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Research Article

Superadditivity, Monotonicity, and Exponential Convexity of the Petrović-Type Functionals

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We consider functionals derived from Petrović-type inequalities and establish their superadditivity, subadditivity, and monotonicity properties on the corresponding real n-tuples. By virtue of established results we also define some related functionals and investigate their properties regarding exponential convexity. Finally, the general results are then applied to some particular settings.

1. Introduction

In this paper we prove some interesting properties of the functionals derived by virtue of the Petrović and related inequalities (see, [1] pages 152–159). For the sake of simplicity these inequalities will be referred to as the Petrović-type inequalities, while the corresponding functionals will be referred to as the Petrović-type functionals.

Therefore, throughout this introduction, we present the above-mentioned Petrović-type inequalities that will be the base in our research and also define the corresponding functionals that will be the subject of our study. We start with the following inequality.

Theorem 1.1. *Let $I = (0, a] \subseteq \mathbb{R}_+$ be an interval, $(x_1, \ldots, x_n) \in I^n$, and let $(p_1, \ldots, p_n) \in \mathbb{R}_+^n$ be a nonnegative real n-tuple such that*

$$\sum_{i=1}^{n} p_i x_i \in I, \qquad \sum_{i=1}^{n} p_i x_i \geq x_j \quad \text{for } j = 1, \ldots, n. \tag{1.1}$$

If $f : I \to \mathbb{R}$ is such that the function $f(x)/x$ is decreasing on I, then

$$f\left(\sum_{i=1}^{n} p_i x_i\right) \le \sum_{i=1}^{n} p_i f(x_i). \tag{1.2}$$

In addition, if $f(x)/x$ is increasing on I, then the sign of inequality in (1.2) *is reversed.*

Remark 1.2. It should be noticed here that, if $f(x)/x$ is strictly increasing function on I, then the equality in (1.2) is valid if and only if we have equalities in (1.1) instead of inequalities, that is, if $x_1 = \cdots = x_n$ and $\sum_{i=1}^{n} p_i = 1$.

Motivated by the above theorem, we define the Petrović-type functional $\mathcal{P}_1$, as a difference between the right-hand side and the left-hand side of inequality (1.2), that is,

$$\mathcal{P}_1(\mathbf{x}, \mathbf{p}; f) = \sum_{i=1}^{n} p_i f(x_i) - f\left(\sum_{i=1}^{n} p_i x_i\right), \tag{1.3}$$

where $\mathbf{x} = (x_1, \ldots, x_n) \in I^n$, $I = (0, a]$, $\mathbf{p} = (p_1, \ldots, p_n) \in \mathbb{R}_+^n$, and f is defined on the interval I.

Remark 1.3. If (1.1) holds and $f(x)/x$ is decreasing on I, then

$$\mathcal{P}_1(\mathbf{x}, \mathbf{p}; f) \ge 0. \tag{1.4}$$

On the other hand, if (1.1) is valid and $f(x)/x$ is increasing on I, then

$$\mathcal{P}_1(\mathbf{x}, \mathbf{p}; f) \le 0. \tag{1.5}$$

The above functional (1.3) will also be considered under slightly altered assumptions on real n-tuples $\mathbf{x}$ and $\mathbf{p}$. For that sake, the following result from [1] will be used in due course.

Theorem 1.4. *Suppose $I = (0, a] \subseteq \mathbb{R}_+$, $(x_1, \ldots, x_n) \in I^n$ is a real n-tuple such that $0 < x_1 \le \cdots \le x_n$, and let $(p_1, \ldots, p_n) \in \mathbb{R}_+^n$. Further, let $f : I \to \mathbb{R}$ be such that $f(x)/x$ is increasing on I.*

(i) *If there exists m $(\le n)$ such that*

$$\overline{P}_1 \ge \overline{P}_2 \ge \cdots \ge \overline{P}_m \ge 1, \qquad \overline{P}_{m+1} = \cdots = \overline{P}_n = 0, \tag{1.6}$$

where $P_k = \sum_{i=1}^{k} p_i$, $\overline{P}_k = P_n - P_{k-1}$, $k = 2, \ldots, n$, and $\overline{P}_1 = P_n$, then (1.2) *holds.*

(ii) *If there exists m $(\le n)$ such that*

$$0 \le \overline{P}_1 \le \overline{P}_2 \le \cdots \le \overline{P}_m \le 1, \qquad \overline{P}_{m+1} = \cdots = \overline{P}_n = 0, \tag{1.7}$$

then the reverse inequality in (1.2) *holds.*

Remark 1.5. If $f(x)/x$ is increasing on I and (1.6) holds, then the Petrović-type functional $\mathcal{P}_1$ is nonnegative, that is, inequality (1.4) is valid. Conversely, if $f(x)/x$ is increasing on I and conditions as in (1.7) are fulfilled, then relation (1.5) holds.

In order to define another Petrović-type functional, we cite the following Petrović-type inequality involving a convex function.

Theorem 1.6. *Let $I = [0, a] \subseteq \mathbb{R}_+$, $(x_1, \ldots, x_n) \in I^n$ and let $(p_1, \ldots, p_n) \in \mathbb{R}_+^n$ fulfill conditions as in* (1.1). *If $f : I \to \mathbb{R}$ is a convex function, then*

$$f\left(\sum_{i=1}^{n} p_i x_i\right) \geq \sum_{i=1}^{n} p_i f(x_i) + \left(1 - \sum_{i=1}^{n} p_i\right) f(0). \tag{1.8}$$

Remark 1.7. If f is a concave function then $-f$ is convex, hence replacing f by $-f$ in Theorem 1.6, we obtain inequality

$$f\left(\sum_{i=1}^{n} p_i x_i\right) \leq \sum_{i=1}^{n} p_i f(x_i) + \left(1 - \sum_{i=1}^{n} p_i\right) f(0). \tag{1.9}$$

Remark 1.8. If the function f from Theorem 1.6 is strictly convex, then the inequality in (1.8) is strict, if all x_i's are not equal or $\sum_{i=1}^{n} p_i \neq 1$.

Now, regarding inequality (1.8) we define another Petrović-type functional $\mathcal{P}_2$ by the formula

$$\mathcal{P}_2(\mathbf{x}, \mathbf{p}; f) = f\left(\sum_{i=1}^{n} p_i x_i\right) - \sum_{i=1}^{n} p_i f(x_i) - \left(1 - \sum_{i=1}^{n} p_i\right) f(0), \tag{1.10}$$

provided that $\mathbf{x} = (x_1, \ldots, x_n) \in I^n$, $I = [0, a]$, $\mathbf{p} = (p_1, \ldots, p_n) \in \mathbb{R}_+^n$, and f is defined on I.

Remark 1.9. If (1.1) holds and $f : I \to \mathbb{R}$ is a convex function, then

$$\mathcal{P}_2(\mathbf{x}, \mathbf{p}; f) \geq 0. \tag{1.11}$$

If (1.1) holds and $f : I \to \mathbb{R}$ is a concave function, then

$$\mathcal{P}_2(\mathbf{x}, \mathbf{p}; f) \leq 0. \tag{1.12}$$

Finally, we will also be concerned with an integral form of the Petrović-type functional, based on the following integral Petrović-type inequality.

Theorem 1.10. *Let $I \subseteq \mathbb{R}$ be an interval, $0 \in I$, and let $f : I \to \mathbb{R}$ be a convex function. Further, suppose $h : [a,b] \to I$ is continuous and monotone with $h(t_0) = 0$, where $t_0 \in [a,b]$ is fixed, and g is a function of bounded variation with*

$$G(t) := \int_a^t dg(x), \qquad \overline{G}(t) := \int_t^b dg(x). \tag{1.13}$$

(a) *If $\int_a^b h(t)dg(t) \in I$ and*

$$0 \leq G(t) \leq 1 \quad \text{for } a \leq t \leq t_0, \qquad 0 \leq \overline{G}(t) \leq 1 \quad \text{for } t_0 \leq t \leq b, \tag{1.14}$$

then

$$\int_a^b f(h(t))dg(t) \geq f\left(\int_a^b h(t)dg(t)\right) + \left(\int_a^b dg(t) - 1\right) f(0). \tag{1.15}$$

(b) *If $\int_a^b h(t)dg(t) \in I$ and either*
there exists an $s \leq t_0$ such that $G(t) \leq 0$ for $t < s$,

$$G(t) \geq 1 \quad \text{for } s \leq t \leq t_0, \qquad \overline{G}(t) \leq 0 \quad \text{for } t > t_0, \tag{1.16}$$

or
there exists an $s \geq t_0$ such that $G(t) \leq 0$ for $t < t_0$,

$$\overline{G}(t) \geq 1 \quad \text{for } t_0 < t < s, \qquad \overline{G}(t) \leq 0 \quad \text{for } t \geq s, \tag{1.17}$$

then the reverse inequality in (1.15) *holds.*

In view of Theorem 1.10, we define the functional

$$\begin{aligned} \mathcal{P}_3(h,g;f) = \int_a^b f(h(t))dg(t) - f\left(\int_a^b h(t)dg(t)\right) \\ - \left(\int_a^b dg(t) - 1\right) f(0), \end{aligned} \tag{1.18}$$

which represents the integral form of the Petrović-type functional.

Remark 1.11. If the functions f, g, and h are defined as in the statement of Theorem 1.10 and (1.14) holds, then the functional $\mathcal{P}_3$ is nonnegative, that is,

$$\mathcal{P}_3(h,g;f) \geq 0. \tag{1.19}$$

Moreover, if either (1.16) or (1.17) holds then

$$\mathcal{P}_3(h, g; f) \leq 0. \tag{1.20}$$

For a comprehensive inspection on the Petrović-type inequalities including proofs and diverse applications, the reader is referred to [1].

The paper is organized in the following way. After this introduction, in Section 2 we prove superadditivity, subadditivity, and monotonicity properties of functionals $\mathcal{P}_1, \mathcal{P}_2$, and $\mathcal{P}_3$. In addition, we also derive some bounds for the functional $\mathcal{P}_1$ via the nonweighted functional of the same type. By virtue of results from Section 2, in Section 3 we study some other classes of Petrović-type functionals and investigate their properties regarding exponential convexity. Finally, in Section 4 we apply our general results to some particular settings.

Convention 1. Throughout this paper $\mathbb{R}$ denotes the set of real numbers, while $\mathbb{R}_+$ denotes the set of nonnegative numbers (including zero). Further, bold letters $\mathbf{p}$, $\mathbf{q}$, and $\mathbf{x}$, respectively, denote real n-tuples $(p_1, p_2, \ldots, p_n)$, $(q_1, q_2, \ldots, q_n)$, and $(x_1, x_2, \ldots, x_n)$. Moreover, $\mathbf{p} \geq \mathbf{q}$ means that $p_i \geq q_i$ for all $i = 1, 2, \ldots, n$.

2. Main Results

In this section we derive some interesting properties of the Petrović-type functionals $\mathcal{P}_1, \mathcal{P}_2$, and $\mathcal{P}_3$, defined in Section 1. More precisely, we establish the conditions under which the appropriate functional is superadditive (subadditive) and increasing (decreasing), with respect to the corresponding n-tuple of real numbers. Our first result refers to the Petrović-type functional $\mathcal{P}_1$ defined by (1.3).

Theorem 2.1. *Let $I = (0, a] \subseteq \mathbb{R}_+$, $\mathbf{x} \in I^n$, and let nonnegative n-tuples $\mathbf{p}$, $\mathbf{q}$ fulfill conditions as in* (1.1). *If $f : I \to \mathbb{R}$ is such that the function $f(x)/x$ is decreasing on I, then the functional* (1.3) *possess the following properties.*

(i) *$\mathcal{P}_1(\mathbf{x}, .; f)$ is superadditive on nonnegative n-tuples, that is,*

$$P_1(\mathbf{x}, \mathbf{p} + \mathbf{q}; f) \geq P_1(\mathbf{x}, \mathbf{p}; f) + P_1(\mathbf{x}, \mathbf{q}; f), \tag{2.1}$$

provided that $\sum_{i=1}^{n} (p_i + q_i) x_i \in I$.

(ii) *If $\mathbf{p}, \mathbf{q} \in \mathbb{R}_+^n$ are such that $\mathbf{p} \geq \mathbf{q}$ and $\sum_{i=1}^{n} (p_i - q_i) x_i \geq x_j$, $j = 1, \ldots, n$, then*

$$\mathcal{P}_1(\mathbf{x}, \mathbf{p}; f) \geq \mathcal{P}_1(\mathbf{x}, \mathbf{q}; f) \geq 0, \tag{2.2}$$

that is, $\mathcal{P}_1(\mathbf{x}, .; f)$ is increasing on nonnegative n-tuples.

(iii) *If $f(x)/x$ is increasing on I, then the signs of inequalities in* (2.1) *and* (2.2) *are reversed, that is, $\mathcal{P}_1(\mathbf{x}, .; f)$ is subadditive and decreasing on nonnegative n-tuples.*

Proof. (i) Using definition (1.3) of the Petrović-type functional $\mathcal{P}_1$ and utilizing the linearity of the sum, we have

$$\begin{aligned}\mathcal{P}_1(\mathbf{x},\mathbf{p}+\mathbf{q};f) &= \sum_{i=1}^{n}(p_i+q_i)f(x_i) - f\left(\sum_{i=1}^{n}(p_i+q_i)x_i\right) \\ &= \sum_{i=1}^{n}p_i f(x_i) + \sum_{i=1}^{n}q_i f(x_i) - f\left(\sum_{i=1}^{n}p_i x_i + \sum_{i=1}^{n}q_i x_i\right).\end{aligned} \tag{2.3}$$

On the other hand, since $f(x)/x$ is decreasing function, Theorem 1.1 in the nonweighted case (for $n=2$) yields inequality

$$f\left(\sum_{i=1}^{n}p_i x_i + \sum_{i=1}^{n}q_i x_i\right) \le f\left(\sum_{i=1}^{n}p_i x_i\right) + f\left(\sum_{i=1}^{n}q_i x_i\right). \tag{2.4}$$

Finally, combining relations (2.3) and (2.4), we obtain

$$\mathcal{P}_1(\mathbf{x},\mathbf{p}+\mathbf{q};f) \ge \sum_{i=1}^{n}p_i f(x_i) + \sum_{i=1}^{n}q_i f(x_i) - f\left(\sum_{i=1}^{n}p_i x_i\right) - f\left(\sum_{i=1}^{n}q_i x_i\right). \tag{2.5}$$

Therefore we have

$$\mathcal{P}_1(\mathbf{x},\mathbf{p}+\mathbf{q};f) \ge \mathcal{P}_1(\mathbf{x},\mathbf{p};f) + \mathcal{P}_1(\mathbf{x},\mathbf{q};f), \tag{2.6}$$

as claimed.

(ii) Monotonicity follows easily from the superadditivity property. Since $\mathbf{p} \ge \mathbf{q} \ge 0$, we can represent $\mathbf{p}$ as the sum of two nonnegative n-tuples, namely, $\mathbf{p} = (\mathbf{p}-\mathbf{q}) + \mathbf{q}$. Now, from relation (2.1) we get

$$\mathcal{P}_1(\mathbf{x},\mathbf{p};f) = \mathcal{P}_1(\mathbf{x},\mathbf{p}-\mathbf{q}+\mathbf{q};f) \ge \mathcal{P}_1(\mathbf{x},\mathbf{p}-\mathbf{q};f) + \mathcal{P}_1(\mathbf{x},\mathbf{q};f). \tag{2.7}$$

Finally, if the conditions as in (ii) are fulfilled, then, taking into account Theorem 1.1 we have that $\mathcal{P}_1(\mathbf{x},\mathbf{p}-\mathbf{q};f) \ge 0$, which implies that $\mathcal{P}_1(\mathbf{x},\mathbf{p};f) \ge \mathcal{P}_1(\mathbf{x},\mathbf{q};f)$.

(iii) The case of increasing function $f(x)/x$ is treated in the same way as in (i) and (ii), taking into account that the sign of the corresponding Petrović-type inequality is reversed. □

By virtue of Theorem 1.4, the above properties of the functional $\mathcal{P}_1$ can also be derived in a slightly different setting.

Theorem 2.2. *Let $I=(0,a] \subseteq \mathbb{R}_+$, $\mathbf{x} \in I^n$, and let real n-tuples $\mathbf{p}$, $\mathbf{q}$ fulfill conditions as in* (1.6)*. If $f: I \to \mathbb{R}$ is such that the function $f(x)/x$ is increasing on I, then the functional $\mathcal{P}_1$ has the following properties.*

(i) $\mathcal{P}_1(\mathbf{x}, .; f)$ *is superadditive on real* n*-tuples, that is,*

$$\mathcal{P}_1(\mathbf{x}, \mathbf{p} + \mathbf{q}; f) \geq \mathcal{P}_1(x, \mathbf{p}; f) + \mathcal{P}_1(\mathbf{x}, \mathbf{q}; f), \tag{2.8}$$

provided that $\sum_{i=1}^{n}(p_i + q_i)x_i \in I$ *and* $0 < \sum_{i=1}^{n} p_i x_i \leq \sum_{i=1}^{n} q_i x_i$.

(ii) *If* $0 < x_1 \leq \cdots \leq x_n$, $\mathbf{p} \geq \mathbf{q}$, *and there exist* m $(\leq n)$ *such that*

$$\begin{gathered} \overline{P}_1 - \overline{Q}_1 \geq \overline{P}_2 - \overline{Q}_2 \geq \cdots \geq \overline{P}_m - \overline{Q}_m \geq 1, \\ \overline{P}_{m+1} = \overline{Q}_{m+1} = \cdots = \overline{P}_n = \overline{Q}_n = 0, \end{gathered} \tag{2.9}$$

where $P_k = \sum_{i=1}^{k} p_i$, $Q_k = \sum_{i=1}^{k} q_i$, $\overline{P}_k - \overline{Q}_k = (P_n + Q_n) - (P_{k-1} + Q_{k-1})$, $k = 2, \ldots, n$, $\overline{P}_1 = P_n$, *and* $\overline{Q}_1 = Q_n$, *then*

$$\mathcal{P}_1(\mathbf{x}, \mathbf{p}; f) \geq \mathcal{P}_1(\mathbf{x}, \mathbf{q}; f) \geq 0, \tag{2.10}$$

that is, $\mathcal{P}_1(\mathbf{x}, .; f)$ *is increasing on real* n*-tuples.*

(iii) *If real* n*-tuples* $\mathbf{p}$ *and* $\mathbf{q}$ *fulfill conditions as in* (1.7), *then the signs of inequalities in* (2.8) *and* (2.10) *are reversed, that is,* $\mathcal{P}_1(\mathbf{x}, \cdot; f)$ *is subadditive and decreasing on real* n*-tuples.*

Proof. (i) The proof follows the same lines as the proof of the previous theorem. Namely, the left-hand side of (2.8) can be rewritten as

$$\begin{aligned} \mathcal{P}_1(\mathbf{x}, \mathbf{p} + \mathbf{q}; f) &= \sum_{i=1}^{n}(p_i + q_i) f(x_i) - f\left(\sum_{i=1}^{n}(p_i + q_i) x_i\right) \\ &= \sum_{i=1}^{n} p_i f(x_i) + \sum_{i=1}^{n} q_i f(x_i) - f\left(\sum_{i=1}^{n} p_i x_i + \sum_{i=1}^{n} q_i x_i\right). \end{aligned} \tag{2.11}$$

Moreover, $f(x)/x$ is increasing, hence Theorem 1.4 for $n = 2$ yields inequality

$$f\left(\sum_{i=1}^{n} p_i x_i + \sum_{i=1}^{n} q_i x_i\right) \leq f\left(\sum_{i=1}^{n} p_i x_i\right) + f\left(\sum_{i=1}^{n} q_i x_i\right). \tag{2.12}$$

Finally, relations (2.11) and (2.12) imply inequality

$$\mathcal{P}_1(\mathbf{x}, \mathbf{p} + \mathbf{q}; f) \geq \sum_{i=1}^{n} p_i f(x_i) + \sum_{i=1}^{n} q_i f(x_i) - f\left(\sum_{i=1}^{n} p_i x_i\right) - f\left(\sum_{i=1}^{n} q_i x_i\right), \tag{2.13}$$

that is, we obtain (2.8).

(ii) Considering $\mathbf{p} \geq \mathbf{q} \geq 0$, the real n-tuple $\mathbf{p}$ can be rewritten as $\mathbf{p} = (\mathbf{p} - \mathbf{q}) + \mathbf{q}$. Now, regarding relation (2.8) we have

$$\mathcal{P}_1(\mathbf{x}, \mathbf{p}; f) = \mathcal{P}_1(\mathbf{x}, \mathbf{p} - \mathbf{q} + \mathbf{q}; f) \geq \mathcal{P}_1(\mathbf{x}, \mathbf{p} - \mathbf{q}; f) + \mathcal{P}_1(\mathbf{x}, \mathbf{q}; f). \tag{2.14}$$

Finally, taking into account conditions as in (2.9), it follows by Theorem 1.4 that $\mathcal{P}_1(\mathbf{x}, \mathbf{p} - \mathbf{q}; f) \geq 0$, that is, $\mathcal{P}_1(\mathbf{x}, \mathbf{p}; f) \geq \mathcal{P}_1(\mathbf{x}, \mathbf{q}; f)$, which completes the proof.

(iii) This case is treated in the same way as in (i) and (ii), taking into account that the sign of the corresponding Petrović-type inequality is reversed. □

Superadditivity and monotonicity properties stated in Theorem 2.1 play an important role in numerous applications of the Petrović-type inequalities. In the sequel we utilize the monotonicity property of the Petrović-type functional $\mathcal{P}_1$. More precisely, we derive some bounds for this functional, expressed in terms of the nonweighted functional of the same type.

Corollary 2.3. *Let $I = (0, a] \subseteq \mathbb{R}_+$, $\mathbf{x} \in I^n$, and let $f : I \to \mathbb{R}$ be such that $f(x)/x$ is decreasing on I. Further, suppose $\mathbf{p} \in \mathbb{R}_+^n$ is such that $\sum_{i=1}^n (p_i - m)x_i \geq x_j$ and $\sum_{i=1}^n (M - p_i)x_i \geq x_j$, $j = 1, 2, \ldots, n$, where $m = \min_{1 \leq i \leq n}\{p_i\}$ and $M = \max_{1 \leq i \leq n}\{p_i\}$.*

If $m > 1$ then the Petrović-type functional $\mathcal{P}_1$ fulfills inequality

$$\mathcal{P}_1(\mathbf{x}, \mathbf{p}; f) \geq m\mathcal{P}_1^0(\mathbf{x}; f), \tag{2.15}$$

while for $M < 1$ one has

$$\mathcal{P}_1(\mathbf{x}, \mathbf{p}; f) \leq M\mathcal{P}_1^0(\mathbf{x}; f), \tag{2.16}$$

where

$$\mathcal{P}_1^0(\mathbf{x}; f) = \sum_{i=1}^n f(x_i) - f\left(\sum_{i=1}^n x_i\right). \tag{2.17}$$

Moreover, if $f(x)/x$ is increasing on I, then the signs of inequalities in (2.15) *and* (2.16) *are reversed.*

Proof. Since $\mathbf{p} = (p_1, \ldots, p_n) \geq \mathbf{m} = (m, m, \ldots, m)$, monotonicity of the Petrović-type functional implies that $\mathcal{P}_1(\mathbf{x}, \mathbf{p}; f) \geq \mathcal{P}_1(\mathbf{x}, \mathbf{m}; f)$.

On the other hand, if $f(x)/x$ is decreasing function, we have

$$f(au) \leq af(u), \quad a > 1, \qquad f(au) \geq af(u), \quad a < 1. \tag{2.18}$$

Now, regarding (2.18) we have

$$\mathcal{P}_1(\mathbf{x}, \mathbf{m}; f) = m\sum_{i=1}^n f(x_i) - f\left(m\sum_{i=1}^n x_i\right) \geq m\sum_{i=1}^n f(x_i) - mf\left(\sum_{i=1}^n x_i\right), \tag{2.19}$$

that is, we obtain (2.15). Inequality (2.16) is derived in a similar way, by using the second inequality in (2.18). □

Our next result provides superadditivity and monotonicity properties of the Petrović-type functional defined by (1.10).

Theorem 2.4. *Let $I = [0, a] \subseteq \mathbb{R}_+$, $\mathbf{x} \in I^n$, and let $\mathbf{p}, \mathbf{q} \in \mathbb{R}_+^n$ fulfill conditions as in* (1.1)*. If $f : I \to \mathbb{R}$ is a convex function, then the functional* (1.10) *has the following properties:*

(i) *$\mathcal{P}_2(\mathbf{x}, \cdot; f)$ is superadditive on nonnegative n-tuples, that is,*

$$\mathcal{P}_2(\mathbf{x}, \mathbf{p} + \mathbf{q}; f) \geq \mathcal{P}_2(\mathbf{x}, \mathbf{p}; f) + \mathcal{P}_2(\mathbf{x}, \mathbf{q}; f), \tag{2.20}$$

provided that $\sum_{i=1}^{n} (p_i + q_i) x_i \in I$.

(ii) *If $\mathbf{p}, \mathbf{q}$ are such that $\mathbf{p} \geq \mathbf{q}$ and $\sum_{i=1}^{n} (p_i - q_i) x_i \geq x_j$, $j = 1, \dots, n$, then*

$$\mathcal{P}_2(\mathbf{x}, \mathbf{p}; f) \geq \mathcal{P}_2(\mathbf{x}, \mathbf{q}; f) \geq 0, \tag{2.21}$$

that is, $\mathcal{P}_2(\mathbf{x}, .; f)$ is increasing on nonnegative n-tuples.

(iii) *If $f : I \to \mathbb{R}$ is a concave function, then the signs of inequalities in* (2.20) *and* (2.21) *are reversed, that is, $\mathcal{P}_2(\mathbf{x}, .; f)$ is subadditive and decreasing on nonnegative n-tuples.*

Proof. (i) The left-hand side of inequality (2.20) can be rewritten as

$$\begin{aligned}
\mathcal{P}_2(\mathbf{x}, \mathbf{p} + \mathbf{q}; f) &= f\left(\sum_{i=1}^{n} (p_i + q_i) x_i\right) - \sum_{i=1}^{n} (p_i + q_i) f(x_i) \\
&\quad - \left(1 - \sum_{i=1}^{n} (p_i + q_i)\right) f(0) \\
&= f\left(\sum_{i=1}^{n} p_i x_i + \sum_{i=1}^{n} q_i x_i\right) - \sum_{i=1}^{n} p_i f(x_i) - \sum_{i=1}^{n} q_i f(x_i) \\
&\quad - \left(1 - \left(\sum_{i=1}^{n} p_i + \sum_{i=1}^{n} q_i\right)\right) f(0).
\end{aligned} \tag{2.22}$$

Further, Theorem 1.6 in the nonweighted case (for $n = 2$) yields inequality

$$f\left(\sum_{i=1}^{n} p_i x_i + \sum_{i=1}^{n} q_i x_i\right) \geq f\left(\sum_{i=1}^{n} p_i x_i\right) + f\left(\sum_{i=1}^{n} q_i x_i\right) - f(0), \tag{2.23}$$

hence combining relations (2.22) and (2.23), we get

$$\begin{aligned}
\mathcal{P}_2(\mathbf{x}, \mathbf{p} + \mathbf{q}; f) \geq{}& f\left(\sum_{i=1}^{n} p_i x_i\right) - \sum_{i=1}^{n} p_i f(x_i) - \left(1 - \sum_{i=1}^{n} p_i\right) f(0) \\
&+ f\left(\sum_{i=1}^{n} p_i x_i\right) - \sum_{i=1}^{n} q_i f(x_i) - \left(1 - \sum_{i=1}^{n} q_i\right) f(0).
\end{aligned} \tag{2.24}$$

Thus, considering definition (1.10) we obtain (2.20), as claimed.

(ii) Monotonicity property follows from the corresponding superadditivity property (2.20), as in Theorem 2.2.

(iii) The case of concave function f follows from the fact that the sign of the corresponding Petrović-type inequality is reversed. □

To conclude this section we also derive the properties of the integral Petrović-type functional, defined by (1.18).

Theorem 2.5. *Suppose $f : I = [0, a] \to \mathbb{R}$ is a convex function, $h : [a, b] \to I$ is continuous and monotone with $h(t_0) = 0$, where $t_0 \in [a, b]$ is fixed, and let g_1, g_2 be functions of bounded variation with*

$$G_i(t) := \int_a^t dg_i(x), \qquad \overline{G}_i(t) := \int_t^b dg_i(x) \quad \text{for } i = 1, 2. \tag{2.25}$$

Then the functional $\mathcal{P}_3$, defined by (1.18), has the following properties.

(i) *$\mathcal{P}_3(h, .; f)$ is subadditive with respect to functions of bounded variation, that is,*

$$\mathcal{P}_3(h, g_1 + g_2; f) \leq \mathcal{P}_3(h, g_1; f) + \mathcal{P}_3(h, g_2; f), \tag{2.26}$$

where $\int_a^b h(t)dg_1(t) \geq 0$, $\int_a^b h(t)dg_2(t) \geq 0$, and $\int_a^b h(t)dg_1(t) + \int_a^b h(t)dg_2(t) \in I$.

(ii) *If $\int_a^b h(t)d(g_1)(t) - \int_a^b h(t)d(g_2)(t) \in I$ and either there exists an $s \leq t_0$ such that $G_1(t) \leq G_2(t)$ for $t < s$, $G_1(t) - G_2(t) \geq 1$ for $s \leq t \leq t_0$, and $\overline{G}_1(t) \leq \overline{G}_2(t)$ for $t > t_0$, or there exists an $s \geq t_0$ such that $G_1(t) \leq G_2(t)$ for $t < t_0$, $\overline{G}_1(t) - \overline{G}_2(t) \geq 1$ for $t_0 < t < s$, and $\overline{G}_1(t) \leq \overline{G}_2(t)$ for $t \geq s$, then*

$$\mathcal{P}_3(h, g_1; f) \leq \mathcal{P}_3(h, g_2; f). \tag{2.27}$$

Proof. (i) Regarding definition (1.18) of the Petrović-type integral functional, we have

$$\begin{aligned} \mathcal{P}_3(h, g_1 + g_2; f) = {} & \int_a^b f(h(t))d(g_1 + g_2)(t) - f\left(\int_a^b h(t)d(g_1 + g_2)(t)\right) \\ & - \left(\int_a^b d(g_1 + g_2)(t) - 1\right) f(0), \end{aligned} \tag{2.28}$$

that is,

$$\begin{aligned}\mathcal{P}_3(h, g_1+g_2; f) = &\int_a^b f(h(t))dg_1(t) + \int_a^b f(h(t))dg_2(t)\\ &- f\left(\int_a^b h(t)dg_1(t) + \int_a^b h(t)dg_2(t)\right)\\ &- \left(\int_a^b dg_1(t) + \int_a^b dg_2(t) - 1\right)f(0),\end{aligned} \tag{2.29}$$

by the linearity of the differential. Now, applying inequality (1.8) to term $f(\int_a^b h(t)dg_1(t) + \int_a^b h(t)dg_2(t))$, we obtain

$$f\left(\int_a^b h(t)dg_1(t) + \int_a^b h(t)dg_2(t)\right) \geq f\left(\int_a^b h(t)dg_1(t)\right) + f\left(\int_a^b h(t)dg_2(t)\right) - f(0). \tag{2.30}$$

Further, inserting (2.30) in (2.29), we have

$$\begin{aligned}\mathcal{P}_3(h, g_1+g_2; f) \leq &\int_a^b f(h(t))dg_1(t) + \int_a^b f(h(t))dg_2(t)\\ &- f\left(\int_a^b h(t)dg_1(t)\right) - f\left(\int_a^b h(t)dg_2(t)\right) + f(0)\\ &- \left(\int_a^b dg_1(t) + \int_a^b dg_2(t) - 1\right)f(0),\end{aligned} \tag{2.31}$$

that is, by rearranging,

$$\mathcal{P}_3(h, g_1+g_2; f) \leq \mathcal{P}_3(h, g_1; f) + \mathcal{P}_3(h, g_2; f). \tag{2.32}$$

(ii) Monotonicity follows from the subadditivity property (2.26). Namely, representing g_1 as $g_1 = (g_1 - g_2) + g_2$, we have

$$\mathcal{P}_3(h, g_1; f) = \mathcal{P}_3(h, (g_1 - g_2) + g_2; f) \leq \mathcal{P}_3(h, g_1 - g_2; f) + \mathcal{P}_3(h, g_2; f). \tag{2.33}$$

Clearly, under assumptions as in the statement of theorem, we have $\mathcal{P}_3(h, g_1 - g_2; f) \leq 0$ (see also Remark 1.11), hence it follows that $\mathcal{P}_3(h, g_1; f) \leq \mathcal{P}_3(h, g_2; f)$, which completes the proof. $\square$

3. n-Exponential Convexity and Exponential Convexity of the Petrović-Type Functionals

By virtue of the results from Section 2, in this section we define several new classes of Petrović-type functionals and investigate their properties regarding exponential convexity.

We start these issues by giving some definitions and notions concerning exponentially convex functions which are frequently used in the results. This is a subclass of convex functions introduced by Bernstein in [2] (see also [3–5]).

Definition 3.1. A function $f : I \to \mathbb{R}$ is n-exponentially convex in the Jensen sense on an interval $I \subseteq \mathbb{R}$, if

$$\sum_{i,j=1}^{n} \xi_i \xi_j f\left(x_i + x_j\right) \geq 0 \tag{3.1}$$

holds for all choices $\xi_i \in \mathbb{R}$ and $x_i + x_j \in I, i, j = 1, \ldots, n$. Function $f : I \to \mathbb{R}$ is n-exponentially convex if it is n-exponentially convex in the Jensen sense and continuous on I.

The following remarks, propositions, and lemmas involving n-exponentially convex functions are well known (see, e.g., papers [6, 7]).

Remark 3.2. It is clear from the definition that 1-exponentially convex functions in the Jensen sense are in fact nonnegative functions. Also, n-exponentially convex functions in the Jensen sense are k-exponentially convex in the Jensen sense for every $k \in \mathbb{N},\ k \leq n$.

By using some linear algebra and definition of the positive semidefinite matrix, we have the following proposition.

Proposition 3.3. *If f is an n-exponentially convex in the Jensen sense then the matrix*

$$\left[f\left(\frac{x_i + x_j}{2}\right)\right]_{i,j=1}^{k} \tag{3.2}$$

is positive semi-definite for all $k \in \mathbb{N},\ k \leq n$. In particular,

$$\det\left[f\left(\frac{x_i + x_j}{2}\right)\right]_{i,j=1}^{k} \geq 0 \tag{3.3}$$

for all $k \in \mathbb{N},\ k \leq n$.

Definition 3.4. A function $f : I \to \mathbb{R}$ is exponentially convex in the Jensen sense on an interval $I \subseteq \mathbb{R}$, if it is n-exponentially convex in the Jensen sense for all $n \in \mathbb{N}$. Moreover, function $f : I \to \mathbb{R}$ is exponentially convex if it is exponentially convex in the Jensen sense and continuous on I.

Remark 3.5. It is known (and easy to show) that $f : I \to \mathbb{R}$ is a log-convex in the Jensen sense if and only if

$$m^2 f(t) + 2mnf\left(\frac{t+r}{2}\right) + n^2 f(r) \geq 0 \tag{3.4}$$

holds for each $m, n \in \mathbb{R}$ and $r, t \in I$. It follows that a function is log-convex in the Jensen sense if and only if it is 2-exponentially convex in the Jensen sense. Also, using the basic convexity theory it follows that the function is log-convex if and only if it is 2-exponentially convex.

We will also need the following result (see, e.g., [1]).

Lemma 3.6. *If Φ is a convex function on an interval I and if $x_1 \leq y_1,\ x_2 \leq y_2,\ x_1 \neq x_2,\ y_1 \neq y_2$, then the following inequality is valid:*

$$\frac{\Phi(x_2) - \Phi(x_1)}{x_2 - x_1} \leq \frac{\Phi(y_2) - \Phi(y_1)}{y_2 - y_1}. \tag{3.5}$$

If the function Φ is concave then the sign of the above inequality is reversed.

Divided differences are found to be very handy and interesting when we have to operate with different functions having different degree of smoothness. Let $f : I \to \mathbb{R}$ be a function, I an interval in $\mathbb{R}$. Then for distinct points $u_i \in I$, $i = 0, 1, 2$ the divided differences of first and second order are defined as follows:

$$\begin{aligned} [u_i; f] &= f(u_i) \quad (i = 0, 1, 2), \\ [u_i, u_{i+1}; f] &= \frac{f(u_{i+1}) - f(u_i)}{u_{i+1} - u_i} \quad (i = 0, 1), \\ [u_0, u_1, u_2; f] &= \frac{[u_1, u_2; f] - [u_0, u_1; f]}{u_2 - u_0}. \end{aligned} \tag{3.6}$$

The values of the divided differences are independent of the order of the points u_0, u_1, u_2 and may be extended to include the cases when some or all points are equal, that is,

$$[u_0, u_0; f] = \lim_{u_1 \to u_0} [u_0, u_1; f] = f'(u_0), \tag{3.7}$$

provided that f' exists.

Now, passing through the limit $u_1 \to u_0$ and replacing u_2 by u in (3.6), we have [1, page 16],

$$[u_0, u_0, u; f] = \lim_{u_1 \to u_0} [u_0, u_1, u; f] = \frac{f(u) - f(u_0) - f'(u_0)(u - u_0)}{(u - u_0)^2}, \quad u \neq u_0, \tag{3.8}$$

provided that f' exists. Also passing to the limit $u_i \to u$ $(i = 0, 1, 2)$ in (3.6), we have

$$[u, u, u; f] = \lim_{u_i \to u} [u_0, u_1, u_2, f] = \frac{f''(u)}{2}, \tag{3.9}$$

provided that f'' exists.

Remark 3.7. One can note that if for all $u_0, u_1 \in I$, $[u_0, u_1, f] \geq 0$ then f is increasing on I and if for all $u_0, u_1, u_2 \in I$, $[u_0, u_1, u_2, f] \geq 0$ then f is convex on I.

Now, we are ready to study some new classes of Petrović-type functionals. For the sake of simplicity and to avoid many notions, we first introduce the following definitions.

(M_1) Under the assumptions of Theorem 1.1 equipped with conditions as in (1.1), we define linear functional as

$$\Phi_1(f) = -\mathcal{P}_1(\mathbf{x}, \mathbf{p}; f). \tag{3.10}$$

(M_2) Under the assumptions of Theorem 1.4 with conditions as in (1.6), we define linear functional as

$$\Phi_2(f) = \Phi_1(f). \tag{3.11}$$

(M_3) Under the assumptions of Theorem 1.4 with conditions as in (1.7), we define linear functional as

$$\Phi_3(f) = -\Phi_1(f). \tag{3.12}$$

(M_4) Under the assumptions of Theorem 2.1 with conditions as in (1.1), and provided that $\sum_{i=1}^{n}(p_i + q_i)x_i \in I$, we define linear functional as

$$\Phi_4(f) = \mathcal{P}_1(\mathbf{x}, \mathbf{p}; f) + \mathcal{P}_1(\mathbf{x}, \mathbf{q}; f) - \mathcal{P}_1(\mathbf{x}, \mathbf{p} + \mathbf{q}; f). \tag{3.13}$$

(M_5) Under the assumptions of Theorem 2.2 with conditions as in (1.6), and provided that $\sum_{i=1}^{n}(p_i + q_i)x_i \in I$, $0 < \sum_{i=1}^{n} p_i x_i \leq \sum_{i=1}^{n} q_i x_i$, we define linear functional as

$$\Phi_5(f) = -\Phi_4(f). \tag{3.14}$$

(M_6) Under the assumptions of Theorem 2.2 with conditions as in (1.7), and provided that $\sum_{i=1}^{n}(p_i + q_i)x_i \in I$, $0 < \sum_{i=1}^{n} p_i x_i \leq \sum_{i=1}^{n} q_i x_i$, we define linear functional as

$$\Phi_6(f) = \Phi_4(f). \tag{3.15}$$

(M_7) Under the assumptions of Theorem 1.6 with conditions as in (1.1), we define linear functional as

$$\Phi_7(f) = \mathcal{P}_2(\mathbf{x}, \mathbf{p}; f). \tag{3.16}$$

(M_8) Under the assumptions of Theorem 1.10 with conditions as in (1.14), we define linear functional as

$$\Phi_8(f) = \mathcal{P}_3(h, g; f). \tag{3.17}$$

(M_9) Under the assumptions of Theorem 1.10 equipped with conditions (1.16) or (1.17), we define linear functional as

$$\Phi_9(f) = -\mathcal{P}_3(h, g; f). \tag{3.18}$$

(M_{10}) Under the assumptions of Theorem 2.4 with conditions as in (1.1), and provided that $\sum_{i=1}^{n}(p_i + q_i)x_i \in I$, we define linear functional as

$$\Phi_{10}(f) = \mathcal{P}_2(\mathbf{x}, \mathbf{p} + \mathbf{q}; f) - \mathcal{P}_2(\mathbf{x}, \mathbf{p}; f) - \mathcal{P}_2(\mathbf{x}, \mathbf{q}; f). \tag{3.19}$$

(M_{11}) Under the assumptions of Theorem 2.5, and provided that

$\int_a^b h(t)dg_1(t) \geq 0$, $\int_a^b h(t)dg_2(t) \geq 0$, $\int_a^b h(t)dg_1(t) + \int_a^b h(t)dg_2(t) \in I$, we define linear functional as

$$\Phi_{11}(f) = \mathcal{P}_3(h, g_1; f) + \mathcal{P}_3(h, g_2; f) - \mathcal{P}_3(h, g_1 + g_2; f). \tag{3.20}$$

Remark 3.8. Considering the assumptions as in (M_k), $k = 1, \ldots, 6$, if $f(u)/u$ is increasing function on I then

$$\Phi_k(f) \geq 0, \quad \text{for } k = 1, \ldots, 6. \tag{3.21}$$

Remark 3.9. Considering the assumptions as in (M_k), $k = 7, \ldots, 11$, if f is convex function on I then

$$\Phi_k(f) \geq 0 \quad \text{for } k = 7, \ldots, 11. \tag{3.22}$$

In order to obtain our main results regarding the exponential convexity, we define different families of functions. Let $I, J \subseteq \mathbb{R}$ be intervals. For distinct points $u_0, u_1, u_2 \in I$, we define the following.

$\mathbf{E}_1 = \{f_t : I \to \mathbb{R} \mid t \in J$ and $t \mapsto [u_0, u_1, F_t]$ is n-exponentially convex in the Jensen sense on J, where $F_t(u) = f_t(u)/u\}$.

$\mathbf{E}_2 = \{f_t : I \to \mathbb{R} \mid t \in J$ and $t \mapsto [u_0, u_1, u_2; f_t]$ is n-exponentially convex in the Jensen sense on $J\}$.

$\mathbf{E}_3 = \{f_t : I \to \mathbb{R} \mid t \in J$ and $t \mapsto [u_0, u_1, F_t]$ is exponentially convex in the Jensen sense on J, where $F_t(u) = f_t(u)/u\}$.

$\mathbf{E}_4 = \{f_t : I \to \mathbb{R} \mid t \in J$ and $t \mapsto [u_0, u_1, u_2; f_t]$ is exponentially convex in the Jensen sense on $J\}$.

$\mathbf{E}_5 = \{f_t : I \to \mathbb{R} \mid t \in J$ and $t \mapsto [u_0, u_1, F_t]$ is 2-exponentially convex in the Jensen sense on J, where $F_t(u) = f_t(u)/u\}$.

$\mathbf{E}_6 = \{f_t : I \to \mathbb{R} \mid t \in J$ and $t \mapsto [u_0, u_1, u_2; f_t]$ is 2-exponentially convex in the Jensen sense on $J\}$.

Theorem 3.10. *Let $\Phi_k(f_t)$ be linear functionals defined as in (M_k), associated with families $\mathbf{E}_1$ and $\mathbf{E}_2$ in such a way that, $f_t \in \mathbf{E}_1$, for $k = 1, \ldots, 6$, and $f_t \in \mathbf{E}_2$, for $k = 7, \ldots, 11$. Then $t \mapsto \Phi_k(f_t)$ is n-exponentially convex function in the Jensen sense on J. If the function $t \mapsto \Phi_k(f_t)$ is continuous on J, then it is n-exponentially convex on J.*

Proof. (a) We first prove n-exponential convexity in the Jensen sense of the function $t \mapsto \Phi_k(f_t)$, for $k = 1, \ldots, 6$. To do this, as we have considered the families of functions defined in $\mathbf{E}_1$, for $\xi_i \in \mathbb{R}$, $i = 1, \ldots, n$, and $t_i \in J$, $i = 1, \ldots, n$, we define the function

$$h(u) = \sum_{i,j=1}^{n} \xi_i \xi_j f_{(t_i+t_j)/2}(u). \tag{3.23}$$

Clearly, we have

$$[u_0, u_1, H] = \sum_{i,j=1}^{n} \xi_i \xi_j \left[u_0, u_1, F_{(t_i+t_j)/2}\right], \tag{3.24}$$

where $H(u) = h(u)/u$ and $F_t(u) = f_t(u)/u$.

Since the function $t \mapsto [u_0, u_1, F_t]$ is n-exponentially convex in the Jensen sense, the right-hand side of the above expression is nonnegative which implies that $h(u)/u$ is increasing on I (see Remark 3.7).

Hence, taking into account Remark 3.8, we have

$$\Phi_k(h) \geq 0, \quad \text{for } k = 1, \ldots, 6, \tag{3.25}$$

that is,

$$\sum_{i,j=1}^{n} \xi_i \xi_j \Phi_k\left(f_{(t_i+t_j)/2}\right) \geq 0. \tag{3.26}$$

Therefore, we conclude that the functions $t \mapsto \Phi_k(f_t)$, $k = 1, 2, \ldots, 6$, are n-exponentially convex on J in the Jensen sense.

It remains to prove the n-exponential convexity in the Jensen sense of the functions $t \mapsto \Phi_k(f_t)$, $k = 7, \ldots, 11$. For that sake, we consider the families of functions defined in $\mathbf{E}_2$. For each $\xi_i \in \mathbb{R}$, $i = 1, \ldots, n$, and $t_i \in J$, $i = 1, \ldots, n$, we consider the function

$$h(u) = \sum_{i,j=1}^{n} \xi_i \xi_j f_{(t_i+t_j)/2}(u). \tag{3.27}$$

Obviously,

$$[u_0, u_1, u_2, h] = \sum_{i,j=1}^{n} \xi_i \xi_j \left[u_0, u_1, u_2, f_{(t_i+t_j)/2}\right]. \tag{3.28}$$

Since $t \mapsto [u_0, u_1, u_2, f_t]$ is n-exponentially convex, the right-hand side of the above expression is nonnegative which implies that $h(u)$ is convex on I. Moreover, taking into account Remark 3.9, we have

$$\Phi_k(h) \geq 0 \quad \text{for } k = 7, \ldots, 11, \tag{3.29}$$

that is,

$$\sum_{i,j=1}^{n} \xi_i \xi_j \Phi_k\left(f_{(t_i+t_j)/2}\right) \geq 0. \tag{3.30}$$

Hence, $t \mapsto \Phi_k(f_t)$ is n-exponentially convex for $k = 7, \ldots, 11$, and the proof is completed. □

The following corollary is an immediate consequence of the above theorem.

Corollary 3.11. *Let $\Phi_k(f_t)$ be linear functionals defined as in (M_k), associated with families $\mathbf{E}_3$ and $\mathbf{E}_4$ in such a way that $f_t \in \mathbf{E}_3$, $k = 1, \ldots, 6$, and $f_t \in \mathbf{E}_4$, $k = 7, \ldots, 11$. Then $t \mapsto \Phi_k(f_t)$ is exponentially convex function in the Jensen sense on J. If $t \mapsto \Phi_k(f_t)$ is continuous on J then it is exponentially convex on J.*

Proof. It follows from the previous theorem. □

Corollary 3.12. *Let $\Phi_k(f_t)$ be linear functionals defined as in (M_k), associated with families $\mathbf{E}_5$ and $\mathbf{E}_6$ in such a way that $f_t \in \mathbf{E}_5$, $k = 1, \ldots, 6$, and $f_t \in \mathbf{E}_6$, $k = 7, \ldots, 11$. Then the following statements hold.*

(i) *If the function $t \mapsto \Phi_k(f_t)$ is continuous on J then it is 2-exponentially convex on J and, thus, log-convex.*

(ii) *If the function $t \mapsto \Phi_k(f_t)$ is strictly positive and differentiable on J, then for all $t, r, u, v \in J$ such that $t \leq u$, $r \leq v$, one has*

$$\mathfrak{B}(t, r; \Phi_k(f_t)) \leq \mathfrak{B}(u, v; \Phi_k(f_t)), \quad k = 1, \ldots, 11, \tag{3.31}$$

where

$$\mathfrak{B}(t,r;\Phi_k(f_t)) = \begin{cases} \left(\dfrac{\Phi_k(f_t)}{\Phi_k(f_r)}\right)^{(1/(t-r))}, & t \neq r, \\ \exp\left(\dfrac{(d/d_t)(\Phi_k(f_t))}{\Phi_k(f_t)}\right), & t = r. \end{cases} \tag{3.32}$$

Proof. (i) This is an immediate consequence of Theorem 3.10 and Remark 3.2.

(ii) By (i), the function $t \mapsto \Phi_k(f_t)$ is log-convex on J, which means that the function $t \mapsto \log \Phi_k(f_t)$ is convex on J. Hence, by using Lemma 3.6 with $t \leq u$, $r \leq v$, $t \neq r$, $u \neq v$, we obtain

$$\frac{\log \Phi_k(f_t) - \log \Phi_k(f_r)}{t-r} \leq \frac{\log \Phi_k(f_u)) - \log \Phi_k(f_v)}{u-v}, \tag{3.33}$$

that is,

$$\mathfrak{B}(t,r;\Phi_k(f_t)) \leq \mathfrak{B}(u,v;\Phi_k(f_t)). \tag{3.34}$$

Finally, if $t = r \leq u$, by taking the limit $\lim_{r\to t}$, we have

$$\mathfrak{B}(t,t;\Phi_k(f_t)) \leq \mathfrak{B}(u,v;\Phi_k(f_t)). \tag{3.35}$$

Other possible cases are treated similarly. □

Remark 3.13. The results given in Theorem 3.10, Corollaries 3.11, and 3.12 are still valid when the points $u_0, u_1 \in I$ coincide, for a family of differentiable functions f_t such that the function $t \mapsto [u_0, u_1, f_t]$ is n-exponentially convex in the Jensen sense (exponentially convex in the Jensen sense, log-convex in the Jensen sense). Note also that the results given in Theorem 3.10, Corollaries 3.11, and 3.12 hold when two of the points $u_0, u_1, u_2 \in I$ coincide, say $u_1 = u_0$, for a family of differentiable functions f_t such that the function $t \mapsto [u_0, u_1, u_2, f_t]$ is n-exponentially convex in the Jensen sense (exponentially convex in the Jensen sense, log-convex in the Jensen sense). Moreover, the above results also hold when all three points coincide for a family of twice differentiable functions with the same property. These results can be proved easily by using the definition of divided differences and Remark 3.7.

4. Examples

We conclude this paper with several examples related to the results from the previous section.

Example 4.1. Let $t > 0$, and let $\zeta_t : (0, \infty) \to \mathbb{R}$ be the function defined by

$$\zeta_t(u) = \begin{cases} \dfrac{ut^{-u}}{-\log t}, & t \neq 1, \\ u^2, & t = 1. \end{cases} \tag{4.1}$$

Obviously, a family of functions $\zeta_t(u)/u$ is increasing for all $t > 0$, hence, by virtue of Theorem 2.1, we obtain that the functional $\mathcal{D}_1(\mathbf{x}, .; \zeta_t)$ is subadditive and decreasing on nonnegative n-tuples.

Moreover, since $(\zeta_t(u)/u)' = t^{-u}$, the mapping $t \mapsto (\zeta_t(u)/u)'$ is exponentially convex (see [8]). Now, regarding Corollary 3.11 and Remark 3.13, we get exponential convexity of the functionals $\Phi_k(\zeta_t)$ for $k = 1, \ldots, 6$.

In addition, Corollary 3.12 provides the log-convexity of these functionals and we have

$$\mathfrak{B}(t, r; \Phi_k(\zeta_t)) = \begin{cases} \left(\dfrac{\Phi_k(\zeta_t)}{\Phi_k(\zeta_r)}\right)^{(1/(t-r))}, & t \neq r, \\ \exp\left(\dfrac{-1}{t \log t} - \dfrac{\Phi_k(u\zeta_t)}{t(\Phi_k(\zeta_t)}\right), & t = r \neq 1, \\ \exp\left(\dfrac{\Phi_k(u\zeta_1)}{-2(\Phi_k(\zeta_1))}\right), & t = r = 1, \end{cases} \tag{4.2}$$

for $k = 1, 2, \ldots, 6$.

Example 4.2. Suppose that $t > 0$ and $\lambda_t : (0, \infty) \to \mathbb{R}$ is the function defined by

$$\lambda_t(u) = \frac{u \exp\left(-u\sqrt{t}\right)}{-\sqrt{t}}. \tag{4.3}$$

Since the function $\lambda_t(u)/u$ is increasing for every $t > 0$, utilizing Theorem 2.1, we obtain that the functional $\mathcal{D}_1(\mathbf{x}, .; \lambda_t)$ is subadditive and decreasing on nonnegative n-tuples.

Further, since $(\lambda_t(u)/u)' = \exp(-u\sqrt{t})$, the mapping $t \mapsto (\lambda_t(u)/u)'$ is exponentially convex (see [8]). Now, by using Corollary 3.11 and Remark 3.13, we get exponential convexity of the functionals $\Phi_k(\lambda_t)$ for $k = 1, \ldots, 6$.

In addition, Corollary 3.12 implies the log-convexity of these functionals and we have

$$\mathfrak{B}(t, r; \Phi_k(\lambda_t)) = \begin{cases} \left(\dfrac{\Phi_k(\lambda_t)}{\Phi_k(\lambda_r)}\right)^{(1/(t-r))}, & t \neq r, \\ \exp\left(\dfrac{-1}{2t} - \dfrac{\Phi_k(u\lambda_t)}{2\sqrt{t}(\Phi_k(\lambda_t))}\right), & t = r, \end{cases} \tag{4.4}$$

for $k = 1, 2, \ldots, 6$.

Example 4.3. Consider the family of functions $\varphi_t : \mathbb{R}_+ \to \mathbb{R}$, $t \in \mathbb{R}_+$, defined by

$$\varphi_t(u) = \begin{cases} \dfrac{u \exp(ut)}{t}, & t \neq 0, \\ u^2, & t = 0. \end{cases} \tag{4.5}$$

It is easy to see that the function $\varphi_t(u)/u$ is increasing on $\mathbb{R}_+$ for all $t \in \mathbb{R}_+$. Hence, by virtue of Theorem 2.1, the functional $\mathcal{D}_1(\mathbf{x}, .; \varphi_t)$ is subadditive and decreasing on nonnegative n-tuples.

In addition, $(\varphi_t(u)/u)' = \exp(ut)$ and the mapping $t \mapsto (\varphi_t(u)/u)'$ is exponentially convex (see [8]). Similarly as in the previous examples, Corollary 3.11 and Remark 3.13 provide exponential convexity of the functionals $\Phi_k(\varphi_t)$ for $k = 1, \ldots, 6$.

Also, by Corollary 3.12, we get log-convexity of these functionals and we have

$$\mathfrak{B}(t, r; \Phi_k(\varphi_t)) = \begin{cases} \left(\dfrac{\Phi_k(\varphi_t)}{\Phi_k(\varphi_r)} \right)^{(1/(t-r))}, & t \neq r, \\ \exp\left(\dfrac{-1}{t} + \dfrac{\Phi_k(u\varphi_t)}{(\Phi_k(\varphi_t)} \right), & t = r \neq 0, \\ \exp\left(\dfrac{\Phi_k(u\varphi_0)}{2(\Phi_k(\varphi_0))} \right), & t = r = 0, \end{cases} \tag{4.6}$$

for $k = 1, 2, \ldots, 6$.

Example 4.4. Let $t > 0$, and let $\beta_t : (0, \infty) \to \mathbb{R}$ be the function defined by

$$\beta_t(u) = \begin{cases} \dfrac{u^t}{t-1}, & t \neq 1, \\ u \log u, & t = 1. \end{cases} \tag{4.7}$$

Obviously, a family of functions $\beta_t(u)/u$ is increasing for $t > 0$, hence, by virtue of Theorem 2.1, we obtain that the functional $\mathcal{D}_1(\mathbf{x}, \cdot; \beta_t)$ is subadditive and decreasing on nonnegative n-tuples.

Further, since $(\beta_t(u)/u)' = u^{t-2} = \exp((t-2)\log u)$, the mapping $t \mapsto (\beta_t(u)/u)'$ is exponentially convex (see [8]). Similarly as in the previous examples, regarding Corollary 3.11 and Remark 3.13, we get exponential convexity of the functionals $\Phi_k(\beta_t)$ for $k = 1, \ldots, 6$.

In addition, Corollary 3.12 provides the log-convexity of these functionals and we have

$$\mathfrak{B}(t, r; \Phi_k(\beta_t)) = \begin{cases} \left(\dfrac{\Phi_k(\beta_t)}{\Phi_k(\beta_r)} \right)^{(1/(t-r))}, & t \neq r, \\ \exp\left(\dfrac{1}{1-t} + \dfrac{\Phi_k(\log u\beta_t)}{(\Phi_k(\beta_t)} \right), & t = r \neq 1, \\ \exp\left(\dfrac{\Phi_k(\log u\beta_1)}{2(\Phi_k(\beta_1))} \right), & t = r = 1, \end{cases} \tag{4.8}$$

for $k = 1, 2, \ldots, 6$.

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