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Research Article

Stochastic Delay Logistic Model under Regime Switching

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This paper is concerned with a delay logistical model under regime switching diffusion in random environment. By using generalized Itô formula, Gronwall's inequality, and Young's inequality, some sufficient conditions for existence of global positive solutions and stochastically ultimate boundedness are obtained, respectively. Also, the relationships between the stochastic permanence and extinction as well as asymptotic estimations of solutions are investigated by virtue of V-function technique, M-matrix method, and Chebyshev's inequality. Finally, an example is given to illustrate the main results.

1. Introduction

The delay differential equation

$$\frac{dx(t)}{dt} = x(t)\left[a - bx(t) + cx(t-\tau)\right] \tag{1.1}$$

has been used to model the population growth of certain species, known as the delay logistic equation. There is an extensive literature concerned with the dynamics of this delay model. We here only mention Gopalsamy [1], Kolmanovskiĭ, and Myshkis [2], Kuang [3] among many others.

In (1.1), the state $x(t)$ denotes the population size of the species. Naturally, we focus on the positive solutions and also require the solutions not to explode at a finite time. To guarantee positive solutions without explosion (i.e., there exists global positive solutions), it is generally assumed that $a > 0$, $b > 0$, and $c < b$ [4] (and the references cited therein).

On the other hand, the population growth is often subject to environmental noise, and the system will change significantly, which may change the dynamical behavior of solutions significantly [5, 6]. It is therefore necessary to reveal how the noise affects on the dynamics of solutions for the delay population model. First of all, let us consider one type of environmental noise, namely, white noise. In fact, recently, many authors have discussed population systems subject to white noise [7–9]. Recall that the parameter a in (1.1) represents the intrinsic growth rate of the population. In practice, we usually estimate it by an average value plus an error term. According to the well-known central limit theorem, the error term follows a normal distribution. In term of mathematics, we can therefore replace the rate a by

$$a + \sigma \dot{w}(t), \tag{1.2}$$

where $\dot{w}(t)$ is a white noise (i.e., $w(t)$ is a Brownian motion) and σ is a positive number representing the intensity of noise. As a result, (1.1) becomes a stochastic differential equation (SDE, in short)

$$dx(t) = x(t)[(a - bx(t) + cx(t - \tau))dt + \sigma dw(t)]. \tag{1.3}$$

We refer to [4] for more details.

To our knowledge, much attention to environmental noise is paid on white noise ([10–14] and the references cited therein). But another type of environmental noise, namely, color noise or say telegraph noise, has been studied by many authors (see, [15–19]). In this context, telegraph noise can be described as a random switching between two or more environmental regimes, which are different in terms of factors such as nutrition or as rain falls [20, 21]. Usually, the switching between different environments is memoryless and the waiting time for the next switch has an exponential distribution. This indicates that we may model the random environments and other random factors in the system by a continuous-time Markov chain $r(t)$, $t \geq 0$ with a finite state space $S = \{1, 2, \ldots, n\}$. Therefore, the stochastic delay logistic (1.3) in random environments can be described by the following stochastic model with regime switching:

$$dx(t) = x(t)[(a(r(t)) - b(r(t))x(t) + c(r(t))x(t - \tau))dt + \sigma(r(t))dw(t)]. \tag{1.4}$$

The mechanism of ecosystem described by (1.4) can be explained as follows. Assume that initially, the Markov chain $r(0) = \iota \in S$, then the ecosystem (1.4) obeys the SDE

$$dx(t) = x(t)[(a(\iota) - b(\iota)x(t) + c(\iota)x(t - \tau))dt + \sigma(\iota)dw(t)], \tag{1.5}$$

until the Markov chain $r(t)$ jumps to another state, say, ς. Then the ecosystem satisfies the SDE

$$dx(t) = x(t)[(a(\varsigma) - b(\varsigma)x(t) + c(\varsigma)x(t - \tau))dt + \sigma(\varsigma)dw(t)], \tag{1.6}$$

for a random amount of time until $r(t)$ jumps to a new state again.

It should be pointed out that the stochastic logistic systems under regime switching have received much attention lately. For instance, the study of stochastic permanence and extinction of a logistic model under regime switching was considered in [18], a new single-species model disturbed by both white noise and colored noise in a polluted environment was developed and analyzed in [22], a general stochastic logistic system under regime switching was proposed and was treated in [23].

Since (1.4) describes a stochastic population dynamics, it is critical to find out whether or not the solutions will remain positive or never become negative, will not explode to infinity in a finite time, will be ultimately bounded, will be stochastically permanent, will become extinct, or have good asymptotic properties.

This paper is organized as follows. In the next section, we will show that there exists a positive global solution with any initial positive value under some conditions. In Sections 3 and 4, we give the sufficient conditions for stochastic permanence or extinction, which show that both have closed relations with the stationary probability distribution of the Markov chain. If (1.4) is stochastically permanent, we estimate the limit of the average in time of the sample path of its solution in Section 5. Finally, an example is given to illustrate our main results.

2. Global Positive Solution

Throughout this paper, unless otherwise specified, let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, P)$ be a complete probability space with a filtration $\{\mathcal{F}_t\}_{t\geq 0}$ satisfying the usual conditions (i.e., it is right continuous and $\mathcal{F}_0$ contains all P-null sets). Let $w(t),\ t \geq 0$, be a scalar standard Brownian motion defined on this probability space. We also denote by R_+ the interval $(0, \infty)$ and denote by $\overline{R}_+$ the interval $[0, \infty)$. Moreover, let $\tau > 0$ and denote by $C([-\tau, 0]; R_+)$ the family of continuous functions from $[-\tau, 0]$ to R_+.

Let $r(t)$ be a right-continuous Markov chain on the probability space, taking values in a finite state space $S = \{1, 2, \ldots, n\}$, with the generator $\Gamma = (\gamma_{uv})$ given by

$$P\{r(t+\delta) = v \mid r(t) = u\} = \begin{cases} \gamma_{uv}\delta + o(\delta), & \text{if } u \neq v, \\ 1 + \gamma_{uv}\delta + o(\delta), & \text{if } u = v, \end{cases} \tag{2.1}$$

where $\delta > 0$, γ_{uv} is the transition rate from u to v and $\gamma_{uv} \geq 0$ if $u \neq v$, while $\gamma_{uu} = -\sum_{v\neq u} \gamma_{uv}$. We assume that the Markov chain $r(\cdot)$ is independent of the Brownian motion $w(\cdot)$. It is well known that almost every sample path of $r(\cdot)$ is a right continuous step function with a finite number of jumps in any finite subinterval of $\overline{R}_+$. As a standing hypothesis we assume in this paper that the Markov chain $r(t)$ is irreducible. This is a very reasonable assumption as it means that the system can switch from any regime to any other regime. Under this condition, the Markov chain has a unique stationary (probability) distribution $\pi = (\pi_1, \pi_2, \ldots, \pi_n) \in R^{1\times n}$ which can be determined by solving the following linear equation:

$$\pi\Gamma = 0, \tag{2.2}$$

subject to

$$\sum_{i=1}^{n} \pi_i = 1, \quad \pi_i > 0,\ \forall\ i \in S. \tag{2.3}$$

We refer to [9, 24] for the fundamental theory of stochastic differential equations.

For convenience and simplicity in the following discussion, define

$$\hat{f} = \min_{i\in S} f(i), \qquad \check{f} = \max_{i\in S} f(i), \qquad \overline{f} = \max_{i\in S} |f(i)|, \tag{2.4}$$

where $\{f(i)\}_{i\in S}$ is a constant vector.

As $x(t)$ in model (1.4) denotes population size at time t, it should be nonnegative. Thus, for further study, we must give some condition under which (1.4) has a unique global positive solution.

Theorem 2.1. *Assume that there are positive numbers $\theta(i)$ $(i = 1, 2, \ldots, n)$ such that*

$$\max_{i\in S}\left(-b(i) + \frac{1}{4\theta(i)}c^2(i) + \check{\theta}\right) \leq 0. \tag{2.5}$$

Then, for any given initial data $\{x(t) : -\tau \leq t \leq 0\} \in C([-\tau, 0]; R_+)$, there is a unique solution $x(t)$ to (1.4) *on $t \geq -\tau$ and the solution will remain in R_+ with probability 1, namely, $x(t) \in R_+$ for all $t \geq -\tau$ a.s.*

Proof. Since the coefficients of the equation are locally Lipschitz continuous, for any given initial data $\{x(t) : -\tau \leq t \leq 0\} \in C([-\tau, 0]; R_+)$, there is a unique maximal local solution $x(t)$ on $t \in [-\tau, \tau_e)$, where τ_e is the explosion time. To show that this solution is global, we need to prove $\tau_e = \infty$ a.s.

Let $k_0 > 0$ be sufficiently large for

$$\frac{1}{k_0} < \min_{-\tau\leq t\leq 0} x(t) \leq \max_{-\tau\leq t\leq 0} x(t) < k_0. \tag{2.6}$$

For each integer $k \geq k_0$, define the stopping time

$$\tau_k = \inf\left\{t \in [0, \tau_e) : x(t) \notin \left(\frac{1}{k}, k\right)\right\}, \tag{2.7}$$

where throughout this paper we set $\inf \emptyset = \infty$ (as usual $\emptyset$ denotes the empty set). Clearly, τ_k is increasing as $k \to \infty$. Set $\tau_\infty = \lim_{k\to\infty} \tau_k$, where $\tau_\infty \leq \tau_e$ a.s. If we can show that $\tau_\infty = \infty$ a.s., then $\tau_e = \infty$ a.s. and $x(t) \in R_+$ a.s. for all $t \geq 0$. In other words, we need to show $\tau_\infty = \infty$ a.s. Define a C^2-function $V : R_+ \to R_+$ by

$$V(x) = x - 1 - \log x, \tag{2.8}$$

which is not negative on $x > 0$. Let $k \geq k_0$ and $T > 0$ be arbitrary. For $0 \leq t \leq \tau_k \wedge T$, it is not difficult to show by the generalized Itô formula that

$$dV(x(t)) = LV(x(t), x(t-\tau), r(t))dt + \sigma(r(t))(x(t) - 1)dw(t), \tag{2.9}$$

where $LV : R_+ \times R_+ \times S \rightarrow R$ is defined by

$$LV(x, y, i) = -a(i) + \frac{1}{2}\sigma^2(i) + (a(i) + b(i))x - c(i)y - b(i)x^2 + c(i)xy. \tag{2.10}$$

Using condition (2.5), we compute

$$-b(i)x^2 + c(i)xy \le -b(i)x^2 + \frac{1}{4\theta(i)}c^2(i)x^2 + \theta(i)y^2 \le -\check{\theta}x^2 + \check{\theta}y^2. \tag{2.11}$$

Moreover, there is clearly a constant $K_1 > 0$ such that

$$-a(i) + \frac{1}{2}\sigma^2(i) + (a(i) + b(i))x - c(i)y \le K_1(1 + x + y). \tag{2.12}$$

Substituting these into (2.10) yields

$$LV(x, y, i) \le K_1(1 + x + y) - \check{\theta}x^2 + \check{\theta}y^2. \tag{2.13}$$

Noticing that $u \le 2(u - 1 - \log u) + 2$ on $u > 0$, we obtain that

$$LV(x, y, i) \le K_2(1 + V(x) + V(y)) - \check{\theta}x^2 + \check{\theta}y^2, \tag{2.14}$$

where K_2 is a positive constant. Substituting these into (2.9) yields

$$\begin{aligned} dV(x(t)) \le{}& \left[K_2(1 + V(x(t)) + V(x(t-\tau))) - \check{\theta}x^2(t) + \check{\theta}x^2(t-\tau)\right]dt \\ &+ \sigma(r(t))(x(t) - 1)dw(t). \end{aligned} \tag{2.15}$$

Now, for any $t \in [0, T]$, we can integrate both sides of (2.15) from 0 to $\tau_k \wedge t$ and then take the expectations to get

$$EV(x(\tau_k \wedge t)) \le V(x(0)) + E\int_0^{\tau_k \wedge t} \left[K_2(1 + V(x(s)) + V(x(s-\tau))) - \check{\theta}x^2(s) + \check{\theta}x^2(s-\tau)\right]ds. \tag{2.16}$$

Compute

$$\begin{aligned} E\int_0^{\tau_k \wedge t} V(x(s-\tau))ds &= E\int_{-\tau}^{\tau_k \wedge t - \tau} V(x(s))ds \\ &\le \int_{-\tau}^{0} V(x(s))ds + E\int_0^{\tau_k \wedge t} V(x(s))ds, \end{aligned} \tag{2.17}$$

and, similarly

$$E\int_0^{\tau_k\wedge t} x^2(s-\tau)ds \le \int_{-\tau}^0 x^2(s)ds + E\int_0^{\tau_k\wedge t} x^2(s)ds. \tag{2.18}$$

Substituting these into (2.16) gives

$$\begin{aligned} EV(x(\tau_k\wedge t)) &\le K_3 + 2K_2E\int_0^{\tau_k\wedge t} V(x(s))ds \\ &\le K_3 + 2K_2E\int_0^t V(x(\tau_k\wedge s))ds \\ &= \ K_3 + 2K_2\int_0^t EV(x(\tau_k\wedge s))ds, \end{aligned} \tag{2.19}$$

where $K_3 = V(x(0)) + K_2T + K_2\int_{-\tau}^0 V(x(s))ds + \check{\theta}\int_{-\tau}^0 x^2(s)ds$.

By the Gronwall inequality, we obtain that

$$EV(x(\tau_k\wedge T)) \le K_3e^{2TK_2}. \tag{2.20}$$

Note that for every $\omega\in\{\tau_k\le T\}$, $x(\tau_k,\omega)$ equals either k or $1/k$, thus

$$V(x(\tau_k,\omega)) \ge \left[(k-1-\log k)\wedge\left(\frac{1}{k}-1+\log k\right)\right]. \tag{2.21}$$

It then follows from (2.20) that

$$\begin{aligned} K_3e^{2TK_2} &\ge E\left[1_{\{\tau_k\le T\}}(\omega)V(x(\tau_k\wedge T,\omega))\right] \\ &= E\left[\mathbf{1}_{\{\tau_k\le T\}}(\omega)V(x(\tau_k,\omega))\right] \\ &\ge P\{\tau_k\le T\}\left[(k-1-\log k)\wedge\left(\frac{1}{k}-1+\log k\right)\right], \end{aligned} \tag{2.22}$$

where $\mathbf{1}_{\{\tau_k\le T\}}$ is the indicator function of $\{\tau_k\le T\}$. Letting $k\to\infty$ gives $\lim_{k\to\infty}P\{\tau_k\le T\}=0$ and hence $P\{\tau_\infty\le T\}=0$. Since $T>0$ is arbitrary, we must have $P\{\tau_\infty<\infty\}=0$, so $P\{\tau_\infty=\infty\}=1$ as required. □

Corollary 2.2. *Assume that there is a positive number θ such that*

$$\max_{i\in S}\left(-b(i)+\frac{1}{4\theta}c^2(i)+\theta\right)\le 0. \tag{2.23}$$

Then the conclusions of Theorem 2.1 hold.

The following theorem is easy to verify in applications, which will be used in the sections below.

Theorem 2.3. *Assume that*

$$-\hat{b} + \bar{c} \le 0. \tag{2.24}$$

Then for any given initial data $\{x(t) : -\tau \le t \le 0\} \in C([-\tau, 0]; R_+)$, *there is a unique solution* $x(t)$ *to* (1.4) *on* $t \ge -\tau$ *and the solution will remain in* R_+ *with probability 1, namely,* $x(t) \in R_+$ *for all* $t \ge -\tau$ *a.s.*

Proof. The proof of this theorem is the same as that of the theorem above. Let

$$V(x) = x - 1 - \log x \quad \text{on } x > 0, \tag{2.25}$$

then we have (2.9) and (2.10). By (2.24), we get

$$\begin{aligned} LV(x, y, i) &\le -a(i) + \frac{1}{2}\sigma^2(i) + (a(i) + b(i))x - c(i)y - b(i)x^2 + c(i)xy \\ &\le K(1 + x + y) + (-b(i) + \bar{c})x^2 - \frac{1}{2}\bar{c}x^2 + \frac{1}{2}\bar{c}y^2 \\ &\le K(1 + x + y) - \frac{1}{2}\bar{c}x^2 + \frac{1}{2}\bar{c}y^2, \end{aligned} \tag{2.26}$$

where K is a positive constant. The rest of the proof is similar to that of Theorem 2.1 and omitted. □

Note that condition (2.5) is used to derive (2.13) from (2.10). In fact, there are several different ways to estimate (2.10), which will lead to different alternative conditions for the positive global solution. For example, we know

$$\begin{aligned} c(i)xy &\le \frac{1}{2\theta(i)}c^2(i)x^2 + \frac{\theta(i)}{2}y^2, \\ -b(i)x^2 + c(i)xy &\le -b(i)x^2 + \frac{1}{2\theta(i)}c^2(i)x^2 + \frac{\theta(i)}{2}y^2 \\ &= \left(-b(i) + \frac{1}{2\theta(i)}c^2(i) + \frac{\check{\theta}}{2}\right)x^2 - \frac{\check{\theta}}{2}x^2 + \frac{\check{\theta}}{2}y^2. \end{aligned} \tag{2.27}$$

Therefore, if we assume that

$$\max_{i\in S}\left(-b(i) + \frac{1}{2\theta(i)}c^2(i) + \frac{\check{\theta}}{2}\right) \le 0, \tag{2.28}$$

then

$$-b(i)x^2 + c(i)xy \leq -\frac{\check{\theta}}{2}x^2 + \frac{\check{\theta}}{2}y^2, \tag{2.29}$$

hence

$$LV(x,y,i) \leq K_1(1+x+y) - \frac{\check{\theta}}{2}x^2 + \frac{\check{\theta}}{2}y^2, \tag{2.30}$$

from which we can show in the same way as in the proof of Theorem 2.1 that the solution of (1.4) is positive and global. In other words, the arguments above can give an alternative result which we describe as a theorem as below.

Theorem 2.4. *Assume that there are positive numbers $\theta(i)$ $(i = 1, 2, \ldots, n)$ such that*

$$\max_{i\in S}\left(-b(i) + \frac{1}{2\theta(i)}c^2(i) + \frac{\check{\theta}}{2}\right) \leq 0. \tag{2.31}$$

Then for any given initial data $\{x(t) : -\tau \leq t \leq 0\} \in C([-\tau,0]; R_+)$, there is a unique solution $x(t)$ to (1.4) on $t \geq -\tau$ and the solution will remain in R_+ with probability 1, namely, $x(t) \in R_+$ for all $t \geq -\tau$ a.s.

Similarly, we can establish a corollary as follows.

Corollary 2.5. *Assume that there is a positive number θ such that*

$$\max_{i\in S}\left(-b(i) + \frac{1}{2\theta}c^2(i) + \frac{\theta}{2}\right) \leq 0. \tag{2.32}$$

Then the conclusions of Theorem 2.4 hold.

3. Asymptotic Bounded Properties

For convenience and simplicity in the following discussion, we list the following assumptions.

(A1) For each $i \in S, b(i) > 0,$ and $-\widehat{b} + \overline{c} \leq 0$.

(A1′) For each $i \in S, b(i) > 0,$ and $-\widehat{b} + \overline{c} < 0$.

(A1″) For each $i \in S, b(i) > 0, c(i) \geq 0$ and $-\widehat{b} + \check{c} < 0$.

(A2) For some $u \in S, \gamma_{iu} > 0$ $(\forall i \neq u)$.

(A3) $\sum_{i=1}^{n} \pi_i[a(i) - (1/2)\sigma^2(i)] > 0$.

(A3′) $\sum_{i=1}^{n} \pi_i[a(i) - (1/2)\sigma^2(i)] < 0$.

(A4) For each $i \in S, a(i) - (1/2)\sigma^2(i) > 0$.

(A4′) For each $i \in S, a(i) - (1/2)\sigma^2(i) < 0$.

Definition 3.1. Equation (1.4) is said to be stochastically permanent if for any $\varepsilon \in (0,1)$, there exist positive constants $H = H(\varepsilon)$, $\delta = \delta(\varepsilon)$ such that

$$\liminf_{t \to +\infty} P\{x(t) \le H\} \ge 1 - \varepsilon, \qquad \liminf_{t \to +\infty} P\{x(t) \ge \delta\} \ge 1 - \varepsilon, \tag{3.1}$$

where $x(t)$ is the solution of (1.4) with any positive initial value.

Definition 3.2. The solutions of (1.4) are called stochastically ultimately bounded, if for any $\varepsilon \in (0,1)$, there exists a positive constant $H = H(\varepsilon)$, such that the solutions of (1.4) with any positive initial value have the property that

$$\limsup_{t \to +\infty} P\{x(t) > H\} < \varepsilon. \tag{3.2}$$

It is obvious that if a stochastic equation is stochastically permanent, its solutions must be stochastically ultimately bounded. So we will begin with the following theorem and make use of it to obtain the stochastically ultimate boundedness of (1.4).

Theorem 3.3. *Let* $(A1')$ *hold and p is an arbitrary given positive constant. Then for any given initial data* $\{x(t) : -\tau \le t \le 0\} \in C([-\tau, 0]; R_+)$, *the solution* $x(t)$ *of* (1.4) *has the properties that*

$$\limsup_{t \to \infty} E(x^p(t)) \le K_1(p), \tag{3.3}$$

$$\limsup_{t \to \infty} \frac{1}{t} \int_0^t E\left(x^{p+1}(s)\right) \le K_2(p), \tag{3.4}$$

where both $K_1(p)$ *and* $K_2(p)$ *are positive constants defined in the proof.*

Proof. By Theorem 2.3, the solution $x(t)$ will remain in R_+ for all $t \ge -\tau$ with probability 1. Let

$$-\lambda = \left(1 + \frac{\bar{c}}{p+1}\right)^{-1} \left(-\hat{b} + \bar{c}\right), \tag{3.5}$$

$$\gamma = \tau^{-1} \log(1 + \lambda). \tag{3.6}$$

Define the function $V : R_+ \times R_+ \rightarrow R_+$ by

$$V(x,t) = e^{\gamma t} x^p. \tag{3.7}$$

By the generalized Itô formula, we have

$$dV(x(t),t) = LV(x(t), x(t-\tau), t, r(t))dt + pe^{\gamma t}\sigma(r(t))x^p \, dw(t), \tag{3.8}$$

where $LV : R_+ \times R_+ \times R_+ \times S \to R$ is defined by

$$LV(x,y,t,i) = e^{\gamma t}\left(\gamma + pa(i) + \frac{1}{2}p(p-1)\sigma^2(i)\right)x^p - pe^{\gamma t}b(i)x^{p+1} + pe^{\gamma t}c(i)x^p y. \tag{3.9}$$

By (3.5) and Young's inequality, we obtain that

$$\begin{aligned} &LV(x,y,t,i) \\ &\quad \le e^{\gamma t}\left(\gamma + pa(i) + \frac{1}{2}p(p-1)\sigma^2(i)\right)x^p - pe^{\gamma t}b(i)x^{p+1} + pe^{\gamma t}\bar{c}\left(\frac{p}{p+1}x^{p+1} + \frac{1}{p+1}y^{p+1}\right) \\ &\quad \le e^{\gamma t}\left[\left(\gamma + p\check{a} + \frac{1}{2}p(p-1)\sigma^2(i)\right)x^p - \lambda p x^{p+1}\right] + e^{\gamma t}\frac{p\bar{c}}{p+1}\left[-(1+\lambda)x^{p+1} + y^{p+1}\right] \\ &\quad \le H_1 e^{\gamma t} + e^{\gamma t}\frac{p\bar{c}}{p+1}\left(-e^{\gamma\tau}x^{p+1} + y^{p+1}\right), \end{aligned} \tag{3.10}$$

where $H_1 = \max_{i\in S}\{\sup_{x\in R_+}[(\gamma + p\check{a} + (1/2)p(p-1)\sigma^2(i))x^p - \lambda p x^{p+1}]\}$. Moreover,

$$\int_0^t e^{\gamma s}x^{p+1}(s-\tau)ds \le e^{\gamma\tau}\int_{-\tau}^0 x^{p+1}(s)ds + e^{\gamma\tau}\int_0^t e^{\gamma s}x^{p+1}(s)ds. \tag{3.11}$$

By (3.10) and (3.11), one has

$$e^{\gamma t}E(x^p(t)) \le x^p(0) + \frac{H_1}{\gamma}(e^{\gamma t} - 1) + \frac{p\bar{c}}{p+1}e^{\gamma\tau}\int_{-\tau}^0 x^{p+1}(s)ds, \tag{3.12}$$

which yields

$$\limsup_{t\to\infty} E(x^p(t)) \le K_1(p), \tag{3.13}$$

where

$$K_1(p) = \max_{i\in S}\left\{\sup_{x\in R_+}\gamma^{-1}\left[\left(\gamma + p\check{a} + \frac{1}{2}p(p-1)\sigma^2(i)\right)x^p - \lambda p x^{p+1}\right]\right\}. \tag{3.14}$$

By the generalized Itô formula, Young's inequality and (3.5) again, it follows

$$
\begin{aligned}
0 &\le E(x^p(t)) \\
&\le x_0^p + E\int_0^t px^p(s)\left[\frac{1}{2}(p-1)\sigma^2(r(s)) + a(r(s))\right] - pb(r(s))x^{p+1}(s) + p\bar{c}x^p(s)x(s-\tau)ds \\
&\le x_0^p + E\int_0^t \left\{\frac{1}{2}p(p-1)\sigma^2(r(s))x^p(s) + pa(r(s))x^p(s)\right. \\
&\qquad \left. + p\left[-\left(1+\frac{\bar{c}}{p+1}\right)\lambda\right]x^{p+1}(s) + \frac{p\bar{c}}{p+1}\left(-x^{p+1}(s) + x^{p+1}(s-\tau)\right)\right\}ds \\
&\le H_2 + E\int_0^t \left\{\frac{1}{2}p(p-1)\sigma^2(r(s))x^p(s) + pa(r(s))x^p + p\left[-\left(1+\frac{\bar{c}}{p+1}\right)\lambda\right]x^{p+1}(s)\right\}ds,
\end{aligned}
\tag{3.15}
$$

where $H_2 = x^p(0) + (p\bar{c}/(p+1))\int_{-\tau}^0 x^{p+1}(s)ds$. This implies

$$
p\lambda E\int_0^t x^{p+1}(s)ds \le H_2 + E\int_0^t \frac{1}{2}p(p-1)\sigma^2(r(s))x^p(s) + pa(r(s))x^p(s) - \frac{\lambda p\bar{c}}{p+1}x^{p+1}(s)ds.
\tag{3.16}
$$

The inequality above implies

$$
\limsup_{t\to\infty}\frac{1}{t}E\int_0^t x^{p+1}(s) \le \frac{H_3}{p\lambda},
\tag{3.17}
$$

where $H_3 = \max_{i\in S}\{\sup_{x\in R_+}[(1/2)p(p-1)\sigma^2(i)x^p + pa(i)x^p - (\lambda p\bar{c}/(p+1))x^{p+1}]\}$ and the desired assertion (3.4) follows by setting $K_2(p) = H_3/p\lambda$. □

Remark 3.4. From (3.3) of Theorem 3.3, there is a $T > 0$ such that

$$
E(x^p(t)) \le 2K_1(p) \quad \forall\ t \ge T.
\tag{3.18}
$$

Since $E(x^p(t))$ is continuous, there is a $\overline{K}_1(p, x_0)$ such that

$$
E(x^p(t)) \le \overline{K}_1(p, x_0) \quad \text{for } t \in [0, T].
\tag{3.19}
$$

Taking $L(p, x_0) = \max(2K_1(p), \overline{K}_1(p, x_0))$, we havefor the fundamental theory of

$$
E(x^p(t)) \le L(p, x_0) \quad \forall\ t \in [0, \infty).
\tag{3.20}
$$

This means that the pth moment of any positive solution of (1.4) is bounded.

Remark 3.5. Equation (3.4) of Theorem 3.3 shows that the average in time of the pth ($p > 1$) moment of solutions of (1.4) is bounded.

Theorem 3.6. *Solutions of* (1.4) *are stochastically ultimately bounded under* ($A1'$).

Proof. This can be easily verified by Chebyshev's inequality and Theorem 3.3. □

Based on the results above, we will prove the other inequality in the definition of stochastic permanence. For convenience, define

$$\beta(i) = a(i) - \frac{1}{2}\sigma^2(i). \tag{3.21}$$

Under (A3), it has $\sum_{i=1}^{n} \pi_i \beta(i) > 0$. Moreover, let G be a vector or matrix. By $G \gg 0$ we mean all elements of G are positive. We also adopt here the traditional notation by letting

$$Z^{n\times n} = \left\{ A = \left(a_{ij}\right)_{n\times n} : a_{ij} \le 0,\ i \ne j \right\}. \tag{3.22}$$

We will also need some useful results.

Lemma 3.7 (see [24]). *If $A \in Z^{n\times n}$, then the following statements are equivalent.*

(1) *A is a nonsingular M-matrix (see [24] for definition of M-matrix).*

(2) *All of the principal minors of A are positive; that is,*

$$\begin{vmatrix} a_{11} & \cdots & a_{1k} \\ \cdots & \cdots & \cdots \\ a_{k1} & \cdots & a_{kk} \end{vmatrix} > 0 \quad \text{for every } k = 1, 2, \ldots, n. \tag{3.23}$$

(3) *A is semipositive, that is, there exists $x \gg 0$ in R^n such that $Ax \gg 0$.*

Lemma 3.8 (see [18]). *(i) Assumptions* ($A2$) *and* ($A3$) *imply that there exists a constant $\theta > 0$ such that the matrix*

$$A(\theta) = \operatorname{diag}(\xi_1(\theta), \xi_2(\theta), \ldots, \xi_n(\theta)) - \Gamma \tag{3.24}$$

is a nonsingular M-matrix, where $\xi_i(\theta) = \theta\beta(i) - (1/2)\theta^2\sigma^2(i),\ \forall i \in S$.

(ii) Assumption (A4) implies that there exists a constant $\theta > 0$ such that the matrix $A(\theta)$ is a nonsingular M-matrix.

Lemma 3.9. *If there exists a constant $\theta > 0$ such that $A(\theta)$ is a nonsingular M-matrix and $c(i) \ge 0$ $(i = 1, 2, \ldots, n)$, then the global positive solution $x(t)$ of* (1.4) *has the property that*

$$\limsup_{t\to\infty} E\left(|x(t)|^{-\theta}\right) \le H, \tag{3.25}$$

where H is a fixed positive constant (defined by (3.35) *in the proof).*

Proof. Let $U(t) = x^{-1}(t)$ on $t \geq 0$. Applying the generalized Itô formula, we have

$$dU(t) = U(t)\Big(-a(r(t)) + \sigma^2(r(t)) + b(r(t))x(t) - c(r(t))x(t-\tau)\Big)dt - \sigma U(t)dw(t). \quad (3.26)$$

By Lemma 3.7, for the given θ, there is a vector $\vec{q} = (q_1, q_2, \ldots, q_n)^T \gg 0$ such that

$$\vec{\lambda} = (\lambda_1, \lambda_2, \ldots, \lambda_n)^T = A(\theta)\vec{q} \gg 0, \quad (3.27)$$

namely,

$$q_i\left(\theta\beta(i) - \frac{1}{2}\theta^2\sigma^2(i)\right) - \sum_{j=1}^{n}\gamma_{ij}q_j > 0 \quad \forall\, 1 \leq i \leq n. \quad (3.28)$$

Define the function $V : R_+ \times S \to R_+$ by $V(U, i) = q_i(1+U)^\theta$. Applying the generalized Itô formula again, we have

$$EV(U(t), r(t)) = V(U(0), r(0)) + E\int_0^t LV(U(s), x(s-\tau), r(s))ds, \quad (3.29)$$

where $LV : R_+ \times R_+ \times S \to R$ is defined by

$$\begin{aligned} LV(U, y, i) = (1+U)^{\theta-2}\Bigg\{ &-U^2\left[q_i\left(\theta\beta(i) - \frac{1}{2}\theta^2\sigma^2(i)\right) - \sum_{j=1}^{n}\gamma_{ij}q_j\right] \\ &+ U\left[q_i\theta\Big(b(i) - a(i) + \sigma^2(i)\Big) + 2\sum_{j=1}^{n}\gamma_{ij}q_j\right] \\ &+ \left[q_i\theta b(i) + \sum_{j=1}^{n}\gamma_{ij}q_j - q_i\theta c(i)(1+U)Uy\right]\Bigg\}. \end{aligned} \quad (3.30)$$

Now, choose a constant $\kappa > 0$ sufficiently small such that

$$\vec{\lambda} - \kappa\vec{q} \gg 0, \quad (3.31)$$

that is,

$$q_i\left(\theta\beta(i) - \frac{1}{2}\theta^2\sigma^2(i)\right) - \sum_{j=1}^{n}\gamma_{ij}q_j - \kappa q_i > 0 \quad \forall\, 1 \leq i \leq n. \quad (3.32)$$

Then, by the generalized Itô formula again,

$$\begin{aligned} &E\left[e^{\kappa t}V(U(t),r(t))\right] \\ &\quad = V(U(0),r(0)) + E\int_0^t \left[\kappa e^{\kappa t}V(U(s),r(s)) + e^{\kappa t}LV(U(s),x(s-\tau),r(s))\right]ds. \end{aligned} \tag{3.33}$$

It is computed that

$$\begin{aligned} &\kappa e^{\kappa t}V(U,i) + e^{\kappa t}LV(U,y,i) \\ &\quad \le e^{\kappa t}(1+U)^{\theta-2}\Bigg\{-U^2\left[q_i\left(\theta\beta(i)-\frac{1}{2}\theta^2\sigma^2(i)\right)-\sum_{j=1}^{n}\gamma_{ij}q_j-\kappa q_i\right] \\ &\qquad + U\left[q_i\theta\left(b(i)-a(i)+\sigma^2(i)\right)+2\sum_{j=1}^{n}\gamma_{ij}q_j+2\kappa q_i\right] \\ &\qquad + q_i\theta b(i)+\sum_{j=1}^{n}\gamma_{ij}q_j+\kappa q_i\Bigg\} \\ &\quad \le \hat{q}\kappa H e^{\kappa t}, \end{aligned} \tag{3.34}$$

where

$$\begin{aligned} H = \frac{1}{\hat{q}\kappa}\max_{i\in S}\Bigg\{\sup_{U\in R_+}(1+U)^{\theta-2}\Bigg\{&-U^2\left[q_i\left(\theta\beta(i)-\frac{1}{2}\theta^2\sigma^2(i)\right)-\sum_{j=1}^{n}\gamma_{ij}q_j-\kappa q_i\right] \\ &+U\left[q_i\theta\left(b(i)-a(i)+\sigma^2(i)\right)+2\sum_{j=1}^{n}\gamma_{ij}q_j+2\kappa q_i\right] \\ &+q_i\theta b(i)+\sum_{j=1}^{n}\gamma_{ij}q_j+\kappa q_i\Bigg\},1\Bigg\}, \end{aligned} \tag{3.35}$$

which implies

$$\hat{q}E\left[e^{\kappa t}(1+U(t))^{\theta}\right] \le \check{q}\left(1+x^{-1}(0)\right)^{\theta} + \hat{q}He^{\kappa t}. \tag{3.36}$$

Then

$$\limsup_{t\to\infty} E\left(U^{\theta}(t)\right) \le \limsup_{t\to\infty} E\left[(1+U(t))^{\theta}\right] \le H. \tag{3.37}$$

Recalling the definition of $U(t)$, we obtain the required assertion. □

Theorem 3.10. *Under* $(A1'')$, $(A2)$, *and* $(A3)$, (1.4) *is stochastically permanent.*

The proof is a simple application of the Chebyshev inequality, Lemmas 3.8 and 3.9, and Theorem 3.6. Similarly, it is easy to obtain the following result.

Theorem 3.11. *Under* $(A1'')$ *and* $(A4)$, (1.4) *is stochastically permanent.*

Remark 3.12. It is well-known that if $a > 0$, $b > 0$ and $0 \leq c < b$, then the solution $x(t)$ of (1.1) is persistent, namely,

$$\liminf_{t \to \infty} x(t) > 0. \tag{3.38}$$

Furthermore, we consider its associated stochastic delay equation (1.4), that is,

$$dx(t) = x(t)[(a(r(t)) - b(r(t))x(t) + c(r(t))x(t-\tau))dt + \sigma(r(t))dw(t)], \tag{3.39}$$

where $a(i) > 0$, $b(i) > 0$, $c(i) \geq 0$, for $i \in S$, and $\check{c} < \hat{b}$. Thus, applying Theorem 3.10 or Theorem 3.11, we can see that (1.4) is stochastically permanent, if the noise intensities are sufficiently small in the sense that

$$\sum_{i=1}^{n} \pi_i \left[a(i) - \frac{1}{2}\sigma^2(i)\right] > 0 \quad \text{or} \quad a(i) - \frac{1}{2}\sigma^2(i) > 0, \quad \text{for each } i \in S. \tag{3.40}$$

Corollary 3.13. *Assume that for some* $i \in S$, $b(i) > 0$, $-b(i) + |c(i)| \leq 0$, *and* $a(i) - (1/2)\sigma^2(i) > 0$. *Then the subsystem*

$$dx(t) = x(t)[(a(i) - b(i)x(t) + c(i)x(t-\tau))dt + \sigma(i)dw(t)] \tag{3.41}$$

is stochastically permanent.

4. Extinction

In the previous sections we have shown that under certain conditions, the original (1.1) and the associated SDE (1.4) behave similarly in the sense that both have positive solutions which will not explode to infinity in a finite time and, in fact, will be ultimately bounded. In other words, we show that under certain condition the noise will not spoil these nice properties. However, we will show in this section that if the noise is sufficiently large, the solution to (1.4) will become extinct with probability 1.

Theorem 4.1. *Assume that* $(A1)$ *holds. Then for any given initial data* $\{x(t) : -\tau \leq t \leq 0\} \in C([-\tau, 0]; R_+)$, *the solution* $x(t)$ *of* (1.4) *has the property that*

$$\limsup_{t \to \infty} \frac{\log x(t)}{t} \leq \sum_{i=1}^{n} \pi_i \left[a(i) - \frac{1}{2}\sigma^2(i)\right] \quad a.s. \tag{4.1}$$

Proof. By Theorem 2.3, the solution $x(t)$ will remain in R_+ for all $t \geq -\tau$ with probability 1. We have by the generalized Itô formula and (A1) that

$$d\log x(t) \leq \left(a(r(t)) - \frac{1}{2}\sigma^2(r(t)) - \widehat{b}x(t) + \widehat{b}x(t-\tau)\right)dt + \sigma(r(t))dw(t), \tag{4.2}$$

where (A1) is used in the last step. Then,

$$\log(V(x(t))) \leq \log(V(x(0))) + \widehat{b}\int_{-\tau}^{0} x(s)ds + \int_0^t \left(a(r(s)) - \frac{1}{2}\sigma^2(r(s))\right)ds + M(t), \tag{4.3}$$

where $M(t) = \int_0^t \sigma(r(t))dw(t)$. The quadratic variation of $M(t)$ is given by

$$\langle M, M\rangle_t = \int_0^t \sigma^2(r(s))ds \leq \overline{\sigma}^2 t. \tag{4.4}$$

Therefore, applying the strong law of large numbers for martingales [24], we obtain

$$\lim_{t\to\infty} \frac{M(t)}{t} = 0 \quad \text{a.s.} \tag{4.5}$$

It finally follows from (4.3) by dividing t on the both sides and then letting $t \to \infty$ that

$$\limsup_{t\to\infty} \frac{\log x(t)}{t} \leq \limsup_{t\to\infty} \frac{1}{t}\int_0^t \left[a(r(s)) - \frac{1}{2}\sigma^2(r(s))\right]ds = \sum_{i=1}^{n} \pi_i \left[a(i) - \frac{1}{2}\sigma^2(i)\right] \quad \text{a.s.}, \tag{4.6}$$

which is the required assertion (4.1). □

Similarly, it is easy to prove the following conclusions.

Theorem 4.2. *Assume that* (A1) *and* (A3′) *hold. Then for any given initial data* $\{x(t) : -\tau \leq t \leq 0\} \in C([-\tau, 0]; R_+)$, *the solution* $x(t)$ *of* (1.4) *has the property that*

$$\limsup_{t\to\infty} \frac{\log x(t)}{t} < 0 \quad \text{a.s.} \tag{4.7}$$

That is, the population will become extinct exponentially with probability 1.

Theorem 4.3. *Assume that (Al) and (A4′) hold. Then for any given initial data* $\{x(t) : -\tau \leq t \leq 0\} \in C([-\tau, 0]; R_+)$, *the solution* $x(t)$ *of* (1.4) *has the property that*

$$\limsup_{t\to\infty} \frac{\log x(t)}{t} \leq -\frac{\varphi}{2} \quad \text{a.s.}, \tag{4.8}$$

where $\varphi = \min_{i\in S}(\sigma^2(i) - 2a(i)) > 0$. *That is, the population will become extinct exponentially with probability 1.*

Remark 4.4. If the noise intensities are sufficiently large in the sense that

$$\sum_{i=1}^{n}\pi_i\left[a(i)-\frac{1}{2}\sigma^2(i)\right]<0 \quad \text{or} \quad a(i)-\frac{1}{2}\sigma^2(i)<0, \quad \text{for each } i\in S, \tag{4.9}$$

then the population $x(t)$ represented by (1.4) will become extinct exponentially with probability 1. However, the original delay equation (1.1) may be persistent without environmental noise.

Remark 4.5. Let A1$''$ and A2 hold, $\sum_{i=1}^{n}\pi_i[a(i)-(1/2)\sigma^2(i)]\neq 0$. Then, SDE (1.4) is either stochastically permanent or extinctive. That is, it is stochastically permanent if and only if $\sum_{i=1}^{n}\pi_i[a(i)-(1/2)\sigma^2(i)]>0$, while it is extinctive if and only if $\sum_{i=1}^{n}\pi_i[a(i)-(1/2)\sigma^2(i)]<0$.

Corollary 4.6. *Assume that for some $i\in S$,*

$$-b(i)+|c(i)|\leq 0, \qquad a(i)-\frac{1}{2}\sigma^2(i)<0. \tag{4.10}$$

Then for any given initial data $\{x(t):-\tau\leq t\leq 0\}\in C([-\tau,0];R_+)$, the solution $x(t)$ of subsystem

$$dx(t)=x(t)[(a(i)-b(i)x(t)+c(i)x(t-\tau))dt+\sigma(i)dw(t)] \tag{4.11}$$

tend to zero a.s.

5. Asymptotic Properties

Lemma 5.1. *Assume that $(A1')$ holds. Then for any given initial data $\{x(t):-\tau\leq t\leq 0\}\in C([-\tau,0];R_+)$, the solution $x(t)$ of* (1.4) *has the property*

$$\limsup_{t\to\infty}\frac{\log(x(t))}{\log t}\leq 1 \quad a.s. \tag{5.1}$$

Proof. By Theorem 2.3, the solution $x(t)$ will remain in R_+ for all $t\geq -\tau$ with probability 1. It is known that

$$\begin{aligned} dx(t) &\leq (\check{a}x(t)+\bar{c}x(t)x(t-\tau))dt+\sigma(r(t))dw(t),\\ E\left(\sup_{t\leq u\leq t+1}x(u)\right) &\leq E(x(t))+\check{a}\int_t^{t+1}E(x(s))ds+\bar{c}\int_t^{t+1}E(x(s)x(s-\tau))ds\\ &\quad+E\left(\sup_{t\leq u\leq t+1}\int_t^{u}\sigma(r(s))x(s)dw(s)\right). \end{aligned} \tag{5.2}$$

From (3.3) of Theorem 3.3, it has

$$\limsup_{t\to\infty} E(x(t)) \le K_1(1),$$
$$\limsup_{t\to\infty} E\left(x^2(t)\right) \le K_1(2). \tag{5.3}$$

By the well-known BDG's inequality [24] and the Hölder's inequality, we obtain

$$\begin{aligned} E\left(\sup_{t\le u\le t+1}\int_t^u \sigma(r(s))x(s)dB(s)\right) &\le 3E\left[\int_t^{t+1}(\sigma(r(s))x(s))^2 ds\right]^{1/2} \\ &\le E\left(\sup_{t\le u\le t+1} x(u)\cdot 9\check{\sigma}\int_t^{t+1} x(s)ds\right)^{1/2} \\ &\le \frac{1}{2}E\left(\sup_{t\le u\le t+1} x(u)\right) + 9\check{\sigma}^2\int_t^{t+1} E(x(s))ds. \end{aligned} \tag{5.4}$$

Note that

$$\int_t^{t+1} E[x(s)x(s-\tau)]ds \le \frac{1}{2}\int_t^{t+1} E\left(x^2(s)\right)ds + \frac{1}{2}\int_t^{t+1} E\left(x^2(s-\tau)\right)ds. \tag{5.5}$$

Therefore,

$$\begin{aligned} E\left(\sup_{t\le u\le t+1} x(u)\right) \le{}& 2E(x(t)) + 2\check{a}\int_t^{t+1} E(x(s))ds + \overline{c}\int_t^{t+1} E\left(x^2(s)\right)ds \\ &+ \overline{c}\int_t^{t+1} E\left(x^2(s-\tau)\right)ds + 18\check{\sigma}^2\int_t^{t+1} E(x(s))ds. \end{aligned} \tag{5.6}$$

This, together with (5.3), yields

$$\limsup_{t\to\infty} E\left(\sup_{t\le u\le t+1} x(u)\right) \le 2\left(1+\check{a}+18\check{\sigma}^2\right)K_1(1) + 2\overline{c}K_1(2). \tag{5.7}$$

From (5.7), there exists a positive constant M such that

$$E\left(\sup_{k\le t\le k+1} x(t)\right) \le M, \quad k = 1,2,\ldots. \tag{5.8}$$

Let $\varepsilon > 0$ be arbitrary. Then, by Chebyshev's inequality,

$$P\left(\sup_{k\le t\le k+1} x(u) > k^{1+\varepsilon}\right) \le \frac{M}{k^{1+\varepsilon}}, \quad k = 1, 2, \ldots. \tag{5.9}$$

Applying the well-known Borel-Cantelli lemma [24], we obtain that for almost all $\omega \in \Omega$

$$\sup_{k\le t\le k+1} x(u) \le k^{1+\varepsilon}, \tag{5.10}$$

for all but finitely many k. Hence, there exists a $k_0(\omega)$, for almost all $\omega \in \Omega$, for which (5.10) holds whenever $k \ge k_0$. Consequently, for almost all $\omega \in \Omega$, if $k \ge k_0$ and $k \le t \le k+1$, then

$$\frac{\log(x(t))}{\log t} \le \frac{(1+\varepsilon)\log k}{\log k} = 1 + \varepsilon. \tag{5.11}$$

Therefore,

$$\limsup_{t\to\infty} \frac{\log(x(t))}{\log t} \le 1 + \varepsilon \quad \text{a.s.} \tag{5.12}$$

Letting $\varepsilon \to 0$, we obtain the desired assertion (5.1). □

Lemma 5.2. *If there exists a constant $\theta > 0$ such that $A(\theta)$ is a nonsingular M-matrix and $c(i) \ge 0$ $(i = 1, 2, \ldots, n)$, then the global positive solution $x(t)$ of SDE* (1.4) *has the property that*

$$\liminf_{t\to\infty} \frac{\log(x(t))}{\log t} \ge -\frac{1}{\theta} \quad a.s. \tag{5.13}$$

Proof. Applying the generalized Itô formula, for the fixed constant $\theta > 0$, we derive from (3.26) that

$$\begin{aligned} d\left[(1+U(t))^{\theta}\right] \le{}& \theta(1+U(t))^{\theta-2}\left[-U^2(t)\left(\widehat{\beta} - \frac{1}{2}\check{\sigma}^2\right) + U(t)\left(\check{b} + \check{\sigma}^2\right) + \check{b}\right]dt \\ &- \theta\sigma(r(t))(1+U(t))^{\theta-1}U(t)dw(t), \end{aligned} \tag{5.14}$$

where $U(t) = 1/x(t)$ on $t > 0$. By (3.37), there exists a positive constant M such that

$$E\left[(1+U(t))^{\theta}\right] \le M \quad \text{on } t \ge 0. \tag{5.15}$$

Let $\delta > 0$ be sufficiently small for

$$\theta\left\{\left[\widehat{\beta} + 2\check{b} + \frac{1}{2}(\theta+2)\check{\sigma}^2\right]\delta + 3\check{a}\delta^{1/2}\right\} < \frac{1}{2}. \tag{5.16}$$

Then (5.14) implies that

$$\begin{aligned}
&E\left[\sup_{(k-1)\delta\le t\le k\delta}(1+U(t))^{\theta}\right]\\
&\quad\le E\left[(1+U((k-1)\delta))^{\theta}\right]\\
&\qquad+E\left\{\sup_{(k-1)\delta\le t\le k\delta}\left|\int_{(k-1)\delta}^{t}\theta(1+U(s))^{\theta-2}\left[-U^{2}(s)\left(\widehat{\beta}-\frac{1}{2}\theta\check{\sigma}^{2}\right)+U(s)\left(\check{b}+\check{\sigma}^{2}\right)+\check{b}\right]ds\right|\right\}\\
&\qquad+E\left\{\sup_{(k-1)\delta\le t\le k\delta}\left|\int_{(k-1)\delta}^{t}\theta\sigma(r(s))(1+U(s))^{\theta-1}U(s)dw(s)\right|\right\}.
\end{aligned}\tag{5.17}$$

By directly computing, we have

$$\begin{aligned}
&E\left\{\sup_{(k-1)\delta\le t\le k\delta}\left|\int_{(k-1)\delta}^{t}\theta(1+U(s))^{\theta-2}\left[-U^{2}(s)\left(\widehat{\beta}-\frac{1}{2}\theta\check{\sigma}^{2}\right)+U(s)\left(\check{b}+\check{\sigma}^{2}\right)+\check{b}\right]ds\right|\right\}\\
&\quad\le\theta E\left\{\int_{(k-1)\delta}^{t}\left[\widehat{\beta}+2\check{b}+\frac{1}{2}(\theta+2)\check{\sigma}^{2}\right](1+U(s))^{\theta}ds\right\}\\
&\quad\le\theta\left[\widehat{\beta}+2\check{b}+\frac{1}{2}(\theta+2)\check{\sigma}^{2}\right]\delta E\left[\sup_{(k-1)\delta\le t\le k\delta}(1+U(t))^{\theta}\right].
\end{aligned}\tag{5.18}$$

By the BDG's inequality, it follows

$$E\left\{\sup_{(k-1)\delta\le t\le k\delta}\left|\int_{(k-1)\delta}^{k\delta}\theta\sigma(r(s))(1+U(s))^{\theta-1}U(s)dw(s)\right|\right\}\le 3\theta\check{\sigma}\delta^{1/2}E\left\{\sup_{(k-1)\delta\le t\le k\delta}(1+U(s))^{\theta}\right\}.\tag{5.19}$$

Substituting this and (5.18) into (5.17) gives

$$\begin{aligned}
&E\left[\sup_{(k-1)\delta\le t\le k\delta}(1+U(t))^{\theta}\right]\\
&\quad\le E\left[(1+U((k-1)\delta))^{\theta}\right]+\theta\left\{\left[\widehat{\beta}+2\check{b}+\frac{1}{2}(\theta+2)\check{\sigma}^{2}\right]\delta+3\check{\sigma}\delta^{1/2}\right\}E\left\{\sup_{(k-1)\delta\le t\le k\delta}(1+U(s))^{\theta}\right\}.
\end{aligned}\tag{5.20}$$

Making use of (5.15) and (5.16), we obtain

$$E\left[\sup_{(k-1)\delta\le t\le k\delta}(1+U(t))^{\theta}\right]\le 2M \quad \text{on } t\ge 0. \tag{5.21}$$

Let $\varepsilon>0$ be arbitrary. Then, we have by Chebyshev's inequality that

$$P\left\{\omega:\sup_{(k-1)\delta\le t\le k\delta}(1+U(t))^{\theta}>(k\delta)^{1+\varepsilon}\right\}\le\frac{2M}{(k\delta)^{1+\varepsilon}},\quad k=1,2,\ldots. \tag{5.22}$$

Applying the Borel-Cantelli lemma, we obtain that for almost all $\omega\in\Omega$,

$$\sup_{(k-1)\delta\le t\le k\delta}(1+U(t))^{\theta}\le(k\delta)^{1+\varepsilon} \tag{5.23}$$

holds for all but finitely many k. Hence, there exists an integer $k_0(\omega)>1/\delta+2$, for almost all $\omega\in\Omega$, for which (5.23) holds whenever $k\ge k_0$. Consequently, for almost all $\omega\in\Omega$, if $k\ge k_0$ and $(k-1)\delta\le t\le k\delta$,

$$\frac{\log(1+U(t))^{\theta}}{\log t}\le\frac{(1+\varepsilon)\log(k\delta)}{\log((k-1)\delta)}\le 1+\varepsilon. \tag{5.24}$$

Therefore,

$$\limsup_{t\to\infty}\frac{\log(1+U(t))^{\theta}}{\log t}\le 1+\varepsilon \quad \text{a.s.} \tag{5.25}$$

Let $\varepsilon\to 0$, we obtain

$$\limsup_{t\to\infty}\frac{\log(1+U(t))^{\theta}}{\log t}\le 1 \quad \text{a.s.} \tag{5.26}$$

Recalling the definition of $U(t)$, this yields

$$\limsup_{t\to\infty}\frac{\log(1/x^{\theta}(t))}{\log t}\le 1 \quad \text{a.s.}, \tag{5.27}$$

which further implies

$$\liminf_{t\to\infty}\frac{\log(x(t))}{\log t}\ge-\frac{1}{\theta} \quad \text{a.s.} \tag{5.28}$$

This is our required assertion (5.13). □

Theorem 5.3. *Assume that* $(A1'')$, $(A2)$, *and* $(A3)$ *hold. Then for any given initial data* $\{x(t) : -\tau \le t \le 0\} \in C([-\tau, 0]; R_+)$, *the solution* $x(t)$ *of* (1.4) *obeys*

$$\limsup_{t\to\infty} \frac{1}{t}\int_0^t x(s)ds \le \frac{1}{\widehat{b}-\overline{c}}\sum_{i=1}^n \pi_i\left(a(i) - \frac{1}{2}\sigma^2(i)\right) \quad a.s., \tag{5.29}$$

$$\liminf_{t\to\infty} \frac{1}{t}\int_0^t x(s)ds \ge \frac{1}{\check{b}}\sum_{i=1}^n \pi_i\left(a(i) - \frac{1}{2}\sigma^2(i)\right) \quad a.s. \tag{5.30}$$

Proof. By Theorem 2.3, the solution $x(t)$ will remain in R_+ for all $t \ge -\tau$ with probability 1. From Lemmas 3.8, 5.1, and 5.2, it follows

$$\lim_{t\to\infty} \frac{\log(x(t))}{t} = 0 \quad \text{a.s.} \tag{5.31}$$

By generalized Itô formula, one has

$$\begin{aligned} \log x(t) = \log x_0 &+ \int_0^t \left(a(r(s)) - \frac{1}{2}\sigma^2(r(s))\right)ds - \int_0^t b(r(s))x(s)ds \\ &+ \int_0^t c(r(s))x(s-\tau)ds + \int_0^t \sigma(r(s))dw(s). \end{aligned} \tag{5.32}$$

Dividing by t on both sides, then we have

$$\begin{aligned} \frac{\log x(t)}{t} \le \frac{\log x_0}{t} &+ \frac{1}{t}\int_0^t \left(a(r(s)) - \frac{1}{2}\sigma^2(r(s))\right)ds + \left(-\widehat{b} + \overline{c}\right)\frac{1}{t}\int_0^t x(s)ds \\ &+ \frac{\overline{c}}{t}\int_{-\tau}^0 x(s)ds + \frac{1}{t}\int_0^t \sigma(r(s))dw(s). \end{aligned} \tag{5.33}$$

Let $t \to \infty$, by the strong law of large numbers for martingales and (5.31), we therefore have

$$\limsup_{t\to\infty} \frac{1}{t}\int_0^t x(s)ds \le \frac{1}{\widehat{b}-\overline{c}}\sum_{i=1}^n \pi_i\left(a(i) - \frac{1}{2}\sigma^2(i)\right) \quad \text{a.s.}, \tag{5.34}$$

which is the required assertions (5.29). And we also have

$$\frac{\log x(t)}{t} \ge \frac{1}{t}\log x_0 + \frac{1}{t}\int_0^t \left(a(r(s)) - \frac{1}{2}\sigma^2(r(s))\right)ds - \frac{\check{b}}{t}\int_0^t x(s)ds + \frac{1}{t}\int_0^t \sigma(r(s))dw(s). \tag{5.35}$$

Let $t \to \infty$, by the strong law of large numbers for martingales and (5.21), we therefore have

$$\liminf_{t\to\infty} \frac{1}{t}\int_0^t x(s)ds \geq \frac{1}{\breve{b}}\sum_{i=1}^{n}\pi_i\left(a(i)-\frac{1}{2}\sigma^2(i)\right) \quad \text{a.s.}, \tag{5.36}$$

which is the required assertions (5.30). □

Similarly, by using Lemmas 3.8, 5.1, and 5.2, it is easy to show the following conclusion.

Theorem 5.4. *Assume that* $(A1'')$ *and* $(A4)$ *hold. Then for any given initial data* $\{x(t) : -\tau \leq t \leq 0\} \in C([-\tau, 0]; R_+)$, *the solution* $x(t)$ *of* (1.4) *obeys*

$$\begin{aligned} &\limsup_{t\to\infty} \frac{1}{t}\int_0^t x(s)ds \leq \frac{1}{\hat{b}-\bar{c}}\sum_{i=1}^{n}\pi_i\left(a(i)-\frac{1}{2}\sigma^2(i)\right) \quad a.s., \\ &\liminf_{t\to\infty} \frac{1}{t}\int_0^t x(s)ds \geq \frac{1}{\breve{b}}\sum_{i=1}^{n}\pi_i\left(a(i)-\frac{1}{2}\sigma^2(i)\right) \quad a.s. \end{aligned} \tag{5.37}$$

Corollary 5.5. *If that for some* $i \in S$,

$$b(i) > 0, \qquad b(i) > |c(i)|, \qquad a(i) - \frac{1}{2}\sigma^2(i) > 0, \tag{5.38}$$

then the solution with positive initial value to subsystem

$$dx(t) = x(t)[(a(i) - b(i)x(t) + c(i)x(t-\tau))dt + \sigma(i)dw(t)] \tag{5.39}$$

has the property that

$$\frac{a(i)-(1/2)\sigma^2(i)}{b(i)} \leq \liminf_{t\to\infty}\frac{1}{t}\int_0^t x(s)ds \leq \limsup_{t\to\infty}\frac{1}{t}\int_0^t x(s)ds \leq \frac{a(i)-(1/2)\sigma^2(i)}{b(i)-|c(i)|} \quad a.s. \tag{5.40}$$

Remark 5.6. If $\bar{c} = 0$, (1.4) will be written by

$$dx(t) = x(t)[(a(r(t)) - b(r(t))x(t))dt + \sigma(r(t))dw(t)], \tag{5.41}$$

which is investigated in [18]. It should be pointed out that (1.4) is more difficult to handle than (5.41). Fortunately, it overcomes the difficulties caused by delay term with the help of Young's inequality. Meanwhile, we get the similar results for $\tau \geq 0$.

6. Examples

Example 6.1. Consider a 2-dimensional stochastic differential equation with Markovian switching of the form

$$dx(t) = x(t)[(a(r(t)) - b(r(t))x(t) + c(r(t))x(t-\tau))dt + \sigma(r(t))dw(t)] \quad \text{on } t \le 0, \tag{6.1}$$

where $r(t)$ is a right-continuous Markov chain taking values in $S = \{1,2\}$, and $r(t)$ and $w(t)$ are independent. Here

$$\begin{aligned} &a(1) = 2, \quad b(1) = 3, \quad c(1) = 1, \quad \sigma(1) = 1, \\ &a(2) = 1, \quad b(2) = 2, \quad c(2) = \frac{3}{2}, \quad \sigma(2) = 2. \end{aligned} \tag{6.2}$$

It can be computed that

$$\widehat{b} = 2; \quad \check{c} = \frac{3}{2}; \quad a(1) - \frac{1}{2}\sigma^2(1) = \frac{3}{2}; \quad a(2) - \frac{1}{2}\sigma^2(2) = -1. \tag{6.3}$$

By Theorem 2.3, the solution $x(t)$ of (6.1) will remain in R_+ for all $t \ge -\tau$ with probability 1.

Case 1. Let the generator of the Markov chain $r(t)$ be

$$\Gamma = \begin{pmatrix} -1 & 1 \\ 2 & -2 \end{pmatrix}. \tag{6.4}$$

By solving the linear equation $\pi\Gamma = 0$, we obtain the unique stationary (probability) distribution

$$\pi = (\pi_1, \pi_2) = \left(\frac{2}{3}, \frac{1}{3}\right). \tag{6.5}$$

Therefore,

$$\sum_{i=1}^{2} \pi_i \left(a(i) - \frac{1}{2}\sigma^2(i) \right) = \frac{2}{3} > 0. \tag{6.6}$$

By Theorems 3.10 and 5.3, (6.1) is stochastically permanent and its solution $x(t)$ with any positive initial value has the following properties:

$$\frac{1}{3} \le \liminf_{t\to\infty} \frac{1}{t}\int_0^t x(s)ds \le \limsup_{t\to\infty} \frac{1}{t}\int_0^t x(s)ds \le \frac{4}{3} \quad \text{a.s.} \tag{6.7}$$

Case 2. Let the generator of the Markov chain $r(t)$ be

$$\Gamma = \begin{pmatrix} -2 & 2 \\ 1 & -1 \end{pmatrix}. \tag{6.8}$$

By solving the linear equation $\pi\Gamma = 0$, we obtain the unique stationary (probability) distribution

$$\pi = (\pi_1, \pi_2) = \left(\frac{1}{3}, \frac{2}{3}\right). \tag{6.9}$$

So,

$$\sum_{i=1}^{2} \pi_i \left(a(i) - \frac{1}{2}\sigma^2(i) \right) = -\frac{1}{6} < 0. \tag{6.10}$$

Applying Theorems 4.2, (6.1) is extinctive.

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