}$. A functional $\rho : X \to [0, +\infty)$ is called a modular if

(i) $\rho(x) = 0$ if and only if $x = \theta$;

(ii) $\rho(\alpha x) = \rho(x)$ if $\alpha \in K$ with $|\alpha| = 1$, for all $x \in X$;

(iii) $\rho(\alpha x + \beta y) \le \rho(x) + \rho(y)$ if $\alpha, \beta \ge 0$ with $\alpha + \beta = 1$, for all $x, y \in X$.

If in place of (iii), we have

(iv) $\rho(\alpha x + \beta y) \le \alpha^s \rho(x) + \beta^s \rho(y)$ for $\alpha, \beta \ge 0$ and $\alpha^s + \beta^s = 1$ with an $s \in (0, 1]$,

then the modular $\rho$ is called an $s$-convex modular, and if $s = 1$, $\rho$ is called convex modular.

A modular $\rho$ defines a corresponding modular space, that is, the space $X_\rho$ given by

$$X_\rho = \{x \in X : \rho(\lambda x) \longrightarrow 0 \text{ as } \lambda \longrightarrow 0\}. \tag{5.1}$$

The subset $A$ of $X_\rho$ is $\rho$-bounded if

$$\sup_{x,y \in A} \rho(x - y) < +\infty. \tag{5.2}$$

The $\rho$-diameter of $A$ is defined by

$$\delta_\rho(A) = \sup_{x,y \in A} \rho(x - y). \tag{5.3}$$

A simple example of a modular space is a $p$-normed space $(X, \|\cdot\|_p)$.

Now, we discuss the existence of solutions for the following perturbed integral equation

$$u(t) = e^{-t} f_0 + \int_0^t e^{r-t}(T+h)u(r)dr, \tag{5.4}$$

in the modular space ($p$-normed space) $C_p = C([0,1], L^p)$, where

$$\begin{aligned} L^p &= L^p[0,1] \\ &= \left\{ f : f : [0,1] \longrightarrow K \text{ is measurable, and } \|f(t)\|_p = \int |f(t)|^p dt < +\infty \right\}, \end{aligned} \tag{5.5}$$

and $C_p = C([0,1], L^p)$ is the space of all continuous functions from $[0,1]$ to $L^p[0,1]$ under the modular (norm)

$$\|g\|_{pc} = \sup_{0 \le t \le 1} \|g(t)\|_p. \tag{5.6}$$

Consider the following assumptions:

(i) $B$ is a bounded closed $s$-convex subset of $L^p$ with $\theta \in B$, where $0 < p \le 1$, $0 < s \le p$;

(ii) $T : B \to B$ is a mapping satisfying

$$\|Tx - Ty\|_p \le k\|x - y\|_p, \tag{5.7}$$

for all $x, y \in B$ where $0 < k < 1$, and $h : B \to B$ is a continuous compact mapping such that $T(B) + h(B) \subseteq B$;

(iii) $f_0$ is a fixed element of $B$.

**Theorem 5.2.** *Suppose that (i)–(iii) are satisfied then the integral equation* (5.4) *has a solution* $u \in C_p$.

*Proof.* Consider $C_p$ with

$$\|u\|_{pc} = \sup_{0 \le t \le 1} \|u(t)\|_p, \tag{5.8}$$

for $u \in D$ with $D = C([0,1], B)$. First we show $S : D \to D$ where $S$ is given by

$$Su(t) = e^{-t} f_0 + \int_0^t e^{r-t}(T+h)u(r)dr. \tag{5.9}$$

Suppose that $t_n, t_0 \in [0,1]$ and $t_n \to t_0$ as $n \to \infty$. Since $T$ and $h$ are continuous, then $(T+h)u$ is continuous at $t_0$. In fact,

$$\|(T+h)u(t_n) - (T+h)u(t_0)\|_p \le \|Tu(t_n) - Tu(t_0)\|_p + \|hu(t_n) - hu(t_0)\|_p. \tag{5.10}$$

Let $n \to \infty$ so we have $\|(T+h)u(t_n) - (T+h)u(t_0)\|_p \to 0$, that is, $(T+h)u$ is continuous at $t_0$. Since $t_0 \in [0,1]$ is arbitrary, $Su$ is continuous from $[0,1]$ into $L^p$. Next, in the complete space $L^p$, we have

$$\int_0^t e^{r-t}(T+h)u(r)dr \in \left(\int_0^t e^{r-t}dr\right)\overline{co_s}(T+h)u(r), \tag{5.11}$$

where $0 \leq r \leq t$. Since $(T+h)(B) \subseteq B$, it follows that

$$\int_0^t e^{r-t}(T+h)u(r)dr \in \left(1-e^{-t}\right)\overline{co_s}(B), \tag{5.12}$$

and since $B$ is closed $s$-convex, we have $\overline{co_s}(B) = \overline{B} = B$. Therefore, $Su(t) \in e^{-t}B+(1-e^{-t})B \subseteq B$ for all $t \in [0,1]$. Thus $S : D \to D$.

Let

$$\begin{gathered} T_1u(t) = e^{-t}f_0 + \int_0^t e^{r-t}Tu(r)dr, \\ h_1u(t) = \int_0^t e^{r-t}hu(r)dr. \end{gathered} \tag{5.13}$$

Thus $S = T_1 + h_1$. We will show that the hypotheses of Theorem 4.11 are satisfied.

Now since $T(B) + h(B) \subset B$ we have

$$T_1u(t) + h_1v(t) \in e^{-t}B + \left(1-e^{-t}\right)B \subset B, \tag{5.14}$$

for any $t \in [0,1]$ and $u, v \in D$. Hence $T_1(D) + h_1(D) \subset D$. For any $u, v \in D$, we have

$$T_1u(t) - T_1v(t) = \int_0^t e^{r-t}(Tu - Tv)(r)dr. \tag{5.15}$$

Fix $t \in [0,1]$. Using the same argument in the proof of Lemma 2.1 [12] we now show that $T_1$ is a contraction. Let $T = \{t_0, t_1, \ldots, t_n\}$ be any subdivision of $[0,t]$. We know that $\sum_{i=0}^{n-1}(t_{i+1} - t_i)e^{t_i - t}x(t_i)$ is convergent to $\int_0^t e^{r-t}x(r)dr$ in $L^p$ when $|T| = \sup\{|t_{i+1} - t_i|,\ i = 0, \ldots, n-1\} \to 0$ as $n \to \infty$. Therefore

$$\left\|\int_0^t e^{r-t}x(r)dr\right\|_p \leq \lim \left\|\sum_{i=0}^{n-1}(t_{i+1} - t_i)e^{t_i-t}x(t_i)\right\|_p. \tag{5.16}$$

On the other hand, since

$$\sum_{i=0}^{n-1}(t_{i+1}-t_i)e^{t_i-t} \leq \int_0^t e^{r-t}dr = 1-e^{-t} \leq 1, \tag{5.17}$$

then we have

$$\begin{aligned}\left\|\sum_{i=0}^{n-1}(t_{i+1}-t_i)e^{t_i-t}x(t_i)\right\|_p &\leq \left|\sum_{i=0}^{n-1}(t_{i+1}-t_i)e^{t_i-t}\right|^p \|x(t_i)\|_p \\ &\leq \|x(t_i)\|_p \\ &\leq \sup \|x(t_i)\|_p \\ &\leq \|x\|_{pc},\end{aligned} \tag{5.18}$$

that is,

$$\left\|\int_0^t e^{r-t}x(r)dr\right\|_p \leq \|x\|_{pc}. \tag{5.19}$$

This implies that

$$\begin{aligned}\|T_1u(t)-T_1v(t)\|_p &= \left\|\int_0^t e^{r-t}(Tu-Tv)(r)dr\right\|_p \\ &\leq \|Tu-Tv\|_{pc}.\end{aligned} \tag{5.20}$$

Note also

$$\begin{aligned}\|Tu-Tv\|_{pc} &= \sup_{t\in[0,1]}\|Tu(t)-Tv(t)\|_p \\ &\leq k\|u-v\|_{pc}.\end{aligned} \tag{5.21}$$

Hence,

$$\|T_1u(t)-T_1v(t)\|_p \leq k\|u-v\|_{pc}, \tag{5.22}$$

and so

$$\|T_1u-T_1v\|_{pc} \leq k\|u-v\|_{pc}, \tag{5.23}$$

for any $u, v \in D$. Therefore, $T_1$ is a contraction.

Now we show that $h_1$ is compact. Let $M \subset D$. Then $h_1(M)$ is equicontinuous. To see this let $u \in M$. Then

$$\begin{aligned} h_1u(t) - h_1u(t^*) &= \int_0^t e^{r-t} hu(r)dr - \int_0^{t^*} e^{r-t^*} hu(r)dr \\ &= e^{-t}\int_0^t e^r hu(r)dr - e^{-t^*}\int_0^{t^*} e^r hu(r)dr \\ &= e^{-t}\int_0^t e^r hu(r)dr - e^{-t^*}\int_0^t e^r hu(r)dr + e^{-t^*}\int_0^t e^r hu(r)dr - e^{-t^*}\int_0^{t^*} e^r hu(r)dr \\ &= \left(e^{-t} - e^{-t^*}\right)\int_0^t e^r hu(r)dr + e^{-t^*}\int_{t^*}^t e^r hu(r)dr. \end{aligned} \tag{5.24}$$

Hence,

$$\begin{aligned} \|h_1u(t) - h_1u(t^*)\|_p &\le \left\|\left(e^{-t} - e^{-t^*}\right)\int_0^t e^r hu(r)dr\right\|_p + \left\|e^{-t^*}\int_{t^*}^t e^r hu(r)dr\right\|_p \\ &= \left|e^{-t} - e^{-t^*}\right|^p \left|\int_0^1 e^r dr\right|^p \delta_{\|\cdot\|_p}(B) + \left|e^{-t^*}\right|^p \left|\int_{t^*}^t e^r dr\right|^p \delta_{\|\cdot\|_p}(B) \\ &\le 2\left|e^{-t} - e^{-t^*}\right|^p \delta_{\|\cdot\|_p}(B) + \left|\int_{t^*}^t e^r dr\right|^p \delta_{\|\cdot\|_p}(B) \\ &= 2\left|e^{-t} - e^{-t^*}\right|^p \delta_{\|\cdot\|_p}(B) + \left|e^{t} - e^{t^*}\right|^p \delta_{\|\cdot\|_p}(B). \end{aligned} \tag{5.25}$$

Recall the functions $t \mapsto e^{-t}$ and $t \mapsto e^{t}$ are uniformly continuous on $[0,1]$. Therefore, for $\varepsilon > 0$, there exists $\eta_1 > 0$ such that if $|t - t^*| < \eta_1$ then

$$\left|e^{-t} - e^{-t^*}\right| \le \left[\frac{\varepsilon}{4\delta_{\|\cdot\|_p}(B)}\right]^{1/p}, \tag{5.26}$$

and there exists $\eta_2 > 0$ such that if $|t - t^*| < \eta_2$ then

$$\left|e^{t} - e^{t^*}\right| \le \left[\frac{\varepsilon}{2\delta_{\|\cdot\|_p}(B)}\right]^{1/p}. \tag{5.27}$$

As a result with $\eta = \min(\eta_1, \eta_2)$ note if $|t - t^*| < \eta$ then

$$\|h_1u(t) - h_1u(t^*)\|_p \le \varepsilon, \tag{5.28}$$

for any $u \in M$. Thus, $h_1(M)$ is equicontinuous. Moreover,

$$\begin{aligned} h_1 u(t) &= \int_0^t e^{r-t} hu(r)dr \in \left(1 - e^{-t}\right)\overline{co_s}\{hu(r),\ 0 \le r \le t\} \\ &\subset \left(1 - e^{-t}\right)\overline{co_s}(h(B)). \end{aligned} \tag{5.29}$$

Hence, for all $t \in [0,1]$, we have

$$h_1(M(t)) \subset \left(1 - e^{-t}\right)\overline{co_s}(h(B)). \tag{5.30}$$

Since, $h(B)$ is compact, this implies that $\overline{co_s}(h(B))$ is compact. Therefore, $\overline{h_1(M(t))}$ is compact for all $t \in [0,1]$. By using the Arzela-Ascoli Theorem, we obtain that $\overline{h_1(M)}$ is compact and also $h_1$ is continuous. Thus $h_1$ is compact. Hence by Theorem 4.11, $S$ has a fixed point. □

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