Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2011, Article ID 714248, 18 pages
doi:10.1155/2011/714248

Research Article

A Two-Species Cooperative Lotka-Volterra System of Degenerate Parabolic Equations

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Received 13 December 2010; Revised 13 February 2011; Accepted 24 February 2011

Academic Editor: Elena Braverman

We consider a cooperating two-species Lotka-Volterra model of degenerate parabolic equations. We are interested in the coexistence of the species in a bounded domain. We establish the existence of global generalized solutions of the initial boundary value problem by means of parabolic regularization and also consider the existence of the nontrivial time-periodic solution for this system.

1. Introduction

In this paper, we consider the following two-species cooperative system:

$$u_t = \Delta u^{m_1} + u^{\alpha}(a - bu + cv), \quad (x,t) \in \Omega \times \mathbb{R}_+, \tag{1.1}$$

$$v_t = \Delta v^{m_2} + v^{\beta}(d + eu - fv), \quad (x,t) \in \Omega \times \mathbb{R}_+, \tag{1.2}$$

$$u(x,t) = 0, \quad v(x,t) = 0, \quad (x,t) \in \partial\Omega \times \mathbb{R}_+, \tag{1.3}$$

$$u(x,0) = u_0(x), \quad v(x,0) = v_0(x), \quad x \in \Omega, \tag{1.4}$$

where $m_1, m_2 > 1$, $0 < \alpha < m_1$, $0 < \beta < m_2$, $1 \le (m_1 - \alpha)(m_2 - \beta)$, $a = a(x,t)$, $b = b(x,t)$, $c = c(x,t)$, $d = d(x,t)$, $e = e(x,t)$, $f = f(x,t)$ are strictly positive smooth functions and periodic in time with period $T > 0$ and $u_0(x)$ and $v_0(x)$ are nonnegative functions and satisfy $u_0^{m_1}, v_0^{m_2} \in W_0^{1,2}(\Omega)$.

In dynamics of biological groups, the system (1.1)-(1.2) can be used to describe the interaction of two biological groups. The diffusion terms Δu^{m_1} and Δv^{m_2} represent the effect of dispersion in the habitat, which models a tendency to avoid crowding and the speed of the diffusion is rather slow. The boundary conditions (1.3) indicate that the habitat is surrounded by a totally hostile environment. The functions u and v represent the spatial densities of the species at time t and a, d are their respective net birth rate. The functions b and f are intra-specific competitions, whereas c and e are those of interspecific competitions.

As famous models for dynamics of population, two-species cooperative systems like (1.1)-(1.2) have been studied extensively, and there have been many excellent results, for detail one can see [1–6] and references therein. As a special case, men studied the following two-species Lotka-Volterra cooperative system of ODEs:

$$\begin{aligned} u'(t) &= u(t)\big(a(t) - b(t)u(t) + c(t)v(t)\big), \\ v'(t) &= v(t)\big(d(t) + e(t)u(t) - f(t)v(t)\big). \end{aligned} \tag{1.5}$$

For this system, Lu and Takeuchi [7] studied the stability of positive periodic solution and Cui [1] discussed the persistence and global stability of it.

When $m_1 = m_2 = \alpha = \beta = 1$, from (1.1)-(1.2) we get the following classical cooperative system:

$$\begin{aligned} u_t &= \Delta u + u(a - bu + cv), \\ v_t &= \Delta v + v(d + eu - fv). \end{aligned} \tag{1.6}$$

For this system, Lin et al. [5] showed the existence and asymptotic behavior of T-periodic solutions when a, b, c, e, d, f are all smooth positive and periodic in time with period $T > 0$. When a, b, c, e, d, f are all positive constants, Pao [6] proved that the Dirichlet boundary value problem of this system admits a unique solution which is uniformly bounded when $ce < bf$, while the blowup solutions are possible when the two species are strongly mutualistic ($ce > bf$). For the homogeneous Neumann boundary value problem of this system, Lou et al. [4] proved that the solution will blow up in finite time under a sufficient condition on the initial data. When $c = e = 0$ and $\alpha = \beta = 1$, from (1.1) we get the single degenerate equation

$$u_t = \Delta u^m + u(a - bu). \tag{1.7}$$

For this equation, Sun et al. [8] established the existence of nontrivial nonnegative periodic solutions by monotonicity method and showed the attraction of nontrivial nonnegative periodic solutions.

In the recent years, much attention has been paid to the study of periodic boundary value problems for parabolic systems; for detail one can see [9–15] and the references therein. Furthermore, many researchers studied the periodic boundary value problem for degenerate parabolic systems, such as [16–19]. Taking into account the impact of periodic factors on the species dynamics, we are also interested in the existence of the nontrivial periodic solutions of the cooperative system (1.1)-(1.2). In this paper, we first show the existence of the global generalized solution of the initial boundary value problem (1.1)–(1.4). Then under the condition that

$$b_l f_l > c_M e_M, \tag{1.8}$$

where $f_M = \sup\{f(x,t) \mid (x,t) \in \Omega \times \mathbb{R}\}$, $f_l = \inf\{f(x,t) \mid (x,t) \in \Omega \times \mathbb{R}\}$, we show that the generalized solution is uniformly bounded. At last, by the method of monotone iteration, we establish the existence of the nontrivial periodic solutions of the system (1.1)-(1.2), which follows from the existence of a pair of large periodic supersolution and small periodic subsolution. At last, we show the existence and the attractivity of the maximal periodic solution.

Our main efforts center on the discussion of generalized solutions, since the regularity follows from a quite standard approach. Hence we give the following definition of generalized solutions of the problem (1.1)–(1.4).

Definition 1.1. A nonnegative and continuous vector-valued function (u,v) is said to be a generalized solution of the problem (1.1)–(1.4) if, for any $0 \le \tau < T$ and any functions $\varphi_i \in C^1(\overline{Q}_\tau)$ with $\varphi_i|_{\partial\Omega\times[0,\tau)} = 0$ $(i = 1,2)$, $\nabla u^{m_1}, \nabla v^{m_2} \in L^2(Q_\tau)$, $\partial u^{m_1}/\partial t$, $\partial v^{m_2}/\partial t \in L^2(Q_\tau)$ and

$$\iint_{Q_\tau} u\frac{\partial\varphi_1}{\partial t} - \nabla u^{m_1}\nabla\varphi_1 + u^\alpha(a - bu + cv)\varphi_1 dx\,dt = \int_\Omega u(x,\tau)\varphi_1(x,\tau)dx - \int_\Omega u_0(x)\varphi_1(x,0)dx,$$
$$\iint_{Q_\tau} v\frac{\partial\varphi_2}{\partial t} - \nabla v^{m_2}\nabla\varphi_2 + v^\beta(d + eu - fv)\varphi_2 dx\,dt = \int_\Omega v(x,\tau)\varphi_2(x,\tau)dx - \int_\Omega v_0(x)\varphi_2(x,0)dx, \tag{1.9}$$

where $Q_\tau = \Omega \times (0,\tau)$.

Similarly, we can define a weak supersolution $(\overline{u},\overline{v})$ (subsolution $(\underline{u},\underline{v})$) if they satisfy the inequalities obtained by replacing "=" with "≤" ("≥") in (1.3), (1.4), and (1.9) and with an additional assumption $\varphi_i \ge 0$ $(i = 1,\ 2)$.

Definition 1.2. A vector-valued function (u,v) is said to be a T-periodic solution of the problem (1.1)–(1.3) if it is a solution in $[0,T]$ such that $u(\cdot,0) = u(\cdot,T)$, $v(\cdot,0) = v(\cdot,T)$ in Ω. A vector-valued function $(\overline{u},\overline{v})$ is said to be a T-periodic supersolution of the problem (1.1)–(1.3) if it is a supersolution in $[0,T]$ such that $\overline{u}(\cdot,0) \ge \overline{u}(\cdot,T)$, $\overline{v}(\cdot,0) \ge \overline{v}(\cdot,T)$ in Ω. A vector-valued function $(\underline{u},\underline{v})$ is said to be a T-periodic subsolution of the problem (1.1)–(1.3), if it is a subsolution in $[0,T]$ such that $\underline{u}(\cdot,0) \le \underline{u}(\cdot,T)$, $\underline{v}(\cdot,0) \le \underline{v}(\cdot,T)$ in Ω.

This paper is organized as follows. In Section 2, we show the existence of generalized solutions to the initial boundary value problem and also establish the comparison principle. Section 3 is devoted to the proof of the existence of the nonnegative nontrivial periodic solutions by using the monotone iteration technique.

2. The Initial Boundary Value Problem

To solve the problem (1.1)–(1.4), we consider the following regularized problem:

$$\frac{\partial u_\varepsilon}{\partial t} = \operatorname{div}\left(\left(mu_\varepsilon^{m_1-1} + \varepsilon\right)\nabla u_\varepsilon\right) + u_\varepsilon^\alpha(a - bu_\varepsilon + cv_\varepsilon), \quad (x,t) \in Q_T, \tag{2.1}$$

$$\frac{\partial v_\varepsilon}{\partial t} = \operatorname{div}\left(\left(mv_\varepsilon^{m_2-1} + \varepsilon\right)\nabla v_\varepsilon\right) + v_\varepsilon^\beta(d + eu_\varepsilon - fv_\varepsilon), \quad (x,t) \in Q_T, \tag{2.2}$$

$$u_\varepsilon(x,t) = 0, \quad v_\varepsilon(x,t) = 0, \quad (x,t) \in \partial\Omega \times (0,T), \tag{2.3}$$

$$u_\varepsilon(x,0) = u_{0\varepsilon}(x), \quad v_\varepsilon(x,0) = v_{0\varepsilon}(x), \quad x \in \Omega, \tag{2.4}$$

where $Q_T = \Omega\times(0,T)$, $0<\varepsilon<1$, $u_{0\varepsilon}, v_{0\varepsilon} \in C_0^\infty(\Omega)$ are nonnegative bounded smooth functions and satisfy

$$\begin{gathered} 0 \le u_{0\varepsilon} \le \|u_0\|_{L^\infty(\Omega)}, \qquad 0 \le v_{0\varepsilon} \le \|v_0\|_{L^\infty(\Omega)}, \\ u_{0\varepsilon}^{m_1} \longrightarrow u_0^{m_1}, \quad v_{0\varepsilon}^{m_2} \longrightarrow v_0^{m_2}, \quad \text{in } W_0^{1,2}(\Omega) \text{ as } \varepsilon \longrightarrow 0. \end{gathered} \tag{2.5}$$

The standard parabolic theory (cf. [20, 21]) shows that (2.1)–(2.4) admits a nonnegative classical solution $(u_\varepsilon, v_\varepsilon)$. So, the desired solution of the problem (1.1)–(1.4) will be obtained as a limit point of the solutions $(u_\varepsilon, v_\varepsilon)$ of the problem (2.1)–(2.4). In the following, we show some important uniform estimates for $(u_\varepsilon, v_\varepsilon)$.

Lemma 2.1. *Let $(u_\varepsilon, v_\varepsilon)$ be a solution of the problem* (2.1)–(2.4).

(1) *If $1 < (m_1-\alpha)(m_2-\beta)$, then there exist positive constants r and s large enough such that*

$$\frac{1}{m_2-\beta} < \frac{m_1+r-1}{m_2+s-1} < m_1-\alpha, \tag{2.6}$$

$$\|u_\varepsilon\|_{L^r(Q_T)} \le C, \qquad \|v_\varepsilon\|_{L^s(Q_T)} \le C, \tag{2.7}$$

where C is a positive constant only depending on $m_1, m_2, \alpha, \beta, r, s$, $|\Omega|$, and T.

(2) *If $1 = (m_1-\alpha)(m_2-\beta)$, then* (2.7) *also holds when $|\Omega|$ is small enough.*

Proof. Multiplying (2.1) by u_ε^{r-1} $(r>1)$ and integrating over Ω, we have that

$$\int_\Omega \frac{\partial u_\varepsilon^r}{\partial t}dx = -\frac{4r(r-1)m_1}{(m_1+r-1)^2}\int_\Omega \left|\nabla u_\varepsilon^{(m_1+r-1)/2}\right|^2 dx + r\int_\Omega u_\varepsilon^{\alpha+r-1}(a-bu_\varepsilon+cv_\varepsilon)dx. \tag{2.8}$$

By Poincaré's inequality, we have that

$$K\int_\Omega u_\varepsilon^{m_1+r-1}dx \le \int_\Omega \left|\nabla u_\varepsilon^{(m_1+r-1)/2}\right|^2 dx, \tag{2.9}$$

where K is a constant depending only on $|\Omega|$ and N and becomes very large when the measure of the domain Ω becomes small. Since $\alpha < m_1$, Young's inequality shows that

$$\begin{gathered} au_\varepsilon^{\alpha+r-1} \le \frac{Kr(r-1)m_1}{(m_1+r-1)^2}u_\varepsilon^{m_1+r-1} + CK^{-(\alpha+r-1)/(m_1-\alpha)}, \\ cu_\varepsilon^{\alpha+r-1}v_\varepsilon \le \frac{Kr(r-1)m_1}{(m_1+r-1)^2}u_\varepsilon^{m_1+r-1} + CK^{-(\alpha+r-1)/(m_1-\alpha)}v_\varepsilon^{(m_1+r-1)/(m_1-\alpha)}. \end{gathered} \tag{2.10}$$

For convenience, here and below, C denotes a positive constant which is independent of ε and may take different values on different occasions. Complying (2.8) with (2.9) and (2.10), we obtain

$$\int_\Omega \frac{\partial u_\varepsilon^r}{\partial t} dx \le -\frac{2Kr(r-1)m_1}{(m_1+r-1)^2} \int_\Omega u_\varepsilon^{m_1+r-1} dx + CK^{-(\alpha+r-1)/(m_1-\alpha)} \int_\Omega v_\varepsilon^{(m_1+r-1)/(m_1-\alpha)} dx + CK^{-(\alpha+r-1)/(m_1-\alpha)}. \tag{2.11}$$

As a similar argument as above, for v_ε and positive constant $s > 1$, we have that

$$\int_\Omega \frac{\partial v_\varepsilon^s}{\partial t} dx \le -\frac{2Ks(s-1)m_2}{(m_2+s-1)^2} \int_\Omega v_\varepsilon^{m_2+s-1} dx + CK^{-(\beta+s-1)/(m_2-\beta)} \int_\Omega u_\varepsilon^{(m_2+s-1)/(m_2-\beta)} dx + CK^{-(\beta+s-1)/(m_2-\beta)}. \tag{2.12}$$

Thus we have that

$$\begin{aligned}\int_\Omega \left(\frac{\partial u_\varepsilon^r}{\partial t} + \frac{\partial v_\varepsilon^s}{\partial t}\right) dx \le &-\frac{2Kr(r-1)m_1}{(m_1+r-1)^2} \int_\Omega u_\varepsilon^{m_1+r-1} dx + CK^{-(\beta+s-1)/(m_2-\beta)} \int_\Omega u_\varepsilon^{(m_2+s-1)/(m_2-\beta)} dx \\ &- \frac{2Ks(s-1)m_2}{(m_2+s-1)^2} \int_\Omega v_\varepsilon^{m_2+s-1} dx + CK^{-(\alpha+r-1)/(m_1-\alpha)} \int_\Omega v_\varepsilon^{(m_1+r-1)/(m_1-\alpha)} dx \\ &+ CK^{-(\alpha+r-1)/(m_1-\alpha)} + CK^{-(\beta+s-1)/(m_2-\beta)}.\end{aligned} \tag{2.13}$$

For the case of $1 < (m_1-\alpha)(m_2-\beta)$, there exist r, s large enough such that

$$\frac{1}{m_1-\alpha} < \frac{m_2+s-1}{m_1+r-1} < m_2-\beta. \tag{2.14}$$

By Young's inequality, we have that

$$\begin{aligned}\int_\Omega u_\varepsilon^{(m_2+s-1)/(m_2-\beta)} dx &\le \frac{r(r-1)m_1 K^{(m_2+s-1)/(m_2-\beta)}}{C(m_1+r-1)^2} \int_\Omega u_\varepsilon^{m_1+r-1} dx + CK^{-\gamma_1}, \\ \int_\Omega v_\varepsilon^{(m_1+r-1)/(m_1-\alpha)} dx &\le \frac{s(s-1)m_2 K^{(m_1+r-1)/(m_1-\alpha)}}{C(m_2+s-1)^{p_2}} \int_\Omega v_\varepsilon^{m_2+s-1} dx + CK^{-\gamma_2},\end{aligned} \tag{2.15}$$

where

$$\begin{aligned}\gamma_1 &= \frac{(m_2+s-1)^2}{[m_2-\beta]\left[(m_2-\beta)(m_1+r-1)-(m_2+s-1)\right]}, \\ \gamma_2 &= \frac{(m_1+r-1)^2}{[m_1-\alpha]\left[(m_1-\alpha)(m_2+s-1)-(m_1+r-1)\right]}.\end{aligned} \tag{2.16}$$

Together with (2.13), we have that

$$\int_\Omega\left(\frac{\partial u_\varepsilon^r}{\partial t}+\frac{\partial v_\varepsilon^s}{\partial t}\right)dx \le -K\int_\Omega\left(u_\varepsilon^{m_1+r-1}+v_\varepsilon^{m_2+s-1}\right)dx + C\left(K^{-\theta_1}+K^{-\theta_2}\right) + CK^{-(\alpha+r-1)/(m_1-\alpha)} + CK^{-(\beta+s-1)/(m_2-\beta)}, \tag{2.17}$$

where

$$\theta_1 = \frac{(m_2+s-1)+(m_1+r-1)(\beta+s-1)}{(m_2-\beta)(m_1+r-1)-(m_2+s-1)}, \qquad \theta_2 = \frac{(m_1+r-1)+(m_2+s-1)(\alpha+r-1)}{(m_1-\alpha)(m_2+s-1)-(m_1+r-1)}. \tag{2.18}$$

Furthermore, by Hölder's and Young's inequalities, from (2.17) we obtain

$$\int_\Omega\left(\frac{\partial u_\varepsilon^r}{\partial t}+\frac{\partial v_\varepsilon^s}{\partial t}\right)dx \le -K\int_\Omega(u_\varepsilon^r+v_\varepsilon^s)dx + C\left(K^{-\theta_1}+K^{-\theta_2}\right)+2K|\Omega| + CK^{-(\alpha+r-1)/(m_1-\alpha)} + CK^{-(\beta+s-1)/(m_2-\beta)}. \tag{2.19}$$

Then by Gronwall's inequality, we obtain

$$\int_\Omega(u_\varepsilon^r+v_\varepsilon^s)dx \le C. \tag{2.20}$$

Now we consider the case of $1=(m_1-\alpha)(m_2-\beta)$. It is easy to see that there exist positive constants r, s large enough such that

$$\frac{1}{m_1-\alpha}=\frac{m_2+s-1}{m_1+r-1}=m_2-\beta. \tag{2.21}$$

Due to the continuous dependence of K upon $|\Omega|$ in (2.9), from (2.13) we have that

$$\int_\Omega\left(\frac{\partial u_\varepsilon^r}{\partial t}+\frac{\partial v_\varepsilon^s}{\partial t}\right)dx \le -K\int_\Omega\left(u_\varepsilon^{m_1+r-1}+v_\varepsilon^{m_2(p_2-1)+s-1}\right)dx + C \tag{2.22}$$

when $|\Omega|$ is small enough. Then by Young's and Gronwall's inequalities we can also obtain (2.20), and thus we complete the proof of this lemma. □

Taking $u_\varepsilon^{m_1}, v_\varepsilon^{m_2}$ as the test functions, we can easily obtain the following lemma.

Lemma 2.2. *Let $(u_\varepsilon, v_\varepsilon)$ be a solution of (2.1)–(2.4); then*

$$\iint_{Q_T}|\nabla u_\varepsilon^{m_1}|^2 dx\,dt \le C, \qquad \iint_{Q_T}|\nabla v_\varepsilon^{m_2}|^2 dx\,dt \le C, \tag{2.23}$$

where C is a positive constant independent of ε.

Lemma 2.3. *Let* $(u_\varepsilon, v_\varepsilon)$ *be a solution of* (2.1)–(2.4), *then*

$$\|u_\varepsilon\|_{L^\infty(Q_T)} \leq C, \qquad \|v_\varepsilon\|_{L^\infty(Q_T)} \leq C, \tag{2.24}$$

where C is a positive constant independent of ε.

Proof. For a positive constant $k > \|u_{0\varepsilon}\|_{L^\infty(\Omega)}$, multiplying (2.1) by $(u_\varepsilon - k)_+^{m_1}\chi_{[t_1,t_2]}$ and integrating the results over Q_T, we have that

$$\begin{aligned}
&\frac{1}{m_1+1}\iint_{Q_T}\frac{\partial(u_\varepsilon - k)_+^{m_1+1}\chi_{[t_1,t_2]}}{\partial t}dx\,dt + \iint_{Q_T}\left|\nabla(u_\varepsilon - k)_+^{m_1}\chi_{[t_1,t_2]}\right|^2 dx\,dt \\
&\quad\leq \iint_{Q_T} a u_\varepsilon^{\alpha+m_1}(a + c v_\varepsilon)dx\,dt,
\end{aligned}\tag{2.25}$$

where $s_+ = \max\{0, s\}$ and $\chi_{[t_1,t_2]}$ is the characteristic function of $[t_1, t_2]$ $(0 \leq t_1 < t_2 \leq T)$. Let

$$I_k(t) = \int_\Omega (u_\varepsilon - k)_+^{m_1+1}dx; \tag{2.26}$$

then $I_k(t)$ is absolutely continuous on $[0, T]$. Denote by σ the point where $I_k(t)$ takes its maximum. Assume that $\sigma > 0$, for a sufficient small positive constant ϵ. Taking $t_1 = \sigma - \epsilon$, $t_2 = \sigma$ in (2.25), we obtain

$$\begin{aligned}
&\frac{1}{(m_1+1)\epsilon}\int_{\sigma-\epsilon}^{\sigma}\int_\Omega \frac{\partial(u_\varepsilon - k)_+^{m_1+1}}{\partial t}dx\,dt + \frac{1}{\epsilon}\int_{\sigma-\epsilon}^{\sigma}\int_\Omega\left|\nabla(u_\varepsilon - k)_+^{m_1}\right|^2 dx\,dt \\
&\quad\leq \frac{1}{\epsilon}\int_{\sigma-\epsilon}^{\sigma}\int_\Omega u_\varepsilon^{\alpha+m_1}(a + c v_\varepsilon)dx\,dt.
\end{aligned}\tag{2.27}$$

From

$$\int_{\sigma-\epsilon}^{\sigma}\int_\Omega \frac{\partial(u_\varepsilon - k)_+^{m_1+1}}{\partial t}dx\,dt = I_k(\sigma) - I_k(\sigma-\epsilon) \geq 0, \tag{2.28}$$

we have that

$$\frac{1}{\epsilon}\int_{\sigma-\epsilon}^{\sigma}\int_\Omega\left|\nabla(u_\varepsilon - k)_+^{m_1}\right|^2 dx\,dt \leq \frac{1}{\epsilon}\int_{\sigma-\epsilon}^{\sigma}\int_\Omega u_\varepsilon^{\alpha+m_1}(a + c v_\varepsilon)dx\,dt. \tag{2.29}$$

Letting $\epsilon \to 0^+$, we have that

$$\int_\Omega \left|\nabla (u_\varepsilon(x,\sigma)-k)_+^{m_1}\right|^2 dx \le \int_\Omega u_\varepsilon^{\alpha+m_1}(x,\sigma)(a+cv_\varepsilon(x,\sigma))dx. \tag{2.30}$$

Denote $A_k(t)=\{x : u_\varepsilon(x,t)>k\}$ and $\mu_k=\sup_{t\in(0,T)}|A_k(t)|$; then

$$\int_{A_k(\sigma)} \left|\nabla (u_\varepsilon-k)_+^{m_1}\right|^2 dx \le \int_{A_k(\sigma)} u_\varepsilon^{\alpha+m_1}(a+cv_\varepsilon)dx. \tag{2.31}$$

By Sobolev's theorem,

$$\left(\int_{A_k(\sigma)} \left((u_\varepsilon-k)_+^{m_1}\right)^p dx\right)^{1/p} \le C\left(\int_{A_k(\sigma)} \left|\nabla (u_\varepsilon-k)_+^{m_1}\right|^2 dx\right)^{1/2}, \tag{2.32}$$

with

$$2<p<\begin{cases} +\infty, & N\le 2, \\ \dfrac{2N}{N-2}, & N>2, \end{cases} \tag{2.33}$$

we obtain

$$\begin{aligned}
\left(\int_{A_k(\sigma)} \left((u_\varepsilon-k)_+^{m_1}\right)^p dx\right)^{2/p} &\le C\int_{A_k(\sigma)} \left|\nabla (u_\varepsilon-k)_+^{m_1}\right|^2 dx \\
&\le C\int_{A_k(\sigma)} u_\varepsilon^{\alpha+m_1}(a+v_\varepsilon)dx \\
&\le C\left(\int_{A_k(\sigma)} u_\varepsilon^r dx\right)^{(m_1+\alpha)/r}\left(\int_{A_k(\sigma)} (a+v_\varepsilon)^{r/(r-m_1-\alpha)}dx\right)^{(r-m_1-\alpha)/r} \\
&\le C\left(\int_{A_k(\sigma)} (a+v_\varepsilon)^{r/(r-m_1-\alpha)}dx\right)^{(r-m_1-\alpha)/r} \\
&\le C\left(\int_{A_k(\sigma)} (a+v_\varepsilon)^s dx\right)^{1/s} |A_k(\sigma)|^{(s(r-m_1-\alpha)-r)/sr} \\
&\le C\mu_k^{(s(r-m_1-\alpha)-r)/sr},
\end{aligned} \tag{2.34}$$

where $r > p(m_1 + \alpha)/(p - 2)$, $s > pr/(p(r - m_1 - \alpha) - 2r)$ and C denotes various positive constants independent of ε. By Hölder's inequality, it yields

$$\begin{aligned} I_k(\sigma) &= \int_\Omega (u_\varepsilon - k)_+^{m_1+1} dx = \int_{A_k(\sigma)} (u_\varepsilon - k)_+^{m_1+1} dx \\ &\leq \left(\int_{A_k(\sigma)} (u_\varepsilon - k)_+^{m_1 p} dx \right)^{(m_1+1)/m_1 p} \mu_k^{1-(m_1+1)/m_1 p} \\ &\leq C\mu_k^{1+[sp(r-m_1-\alpha)-pr-2sr](m_1+1)/2psrm_1}. \end{aligned} \tag{2.35}$$

Then

$$I_k(t) \leq I_k(\sigma) \leq C\mu_k^{1+[sp(r-m_1-\alpha)-pr-2sr](m_1+1)/2psrm_1}, \quad t \in [0,T]. \tag{2.36}$$

On the other hand, for any $h > k$ and $t \in [0,T]$, we have that

$$I_k(t) \geq \int_{A_k(t)} (u_\varepsilon - k)_+^{m_1+1} dx \geq (h-k)^{m_1+1} |A_h(t)|. \tag{2.37}$$

Combined with (2.35), it yields

$$(h-k)^{m_1+1} \mu_h \leq C\mu_k^{1+[sp(r-m_1-\alpha)-pr-2sr](m_1+1)/2psrm_1}, \tag{2.38}$$

that is,

$$\mu_h \leq \frac{C}{(h-k)^{m_1+1}} \mu_k^{1+[sp(r-m_1-\alpha)-pr-2sr](m_1+1)/2psrm_1}. \tag{2.39}$$

It is easy to see that

$$\gamma = 1 + \frac{\left[sp(r - m_1 - \alpha) - pr - 2sr\right](m_1 + 1)}{2psrm_1} > 1. \tag{2.40}$$

Then by the De Giorgi iteration lemma [22], we have that

$$\mu_{l+d} = \sup|A_{l+d}(t)| = 0, \tag{2.41}$$

where $d = C^{1/(m_1+1)} \mu_l^{(\gamma-1)/(m_1+1)} 2^{\gamma/(\gamma-1)}$. That is,

$$u_\varepsilon \leq l + d \quad \text{a.e. in } Q_T. \tag{2.42}$$

It is the same for the second inequality of (2.24). The proof is completed. □

Lemma 2.4. *The solution* $(u_\varepsilon, v_\varepsilon)$ *of* (2.1)–(2.4) *satisfies the following:*

$$\iint_{Q_T} \left| \frac{\partial u_\varepsilon^{m_1}}{\partial t} \right|^2 dx\,dt \le C, \qquad \iint_{Q_T} \left| \frac{\partial v_\varepsilon^{m_2}}{\partial t} \right|^2 dx\,dt \le C, \tag{2.43}$$

where C *is a positive constant independent of* ε.

Proof. Multiplying (2.1) by $(\partial/\partial t)u_\varepsilon^{m_1}$ and integrating over Ω, by (2.3), (2.4) and Young's inequality we have that

$$\begin{aligned}
&\frac{4m_1}{(m_1+1)^2}\iint_{Q_T}\left|\frac{\partial}{\partial t}u_\varepsilon^{(m_1+1)/2}\right|^2 dx\,dt \\
&\quad= \iint_{Q_T}\frac{\partial u_\varepsilon}{\partial t}\frac{\partial u_\varepsilon^{m_1}}{\partial t}dx\,dt \\
&\quad= \frac{1}{2}\int_\Omega |\nabla u_\varepsilon^{m_1}(x,0)|^2 dx - \frac{1}{2}\int_\Omega |\nabla u_\varepsilon^{m_1}(x,T)|^2 dx \\
&\qquad+ \iint_{Q_T} m_1 u_\varepsilon^{\alpha+m_1-1}(a - bu_\varepsilon + cv_\varepsilon)\frac{\partial u_\varepsilon}{\partial t}dx\,dt \\
&\quad= \frac{1}{2}\int_\Omega |\nabla u_\varepsilon^{m_1}(x,0)|^2 dx - \frac{1}{2}\int_\Omega |\nabla u_\varepsilon^{m_1}(x,T)|^2 dx \\
&\qquad+ \iint_{Q_T} \frac{2m_1}{m_1+1} u_\varepsilon^{(2\alpha+m_1-1)/2}(a - bu_\varepsilon + cv_\varepsilon)\frac{\partial u_\varepsilon^{(m_1+1)/2}}{\partial t}dx\,dt \\
&\quad\le \frac{1}{2}\int_\Omega |\nabla u_\varepsilon^{m_1}(x,0)|^2 dx + 2m_1\iint_{Q_T} u_\varepsilon^{2\alpha+m_1-1}(a - bu_\varepsilon + cv_\varepsilon)^2 dx\,dt \\
&\qquad+ \frac{2m_1}{(m_1+1)^2}\iint_{Q_T}\left|\frac{\partial}{\partial t}u_\varepsilon^{(m_1+1)/2}\right|^2 dx\,dt,
\end{aligned} \tag{2.44}$$

which together with the bound of $a, b, c, u_\varepsilon, v_\varepsilon$ shows that

$$\iint_{Q_T}\left|\frac{\partial u_\varepsilon^{(m_1+1)/2}}{\partial t}\right|^2 dx\,dt \le C, \tag{2.45}$$

where C is a positive constant independent of ε. Noticing the bound of u_ε, we have that

$$\iint_{Q_T}\left|\frac{\partial u_\varepsilon^{m_1}}{\partial t}\right|^2 dx\,dt = \frac{4m_1^2}{(m_1+1)^2}\iint_{Q_T} u_\varepsilon^{m_1-1}\left|\frac{\partial}{\partial t}u_\varepsilon^{(m_1+1)/2}\right|^2 dx\,dt \le C. \tag{2.46}$$

It is the same for the second inequality. The proof is completed. □

From the above estimates of $u_\varepsilon, v_\varepsilon$, we have the following results.

Theorem 2.5. *The problem* (1.1)–(1.4) *admits a generalized solution.*

Proof. By Lemmas 2.2, 2.3, and 2.4, we can see that there exist subsequences of $\{u_\varepsilon\}, \{v_\varepsilon\}$ (denoted by themselves for simplicity) and functions u, v such that

$$\begin{gathered} u_\varepsilon \longrightarrow u, \quad v_\varepsilon \longrightarrow v, \quad \text{a.e in } Q_T, \\ \frac{\partial u_\varepsilon^{m_1}}{\partial t} \longrightarrow \frac{\partial u^{m_1}}{\partial t}, \quad \frac{\partial v_\varepsilon^{m_2}}{\partial t} \longrightarrow \frac{\partial v^{m_2}}{\partial t}, \quad \text{weakly in } L^2(Q_T), \\ \nabla u_\varepsilon^{m_1} \longrightarrow \nabla u^{m_1}, \quad \nabla v_\varepsilon^{m_2} \longrightarrow \nabla v^{m_2}, \quad \text{weakly in } L^2(Q_T), \end{gathered} \tag{2.47}$$

as $\varepsilon \to 0$. Then a rather standard argument as [23] shows that (u, v) is a generalized solution of (1.1)–(1.4) in the sense of Definition 1.1. □

In order to prove that the generalized solution of (1.1)–(1.4) is uniformly bounded, we need the following comparison principle.

Lemma 2.6. *Let* $(\underline{u}, \underline{v})$ *be a subsolution of the problem* (1.1)–(1.4) *with the initial value* $(\underline{u}_0, \underline{v}_0)$ *and* $(\overline{u}, \overline{v})$ *a supersolution with a positive lower bound of the problem* (1.1)–(1.4) *with the initial value* $(\overline{u}_0, \overline{v}_0)$. *If* $\underline{u}_0 \le \overline{v}_0$, $\underline{u}_0 \le \overline{v}_0$, *then* $\underline{u}(x,t) \le \overline{u}(x,t)$, $\underline{v}(x,t) \le \overline{v}(x,t)$ *on* Q_T.

Proof. Without loss of generality, we might assume that $\|\underline{u}(x,t)\|_{L^\infty(Q_T)}$, $\|\overline{u}(x,t)\|_{L^\infty(Q_T)}$, $\|\underline{v}(x,t)\|_{L^\infty(Q_T)}$, $\|\overline{v}(x,t)\|_{L^\infty(Q_T)} \le M$, where M is a positive constant. By the definitions of subsolution and supersolution, we have that

$$\begin{aligned} &\int_0^t \int_\Omega -\underline{u}\frac{\partial \varphi}{\partial t} + \nabla \underline{u}^{m_1} \nabla \varphi dx\, d\tau + \int_\Omega \underline{u}(x,t)\varphi(x,t)dx - \int_\Omega \underline{u}_0(x)\varphi(x,0)dx \\ &\qquad \le \int_0^t \int_\Omega \underline{u}^\alpha (a - b\underline{u} + c\underline{v})\varphi dx\, d\tau, \\ &\int_0^t \int_\Omega -\overline{u}\frac{\partial \varphi}{\partial t} + \nabla \overline{u}^{m_1} \nabla \varphi dx\, d\tau + \int_\Omega \overline{u}(x,t)\varphi(x,t)dx - \int_\Omega \overline{v}_0(x)\varphi(x,0)dx \\ &\qquad \ge \int_0^t \int_\Omega \overline{u}^\alpha (a - b\overline{u} + c\overline{v})\varphi dx\, d\tau. \end{aligned} \tag{2.48}$$

Take the test function as

$$\varphi(x,t) = H_\varepsilon\left(\underline{u}^{m_1}(x,t) - \overline{u}^{m_1}(x,t)\right), \tag{2.49}$$

where $H_\varepsilon(s)$ is a monotone increasing smooth approximation of the function $H(s)$ defined as follows:

$$H(s) = \begin{cases} 1, & s > 0, \\ 0, & \text{otherwise}. \end{cases} \tag{2.50}$$

It is easy to see that $H_\varepsilon'(s) \to \delta(s)$ as $\varepsilon \to 0$. Since $\partial \underline{u}^{m_1}/\partial t$, $\partial \overline{u}^{m_1}/\partial t \in L^2(Q_T)$, the test function $\varphi(x,t)$ is suitable. By the positivity of a, b, c we have that

$$\begin{aligned}
&\int_\Omega (\underline{u}-\overline{u}) H_\varepsilon(\underline{u}^{m_1}-\overline{u}^{m_1})\,dx - \int_0^t\int_\Omega (\underline{u}-\overline{u}) \frac{\partial H_\varepsilon(\underline{u}^{m_1}-\overline{u}^{m_1})}{\partial t}\,dx\,d\tau \\
&\quad + \int_0^t\int_\Omega H_\varepsilon'(\underline{u}^{m_1}-\overline{u}^{m_1}) \left|\nabla(\underline{u}^{m_1}-\overline{u}^{m_1})\right|^2 dx\,d\tau \\
&\quad \le \int_0^t\int_\Omega a(\underline{u}^\alpha-\overline{u}^\alpha) H_\varepsilon(\underline{u}^{m_1}-\overline{u}^{m_1}) + c(\underline{u}^\alpha\underline{v}-\overline{u}^\alpha\overline{v}) H_\varepsilon(\underline{u}^{m_1}-\overline{u}^{m_1})\,dx\,d\tau,
\end{aligned} \tag{2.51}$$

where C is a positive constant depending on $\|a(x,t)\|_{C(Q_t)}$, $\|c(x,t)\|_{C(Q_t)}$. Letting $\varepsilon \to 0$ and noticing that

$$\int_0^t\int_\Omega H_\varepsilon'(\underline{u}^{m_1}-\overline{u}^{m_1}) \left|\nabla(\underline{u}^{m}-\overline{u}^{m})\right|^2 dx\,d\tau \ge 0, \tag{2.52}$$

we arrive at

$$\int_\Omega [\underline{u}(x,t)-\overline{u}(x,t)]_+ dx \le C\int_0^t\int_\Omega (\underline{u}^\alpha-\overline{u}^\alpha)_+ + \underline{v}(\underline{u}^\alpha-\overline{u}^\alpha)_+ + \overline{u}^\alpha(\underline{v}-\overline{v})_+ dx\,d\tau. \tag{2.53}$$

Let $(\overline{u},\overline{v})$ be a supsolution with a positive lower bound σ. Noticing that

$$\begin{aligned}
&(x^\alpha-y^\alpha)_+ \le C(\alpha)(x-y)_+, \quad \text{for } \alpha \ge 1, \\
&(x^\alpha-y^\alpha)_+ \le x^{\alpha-1}(x-y)_+ \le y^{\alpha-1}(x-y)_+, \quad \text{for } \alpha < 1,
\end{aligned} \tag{2.54}$$

with $x, y > 0$, we have that

$$\int_0^t\int_\Omega (\underline{u}^\alpha-\overline{u}^\alpha)_+ + \underline{v}(\underline{u}^\alpha-\overline{u}^\alpha)_+ + \overline{u}^\alpha(\underline{v}-\overline{v})_+ dx\,d\tau \le C\int_0^t\int_\Omega (\underline{u}-\overline{u})_+ + (\underline{v}-\overline{v})_+ dx\,d\tau, \tag{2.55}$$

where C is a positive constant depending upon α, σ, M.

Similarly, we also have that

$$\int_\Omega [\underline{v}(x,t)-\overline{v}(x,t)]_+ dx \le C\int_0^t\int_\Omega (\underline{u}-\overline{u})_+ + (\underline{v}-\overline{v})_+ dx\,d\tau. \tag{2.56}$$

Combining the above two inequalities, we obtain

$$\int_{\Omega} [\underline{u}(x,t) - \overline{u}(x,t)]_+ + [\underline{v}(x,t) - \overline{v}(x,t)]_+ dx \le C \int_0^t \int_{\Omega} (\underline{u} - \overline{u})_+ + (\underline{v} - \overline{v})_+ dx\, d\tau. \quad (2.57)$$

By Gronwall's lemma, we see that $\underline{u} \le \overline{u}$, $\underline{v} \le \overline{v}$. The proof is completed. □

Corollary 2.7. *If $b_l f_l > c_M e_M$, then the problem* (1.1)–(1.4) *admits at most one global solution which is uniformly bounded in $\overline{\Omega} \times [0,\infty)$.*

Proof. The uniqueness comes from the comparison principle immediately. In order to prove that the solution is global, we just need to construct a bounded positive supersolution of (1.1)–(1.4).

Let $\rho_1 = (a_M f_l + d_M c_M)/(b_l f_l - c_M e_M)$ and $\rho_2 = (a_M e_M + d_M b_l)/(b_l f_l - c_M e_M)$, since $b_l f_l > c_M e_M$; then $\rho_1, \rho_2 > 0$ and satisfy

$$a_M - b_l \rho_1 + c_M \rho_2 = 0, \qquad d_M + e_M \rho_1 - f_l \rho_2 = 0. \quad (2.58)$$

Let $(\overline{u}, \overline{v}) = (\eta\rho_1, \eta\rho_2)$, where $\eta > 1$ is a constant such that $(u_0, v_0) \le (\eta\rho_1, \eta\rho_2)$; then we have that

$$\overline{u}_t - \Delta \overline{u}^{m_1} = 0 \ge \overline{u}^{\alpha}(a - b\overline{u} + c\overline{v}), \qquad \overline{v}_t - \Delta \overline{v}^{m_2} = 0 \ge \overline{v}^{\beta}(d + e\overline{u} - f\overline{v}). \quad (2.59)$$

That is, $(\overline{u}, \overline{v}) = (\eta\rho_1, \eta\rho_2)$ is a positive supersolution of (1.1)–(1.4). Since $\overline{u}, \overline{v}$ are global and uniformly bounded, so are u and v. □

3. Periodic Solutions

In order to establish the existence of the nontrivial nonnegative periodic solutions of the problem (1.1)–(1.3), we need the following lemmas. Firstly, we construct a pair of T-periodic supersolution and T-periodic subsolution as follows.

Lemma 3.1. *In case of $b_l f_l > c_M e_M$, there exists a pair of T-periodic supersolution and T-periodic subsolution of the problem* (1.1)–(1.3).

Proof. We first construct a T-periodic subsolution of (1.1)–(1.3). Let λ be the first eigenvalue and ϕ be the uniqueness solution of the following elliptic problem:

$$-\Delta\phi = \lambda\phi, \quad x \in \Omega, \quad \phi = 0, \quad x \in \partial\Omega; \quad (3.1)$$

then we have that

$$\lambda > 0, \quad \phi(x) > 0 \quad \text{in } \Omega, \quad |\nabla\phi| > 0 \quad \text{on } \partial\Omega, \quad M = \max_{x\in\overline{\Omega}} \phi(x) < \infty. \quad (3.2)$$

Let

$$(\underline{u}, \underline{v}) = \left(\varepsilon\phi^{2/m_1}(x), \varepsilon\phi^{2/m_2}(x)\right), \tag{3.3}$$

where $\varepsilon > 0$ is a small constant to be determined. We will show that $(\underline{u}, \underline{v})$ is a (time independent, hence T-periodic) subsolution of (1.1)–(1.3).

Taking the nonnegative function $\varphi_1(x,t) \in C^1(\overline{Q}_T)$ as the test function, we have that

$$\begin{aligned}
&\iint_{Q_T}\left(\underline{u}\frac{\partial\varphi_1}{\partial t} + \Delta\underline{u}^{m_1}\varphi_1 + \underline{u}^{\alpha}(a - b\underline{u} + c\underline{v})\varphi_1\right)dx\,dt \\
&\qquad + \int_{\Omega}\underline{u}(x,0)\varphi_1(x,0) - \underline{u}(x,T)\varphi_1(x,T)dx \\
&\quad = \iint_{Q_T}\left(\underline{u}^{\alpha}(a - b\underline{u} + c\underline{v}) + \Delta\underline{u}^{m_1}\right)\varphi_1 dx\,dt \\
&\quad = \iint_{Q_T}\underline{u}^{\alpha}(a - b\underline{u} + c\underline{v})\varphi_1 dx\,dt - \iint_{Q_T}\nabla\underline{u}^{m_1}\nabla\varphi_1 dx\,dt \\
&\quad = \iint_{Q_T}\underline{u}^{\alpha}(a - b\underline{u} + c\underline{v})\varphi_1 dx\,dt - 2\varepsilon^{m_1}\iint_{Q_T}\phi\nabla\phi\cdot\nabla\varphi_1 dx\,dt \\
&\quad = \iint_{Q_T}\underline{u}^{\alpha}(a - b\underline{u} + c\underline{v})\varphi_1 dx\,dt - 2\varepsilon^{m_1}\iint_{Q_T}\nabla\phi\nabla(\phi\varphi_1) - |\nabla\phi|^2\varphi_1 dx\,dt \\
&\quad = \iint_{Q_T}\underline{u}^{\alpha}(a - b\underline{u} + c\underline{v})\varphi_1 dx\,dt - 2\varepsilon^{m_1}\iint_{Q_T} - \operatorname{div}(\nabla\phi)\phi\varphi_1 - |\nabla\phi|^2\varphi_1 dx\,dt \\
&\quad = \iint_{Q_T}\underline{u}^{\alpha}(a - b\underline{u} + c\underline{v})\varphi_1 dx\,dt - 2\varepsilon^{m_1}\iint_{Q_T}\left(\lambda\phi^2 - |\nabla\phi|^2\right)\varphi_1 dx\,dt.
\end{aligned} \tag{3.4}$$

Similarly, for any nonnegative test function $\varphi_2(x,t) \in C^1(\overline{Q}_T)$, we have that

$$\begin{aligned}
&\iint_{Q_T}\left(\underline{v}\frac{\partial\varphi_2}{\partial t} + \Delta\underline{v}^{m_2}\varphi_2 + \underline{v}^{\beta}(d + e\underline{u} - f\underline{v})\varphi_2\right)dx\,dt + \int_{\Omega}\underline{v}(x,0)\varphi_2(x,0) - \underline{v}(x,T)\varphi_2(x,T)dx \\
&\quad = \iint_{Q_T}\underline{v}^{\beta}(d + e\underline{u} - f\underline{v})\varphi_2 dx\,dt - 2\varepsilon^{m_2}\iint_{Q_T}\left(\lambda\phi^2 - |\nabla\phi|^2\right)\varphi_2 dx\,dt.
\end{aligned} \tag{3.5}$$

We just need to prove the nonnegativity of the right-hand side of (3.4) and (3.5).

Since $\phi_1 = \phi_2 = 0$, $|\nabla\phi_1|, |\nabla\phi_2| > 0$ on $\partial\Omega$, then there exists $\delta > 0$ such that

$$\lambda\phi^2 - |\nabla\phi|^2 \le 0, \qquad x \in \overline{\Omega}_{\delta}, \tag{3.6}$$

where $\overline{\Omega}_\delta = \{x \in \Omega \mid \operatorname{dist}(x, \partial\Omega) \le \delta\}$. Choosing

$$\varepsilon \le \min\left\{ \frac{a_l}{b_M M^{2/m_1}}, \frac{d_l}{f_M M^{2/m_2}} \right\}, \tag{3.7}$$

then we have that

$$\begin{aligned} &2\varepsilon^{m_1} \int_0^T \int_{\Omega_\delta} \left(\lambda\phi^2 - |\nabla\phi|^2\right)\varphi_1 dx\, dt \le 0 \le \int_0^T \int_{\Omega_\delta} \underline{u}^\alpha (a - b\underline{u} + c\underline{v})\varphi_1 dx\, dt, \\ &2\varepsilon^{m_2} \int_0^T \int_{\Omega_\delta} \left(\lambda\phi^2 - |\nabla\phi|^2\right)\varphi_2 dx\, dt \le 0 \le \int_0^T \int_{\Omega_\delta} \underline{v}^\beta (d + e\underline{u} - f\underline{v})\varphi_2 dx\, dt, \end{aligned} \tag{3.8}$$

which shows that $(\underline{u}, \underline{v})$ is a positive (time independent, hence T-periodic) subsolution of (1.1)–(1.3) on $\overline{\Omega}_\delta \times (0, T)$.

Moreover, we can see that, for some $\sigma > 0$,

$$\phi(x) \ge \sigma > 0, \quad x \in \Omega \setminus \overline{\Omega}_\delta. \tag{3.9}$$

Choosing

$$\varepsilon \le \min\left\{ \frac{a_l}{2b_M M^{2/m_1}}, \left(\frac{a_l}{4\lambda M^{2(m_1-\alpha)/m_1}}\right)^{1/(m_1-\alpha)}, \frac{d_l}{2f_M M^{2/m_2}}, \left(\frac{d_l}{4\lambda M^{2(m_2-\beta)/m_2}}\right)^{1/(m_2-\beta)} \right\}, \tag{3.10}$$

then

$$\begin{aligned} &\varepsilon^\alpha \phi^{2\alpha/m_1} a - b\varepsilon^{\alpha+1}\phi^{2(\alpha+1)/m_1} + c\varepsilon^\alpha \phi^{2\alpha/m_1}\varepsilon\phi^{2/m_2} - 2\varepsilon^{m_1}\lambda\phi^2 \ge 0, \\ &\varepsilon^\beta \phi^{2\beta/m_2} d + e\varepsilon\phi^{2/m_1}\varepsilon^\beta\phi^{2\beta/m_2} - f\varepsilon^{\beta+1}\phi^{2(\beta+1)/m_2} - 2\varepsilon^{m_2}\lambda\phi^2 \ge 0 \end{aligned} \tag{3.11}$$

on Q_T, that is

$$\begin{aligned} &\iint_{Q_T} \underline{u}^\alpha (a - b\underline{u} + c\underline{v})\varphi_1 dx\, dt - 2\varepsilon^{m_1} \iint_{Q_T} \left(\lambda\phi^2 - |\nabla\phi|^2\right)\varphi_1 dx\, dt \ge 0, \\ &\iint_{Q_T} \underline{v}^\beta (d + e\underline{u} - f\underline{v})\varphi_2 dx\, dt - 2\varepsilon^{m_2} \iint_{Q_T} \left(\lambda\phi^2 - |\nabla\phi|^2\right)\varphi_2 dx\, dt \ge 0. \end{aligned} \tag{3.12}$$

These relations show that $(\underline{u}, \underline{v}) = (\varepsilon\phi_1^{2/m_1}(x), \varepsilon\phi_2^{2/m_2}(x))$ is a positive (time independent, hence T-periodic) subsolution of (1.1)–(1.3).

Letting $(\overline{u}, \overline{v}) = (\eta\rho_1, \eta\rho_2)$, where η, ρ_1, ρ_2 are taken as those in Corollary 2.7, it is easy to see that $(\overline{u}, \overline{v})$ is a positive (time independent, hence T-periodic) subsolution of (1.1)–(1.3).

Obviously, we may assume that $\underline{u}(x,t) \le \overline{u}(x,t)$, $\underline{v}(x,t) \le \overline{v}(x,t)$ by changing η, ε appropriately. $\square$

Lemma 3.2 (see [24, 25]). *Let u be the solution of the following Dirichlet boundary value problem*

$$\frac{\partial u}{\partial t} = \Delta u^m + f(x,t), \quad (x,t) \in \Omega \times (0,T),$$
$$u(x,t) = 0, \quad (x,t) \in \partial\Omega \times (0,T), \tag{3.13}$$

where $f \in L^\infty(\Omega \times (0,T))$; then there exist positive constants K and $\alpha \in (0,1)$ depending only upon $\tau \in (0,T)$ and $\|f\|_{L^\infty(\Omega\times(0,T))}$, such that, for any $(x_i,t_i) \in \Omega \times [\tau,T]$ $(i=1,2)$,

$$|u(x_1,t_1) - u(x_2,t_2)| \le K\left(|x_1-x_2|^\alpha + |t_1-t_2|^{\alpha/2}\right). \tag{3.14}$$

Lemma 3.3 (see [26]). *Define a Poincaré mapping*

$$P_t : L^\infty(\Omega) \times L^\infty(\Omega) \longrightarrow L^\infty(\Omega) \times L^\infty(\Omega),$$
$$P_t(u_0(x), v_0(x)) := (u(x,t), v(x,t)) \quad (t > 0), \tag{3.15}$$

where $(u(x,t),v(x,t))$ is the solution of (1.1)–(1.4) with initial value $(u_0(x),v_0(x))$. According to Lemmas 2.6 and 3.2 and Theorem 2.5, the map P_t has the following properties:

(i) *P_t is defined for any $t > 0$ and order preserving;*

(ii) *P_t is order preserving;*

(iii) *P_t is compact.*

Observe that the operator P_T is the classical Poincaré map and thus a fixed point of the Poincaré map gives a T-periodic solution setting. This will be made by the following iteration procedure.

Theorem 3.4. *Assume that $b_l f_l > c_M e_M$ and there exists a pair of nontrivial nonnegative T-periodic subsolution $(\underline{u}(x,t), \underline{v}(x,t))$ and T-periodic supersolution $(\overline{u}(x,t), \overline{v}(x,t))$ of the problem (1.1)–(1.3) with $\underline{u}(x,0) \le \overline{u}(x,0)$; then the problem (1.1)–(1.3) admits a pair of nontrivial nonnegative periodic solutions $(u_*(x,t), v_*(x,t))$, $(u^*(x,t), v^*(x,t))$ such that*

$$\underline{u}(x,t) \le u_*(x,t) \le u^*(x,t) \le \overline{u}(x,t), \quad \underline{v}(x,t) \le v_*(x,t) \le v^*(x,t) \le \overline{v}(x,t), \quad \text{in } Q_T. \tag{3.16}$$

Proof. Taking $\overline{u}(x,t), \underline{u}(x,t)$ as those in Lemma 3.1 and choosing suitable $B(x_0,\delta), B(x_0,\delta'), \Omega'$, k_1, k_2, and K, we can obtain $\underline{u}(x,0) \le \overline{u}(x,0)$. By Lemma 2.6, we have that $P_T(\underline{u}(\cdot,0)) \ge \underline{u}(\cdot,T)$. Hence by Definition 1.2 we get $P_T(\underline{u}(\cdot,0)) \ge \underline{u}(\cdot,0)$, which implies $P_{(k+1)T}(\underline{u}(\cdot,0)) \ge P_{kT}(\underline{u}(\cdot,0))$ for any $k \in \mathbb{N}$. Similarly we have that $P_T(\overline{u}(\cdot,0)) \le \overline{u}(\cdot,T) \le \overline{u}(\cdot,0)$, and hence $P_{(k+1)T}(\overline{u}(\cdot,0)) \le P_{kT}(\overline{u}(\cdot,0))$ for any $k \in \mathbb{N}$. By Lemma 2.6, we have that $P_{kT}(\underline{u}(\cdot,0)) \le P_{kT}(\overline{u}(\cdot,0))$ for any $k \in \mathbb{N}$. Then

$$u_*(x,0) = \lim_{k\to\infty} P_{kT}(\underline{u}(x,0)), \qquad u^*(x,0) = \lim_{k\to\infty} P_{kT}(\overline{u}(x,0)) \tag{3.17}$$

exist for almost every $x \in \Omega$. Since the operator P_T is compact (see Lemma 3.3), the above limits exist in $L^\infty(\Omega)$, too. Moreover, both $u_*(x,0)$ and $u^*(x,0)$ are fixed points of P_T. With the similar method as [26], it is easy to show that the even extension of the function $u_*(x,t)$, which is the solution of the problem (1.1)–(1.4) with the initial value $u_*(x,0)$, is indeed a nontrivial nonnegative periodic solution of the problem (1.1)–(1.3). It is the same for the existence of $u^*(x,t)$. Furthermore, by Lemma 2.6, we obtain (3.16) immediately, and thus we complete the proof. □

Furthermore, by De Giorgi iteration technique, we can also establish a prior upper bound of all nonnegative periodic solutions of (1.1)–(1.3). Then with a similar method as [18], we have the following remark which shows the existence and attractivity of the maximal periodic solution.

Remark 3.5. If $b_l f_l > c_M e_M$, the problem (1.1)–(1.3) admits a maximal periodic solution (U,V). Moreover, if (u,v) is the solution of the initial boundary value problem (1.1)–(1.4) with nonnegative initial value (u_0,v_0), then, for any $\varepsilon > 0$, there exists t depending on u_0, v_0, and ε, such that

$$0 \le u \le U + \varepsilon, \quad 0 \le v \le V + \varepsilon, \quad \text{for } x \in \Omega,\ t \ge t. \tag{3.18}$$

Acknowledgments

This work was supported by NSFC (10801061), the Fundamental Research Funds for the Central Universities (Grant no. HIT. NSRIF. 2009049), Natural Sciences Foundation of Heilongjiang Province (Grant no. A200909), and also the 985 project of Harbin Institute of Technology.

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