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*Research Article*

# Solvability of a Class of Integral Inclusions

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This paper presents sufficient conditions for the existence of positive solutions for a class of integral inclusions. Our results are obtained via a new fixed point theorem for multivalued operators developed in the paper, in which some nonnegative function is used to describe the cone expansion and compression instead of the classical norm-type, and lead to new existence principles.

## 1. Introduction

Let $(E, \|\cdot\|)$ be a Banach space. A nonempty convex closed set $P \subset E$ is called a cone of $E$ if the following conditions hold:

$$x \in P, \quad \lambda \geq 0, \quad \text{implies } \lambda x \in P, \qquad x \in P, \quad -x \in P, \quad \text{implies } x = \theta, \tag{1.1}$$

where $\theta$ stands for the zero element of $E$. A cone $P$ is said to be normal if there exists a positive constant $N$, which is called the normal constant of $P$, such that $\theta \leq x \leq y$ $(x, y \in E)$ implies that $\|x\| \leq N\|y\|$. Here, the partially order "$\leq$" in $E$ is introduced as follows: $x \leq y$ if and only if $y - x \in P$ for any $x$, $y \in E$, $x < y$ if and only if $x \leq y$ and $x \neq y$.

Given a cone $P$ of $E$, denote that $P^+ = P \setminus \{\theta\}$. For $u_0 \in P^+$, denote that

$$P(u_0) = \{x \in P : \lambda u_0 \leq x, \ \text{for some } \lambda > 0\}. \tag{1.2}$$

For notational purposes for $\eta > 0$, let

$$\Omega_\eta = \{y \in E : \|y\| < \eta\}, \quad \partial\Omega_\eta = \{y \in E : \|y\| = \eta\},$$
$$\overline{\Omega}_\eta = \{y \in E : \|y\| \le \eta\}, \quad \partial\Omega \text{ denote the boundary of set } \Omega. \tag{1.3}$$

This paper is concerned with the existence of solutions for the following multivalued integral inclusion:

$$x(t) = f(t,x) \int_0^{+\infty} u_x(t,s)\, ds, \tag{1.4}$$

where $f : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ is a single-valued map, $U : H \times \mathbb{R} \multimap 2^{\mathbb{R}}$ is a multivalued map, and $u_x \in S_{U,x}$. Here, $\mathbb{R}_+ = [0, +\infty)$, $H = \mathbb{R}_+ \times \mathbb{R}_+$, and the set of $L^1$-selections $S_{U,x}$ of the multivalued map $U$ is defined by

$$S_{U,x} := \left\{ f_x \in L^1(\mathbb{R}_+, \mathbb{R}) : f_x(t,s) \in U(t,s,x(s)) \text{ a.e. for } \ t \ge 0 \right\}. \tag{1.5}$$

Some problems considered in the vehicular traffic theory, biology, and queuing theory lead to the following nonlinear functional-integral equation:

$$x(t) = f(t, x(t)) \int_0^1 u(t,s,x(s))\, ds, \quad t \in [0,1]. \tag{1.6}$$

(cf. [1]). The Volterra counterpart of the above equation on unbounded interval was studied by [2]. Namely, in [2], the existence of solutions of the following integral equation:

$$x(t) = f(t, x(t)) \int_0^t u(t,s,x(s))\, ds, \quad t \ge 0 \tag{1.7}$$

was proved by using the technique associated with measures of noncompactness, and the functions were assumed continuous and bounded on $\mathbb{R}_+$. The sufficient conditions for the existence of solutions to this equation, under the assumption of $u$ being a multivalued map, was presented by [3] via a fixed-point theorem due to Martelli [4] on ordered Banach spaces, [5] via expansion and compression fixed point theorems for multivalued mapping due to Agarwal and O'Regan [6]. When $f(t,x) = 1$, also, [7] established the existence of solutions to the multivalued problem (1.4) in Fréchet spaces. In this paper, we give existence results of positive solutions for system (1.4).

The fundamental tool used in the proof of our main results is essentially the fixed point theorem (see Theorem 2.3) based on expansion and compression fixed point theorems for multivalued mappings. However, the hypotheses imposed on functions on the right-hand side of (1.4) and methods of the proof in this paper are different from the above-cited works.

Cone compression and expansion fixed point theorems are frequently used tools for studying the existence of positive solutions for boundary value problems of integral and differential equations. For instance, in [8–10], authors considered the existence of positive solutions for singular second-order $m$-point boundary value problem, in [11] Leggett and Williams discussed the nonlinear equation modelling certain infectious diseases. In [12] Zima discussed a three-point boundary value problem for second-order ordinary differential equations. In [13, 14] the authors proved multiplicity of positive radial solutions for an elliptic system on an annulus and so on. The original result of Krasnoselskii fixed point theorem concerning cone compression and expansion was obtained by Krasnoselskii [15]. Afterward, a lot of generalization of this theorem has appeared (see, e.g., [8, 11, 12, 16, 17]). For instance, in [16] Guo and Lashmikantham gave the result of the norm type, and in [17] Anderson and Avery obtained a generalization of the norm type by applying conditions formulated in the terms of two functionals replacing the norm type assumptions. In [8] Zhang and Sun obtained an extension, in which the norm is replayed with some uniformly continuous convex function (see [8], Corollary 2.1). On the other hand, in [11], Leggett and Williams obtained another generalization of Krasnoselskiis original result. In [18] one can find some refinements of [11]. In [12] Zima proved another result via replacing Leggett and Williams type-ordering conditions by the conditions of the norm type (see [12], Theorem 2.1). In addition, Agarwal and O'Regan [6] extended Krasnoselskii's fixed point theorem of norm type to multivalued operator problems and obtained fixed point theorems for $k$-set contractive multivalued operators (see [6], Theorems 2.4 and 2.8). In general, while the expansion may be easily verified for a large class of nonlinear integral operators, the compression is a rather stringent condition and is usually not easily verified. By improving the compression of the cone theorem via replacing the cone $P$ with the set $P(u_0)$, the result of Leggett and Williams [11] has the advantage which consists in its usually being easier to apply even when the compression of the cone theorem is also applicable to a large class of operators. In this paper we will extend Leggett and Williams fixed point theorem to multivalued operator problems and obtain a fixed point theorem for $k$-set-contractive multivalued operators, in which the norm of [11] will be replayed with some nonnegative function. Our result is not only the fundamental tool to prove our main theorem, but also a generalization of corresponding results in [6, 8, 11, 12].

## 2. Preliminaries

We begin this section with gathering together some definitions and known facts. For two subsets $C$, $D$ of $E$, we write $C \le D$ (or $D \ge C$) if

$$\forall p \in D, \qquad \exists q \in C \ \text{ such that } q \le p. \tag{2.1}$$

A multivalued operator $A$ is called upper semicontinuous (u.s.c.) on $E$ if for each $x \in E$ the set $A(x)$ is a nonempty closed subset of $E$, and if for each open set $B$ of $E$ containing $A(x)$, there exists an open neighborhood $V$ of $x$ such that $A(V) \subseteq B$.

$A$ is called a $k$-set contraction if $\gamma(A(D)) \le k\gamma(D)$ for all bounded sets $D$ of $E$ and $A(D)$ is bounded, where $\gamma$ denotes the Kuratowskii measure of noncompactness.

Throughout this paper, we denote by $CK(C)$ the family of nonempty, compact, and convex subsets of set $C$ and denote by $K_{\partial U}(\overline{U}, C)$ the set of all u.s.c., $k$-set-contractive maps $A : \overline{U} \to CK(C)$ with $x \notin A(x)$ for $x \in \partial U$.

The nonzero fixed point theorems of multivalued operators (see [6], Theorems 2.3 and 2.7) will play an important role in this section. It is not hard to extend these results on open sets, so we have the following.

**Lemma 2.1.** *Let $E$ be an ordered Banach space and $P$ a cone in $E$, and let $\Omega_1$ and $\Omega_2$ be bounded open sets in $E$ such that $\theta \in \Omega_1$ and $\overline{\Omega}_1 \subset \Omega_2$. Assume that $A : \overline{\Omega}_2 \to CK(P)$ is a u.s.c., $k$-set contractive (here $0 \leq k < 1$) map and assume one of the following conditions hold:*

$$x \notin \lambda Ax, \quad \forall \lambda \in [0,1), \qquad x \in \partial\Omega_2 \cap P, \tag{2.2}$$

$$\text{there exists a } v \in P^+ \text{ with } x \notin Ax + \delta v \text{ for } x \in \partial\Omega_1 \cap P, \quad \delta \geq 0. \tag{2.3}$$

*Or*

$$x \notin \lambda Ax, \quad \forall\ \lambda \in [0,1), \qquad x \in \partial\Omega_1 \cap P, \tag{2.4}$$

$$\text{there exists a } v \in P^+ \text{ with } x \notin Ax + \delta v \text{ for } x \in \partial\Omega_2 \cap P, \quad \delta \geq 0. \tag{2.5}$$

*Then $A$ has at least one fixed point $y$ with $y \in (\overline{\Omega}_2 \setminus \Omega_1) \cap P$.*

**Lemma 2.2** (see [19]). *Let $E$ be a Banach space, $D$ a closed convex subset of $E$, and $U$ an open subset of $D$ with $\theta \in U$. Suppose that $A : \overline{U} \multimap CK(D)$ is u.s.c, $k$-set-contractive (here $0 \leq k < 1$). Then either*

(h1) *there exists $x \in \overline{U}$ with $x \in Ax$, or*

(h2) *there exists $u \in \partial U$ and $\lambda \in (0,1)$ with $u \in \lambda Ax$.*

The proof of the following theorem is not complicated but it is essential to prove our main results.

**Theorem 2.3.** *Assume that $\Omega_1$ and $\Omega_2$ are bounded open sets in $E$ such that $\theta \in \Omega_1$ and $\overline{\Omega}_1 \subset \Omega_2$. Let $A : P \cap \overline{\Omega}_2 \to CK(P)$ be a u.s.c, $k$-set-contractive (here $0 \leq k < 1$) operator, $u_0 \in P^+$, and $\rho : P \to [0,+\infty)$ a nondecreasing function with $\rho(\theta) = 0$ and $\rho(x) > 0$ for $x \in P^+$. Moreover,*

(h) *$\rho(\lambda x) \leq \lambda\rho(x)$, for all $x \in P$ and $\lambda \in [0,1]$.*

*If one of the following two conditions holds:*

(H1) (i) *$\rho(y) > \rho(x)$ for all $y \in A(x)$ and $x \in P(u_0) \cap \partial\Omega_1$,*

(ii) *$\rho(y) \leq \rho(x)$ for all $y \in A(x)$ and $x \in P \cap \partial\Omega_2$;*

(H2) (i) *$\rho(y) \leq \rho(x)$ for all $y \in A(x)$ and $x \in P \cap \partial\Omega_1$,*

(ii) *$\rho(y) > \rho(x)$ for all $y \in A(x)$ and $x \in P(u_0) \cap \partial\Omega_2$,*

*then $A$ has a positive fixed point in the set $P \cap (\overline{\Omega}_2 \setminus \Omega_1)$.*

*Proof.* We seek to apply Lemma 2.1. It is sufficient to check that $A$ satisfies the conditions (2.2) and (2.3) in $\Omega_1$ and in $\Omega_2$, respectively, provided that the condition (H1) holds. First, (H1)(ii) with $x \in P \cap \partial\Omega_2$ implies that (2.2) is true. To see this suppose that there exist $x \in P \cap \partial\Omega_2$ and $\lambda \in [0,1)$ with $x \in \lambda A(x)$. Then there exists $y \in A(x)$ with $x = \lambda y$. Therefore, by the condition (h), we have

$$0 < \rho(x) = \rho(\lambda y) \leq \lambda\rho(y) < \rho(y) \leq \rho(x), \tag{2.6}$$

a contradiction. Next, we will prove that for any $x \in P \cap \partial\Omega_1$ and any $\delta \geq 0$,

$$x \notin A(x) + \delta u_0. \tag{2.7}$$

Suppose, on the contrary, that there exist $x_0 \in P \cap \partial\Omega_1$ and $t \geq 0$ such that $x_0 \in A(x_0) + tu_0$, that is, there exists $y_0 \in A(x_0)$ such that $x_0 = y_0 + tu_0$. Hence,

$$x_0 - tu_0 = y_0 \in A(x_0). \tag{2.8}$$

Clearly, $t \neq 0$ (otherwise, this proof is completed). Noting that $A(x_0) \subset P$, we conclude that

$$tu_0 \leq tu_0 + y \tag{2.9}$$

for all $y \in A(x_0)$. Then, combining (2.8), we get that $x_0 \in P(u_0)$. Since $x_0 - tu_0 = y_0$, we have

$$\theta \leq y_0 \leq x_0, \quad x_0 \neq y_0. \tag{2.10}$$

In virtue of the monotonicity of $\rho$, we have

$$\rho(y_0) \leq \rho(x_0). \tag{2.11}$$

Since $x_0 \in P(u_0) \cap \partial\Omega_1$, (2.11) contradicts (H1)(i). Hence, (2.7) is true. This implies that (2.3) is true. The result of Theorem 2.3 now follows from Lemma 2.1.

Similarly, we can prove that the result of Theorem 2.3 follows if (H2) holds. This proof is completed. □

**Corollary 2.4.** *Assume that $\Omega_1$, $\Omega_2$ and the multivalued mapping $A$ are given as in Theorem 2.3, $u_0 \in P^+$, and a function $\rho : P \to [0, +\infty)$ satisfies the condition* (h), *$\rho(\theta) = 0$ and $\rho(x) > 0$ for $x \in P^+$. Moreover, there exists a constant $N > 0$ such that*

(h′) *$\theta \leq x \leq y$ with $x, y \in E$ implies that $\rho(x) \leq N\rho(y)$.*

*If either*

(H′1) (i) *$\rho(y) > N\rho(x)$ for all $y \in A(x)$ and $x \in P(u_0) \cap \partial\Omega_1$,*

(ii) *$\rho(y) \leq \rho(x)$ for all $y \in A(x)$ and $x \in P \cap \partial\Omega_2$, or*

(H′2) (i) *$\rho(y) \leq \rho(x)$ for all $y \in A(x)$ and $x \in P \cap \partial\Omega_1$,*

(ii) *$\rho(y) > N\rho(x)$ for all $y \in A(x)$ and $x \in P(u_0) \cap \partial\Omega_2$*

*is satisfied, then $A$ has a positive fixed point in $P \cap (\overline{\Omega_2} \setminus \Omega_1)$.*

*Proof.* We seek to apply Lemma 2.1. The hypothesis (2.2) is true, the proof of which is the same as Theorem 2.3. Next, we will prove that (2.7) is satisfied for any $x \in P \cap \partial\Omega_1$ and any $\delta \geq 0$. Suppose, on the contrary, that there exist $x_0 \in P \cap \partial\Omega_1$ and $t \geq 0$ such that $x_0 \in A(x_0) + tu_0$, that is, (2.8) holds. Similarly, we have $x_0 \in P(u_0)$ and $\theta \le y_0 \le x_0$ with $x_0 - tu_0 = y_0 \in A(x_0)$. In virtue of the condition (h′), we have

$$\rho(y_0) \le N\rho(x_0). \tag{2.12}$$

Since $x_0 \in P \cap \partial\Omega_1$, (2.12) contradicts (H′1)(i). Hence, (2.7) is true. This shows that the conditions of Lemma 2.1 are satisfied.

Similarly, we can prove that the result of Corollary 2.4 follows if (H′2 ) holds. This proof is completed. □

*Remark 2.5.* If the function $\rho$ is convex on $P$, namely, $\rho(tx + (1-t)y) \le t\rho(x) + (1-t)\rho(y)$ for all $x, y \in P$ and $t \in [0,1]$, then the condition (h) holds provided that $\rho(\theta) = 0$. From this point of view, we extend the corresponding result of [8]. Let $\rho(x) = \|x\|$. Then $\rho(x)$ is a convex function with $\rho(\theta) = 0$, $\rho(x) > 0$ for $x \neq 0$, and the condition (h) is satisfied. Obviously, $\rho$ is nondecreasing if $\|\cdot\|$ be increasing with respect to $P$. This shows that Theorem 2.3 contains the corresponding result of [6]. In addition, the condition (h′) holds if $P$ is a normal cone. Hence, Corollary 2.4 extends and improves the corresponding result of [11].

*Remark 2.6.* Let $E = C[0,1]$ with the norm $\|\varphi\| = \max_{0\le t\le 1}|\varphi(t)|$ and the cone

$$P_0 = \{\varphi \in C[0,1] : \varphi(t) \ge 0,\ \forall t \in [0,1]\}. \tag{2.13}$$

Define $\varphi_1 \le \varphi_2$ if and only if $\varphi_1(t) \le \varphi_2(t)$ for every $t \in [0,1]$. Then the function $\rho : P_0 \to [0,+\infty)$ defined by

$$\rho(\varphi) = \left(\int_0^1 \varphi^p(t)\,dt\right)^{1/p}, \quad (p \ge 1) \tag{2.14}$$

is nondecreasing convex and $\rho(\theta) = 0$, $\rho(\varphi) > 0$ for $\varphi \neq 0$, and $\rho$ yields the condition (h).

In what follows, we combine Lemma 2.2 and Theorem 2.3 to establish existence of multiple fixed points.

**Theorem 2.7.** *Assume that the conditions of Theorem 2.3 hold and*

$$x \notin A(x), \quad \forall x \in \partial\Omega_1 \cap P. \tag{2.15}$$

*Then $A$ has at least two fixed points $x_1$ and $x_2$ with $x_1 \in \Omega_1 \cap P$ and $x_2 \in P \cap (\overline{\Omega}_2 \setminus \Omega_1)$.*

*Proof.* Theorem 2.3 guarantees that $A$ has at least one fixed point $x_2$ with $x_2 \in P \cap (\overline{\Omega}_2 \setminus \Omega_1)$. In addition, we obtain in the proof of Theorem 2.3 that $x \notin \lambda A(x)$ for all $\lambda \in [0,1)$ and $x \in \partial\Omega_1 \cap P$. Hence, we combine (2.15) and Lemma 2.2 to conclude that $A$ has a fixed point $x_1 \in \Omega_1 \cap P$. This completes the proof of Theorem 2.7. □

For constants $L, r, R$ with $0 < r < L < R$, let us suppose that

(H3) $x \notin A(x)$ for all $x \in \partial\Omega_L \cap P$;

(H4) (i) $\rho(x) < \rho(y)$ for all $y \in A(x)$ and $x \in P(u_0) \cap \partial\Omega_r$,

(ii) $\rho(y) \leq \rho(x)$ for all $y \in A(x)$ and $x \in P \cap \partial\Omega_L$,

(iii) $\rho(x) < \rho(y)$ for all $y \in A(x)$ and $x \in P(u_0) \cap \partial\Omega_R$;

(H5) (i) $\rho(y) \leq \rho(x)$ for all $y \in A(x)$ and $x \in P \cap \partial\Omega_r$,

(ii) $\rho(x) < \rho(y)$ for all $y \in A(x)$ and $x \in P(u_0) \cap \partial\Omega_L$,

(iii) $\rho(y) \leq \rho(x)$ for all $y \in A(x)$ and $x \in P \cap \partial\Omega_R$.

**Theorem 2.8.** *Let $A : \overline{\Omega}_R \cap P \to CK(P)$ be a u.s.c., $k$-set-contractive (here $0 \leq k < 1$) operator and the function $\rho$ be given as in Theorem 2.3. If either the conditions* (H3) *and* (H4) *or the conditions* (H3) *and* (H5) *hold, then $A$ has at least two positive fixed points $x_1$ and $x_2$ with $x_1 \in P \cap (\Omega_L \setminus \Omega_r)$ and $x_2 \in P \cap (\overline{\Omega}_R \setminus \overline{\Omega}_L)$.*

*Proof.* Theorem 2.3 implies that $A$ has a fixed point $x_1 \in P \cap (\overline{\Omega}_L \setminus \Omega_r)$. (H3) shows that $x_1 \notin \partial\Omega_L$. Hence, $x_1 \in P \cap (\overline{\Omega}_L \setminus \Omega_r)$. Again, Theorem 2.3 guarantees the existence of $x_2$. This proof is completed. □

## 3. Main Results

In this section, we shall discuss the existence of solutions of integral inclusion (1.4) by using fixed point theorems involved in Section 2. Let us start by defining that a function $x \in C(\mathbb{R}_+, \mathbb{R})$ is said to be a solution of (1.4) if it satisfies (1.4).

By $BC := BC(\mathbb{R}_+, \mathbb{R})$, we mean the Banach algebra consisting of all functions defined, bounded, and continuous on $\mathbb{R}_+$ with the norm

$$\|x\| = \sup\{|x(t)| : t \geq 0\}. \tag{3.1}$$

For any $x, y \in BC$, define that $x \leq y$ if and only if $x(t) \leq y(t)$ for each $t \geq 0$, $x < y$, if and only if $x \leq y$ and there exists some $t \geq 0$ such that $x(t) \neq y(t)$.

In following Theorem 3.1, we need impose the following hypotheses on the single valued map $f$ and the multivalued map $U$.

(S1) $f : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ is a continuous function.

(S2) There exists a bounded continuous function $g : \mathbb{R}_+ \to \mathbb{R}_+$, such that

$$\left|f(t, x)\right| \leq g(t), \qquad \text{for any } t \geq 0,\ x \geq 0. \tag{3.2}$$

(S3) There exist positive constants $\mathcal{C}$, $\delta$, $\xi$ with $\mathcal{C} > 0$ and $0 < \delta < \xi < +\infty$ such that $|f(t, x)| \geq \mathcal{C}$ for $t \in [\delta, \xi]$, $x \geq 0$.

(S4) $U : H \times \mathbb{R}_+ \to CK(\mathbb{R}_+)$ is $L^1$-Carathéodory, that is,

$(t, s) \to U(t, s, x)$ is measurable for every $x \in \mathbb{R}_+$;

$x \to U(t, s, x)$ is u.s.c. for a.e. $(t, s) \in H$.

In addition, the set $S_{U,x}$ is nonempty for each fixed $x \in BC$.

(S5) There exist a bounded, continuous, and nondecreasing function $\beta : \mathbb{R}_+ \to \mathbb{R}_+$, a function $\alpha \in L^1(\mathbb{R}_+, \mathbb{R}_+)$ with $\int_\delta^\xi \alpha(t)dt > 0$, and a continuous function $\gamma : [\delta, \xi] \to (0, +\infty)$ such that

$$\begin{gathered} |u_x(t,s)| \le \alpha(s)\beta(x(s)), \quad \text{for a.e. } (t,s) \in H,\ x \ge 0,\ u_x \in S_{U,x}; \\ |u_x(t,s)| \ge \gamma(t)\alpha(s)\beta(x(s)), \quad \text{for a.e. } (t,s) \in [\delta,\xi] \times [0,+\infty),\ x \ge 0,\ u_x \in S_{U,x}; \\ \lim_{\varepsilon \to 0^+} \frac{\beta(\varepsilon)}{\varepsilon} = +\infty. \end{gathered} \tag{3.3}$$

(S6) There exists a positive number $R > 0$ such that $\beta(R) > 0$ and

$$\frac{R}{\beta(R) \int_\delta^\xi g(s)ds \int_0^{+\infty} \alpha(\tau)d\tau} \ge \frac{M}{\mathcal{C}\mathcal{M}}, \tag{3.4}$$

where $M = \sup_{t \ge 0} g(t)$, $\mathcal{M} = \min_{\delta \le t \le \xi} \gamma(t)$.

**Theorem 3.1.** *If the conditions (S1)–(S6) hold, then* (1.4) *has at least one (positive) solution* $x \in BC(\mathbb{R}_+, \mathbb{R})$ *with* $x \ge 0$ *on* $\mathbb{R}_+$ *and with* $r < \|x\| \le R$ *for given* $0 < r < R$.

*Proof.* Let us define the multivalued map $A$ on the space $BC$ by the following:

$$(A(x))(t) = \{f(t, x(t))\} \int_0^{+\infty} u_x(t,s)ds, u_x \in S_{U,x}, t \ge 0. \tag{3.5}$$

We will show that $A$ has a fixed point recurring to Theorem 2.3. Define the function $\rho : BC(\mathbb{R}_+, \mathbb{R}) \to [0, +\infty)$ by

$$\rho(x) = \max_{\delta \le t \le \xi} |x(t)| \tag{3.6}$$

and the set $X_+$ by

$$X_+ = \left\{ x \in BC : x(t) \ge 0, \text{ for } t \ge 0, x(t) \ge \frac{\mathcal{C}\mathcal{M}}{M}\|x\|, \text{ for } t \in [\delta, \xi] \right\}. \tag{3.7}$$

It is easy to see that $X_+$ is a cone of $BC$ and $\rho : X_+ \to [0, +\infty)$ given in (3.6) is a nondecreasing convex function with $\rho(\theta) = 0$ and $\rho(\varphi) > 0$ for $\varphi \ne \theta$.

First we point out that $A(x) \in CK(X_+)$ for each fixed $x \in \overline{\Omega}_p \cap X_+$ with $p > 0$. In fact, for any $y \in A(x)$, there exists $u_x \in S_{U,x}$ such that $y(t) = f(t,x)\int_0^\infty u_x(t,s)ds$ for $t \geq 0$. (S1) and (S4) imply that $y(t) \geq 0$ for $t \geq 0$ and $\alpha\beta(x) \in L^1(\mathbb{R}_+, \mathbb{R}_+)$, where $\beta(x)$ is defined by $(\beta(x))(t) = \beta(x(t))$. Applying our assumptions we have the following estimate:

$$\begin{aligned} |y(t)| &\leq |f(t,x)| \int_0^{+\infty} |u_x(t,s)| ds \leq g(t) \int_0^{+\infty} \alpha(s)\beta(x(s))\, ds \\ &\leq \sup_{t\geq 0} g(t) \int_0^{\infty} \alpha(s)\beta(x(s))\, ds, \end{aligned} \tag{3.8}$$

and so

$$\|y\| \leq M \int_0^{+\infty} \alpha(s)\beta(x(s))\, ds. \tag{3.9}$$

In addition (S2) and (S5), together with (3.9), guarantee that

$$\begin{aligned} y(t) &= f(t,s) \int_0^{+\infty} u_x(t,s)\, ds \geq C \int_0^{+\infty} \gamma(t)\alpha(s)\beta(x(s))\, ds \\ &\geq C \min_{\delta\leq t\leq \xi} \gamma(t) \int_0^{+\infty} \alpha(s)\beta(x(s))\, ds \geq \frac{C\mathcal{M}}{M}\|y\|. \end{aligned} \tag{3.10}$$

This implies that, by the arbitrariness of $y$, $A(x)$ is bounded and $A(x) \subset X_+$ for each $x \in \overline{\Omega}_p \cap X_+$. Similar to [20] we can infer that $A(x)$ is convex for each $x \in \overline{\Omega}_p \cap X_+$. In the light of our assumptions and the Lebesgue dominated convergence theorem, we can see that $A(x)$ is compact for each $x \in \overline{\Omega}_p \cap X_+$. Hence, $A$ maps $\overline{\Omega}_p \cap X_+$ into $CK(X_+)$.

Next, we prove that $A$ has closed graph. Take $x_k \to x^*$, $h_k \in A(x_k)$ and $h_k \to h^*$ as $k \to \infty$. We shall prove that $h^* \in A(x^*)$. $h_k \in A(x_k)$ means that there exists $u_k \in S_{U,x_k}$ such that for each $t \geq 0$,

$$h_k(t) = f(t,x_k) \int_0^{+\infty} u_k(t,s)\, ds, \quad k = 1,2,\ldots. \tag{3.11}$$

Let $h^*(t) = f(t,x^*)h(t)$. From the continuity of $f$, it follows that $f(t,x_k) \to f(t,x^*)$. From (S4) it follows that $u_k \to u_{x^*} \in S_{U,x^*}$ as $k \to \infty$. From (S5) and the Lebesgue dominated convergence theorem it follows that $\int_0^{+\infty} u_k(t,s)ds \to \int_0^{+\infty} u_{x^*}(t,s)ds$. It is easy to see that $\int_0^{+\infty} u_{x^*}(t,s)ds = h(t)$, that is, for each $t \geq 0$,

$$h^*(t) = f(t,x^*) \int_0^{+\infty} u_{x^*}(t,s)\, ds. \tag{3.12}$$

This implies that $h^* \in A(x^*)$. We want to point out that u.s.c. is equivalent to the condition of being a closed graph multivalued map when the map has nonempty compact values; that is, we have shown that $A$ is u.s.c. It is clear that $A$ is a $k$-set-contractive multivalued map with $k = 0$.

It remains to prove (in virtue of Theorem 2.3) that the condition (H1) holds to conclude that $A$ has a fixed point in $X_+$, that is, that (1.4) has a positive solution. Given $x \in \partial\Omega_R \cap X_+$ with $R$ satisfying the condition (S6), for any $y \in A(x)$, there exists $u_x \in S_{U,x}$ such that $y(t) = f(t, x(t)) \int_0^\infty u_x(t,s)\,ds$, $t \geq 0$. Hence,

$$\begin{aligned} \rho(y) &= \max_{\delta\leq t\leq\xi} |y(t)| = \max_{\delta\leq t\leq\xi} \left| f(t,x) \int_0^{+\infty} u_x(t,s)\,ds \right| \\ &\leq \max_{\delta\leq t\leq\xi} g(t) \int_0^{+\infty} \alpha(s)\beta(x(s))ds \leq \beta(R) \max_{\delta\leq t\leq\xi} g(t) \int_0^{+\infty} \alpha(s)\,ds. \end{aligned} \tag{3.13}$$

In addition (S6) shows that

$$\beta(x(\tau)) \leq \beta(R) \leq \frac{\mathcal{C}\mathcal{M}R}{M\max_{\delta\leq t\leq\xi} g(t) \int_0^{+\infty} \alpha(s)\,ds}, \tag{3.14}$$

and this together with (3.13) gives the following:

$$\rho(y) \leq \frac{\mathcal{C}\mathcal{M}R}{M\max_{\delta\leq t\leq\xi} g(t) \int_0^{+\infty} \alpha(s)\,ds} \max_{\delta\leq t\leq\xi} g(t) \int_0^{+\infty} \alpha(s)\,ds = \frac{\mathcal{C}\mathcal{M}R}{M} \leq \rho(x). \tag{3.15}$$

Thus, $A$ satisfies condition (H1)(ii).

Take $\mathcal{K} := M/\mathcal{C}^2\mathcal{M}\min_{\delta\leq t\leq\xi} \gamma(t) \int_\delta^\xi \alpha(s)\,ds$. (S5) shows that there exists a positive number $r < R$ small enough such that

$$\beta(\varepsilon) > \mathcal{K}\varepsilon, \quad 0 < \varepsilon \leq r. \tag{3.16}$$

Let $u_0(t) \equiv 1$. To prove that (H1)(i) is true, let $y \in Ax$ with $y \neq x$ and $x \in \partial\Omega_r \cap X_+(1)$. Then there exists $u_x \in S_{U,x}$ with $y(t) = f(t,x) \int_0^{+\infty} u_x(t,s)\,ds$. In virtue of the definition of $X_+(1)$, there exists $0 < \lambda < 1$ such that

$$\lambda \leq x(t) \leq r, \quad t \geq 0. \tag{3.17}$$

Note that there exists $\eta \in [\delta, \xi]$ such that $\rho(y) = y(\eta)$. Now our assumptions imply that

$$\begin{aligned} \rho(y) &= y(\eta) = f(\eta, x) \int_0^{+\infty} u_x(\eta, s)\,ds \\ &\geq \mathcal{C} \int_0^{+\infty} \gamma(\eta)\alpha(s)\beta(x(s))ds \geq \mathcal{C} \min_{\delta\leq t\leq\xi} \gamma(t) \int_\delta^\xi \alpha(s)\beta(x(s))\,ds \\ &> \frac{\mathcal{C}M\min_{\delta\leq t\leq\xi}\gamma(t)}{\mathcal{C}^2\mathcal{M}\min_{\delta\leq t\leq\xi}\gamma(t) \int_\delta^\xi \alpha(s)ds} \int_\delta^\xi \alpha(s)x(s)\,ds \geq \|x\| \geq \rho(x). \end{aligned} \tag{3.18}$$

So $\rho(y) > \rho(x)$ for all $y \in A(x)$. This shows that (H1)(i) is satisfied. Conclusively, Theorem 2.3 guarantees that $A$ has a fixed point $y$ with $r \leq \|y\| \leq R$. This proof is completed. □

**Theorem 3.2.** *Suppose that conditions (S1)–(S6) hold. Then* (1.4) *has at least two positive solutions if the following conditions are satisfied:*

$$\lim_{\eta \to +\infty} \frac{\beta(\eta)}{\eta} = +\infty. \tag{3.19}$$

*Example 3.3.* Let $\delta = 1/4$, $\xi = 3/4$, $\gamma(t) = 1/2$ for $t \in [\delta, \xi]$,

$$\alpha(s) = \begin{cases} 1, & s \in [0,1], \\ s, & s = 2, 3, \ldots, \\ s^{-2}, & \text{others}, \end{cases} \tag{3.20}$$

$\beta(x) = \sqrt{x}$ and $U(t,s,x) = [(1/2)\alpha(s)\sqrt{x}, \alpha(s)\sqrt{x}]$. Let $f(t,x) = e^{-t}(\sin x + 1)$, $g(t) = 2e^{-t}$, $\mathcal{L} = (1/2e)$, $R = 32e^{3/4}$. It is clear that conditions (S1)–(S6) are satisfied. Hence, Theorem 3.1 guarantees the problem

$$x(t) = e^{-t}(\sin x + 1) \int_0^\infty u_x(t,s)\, ds \tag{3.21}$$

with $u_x \in [(1/2)\alpha(s)\sqrt{x}, \alpha(s)\sqrt{x}]$ having at least a positive solution $x$ with $\|x\| \le 32e^{3/4}$.

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## References

[1] K. Deimling, *Nonlinear Functional Analysis*, Springer, Berlin, Germany, 1985.

[2] J. Banaś and B. Rzepka, “On existence and asymptotic stability of solutions of a nonlinear integral equation,” *Journal of Mathematical Analysis and Applications*, vol. 284, no. 1, pp. 165–173, 2003.

[3] S. H. Hong and L. Wang, “Existence of solutions for integral inclusions,” *Journal of Mathematical Analysis and Applications*, vol. 317, no. 2, pp. 429–441, 2006.

[4] M. Martelli, “A Rothe’s type theorem for non-compact acyclic-valued maps,” vol. 11, no. 3, supplement 3, pp. 70–76, 1975.

[5] S. H. Hong, “Multiple positive solutions for a class of integral inclusions,” *Journal of Computational and Applied Mathematics*, vol. 214, no. 1, pp. 19–29, 2008.

[6] R. P. Agarwal and D. O’Regan, “A note on the existence of multiple fixed points for multivalued maps with applications,” *Journal of Differential Equations*, vol. 160, no. 2, pp. 389–403, 2000.

[7] R. P. Agarwal and D. O’Regan, “Cone compression and expansion fixed point theorems in Fréchet spaces with applications,” *Journal of Differential Equations*, vol. 171, no. 2, pp. 412–429, 2001.

[8] G. Zhang and J. Sun, “A generalization of the cone expansion and compression fixed point theorem and applications,” *Nonlinear Analysis*, vol. 67, no. 2, pp. 579–586, 2007.

[9] D. Jiang, J. Chu, and M. Zhang, “Multiplicity of positive periodic solutions to superlinear repulsive singular equations,” *Journal of Differential Equations*, vol. 211, no. 2, pp. 282–302, 2005.

[10] J. R. Graef and B. Yang, “Positive solutions to a multi-point higher order boundary value problem,” *Journal of Mathematical Analysis and Applications*, vol. 316, no. 2, pp. 409–421, 2006.

[11] R. W. Leggett and L. R. Williams, "A fixed point theorem with application to an infectious disease model," *Journal of Mathematical Analysis and Applications*, vol. 76, no. 1, pp. 91–97, 1980.

[12] M. Zima, "Fixed point theorem of Leggett-Williams type and its application," *Journal of Mathematical Analysis and Applications*, vol. 299, no. 1, pp. 254–260, 2004.

[13] D. R. Dunninger and H. Wang, "Existence and multiplicity of positive solutions for elliptic systems," *Nonlinear Analysis*, vol. 29, no. 9, pp. 1051–1060, 1997.

[14] S. H. Hong and J. Chen, "Multiplicity of positive radial solutions for an elliptic inclusion system on an annulus," *Journal of Computational and Applied Mathematics*, vol. 221, no. 1, pp. 66–75, 2008.

[15] M. A. Krasnoselskii, *Positive Solutions of Operator Equations*, P. Noordhoff, Groningen, The Netherlands, 1964.

[16] D. J. Guo and V. Lakshmikantham, *Nonlinear Problems in Abstract Cones*, vol. 5, Academic Press, Boston, Mass, USA, 1988.

[17] D. R. Anderson and R. I. Avery, "Fixed point theorem of cone expansion and compression of functional type," *Journal of Difference Equations and Applications*, vol. 8, no. 11, pp. 1073–1083, 2002.

[18] P. J. Torres, "Existence of one-signed periodic solutions of some second-order differential equations via a Krasnoselskii fixed point theorem," *Journal of Differential Equations*, vol. 190, no. 2, pp. 643–662, 2003.

[19] D. O'Regan, "Fixed points for set-valued mappings in locally convex linear topological spaces," *Mathematical and Computer Modelling*, vol. 28, no. 1, pp. 45–55, 1998.

[20] R. P. Agarwal and D. O'Regan, "Periodic solutions to nonlinear integral equations on the infinite interval modelling infectious disease," *Nonlinear Analysis*, vol. 40, pp. 21–35, 2000.