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Research Article

Generalized Hyperbolic Function Solution to a Class of Nonlinear Schrödinger-Type Equations

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With the help of the generalized hyperbolic function, the subsidiary ordinary differential equation method is improved and proposed to construct exact traveling wave solutions of the nonlinear partial differential equations in a unified way. A class of nonlinear Schrödinger-type equations including the generalized Zakharov system, the Rangwala-Rao equation, and the Chen-Lee-Liu equation are investigated and the exact solutions are derived with the aid of the homogenous balance principle and generalized hyperbolic functions. We study the effect of the generalized hyperbolic function parameters p and q in the obtained solutions by using the computer simulation.

1. Introduction

The nonlinear Schrödinger (NLS) equation is a ubiquitous and significant model that naturally arises in many fields of physics, such as nonlinear optical systems, plasmas, fluid dynamics, and Bose-Einstein condensation. In the last three decades, great progress has been made on the construction of exact solutions of NLS equation. Many significant methods have been established by mathematicians and physicists to obtain special solutions of nonlinear partial differential equations (NLPDEs), including the inverse scattering method, Darboux transformation, Hirota's bilinear method, homogeneous balance method, Jacobi elliptic function method, variational iteration method, the sine-cosine method, the (G'/G) expansion method, tanh-function method, F-expansion method, Lucas Riccati method, auxiliary equation method, algebraic method, and others [1–36]. The last five methods mentioned above belong to a class of method called subsidiary ordinary differential equation (sub-ODE) method. The key ideas of the sub-ODE method are that the travelling wave solutions of the complicated NLPDE can be expressed as a polynomial, the variable of which is one of the solutions of simple and solvable ODE that called the sub-ODE. The sub-ODEs which were often used are the Riccati equation, Jacobi elliptic equation, projective Riccati equations, and so forth. With the development of computer science, recently, the sub-ODEs with nonlinear terms of high order have attracted much attention [34, 35]. This is due to the availability of symbolic computation systems like Mathematica or Maple which enable us to perform the complex and tedious computation on computers. Recently, Sirendaoreji [21–23] introduced the auxiliary equation method by using the six degree first-order nonlinear differential equation, Wang et al. [24] proposed new sub-ODE equation involving an arbitrary positive power to construct exact travelling wave solutions of NLPDEs in a unified way. In this study, we improve the method presented by Wang et al. [24] and introduce some solutions for the sub-ODE equation in terms of the generalized hyperbolic functions (GHFs) [36].

The rest of this paper is organized as follows: in the following section, we introduce the improved sub-ODE method to construct exact solutions of some NLPDEs in terms of the GHFs. In Section 3, we apply this method to the generalized Zakharov system, the Rangwala-Rao equation and the Chen-Lee-Liu equation. Finally, we conclude the paper and give some futures and comments.

2. The Improved Subsidiary Ordinary Differential Equation Method

The main idea of this method is to express the solutions of NLPDEs as polynomials in the solution of sub-ODE involving an arbitrary positive power of dependent variable that the GHFs satisfy (see the appendix). Consider a given NLPDE

$$H(u, u_t, u_x, u_{tt}, u_{tx}, u_{xx}, \ldots) = 0. \tag{2.1}$$

The improved sub-ODE method for solving NLPDE (2.1) proceeds in the following four steps [36].

Step 1. We seek its traveling wave solution of (2.1) in the form

$$u(x,t) = u(\xi), \qquad \xi = kx - \omega t, \tag{2.2}$$

where k and ω are constants to be determined later. Substituting (2.2) into (2.1) yields an ordinary differential equation

$$\widetilde{H}\ (u, u', u'', u''', \ldots) = 0, \qquad u' = \frac{du}{d\xi}, \ldots, \tag{2.3}$$

where $\widetilde{H}$ is a polynomial of u and its various derivatives. If $\widetilde{H}$ is not a polynomial of u and its various derivatives, then we may use new variables $v = v(\xi)$ which makes $\widetilde{H}$ become polynomial of v and its various derivatives.

Step 2. Suppose that $u(\xi)$ can be expressed by a finite power series of $F(\xi)$

$$u(\xi) = a_0 + \sum_{j=1}^{n} a_j F^j(\xi), \qquad a_n \neq 0, \tag{2.4}$$

where n is a positive integer which can be determined by balancing the highest derivative term with the highest nonlinear term in (2.3) and $a_j(j = 0, 1, \ldots, n)$ are some parameters to be determined. The function $F(\xi)$ satisfies the sub-ODE with an arbitrary positive power

$$F'^2(\xi) = AF^2(\xi) + BF^{2+r}(\xi) + CF^{2+2r}(\xi), \quad r > 0, \tag{2.5}$$

where A, B, and C are parameters to be determined.

Step 3. Substituting (2.4) and (2.5) into the ODE (2.3), then the left-hand side of (2.3) can be converted into a polynomial in $F(\xi)$. Setting all coefficients of the polynomial to zero yields system of algebraic equations for $a_j(j = 0, 1, \ldots, n)$, k, and ω.

Step 4. Solving this system obtained in Step 3, then $a_j(j = 0, 1, \ldots, n)$, k, and ω can be expressed by A, B, and C. Substituting these results into (2.4), then the general formulae of the travelling wave solution of (2.1) can be obtained. Selecting the values of A, B, C, and the corresponding GHF solution $F(\xi)$ of (2.5) given bellow to obtain the exact solutions of (2.1). The definition and proprieties of the GHFs are given in the appendix.

We list various types of exact solutions of (2.5) as follows.

Case 1. If $A > 0$, $B = 2\sigma A$, and $C = (\sigma^2 - pq)A$, then (2.5) admits the following positive solution:

$$F(\xi) = \left[\frac{1}{\cosh_{pq}\left(r\sqrt{A}\,\xi\right) - \sigma}\right]^{1/r}, \quad \sigma < 1. \tag{2.6}$$

Case 2. If $A > 0$, $B = 2\sigma A$, and $C = (\sigma^2 + pq)A$, then (2.5) admits the following positive solution:

$$F(\xi) = \left[\frac{1}{\sinh_{pq}\left(r\sqrt{A}\,\xi\right) - \sigma}\right]^{1/r}, \quad \sigma < 1. \tag{2.7}$$

Case 3. If $A > 0$, $B = -2\sqrt{AC}$, and $C > 0$, then (2.5) admits the following positive solution:

$$F(\xi) = \left[\frac{1}{2}\sqrt{\frac{A}{C}}\left(1 \pm \tanh_{pq}\left(\frac{r}{2}\sqrt{A}\,\xi\right)\right)\right]^{1/r},$$
$$F(\xi) = \left[\frac{1}{2}\sqrt{\frac{A}{C}}\left(1 \pm \coth_{pq}\left(\frac{r}{2}\sqrt{A}\,\xi\right)\right)\right]^{1/r}. \tag{2.8}$$

Case 4. If $A = 0$, $B = 4/r^2$, and $C = -4\sigma/r^2$, then (2.5) admits the following positive solution:

$$F(\xi) = \left[\frac{1}{\xi^2 + \sigma}\right]^{1/r}, \qquad \sigma > 0. \tag{2.9}$$

3. Applications

In the following we use the improved sub-ODE method to seek exact traveling wave solutions of the class of nonlinear Schrödinger-type equations which are of interest in plasma physics, wave propagation in nonlinear optical fibers, Ginzburg-Landau theory of superconductivity, and so forth.

3.1. Generalized Zakharov System

In the interaction of laser-plasma the system of Zakharov equation plays an important role. This system has wide interest and attention for many scientists. Let us consider the generalized Zakharov system [18]:

$$u_{tt} - c_s^2 u_{xx} = \beta\left(|E|^2\right)_{xx},$$
$$iE_t + \alpha E_{xx} - \delta_1 uE + \delta_2 |E|^2 E + \delta_3 |E|^4 E = 0. \tag{3.1}$$

When $\delta_2 = \delta_3 = 0$, the generalized Zakharov system reduce to the famous Zakharov system which describe the propagation Langmuir waves in plasmas. The real unknown function $u(x, t)$ is the fluctuation in the ion density about its equilibrium value, and the complex unknown function $E(x, t)$ is the slowly varying envelope of highly oscillatory electron field. The parameters $\delta_1, \delta_2, \delta_3$ and c_s are real numbers, where c_s is proportional to the ion acoustic speed (or electron sound speed). Here we seek its traveling wave solution in the form

$$E(x,t) = H(\xi)\, e^{i(kx - \omega t)}, \qquad u(x, t) = u(\xi),\ \xi = x - ct, \tag{3.2}$$

where $k,\ \omega$, and c are constants and $H(\xi)$ is real function. Therefore, system (3.1) is reduced to

$$\begin{gathered}\left(c^2-c_s^2\right)u''=\beta\left(H^2\right)'',\\ \alpha H''+i(2\alpha k-c)H'+\left(\omega-\alpha k^2\right)H-\delta_1 uH+\delta_2 H^3+\delta_3 H^5=0.\end{gathered}\tag{3.3}$$

Integrating the first equation of (3.3) with respect to ξ and taking the integration constants to zero yields

$$u=\frac{\beta}{c^2-c_s^2}H^2,\qquad c^2-c_s^2\neq 0.\tag{3.4}$$

Substituting (3.4) into (3.3) results in

$$H''+\frac{1}{\alpha}\left[\left(\omega-\alpha k^2\right)H+\left(\delta_2-\frac{\delta_1\beta}{c^2-c_s^2}\right)H^3+\delta_3 H^5\right]=0,\qquad c=2\alpha\ k,\ \alpha\neq 0.\tag{3.5}$$

Now, by balancing the higher order derivative term H'' to the higher power nonlinear term H^5 in (3.5), we have $n+2r=5n$ which gives $2r=4n$. Since n must be a positive integer, thus $r=2,4,6,\ldots$ this gives $n=1,2,3,\ldots$. For simplicity, we take $r=2$ and $n=1$. We suppose that

$$H(\xi)=a_0+a_1F(\xi).\tag{3.6}$$

Substituting (3.6) with (2.5) into (3.5) and equating each of the coefficients of $F(\xi)$ to zero, we obtain system of algebraic equations. Solving this system by using Maple, yields

$$a_0=0,\qquad c=2\alpha k,\qquad \omega=\alpha\left(k^2-A\right),\qquad a_1=\pm\sqrt{\frac{3C}{2\delta_3 B}\left(\delta_2-\frac{\beta\delta_1}{4\alpha^2k^2-c_s^2}\right)}.\tag{3.7}$$

Now based on the solutions of (2.5), one can obtain new types of solitary wave solution of the generalized Zakharov system. The general formulae take the form

$$\begin{gathered}u(x,\ t)=\frac{3C\beta}{2\delta_3 B\ \left(4\alpha^2k^2-c_s^2\right)}\left(\delta_2-\frac{\beta\delta_1}{4\alpha^2k^2-c_s^2}\right)F^2(x-2\alpha kt),\\ E(x,\ t)=\pm\sqrt{\frac{3C}{2\delta_3 B}\left(\delta_2-\frac{\beta\delta_1}{4\alpha^2k^2-c_s^2}\right)}F(x-2\alpha kt)e^{i(kx-\omega t)}.\end{gathered}\tag{3.8}$$

By selecting the special values of the A, B, C and the corresponding function $F(\xi)$, we have the following solutions of the generalized Zakharov system (3.1):

$$u_1 = \frac{3\beta(\sigma^2 - pq)}{4\delta_3\sigma(4\alpha^2k^2 - c_s^2)}\left(\delta_2 - \frac{\beta\delta_1}{4\alpha^2k^2 - c_s^2}\right)\left[\frac{1}{\cosh_{pq}\left(2\sqrt{A}\,(x - 2\alpha kt)\right) - \sigma}\right],$$
$$E_1 = \pm\sqrt{\frac{3(\sigma^2 - pq)}{4\delta_3\sigma}\left(\delta_2 - \frac{\beta\delta_1}{4\alpha^2k^2 - c_s^2}\right)\left[\frac{1}{\cosh_{pq}\left(2\sqrt{A}\,(x - 2\alpha kt)\right) - \sigma}\right]}\,e^{i(kx - \omega t)}, \tag{3.9}$$

$$u_2 = \frac{3\beta(\sigma^2 + pq)}{4\delta_3\sigma\,(4\alpha^2k^2 - c_s^2)}\left(\delta_2 - \frac{\beta\delta_1}{4\alpha^2k^2 - c_s^2}\right)\left[\frac{1}{\sinh_{pq}\left(2\sqrt{A}(x - 2\alpha kt)\right) - \sigma}\right],$$
$$E_2 = \pm\sqrt{\frac{3(\sigma^2 + pq)}{4\delta_3\sigma}\left(\delta_2 - \frac{\beta\delta_1}{4\alpha^2k^2 - c_s^2}\right)}\left[\frac{1}{\sinh_{pq}\left(2\sqrt{A}(x - 2\alpha kt)\right) - \sigma}\right]^{1/2} e^{i(kx - \omega t)}, \tag{3.10}$$

$$u_3 = \frac{3\beta C}{8\delta_3A(4\alpha^2k^2 - c_s^2)}\left(\frac{\beta\delta_1}{4\alpha^2k^2 - c_s^2} - \delta_2\right)\left[1 \pm \tanh_{pq}\left(\sqrt{A}\,(x - 2\alpha kt)\right)\right],$$
$$E_3 = \pm\frac{1}{2}\sqrt{\frac{3C}{2\delta_3A}\left(\frac{\beta\delta_1}{4\alpha^2k^2 - c_s^2} - \delta_2\right)\left[1 \pm \tanh_{pq}\left(\sqrt{A}(x - 2\alpha kt)\right)\right]}\,e^{i(kx - \omega t)}, \tag{3.11}$$

$$u_4 = \frac{3\beta C}{8\delta_3A\,(4\,\alpha^2k^2 - c_s^2)}\left(\frac{\beta\delta_1}{4\alpha^2k^2 - c_s^2} - \delta_2\right)\left[1 \pm \coth_{pq}\left(\sqrt{A}(x - 2\alpha kt)\right)\right],$$
$$E_4 = \pm\frac{1}{2}\sqrt{\frac{3C}{2\delta_3A}\left(\frac{\beta\delta_1}{4\alpha^2k^2 - c_s^2} - \delta_2\right)\left[1 \pm \coth_{pq}\left(\sqrt{A}(x - 2\alpha kt)\right)\right]}\,e^{i(kx - \omega t)}, \tag{3.12}$$

$$u_5 = \frac{3\sigma\beta}{2\delta_3(4\alpha^2k^2 - c_s^2)}\left(\frac{\beta\delta_1}{4\alpha^2k^2 - c_s^2} - \delta_2\right)\left[\frac{1}{(x - 2\alpha kt)^2 + \sigma}\right],$$
$$E_5 = \pm\sqrt{\frac{3}{2\delta_3}\left(\frac{\beta\delta_1}{4\alpha^2k^2 - c_s^2} - \delta_2\right)\left[\frac{1}{(x - 2\alpha kt)^2 + \sigma}\right]}\,e^{i(kx - \omega t)}\,. \tag{3.13}$$

In order to understand the significance of these solutions expressed by (3.9)–(3.13), the main features of them are investigated by using direct computer simulations with the accuracy as high as 10^{-9}. We study the effect of the parameters p and q of the GHF in the solution u_1 given by the first equation of (3.9) by choosing different values of the parameters p and q, for $A = \delta_1 = \delta_2 = \delta_3 = \alpha = \beta = k = 1$, $\sigma = 0.9$, and $c_s = 0.4$, by means of Figures 1, 2, and 3 over same region $\{(x,\ t) : \|x\| \leq 5,\ \|t\| \leq 5\}$. From Figures 1, 2 and 3, we see that the solution u_1 describes dark soliton solution (increase, decrease, and have singularity with the change of the two parameters p and q). Also, we discuss u_2 given by the first equation of (3.10), by taking the same value of the parameters by means of Figures 4, 5, and 6 over

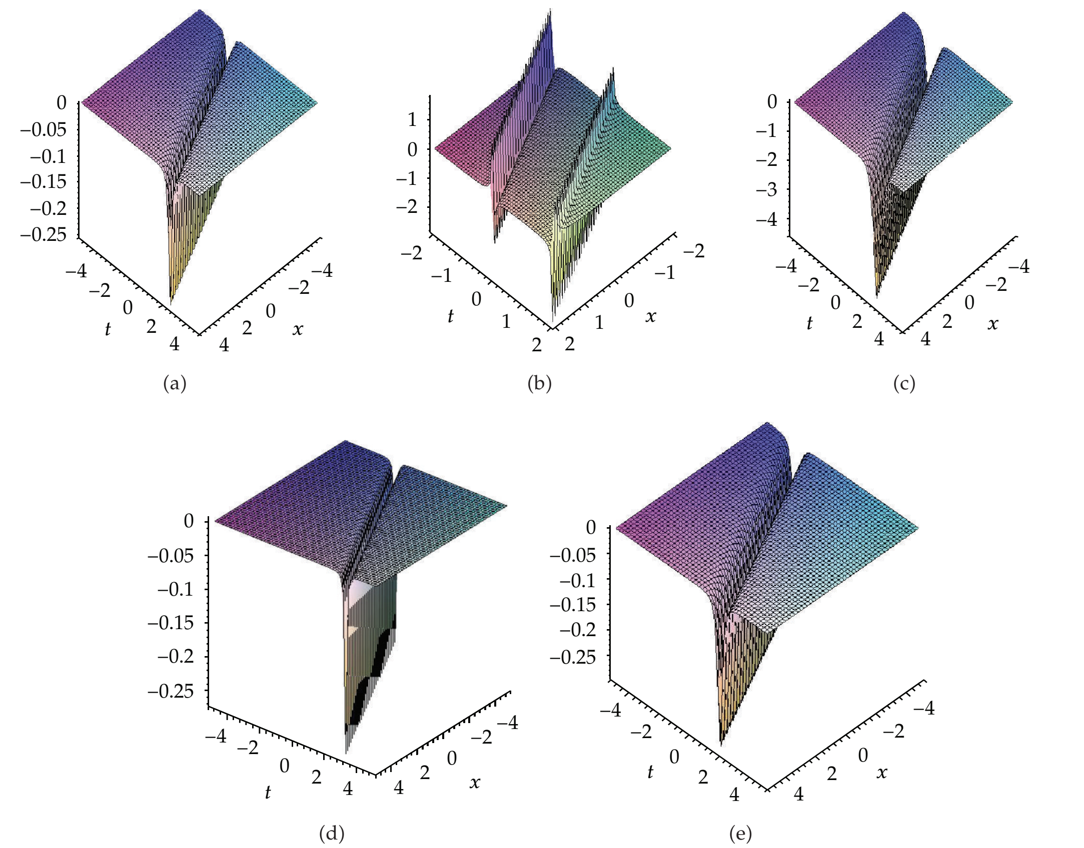


Figure 1: Evolutional behaviour of u_1 describes dark soliton: (a) $p = q = 1$; (b) $p = 1/32$, $q = 1/33$, (singular wave solution); (c) $p = 23$, $q = 32$; (d) $p = 1/13$, $q = 11$; (e) $p = 13$, $q = 1/11$.

the region $\{(x, t) : \|x\| \le 5, \|t\| \le 5\}$. From Figures 4, 5, and 6, we see that the solution u_2 describes singular wave solution changed with different value of the two parameters p and q. Of course, we can plot the other figures of the exact solutions of (3.1), we omit them here for convenience. As a result, we find that the parameters p and q affect the solution structure.

3.2. Rangwala-Rao Equation

The Rangwala-Rao equation [19]

$$u_{xt} - \beta_1 u_{xx} + u + iT\beta_2 |u|^2 u_x = 0, \quad T = \pm 1, \tag{3.14}$$

where β_1 and β_2 are real constants. Rangwala and Rao introduced (3.14) as the integrability condition when they studied the mixed derivative nonlinear Schrödinger equations and looked for the Bäckland transformation and solitary wave solutions.

Suppose the exact solutions of (3.14) is of the form

$$u(x, t) = e^{-\omega t} e^{i\varphi(x-ct)} H(x - ct), \tag{3.15}$$

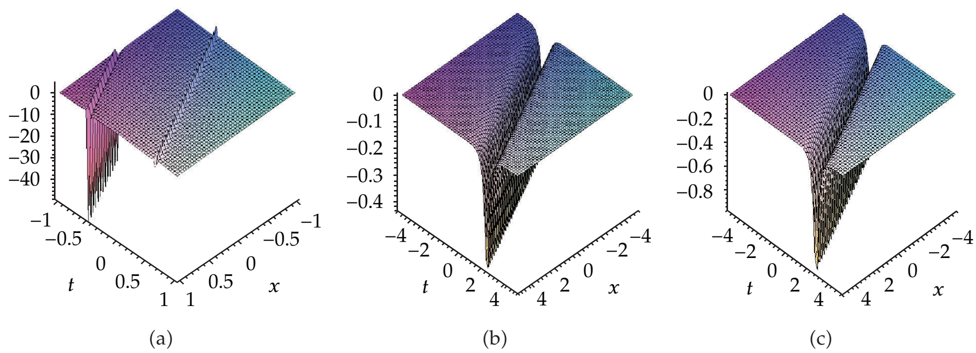


Figure 2: Evolutional behaviour of u_1 describes dark soliton: (a) $p = 1$, $q = 1/32$ (singular wave solution); (b) $p = 1$, $q = 3$; (c) $p = 1$, $q = 27$.

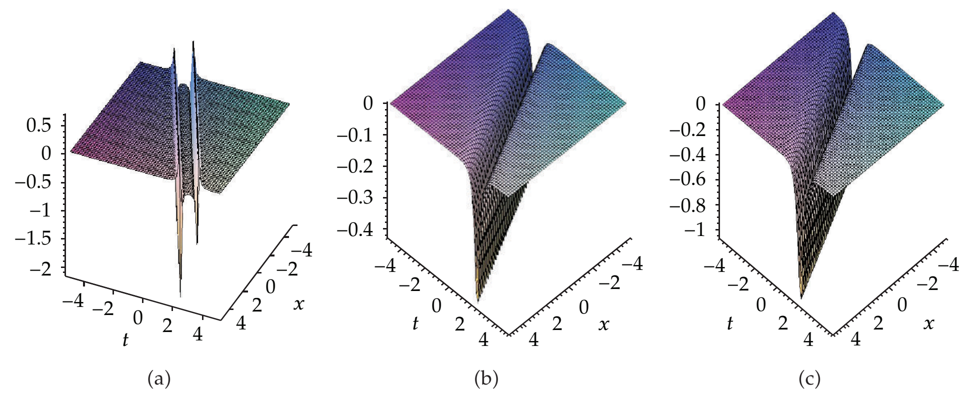


Figure 3: Evolutional behaviour of u_1 describes dark soliton: (a) $p = 1/32$, $q = 1$; (singular wave solution); (b) $p = 3$, $q = 1$; (c) $p = 27$, $q = 1$.

where ω, c are constants determined later and ψ, H are undetermined real functions with one variable only. Set the relation of ψ, H as

$$\psi'(\xi) = \frac{\omega}{2(c+\beta_1)} + \frac{T\beta_2}{4(c+\beta_1)} H^2(\xi), \qquad ' = \frac{d}{d\xi}, \quad \xi = x - ct, \tag{3.16}$$

substituting (3.15) with (3.16) into (3.14) simultaneously yields

$$H'' - \frac{4(c+\beta_1) - \omega^2}{4(c+\beta_1)^2} H - \frac{T\beta_2\omega}{2(c+\beta_1)^2} H^3 + \frac{3T^2\beta_2^2}{16(c+\beta_1)^2} H^5 = 0. \tag{3.17}$$

Applying the homogeneous balance principle, to (3.17) we have $r = 2, 4, 6, \ldots$ this gives $n = 1, 2, 3, \ldots$. For simplicity, we suppose

$$H(\xi) = a_0 + a_1 F(\xi). \tag{3.18}$$

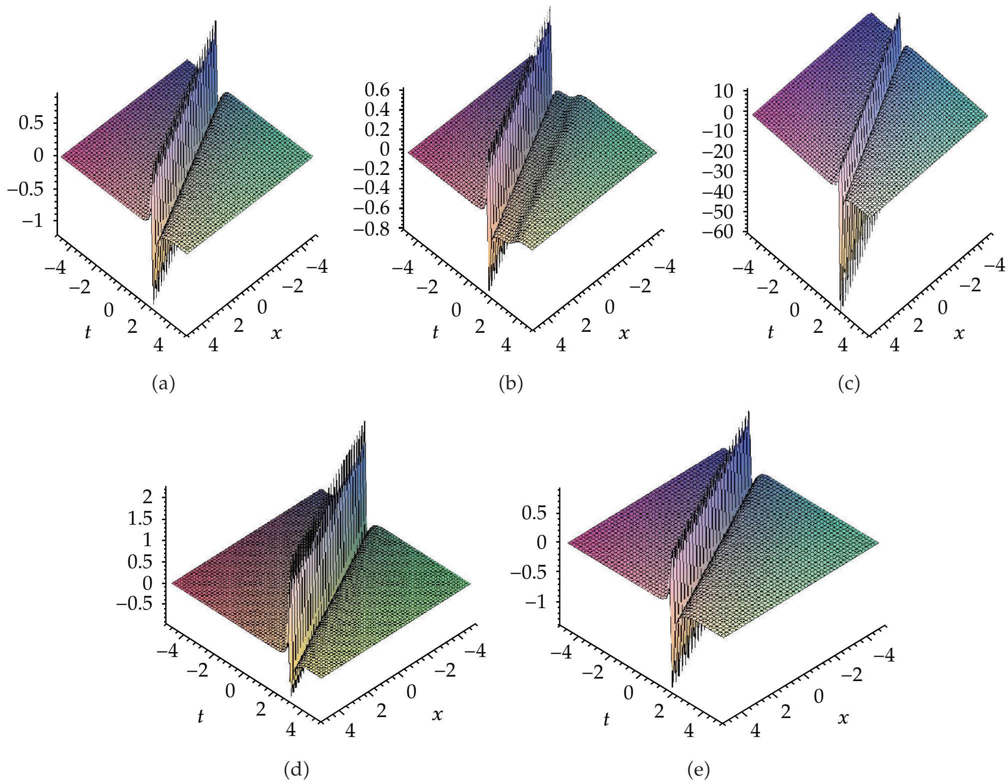


Figure 4: Evolutional behaviour of u_2 describes singular wave solution: (a) $p = q = 1$; (b) $p = 1/32$, $q = 1/33$; (c) $p = 23$, $q = 32$; (d) $p = 1/13$, $q = 11$; (e) $p = 13$, $q = 1/11$.

Substituting (3.18) with (2.5) into (3.17) and equating each of the coefficients of $F(\xi)$ to zero, we obtain system of algebraic equations. Solving this system with the aid of Maple, we obtain the following solution:

$$a_0 = 0, \omega = \pm 2\sqrt{(c+\beta_1)\left[1-(c+\beta_1)A\right]}, \qquad a_1 = \pm\sqrt{-\frac{8C\omega}{3TB\beta_2}}. \tag{3.19}$$

The general formulae of the solution of Rangwala-Rao equation

$$u(x, t) = \pm\sqrt{-\frac{8C\omega}{3TB\beta_2}}\, F(x-ct)\, e^{-i\omega t} e^{i\varphi(x-ct)}, \tag{3.20}$$

with $\varphi(\xi) = \omega/(6\, B\, (c+\beta_1)) \int[3B + 4CF^2(\xi)]\, d\xi$, $\omega = \pm 2\sqrt{(c+\beta_1)[1-(c+\beta_1)A]}$. By selecting the special values of the A, B, C and the corresponding function $F(\xi)$yields

$$|u_1(x, t)|^2 = -\frac{4\,(\sigma - pq)\omega}{3T\sigma\beta_2\left[\cosh_{pq}\left(2\sqrt{A}\,(x-ct)\right)-\sigma\right]}, \tag{3.21}$$

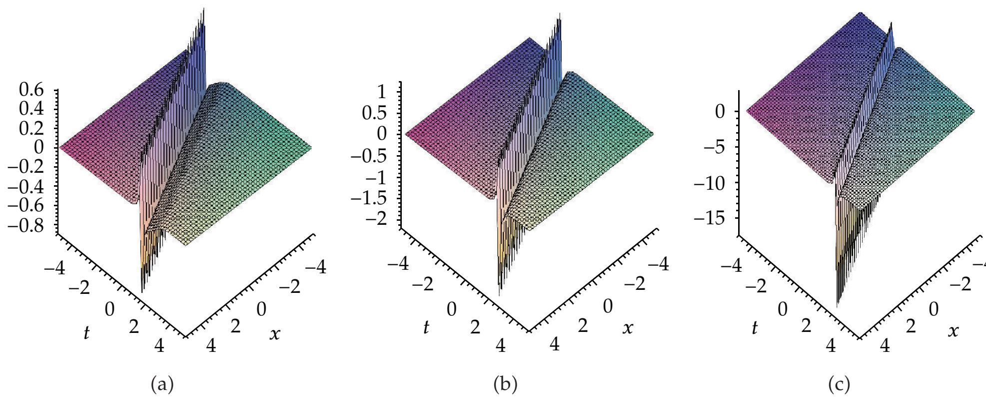


Figure 5: Evolutional behaviour of $u_2$describes singular wave solution: (a) $p = 1$, $q = 1/32$; (b) $p = 1$, $q = 3$; (c) $p = 1$, $q = 27$.

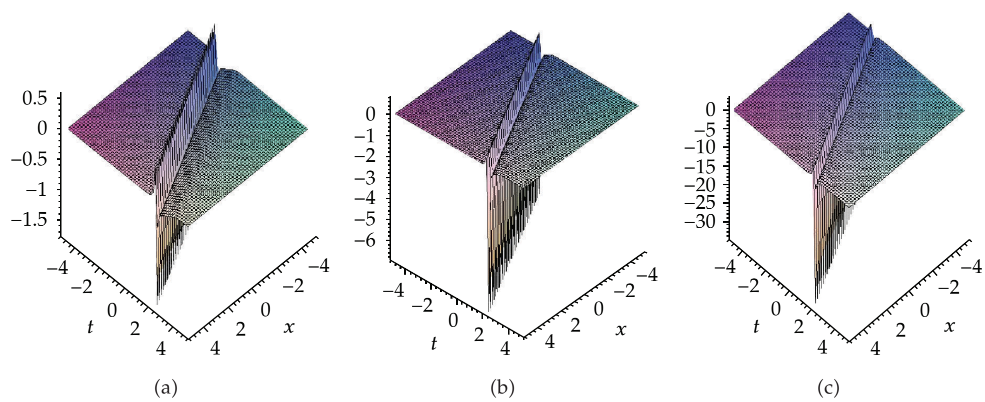


Figure 6: Evolutional behaviour of u_2 describes singular wave solution: (a) $p = 1/32$, $q = 1$; (b) $p = 3$, $q = 1$; (c) $p = 27$, $q = 1$.

$$|u_2(x,\ t)|^2 = -\frac{4\ (\sigma + pq)\omega}{3T\sigma\beta_2\ \left[\sinh_{pq}\left(2\sqrt{A}\ (x - ct)\right) -\ \sigma\right]}. \tag{3.22}$$

We omitted the reminder solutions for simplicity. Now, we study the affect of the parameters p and q of the GHF in the intensity u_1 given by the (3.21) by choosing different values of the parameters p and q, for $A = \delta = c = \beta_2 = T = 1$, $\omega = 0.4$, and $\sigma = 0.9$, by means of Figure 7 over the region $\{(x,\ t) : \|x\| \leq\ 5,\ \|t\| \leq 5\}$. We have bright soliton which are increase, decrease and have singularity with the different choose of the two parameters p and q. This means that the parameters p and q affect in the solution structure.

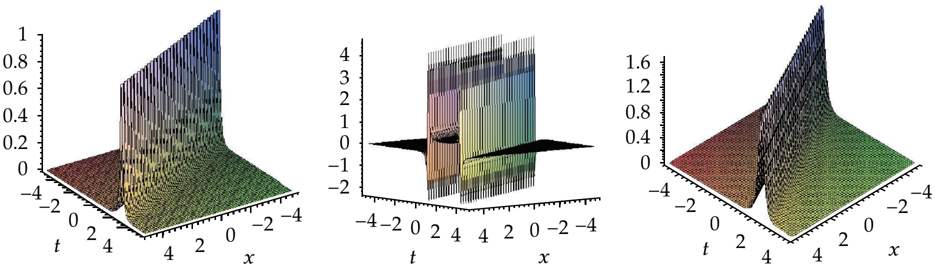


Figure 7: Evolutional behaviour of the intensity $|u_1|^2$ describes the bright soliton: (a) $p = 1, q = 1$; (b) $p = 1/332, q = 1/333$ (singular wave solution); (c) $p = 2.132, q = 1.723$.

3.3. *Chen-Lee-Liu Equation*

The Chen-Lee-Liu equation [19]

$$iu_t + u_{xx} + i\delta|u|^2 u_x = 0, \tag{3.23}$$

were δ is a real constant. Similar as before, we suppose that the exact solution of (3.23)is of the form

$$u(x,\ t) = e^{-\omega t} e^{i\varphi(x-ct)}\ H(x-ct). \tag{3.24}$$

Set the relation of $\varphi,\ H$ as

$$\varphi'(\xi) = \frac{c}{2} + \frac{\delta}{4}\ H^2(\xi), \qquad ' = \frac{d}{d\xi}, \quad \xi = x - ct, \tag{3.25}$$

substituting (3.25) with (3.24) into (3.23) simultaneously yields

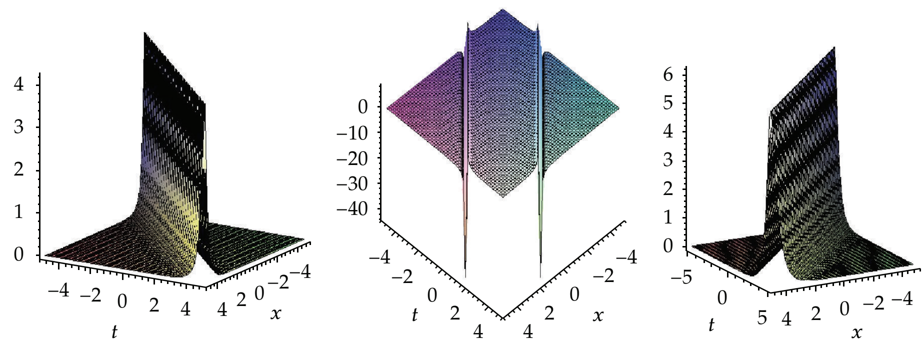
$$H'' + \frac{1}{4}\left(4\omega + c^2\right)H - \frac{c\delta}{2}\ H^3 + \frac{3\delta^2}{16}\ H^5 = 0. \tag{3.26}$$

According the homogeneous balance principle, we have $r = 2, 4, 6, \ldots$ and $n = 1, 2, 3, \ldots$, we let

$$H(\xi) = a_0 + a_1 F(\xi). \tag{3.27}$$

Substituting (3.27) with (2.5) into (3.26) and equating each of the coefficients of $F(\xi)$ to zero, we obtain system of algebraic equations. Solving this system with the aid of Maple, we obtain the following solution:

$$a_0 = 0, \qquad \omega = A - \frac{c^2}{4}, \quad a_1 = \pm 2\sqrt{-\frac{cC}{\delta\beta_2}}. \tag{3.28}$$



(a) (b) (c)

Figure 8: Evolutional behaviour of the intensity $|u_1|^2$ describes the bright soliton: (a) $p = 1, q = 1$; (b) $p = 1/332, q = 1/333$ (singular wave solution); (c) $p = 2.132, q = 1.723$.

The general formulae of the solution of Chen-Lee-Liu equation

$$u(x,t) = \pm 2\sqrt{-\frac{cC}{\delta\beta_2}}\; F(x-ct)e^{-i\omega t}e^{i\,\varphi(x-ct)}, \tag{3.29}$$

with $\varphi(\xi) = (c/2B)\ \int[B + 2CF^2(\xi)]\ d\xi,\ \omega = A - (c^2/4)$. By selecting the special values of the A, B, C and the corresponding function $F(\xi)$ yields

$$|u_1(x,t)|^2 = -\frac{2c(\sigma - pq)}{\delta\sigma\left[\cosh_{pq}\left(2\sqrt{A}\ (x-ct)\right) - \sigma\right]}, \tag{3.30}$$

$$|u_2(x,t)|^2 = -\frac{2c(\sigma + pq)}{\delta\sigma\left[\sinh_{pq}\left(2\sqrt{A}\ (x-ct)\right) - \sigma\right]}. \tag{3.31}$$

We omitted the reminder solutions for simplicity. Now, we show the effect of the parameters p and q of the GHF in the intensity u_1 given by the (3.30) by choosing different values of the parameters p and q, for $A = \delta = c = 1$, and $\sigma = 0.9$, by means of Figure 8 over the region $\{(x,\ t) :\ |x| \ \le\ 5,\ |t|\ \le 5\}$. We have bright soliton which are increase, decrease and have singularity with the different choose of the two parameters p and q. This means that the parameters p and q affect in the solution structure.

Besides the solutions obtained above, the ODE equation (2.5), albeit with different parameters, has been studied in the different context [20–24]. It has been shown that this equation possesses abundant solutions, including Weierstrass function solutions, kink solutions and periodic solutions, and so forth. To the best of our knowledge, some of our explicit solutions are new.

Notice that the GHFs are generalization of the hyperbolic functions as stated in the appendix. Also, the two parameters p and q describe the degree of the wave energy localization in the obtained solutions. Referring to [18, 19], it is easy to see that the obtained results in this paper are new and the improved method is valuable.

4. Summary and Discussion

In this paper, a set of sub-ODEs are introduced by using GHFs. Also, we have obtained many families of exact traveling wave solutions of a class of nonlinear Schrödinger equations including the generalized Zakharov system, the Rangwala-Rao equation, and the chen-Lee-Liu equation. We study and analyze the properties of these solutions by taking different parameter values of the generalized hyperbolic functions p and q. As a result, we find that these parameter values affect the solution structure. These solutions include the GHF solution, hyperbolic function solution, q deformed hyperbolic function solution, and others. The obtained solutions which depend on p and q may be of important significance for the explanation of some practical physical problems. We believe that one can apply this method to many other nonlinear differential equations in mathematical physics.

Appendix

The generalized hyperbolic sine, cosine, and tangent functions are

$$\sinh_{pq}(\xi) = \frac{pe^{\xi} - qe^{-\xi}}{2}, \quad \cosh_{pq}(\xi) = \frac{pe^{\xi} + qe^{-\xi}}{2}, \quad \tanh_{pq}(\xi) = \frac{pe^{\xi} - qe^{-\xi}}{pe^{\xi} + qe^{-\xi}}, \tag{A.1}$$

where ξ is an independent variable, p and q are arbitrary constants greater than zero and called deformation parameters. The generalized hyperbolic cotangent function is $\coth_{pq}(\xi) = 1/\tanh_{pq}(\xi)$, the generalized hyperbolic secant function is $\operatorname{sech}_{pq}(\xi) = 1/\cosh_{pq}(\xi)$, the generalized hyperbolic cosecant function is $\operatorname{csch}_{pq}(\xi) = 1/\sinh_{pq}(\xi)$, the above six kinds of functions are said GHFs. These functions satisfy the following relations [9, 12]:

$$\begin{gathered}\cosh^2_{pq}(\xi) - \sinh^2_{pq}(\xi) = pq, \qquad \cosh_{pq}(\xi) = \sqrt{pq}\cosh\left(\xi - \frac{1}{2}\ln\frac{q}{p}\right), \\ \left(\sinh_{pq}(\xi)\right)' = \cosh_{pq}(\xi), \qquad \left(\cosh_{pq}(\xi)\right)' = \sinh_{pq}(\xi).\end{gathered} \tag{A.2}$$

Note that if $p,\ q \neq 1$ then $\sinh_{pq}(\xi)$ is not odd and $\cos\mathrm{h}_{pq}(\xi)$ is not even:

$$\sinh_{pq}(-\xi) = -pq\ \sinh_{(1/p)(1/q)}(\xi), \qquad \cosh_{pq}(-\xi) = pq\ \cosh_{(1/p)(1/q)}(\xi). \tag{A.3}$$

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