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Research Article

Some New Delay Integral Inequalities in Two Independent Variables on Time Scales

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Some new Gronwall-Bellman type delay integral inequalities in two independent variables on time scales are established, which can be used as a handy tool in the research of boundedness of solutions of delay dynamic equations on time scales. Some of the established results are 2D extensions of several known results in the literature, while some results unify existing continuous and discrete analysis.

1. Introduction

In the research of solutions of certain differential and difference equations, if the solutions are unknown, then it is necessary to study their qualitative and quantitative properties such as boundedness, uniqueness, and continuous dependence on initial data. The Gronwall-Bellman inequality [1, 2] and its various generalizations, which provide explicit bounds, play a fundamental role in the research of this domain. During the past decades, much effort has been done for developing such inequalities (e.g., see [3–15] and the references therein). On the other hand, Hilger [16] initiated the theory of time scales as a theory capable to contain both difference and differential calculus in a consistent way. Since then many authors have expounded on various aspects of the theory of dynamic equations on time scales (e.g., see [17–19] and the references therein). In these investigations, integral inequalities on time scales have been paid much attention by many authors, which play a fundamental role in the research of quantitative as well as qualitative properties of solutions of certain dynamic equations on time scales. A lot of integral inequalities on time scales have been established (e.g., see [20–26]), which have been designed to unify continuous and discrete analysis. But to our best knowledge, the Gronwall-Bellman-type delay integral inequalities on time scales have been paid little attention in the literature so far. Recent results in this direction include the work of Li [27] and that of Ma and Pečarić [28]. Furthermore, nobody has studied the Gronwall-Bellman-type delay integral inequalities in two independent variables on time scales.

The aim of this paper is to establish some new Gronwall-Bellman-type delay integral inequalities in two independent variables on time scales, which provide new bounds for the unknown functions concerned. Some of our results are 2D extensions of many known inequalities in the literature, while some results unify existing continuous and discrete analysis. For illustrating the validity of the established results, we will present some applications of them.

First we will give some preliminaries on time scales and some universal symbols for further use.

Throughout this paper, $\mathbb{R}$ denotes the set of real numbers and $\mathbb{R}_+ = [0, \infty)$, while $\mathbb{Z}$ denotes the set of integers. For two given sets G, H, we denote the set of maps from G to H by (G, H).

A time scale is an arbitrary nonempty closed subset of the real numbers. In this paper, $\mathbb{T}$ denotes an arbitrary time scale. On $\mathbb{T}$ we define the forward and backward jump operators $\sigma \in (\mathbb{T}, \mathbb{T})$ and $\rho \in (\mathbb{T}, \mathbb{T})$ such that $\sigma(t) = \inf\{s \in \mathbb{T}, s > t\}$, $\rho(t) = \sup\{s \in \mathbb{T}, s < t\}$.

Definition 1.1. The graininess $\mu \in (\mathbb{T}, \mathbb{R}_+)$ is defined by $\mu(t) = \sigma(t) - t$.

Remark 1.2. Obviously, $\mu(t) = 0$ if $\mathbb{T} = \mathbb{R}$ while $\mu(t) = 1$ if $\mathbb{T} = \mathbb{Z}$.

Definition 1.3. A point $t \in \mathbb{T}$ is said to be left-dense if $\rho(t) = t$ and $t \neq \inf \mathbb{T}$, right-dense if $\sigma(t) = t$ and $t \neq \sup \mathbb{T}$, left-scattered if $\rho(t) < t$, and right-scattered if $\sigma(t) > t$.

Definition 1.4. The set $\mathbb{T}^\kappa$ is defined to be $\mathbb{T}$ if $\mathbb{T}$ does not have a left-scattered maximum; otherwise it is $\mathbb{T}$ without the left-scattered maximum.

Definition 1.5. A function $f \in (\mathbb{T}, \mathbb{R})$ is called rd-continuous if it is continuous at right-dense points and if the left-sided limits exist at left-dense points, while f is called regressive if $1 + \mu(t)f(t) \neq 0$. C_{rd} denotes the set of rd-continuous functions, while $\mathfrak{R}$ denotes the set of all regressive and rd-continuous functions, and $\mathfrak{R}^+ = \{f \mid f \in \mathfrak{R}, 1 + \mu(t)f(t) > 0,\ \forall t \in \mathbb{T}\}$.

Definition 1.6. For some $t \in \mathbb{T}^\kappa$ and a function $f \in (\mathbb{T}, \mathbb{R})$, the *delta derivative* of f at t is denoted by $f^\Delta(t)$ (provided it exists) with the property such that for every $\varepsilon > 0$ there exists a neighborhood $\mathfrak{U}$ of t satisfying

$$\left| f(\sigma(t)) - f(s) - f^\Delta(t)(\sigma(t) - s) \right| \leq \varepsilon |\sigma(t) - s| \quad \forall s \in \mathfrak{U}. \tag{1.1}$$

Similarly, for some $y \in \mathbb{T}^\kappa$ and a function $f \in (\mathbb{T} \times \mathbb{T},\ \mathbb{R})$, the *partial delta* of f with respect to y is denoted by $(f(x, y))_y^\Delta$ or $f_y^\Delta(x, y)$ and satisfies

$$\left| f(x, \sigma(y)) - f(x, s) - f_y^\Delta(x, y)(\sigma(y) - s) \right| \leq \varepsilon \left| \sigma(y) - s \right| \quad \forall \varepsilon > 0, \tag{1.2}$$

where $s \in \mathfrak{U}$ and $\mathfrak{U}$ is a neighborhood of y.

Remark 1.7. If $\mathbb{T} = \mathbb{R}$, then $f^\Delta(t)$ becomes the usual derivative $f'(t)$, while $f^\Delta(t) = f(t+1) - f(t)$ if $\mathbb{T} = \mathbb{Z}$, which represents the forward difference.

For more details about the calculus of time scales, see [29]. In the rest of this paper, for the convenience of notation, we always assume that $\mathbb{T}_0 = [x_0, \infty) \bigcap \mathbb{T}, \widetilde{\mathbb{T}}_0 = [y_0, \infty) \bigcap \mathbb{T}$, where $x_0, y_0 \in \mathbb{T}^\kappa$ and furthermore assume $\mathbb{T}_0 \subseteq \mathbb{T}^\kappa, \widetilde{\mathbb{T}}_0 \subseteq \mathbb{T}^\kappa$.

2. Main Results

We will give some lemmas for further use.

Lemma 2.1. *Suppose $X \in \mathbb{T}_0$ is a fixed number, and $u(X,y)$, $a(X,y)$, $b(X,y) \in C_{\mathrm{rd}}$, $m(X,y) \in \mathfrak{R}_+$ with respect to y, $m(X,y) \geq 0$, then*

$$u(X,y) \leq a(X,y) + b(X,y) \int_{y_0}^{y} m(X,t)u(X,t)\Delta t, \quad y \in \widetilde{\mathbb{T}}_0, \tag{2.1}$$

implies

$$u(X,y) \leq a(X,y) + b(X,y) \int_{y_0}^{y} e_{\overline{m}}(y, \sigma(t))a(X,t)m(X,t)\Delta t, \quad y \in \widetilde{\mathbb{T}}_0, \tag{2.2}$$

where $\overline{m}(X,y) = m(X,y)b(X,y)$: and $e_{\overline{m}}(y, y_0)$ is the unique solution of the following equation

$$z_y^{\Delta}(X,y) = m(X,y)z(X,y), z(X,y_0) = 1. \tag{2.3}$$

The proof of Lemma 2.1 is similar to [26, Theorem 5.6].

Lemma 2.2. *Under the conditions of Lemma 2.1, and furthermore assuming $a(x,y)$ is nondecreasing in y for every fixed x, $b(x,y) \equiv 1$, then one has*

$$u(X,y) \leq a(X,y)e_m(y, y_0). \tag{2.4}$$

Proof. Since $a(x,y)$ is nondecreasing in y for every fixed x, then from Lemma 2.1 we have

$$u(X,y) \leq a(X,y) + \int_{y_0}^{y} e_m(y, \sigma(t))a(X,t)m(X,t)\Delta t \leq a(X,y)\left[1 + \int_{y_0}^{y} e_m(y, \sigma(t))m(X,t)\Delta t\right]. \tag{2.5}$$

On the other hand, from [29, Theorems 2.39 and 2.36 (i)] we have $1+\int_{y_0}^{y} e_m(y, \sigma(t))m(X,t)\Delta t = e_m(y, y_0)$. Then collecting the above information, we can obtain the desired inequality. □

Lemma 2.3 (see [11]). *Assume that $a \geq 0$, $p \geq q \geq 0$, and $p \neq 0$; then, for any $K > 0$*

$$a^{q/p} \leq \frac{q}{p} K^{(q-p)/p} a + \frac{p-q}{p} K^{q/p}. \tag{2.6}$$

Lemma 2.4. *Let $h : \mathbb{T} \times \mathbb{R} \to \mathbb{R}$ be continuous and nondecreasing in the second variable, and assume X is a fixed number in $\mathbb{T}$. Suppose $v(X, y)$ and $w(X, y)$ satisfy the dynamics inequalities:*

$$v_y^{\Delta} \leq h(y, v), \qquad w_y^{\Delta} \geq h(y, w). \tag{2.7}$$

Then $v(X, y_0) \leq w(X, y_0)$ for some $y_0 \in \mathbb{T}$ implies $v(X, y) \leq w(X, y)$ for all $y \in \mathbb{T}$.

The proof of Lemma 2.4 is similar to [26, Theorem 5.7].

Theorem 2.5. *Suppose $u, a, b, f \in C_{\mathrm{rd}}(\mathbb{T}_0 \times \widetilde{\mathbb{T}}_0, \mathbb{R}_+)$, and $a(x, y)$, $b(x, y)$ are nondecreasing. p is a constant, and $p \geq 1$. $\tau_1 \in (\mathbb{T}_0, \mathbb{T})$, $\tau_1(x) \leq x, -\infty < \alpha = \inf\{\tau_1(x), x \in \mathbb{T}_0\} \leq x_0$. $\tau_2 \in (\widetilde{\mathbb{T}}_0, \mathbb{T})$, $\tau_2(y) \leq y, -\infty < \beta = \inf\{\tau_2(y), y \in \widetilde{\mathbb{T}}_0\} \leq y_0$. $\phi \in C_{rd}(([\alpha, x_0] \times [\beta, y_0]) \bigcap \mathbb{T}^2, \mathbb{R}_+)$. If for $(x, y) \in \mathbb{T}_0 \times \widetilde{\mathbb{T}}_0$, $u(x, y)$ satisfies the following inequality:*

$$u^p(x, y) \leq a(x, y) + b(x, y) \int_{y_0}^{y} \int_{x_0}^{x} [f(s, t) u(\tau_1(s), \tau_2(t))] \Delta s \, \Delta t, \tag{2.8}$$

with the initial condition

$$\begin{gathered} u(x, y) = \phi(x, y), \quad \text{if } x \in [\alpha, x_0] \bigcap \mathbb{T} \text{ or } y \in [\beta, y_0] \bigcap \mathbb{T}, \\ \phi(\tau_1(x), \tau_2(y)) \leq a^{1/p}(x, y), \quad \forall (x, y) \in \mathbb{T}_0 \times \widetilde{\mathbb{T}}_0, \text{ if } \tau_1(x) \leq x_0 \text{ or } \tau_2(y) \leq y_0, \end{gathered} \tag{2.9}$$

then

$$u(x, y) \leq \left[H_1(x, y) + b(x, y) \int_{y_0}^{y} e_{\overline{H}_2}(y, \sigma(t)) H_2(x, t) H_1(x, t) \Delta t \right]^{1/p}, \quad (x, y) \in \mathbb{T}_0 \times \widetilde{\mathbb{T}}_0, \tag{2.10}$$

where

$$\begin{aligned} H_1(x, y) &= a(x, y) + b(x, y) \int_{y_0}^{y} \int_{x_0}^{x} f(s, t) \frac{p-1}{p} K^{1/p} \Delta s \, \Delta t, \quad \forall K > 0, \\ H_2(x, y) &= \int_{x_0}^{x} f(s, y) \frac{1}{p} K^{(1-p)/p} \Delta s, \\ \overline{H}_2(x, y) &= b(x, y) H_2(x, y). \end{aligned} \tag{2.11}$$

Proof. Fix $X \in \mathbb{T}_0$, and $x \in [x_0, X] \bigcap \mathbb{T}$, $y \in \widetilde{\mathbb{T}}_0$. Let

$$v(x, y) = a(x, y) + b(x, y) \int_{y_0}^{y} \int_{x_0}^{x} [f(s, t) u(\tau_1(s), \tau_2(t))] \Delta s \, \Delta t. \tag{2.12}$$

Then

$$u(x,y) \le v^{1/p}(x,y) \le v^{1/p}(X,y), \quad \forall x \in [x_0, X] \bigcap \mathbb{T},\ y \in \widetilde{\mathbb{T}}_0. \tag{2.13}$$

If $\tau_1(x) \ge x_0$ and $\tau_2(y) \ge y_0$, then $\tau_1(x) \in [x_0, X] \bigcap \mathbb{T},\ \tau_2(y) \in \widetilde{\mathbb{T}}_0$, and

$$u(\tau_1(x), \tau_2(y)) \le v^{1/p}(\tau_1(x), \tau_2(y)) \le v^{1/p}(x,y). \tag{2.14}$$

If $\tau_1(x) \le x_0$ or $\tau_2(y) \le y_0$, then from (2.9) we have

$$u(\tau_1(x), \tau_2(y)) = \phi(\tau_1(x), \tau_2(y)) \le a^{1/p}(x,y) \le v^{1/p}(x,y). \tag{2.15}$$

From (2.14) and (2.15) we always have

$$u(\tau_1(x), \tau_2(y)) \le v^{1/p}(x,y), \quad x \in [x_0, X] \bigcap \mathbb{T},\ y \in \widetilde{\mathbb{T}}_0. \tag{2.16}$$

Moreover

$$\begin{aligned} v(X,y) &= a(X,y) + b(X,y) \int_{y_0}^{y} \int_{x_0}^{X} [f(s,t) u(\tau_1(s), \tau_2(t))] \Delta s\, \Delta t \\ &\le a(X,y) + b(X,y) \int_{y_0}^{y} \int_{x_0}^{X} f(s,t) v^{1/p}(s,t) \Delta s\, \Delta t. \end{aligned} \tag{2.17}$$

From Lemma 2.3, we have

$$v^{1/p}(s,t) \le \frac{1}{p} K^{(1-p)/p} v(s,t) + \frac{p-1}{p} K^{1/p}, \quad \forall K > 0. \tag{2.18}$$

So

$$\begin{aligned} v(X,y) &\le a(X,y) + b(X,y) \int_{y_0}^{y} \int_{x_0}^{X} f(s,t) \left[\frac{1}{p} K^{(1-p)/p} v(s,t) + \frac{p-1}{p} K^{1/p} \right] \Delta s\, \Delta t \\ &\le a(X,y) + b(X,y) \int_{y_0}^{y} \int_{x_0}^{X} f(s,t) \frac{p-1}{p} K^{1/p} \Delta s\, \Delta t \\ &\quad + b(X,y) \int_{y_0}^{y} \left[\int_{x_0}^{X} f(s,t) \frac{1}{p} K^{(1-p)/p} \Delta s \right] v(X,t) \Delta t \\ &= H_1(X,y) + b(X,y) \int_{y_0}^{y} H_2(X,t) v(X,t) \Delta t. \end{aligned} \tag{2.19}$$

Then applying Lemma 2.1 to (2.19), we obtain

$$v(X,y) \leq H_1(X,y) + b(X,y) \int_{y_0}^{y} e_{\overline{H}_2}(y,\sigma(t)) H_2(X,t) H_1(X,t) \Delta t. \tag{2.20}$$

So

$$u(x,y) \leq v^{1/p}(X,y) \leq \left[H_1(X,y) + b(X,y) \int_{y_0}^{y} e_{\overline{H}_2}(y,\sigma(t)) H_2(X,t) H_1(X,t) \Delta t \right]^{1/p}, \quad x \in [x_0, X] \bigcap \mathbb{T}, \quad y \in \widetilde{\mathbb{T}}_0. \tag{2.21}$$

Setting $x = X$ in (2.21), it follows that

$$u(X,y) \leq \left[H_1(X,y) + b(X,y) \int_{y_0}^{y} e_{\overline{H}_2}(y,\sigma(t)) H_2(X,t) H_1(X,t) \Delta t \right]^{1/p}. \tag{2.22}$$

Replacing X with x in (2.22), we obtain the desired inequality. □

Remark 2.6. Theorem 2.5 is the 2D extension of [27, Theorem 1]. For its special case $\mathbb{T} = \mathbb{R}$, the established bound for $u(x,y)$ in (2.10) is a new bound compared with the result in [12, Theorem 2.2].

Remark 2.7. Assume $b(x,y) \equiv 1$ in Theorem 2.5. If we apply Lemma 2.2 instead of Lemma 2.1 to (2.19) in the proof of Theorem 2.5, then we obtain another bound for $u(x,y)$ as follows:

$$u(x,y) \leq \left[H_1(x,y) e_{H_2}(y,y_0) \right]^{1/p}, \quad (x,y) \in \mathbb{T}_0 \times \widetilde{\mathbb{T}}_0. \tag{2.23}$$

Now we will establish a more general inequality than that in Theorem 2.5.

Theorem 2.8. *Suppose $u, a, b, f, \phi, \tau_1, \tau_2, \alpha, \beta$ are the same as in Theorem 2.5, and $g \in C_{rd}(\mathbb{T}_0 \times \widetilde{\mathbb{T}}_0, \mathbb{R}_+)$. p, q, r are constants, and $p \geq q \geq 0$, $p \geq r \geq 0, \rho \neq 0$. If for $(x,y) \in \mathbb{T}_0 \times \widetilde{\mathbb{T}}_0$, $u(x,y)$ satisfies the following inequality:*

$$u^p(x,y) \leq a(x,y) + b(x,y) \int_{y_0}^{y} \int_{x_0}^{x} \left[f(s,t) u^q(\tau_1(s), \tau_2(t)) + g(s,t) u^r(\tau_1(s), \tau_2(t)) \right] \Delta s \, \Delta t, \tag{2.24}$$

with the initial condition (2.9), then

$$u(x,y) \leq \left[\widetilde{H}_1(x,y) + b(x,y) \int_{y_0}^{y} e_{\widetilde{\overline{H}}_2}(y,\sigma(t)) \widetilde{H}_2(x,t) \widetilde{H}_1(x,t) \Delta t \right]^{1/p}, \quad (x,y) \in \mathbb{T}_0 \times \widetilde{\mathbb{T}}_0, \tag{2.25}$$

where

$$\widetilde{H}_1(x,y) = a(x,y) + b(x,y)\int_{y_0}^{y}\int_{x_0}^{x}\left[f(s,t)\frac{p-q}{p}K^{q/p} + g(s,t)\frac{p-r}{p}K^{r/p}\right]\Delta s\,\Delta t,$$
$$\widetilde{H}_2(x,y) = \int_{x_0}^{x}\left[f(s,y)\frac{q}{p}K^{(q-p)/p} + g(s,y)\frac{r}{p}K^{(r-p)/p}\right]\Delta s, \tag{2.26}$$
$$\overline{\widetilde{H}}_2(x,y) = b(x,y)\widetilde{H}_2(x,y).$$

Proof. Fix $X \in \mathbb{T}_0$, and $x \in [x_0, X]\bigcap\mathbb{T},\ y \in \widetilde{\mathbb{T}}_0$. Let

$$v(x,y) = a(x,y) + b(x,y)\int_{y_0}^{y}\int_{x_0}^{x}[f(s,t)u^q(\tau_1(s),\tau_2(t)) + g(s,t)u^r(\tau_1(s),\tau_2(t))]\Delta s\,\Delta t. \tag{2.27}$$

Then

$$u(x,y) \le v^{1/p}(x,y) \le v^{1/p}(X,y), \quad \forall x \in [x_0, X]\bigcap\mathbb{T},\ y \in \widetilde{\mathbb{T}}_0. \tag{2.28}$$

Similar to (2.14)–(2.16), we obtain

$$u(\tau_1(x),\tau_2(y)) \le v^{1/p}(x,y), \quad x \in [x_0, X]\bigcap\mathbb{T},\ y \in \widetilde{\mathbb{T}}_0. \tag{2.29}$$

So

$$\begin{aligned} v(X,y) &= a(X,y) + b(X,y)\int_{y_0}^{y}\int_{x_0}^{X}[f(s,t)u^q(\tau_1(s),\tau_2(t)) + g(s,t)u^r(\tau_1(s),\tau_2(t))]\Delta s\,\Delta t \\ &\le a(X,y) + b(X,y)\int_{y_0}^{y}\int_{x_0}^{X}\left[f(s,t)v^{q/p}(s,t) + g(s,t)v^{r/p}(s,t)\right]\Delta s\,\Delta t. \end{aligned} \tag{2.30}$$

From Lemma 2.3, we have

$$\begin{aligned} v^{q/p}(s,t) &\le \frac{q}{p}K^{(q-p)/p}v(s,t) + \frac{p-q}{p}K^{q/p}, \quad \forall K > 0, \\ v^{r/p}(s,t) &\le \frac{r}{p}K^{(r-p)/p}\,v(s,t) + \frac{p-r}{p}K^{r/p}, \quad \forall K > 0. \end{aligned} \tag{2.31}$$

Combining (2.30) and (2.31) we get that

$$\begin{aligned}
v(X,y) &\le a(X,y) + b(X,y)\int_{y_0}^{y}\int_{x_0}^{X}\left[f(s,t)\left(\frac{q}{p}K^{(q-p)/p}v(s,t)+\frac{p-q}{p}K^{q/p}\right)\right.\\
&\quad\left.+g(s,t)\left(\frac{r}{p}K^{(r-p)/p}v(s,t)+\frac{p-r}{p}K^{r/p}\right)\right]\Delta s\,\Delta t\\
&\le a(X,y)+b(X,y)\int_{y_0}^{y}\int_{x_0}^{X}\left[f(s,t)\frac{p-q}{p}K^{q/p}+g(s,t)\frac{p-r}{p}K^{r/p}\right]\Delta s\,\Delta t\\
&\quad+b(X,y)\int_{y_0}^{y}\left\{\int_{x_0}^{X}\left[f(s,t)\frac{q}{p}K^{(q-p)/p}+g(s,t)\frac{r}{p}K^{(r-p)/p}\right]\Delta s\right\}v(X,t)\Delta t\\
&=\widetilde{H}_1(X,y)+b(X,y)\int_{y_0}^{y}\widetilde{H}_2(X,t)v(X,t)\Delta t.
\end{aligned}\tag{2.32}$$

Applying Lemma 2.1 to (2.32) yields

$$v(X,y)\le\widetilde{H}_1(X,y)+b(X,y)\int_{y_0}^{y}e_{\overline{\widetilde{H}_2}}(y,\sigma(t))\widetilde{H}_2(X,t)\widetilde{H}_1(X,t)\Delta t.\tag{2.33}$$

Then

$$\begin{gathered}
u(x,y)\le v^{1/p}(X,y)\le\left[\widetilde{H}_1(X,y)+b(X,y)\int_{y_0}^{y}e_{\overline{\widetilde{H}_2}}(y,\sigma(t))\widetilde{H}_2(X,t)\widetilde{H}_1(X,t)\Delta t\right]^{1/p},\\
x\in[x_0,X]\bigcap\mathbb{T},\quad y\in\widetilde{\mathbb{T}}_0.
\end{gathered}\tag{2.34}$$

Setting $x=X$ in (2.34) yields

$$u(X,y)\le\left[\widetilde{H}_1(X,y)+b(X,y)\int_{y_0}^{y}e_{\overline{\widetilde{H}_2}}(y,\sigma(t))\widetilde{H}_2(X,t)\widetilde{H}_1(X,t)\Delta t\right]^{1/p}.\tag{2.35}$$

Considering $X\in\mathbb{T}_0$ is arbitrary and replacing X with x in (2.35), we obtain the desired inequality. □

Remark 2.9. Assume $b(x,y)\equiv 1$ in Theorem 2.8. If we apply Lemma 2.2 instead of Lemma 2.1 to (2.32) in the proof of Theorem 2.8, then we obtain another bound for $u(x,y)$ as follows:

$$u(x,y)\le\left[\widetilde{H}_1(x,y)e_{\widetilde{H}_2}(y,y_0)\right]^{1/p},\quad(x,y)\in\mathbb{T}_0\times\widetilde{\mathbb{T}}_0.\tag{2.36}$$

Remark 2.10. Theorem 2.8 is the 2D extension of [27, Theorem 3].

Theorem 2.11. *Suppose $u, f, \alpha, \beta, \phi, \tau_1, \tau_2$ are the same as in Theorem 2.5, and $C > 0$ is a constant. If for $(x, y) \in (\mathbb{T}_0 \times \widetilde{\mathbb{T}}_0)$, $u(x, y)$ satisfies the following inequality:*

$$u^2(x,y) \le C + \int_{y_0}^{y}\int_{x_0}^{x} f(s,t)[u(\tau_1(s),\tau_2(t)) + u(\tau_1(s),\sigma(\tau_2(t)))]\Delta s\,\Delta t \tag{2.37}$$

with the initial condition

$$\begin{gathered} u(x,y) = \phi(x,y), \quad \text{if } x \in [\alpha, x_0] \bigcap \mathbb{T} \text{ or } y \in [\beta, y_0] \bigcap \mathbb{T}, \\ \phi(\tau_1(x),\tau_2(y)) \le C^{1/2}, \quad \forall (x,y) \in \mathbb{T}_0 \times \widetilde{\mathbb{T}}_0, \text{ if } \tau_1(x) \le x_0 \text{ or } \tau_2(y) \le y_0, \end{gathered} \tag{2.38}$$

then

$$u(x,y) \le \sqrt{C} + \int_{y_0}^{y}\int_{x_0}^{x} f(s,t)\Delta s\,\Delta t, \quad (x,y) \in \left(\mathbb{T}_0 \times \widetilde{\mathbb{T}}_0\right). \tag{2.39}$$

Proof. Let the right side of (2.37) be $v^2(x, y)$. Then

$$u(x,y) \le v(x,y), \quad \forall (x,y) \in \left(\mathbb{T}_0 \times \widetilde{\mathbb{T}}_0\right). \tag{2.40}$$

For $(x, y) \in (\mathbb{T}_0 \times \widetilde{\mathbb{T}}_0)$, if $\tau_1(x) \ge x_0$ and $\tau_2(y) \ge y_0$, then $\tau_1(x) \in \mathbb{T}_0$ and $\tau_2(y) \in \widetilde{\mathbb{T}}_0$, and from (2.40) we have

$$u(\tau_1(x),\tau_2(y)) \le v(\tau_1(x),\tau_2(y)) \le v(x,y). \tag{2.41}$$

If $\tau_1(x) \le x_0$ or $\tau_2(y) \le y_0$, from (2.38) we have

$$u(\tau_1(x),\tau_2(y)) = \phi(\tau_1(x),\tau_2(y)) \le a^{1/2}(\tau_1(x),\tau_2(y)) \le a^{1/2}(x,y) \le v(x,y). \tag{2.42}$$

So from (2.41) and (2.42), we always have

$$u(\tau_1(x),\tau_2(y)) \le v(x,y), \quad \forall (x,y) \in \left(\mathbb{T}_0 \times \widetilde{\mathbb{T}}_0\right). \tag{2.43}$$

Similarly, when $\tau_1(x) \ge x_0$ and $\sigma(\tau_2(y)) \ge y_0$, then $\tau_1(x) \in \mathbb{T}_0$ and $\sigma(\tau_2(y)) \in \widetilde{\mathbb{T}}_0$, and from (2.40) we have

$$u(\tau_1(x),\sigma(\tau_2(y))) \le v(\tau_1(x),\sigma(\tau_2(y))) \le v(x,\sigma(y)). \tag{2.44}$$

When $\tau_1(x) \le x_0$ or $\sigma(\tau_2(y)) \le y_0$, considering $\sigma(\tau_2(y)) \ge \tau_2(y) \ge \beta$, from (2.38) it follows that

$$u(\tau_1(x),\sigma(\tau_2(y))) = \phi(\tau_1(x),\sigma(\tau_2(y))) \le C^{1/2} \le v(x,y) \le v(x,\sigma(y)). \tag{2.45}$$

Combining (2.44) and (2.45), we always have

$$u(\tau_1(x), \sigma(\tau_2(y))) \le v(x, \sigma(y)), \quad \forall (x, y) \in \left(\mathbb{T}_0 \times \widetilde{\mathbb{T}}_0\right). \tag{2.46}$$

By (2.43) and (2.46), we obtain

$$v^2(x, y) \le C + \int_{y_0}^{y} \int_{x_0}^{x} f(s, t)[v(s, t) + v(s, \sigma(t))]\Delta s\, \Delta t, \quad x \in \mathbb{T}_0,\ y \in \widetilde{\mathbb{T}}_0. \tag{2.47}$$

Let the right side of (2.47) be $z^2(x, y)$. Then

$$v(x, y) \le z(x, y), \quad \forall (x, y) \in \left(\mathbb{T}_0 \times \widetilde{\mathbb{T}}_0\right), \tag{2.48}$$

$$\begin{aligned}
\left(z^2(x, y)\right)_y^{\Delta} &= \int_{x_0}^{x} f(s, y)[v(s, y) + v(s, \sigma(y))]\Delta s \\
&\le \left(\int_{x_0}^{x} f(s, y)\Delta s\right)[v(x, y) + v(x, \sigma(y))] \\
&\le \left(\int_{x_0}^{x} f(s, y)\Delta s\right)[z(x, y) + z(x, \sigma(y))].
\end{aligned} \tag{2.49}$$

Considering $z(x, y) + z(x, \sigma(y)) \ge z(x_0, y_0) = C > 0$, and $(z^2(x, y))_y^{\Delta} = [z(x, y) + z(x, \sigma(y))](z(x, y))_y^{\Delta}$, from (2.49) it follows that

$$(z(x, y))_y^{\Delta} \le \int_{x_0}^{x} f(s, y)\Delta s. \tag{2.50}$$

An integration of (2.50) with respect to y from y_0 to y yields $z(x, y) - z(x, y_0) \le \int_{y_0}^{y} \int_{x_0}^{x} f(s, t)\Delta s\, \Delta t$.

Considering $z(x, y_0) = \sqrt{C}$, it follows that

$$z(x, y) \le \sqrt{C} + \int_{y_0}^{y} \int_{x_0}^{x} f(s, t)\Delta s\, \Delta t. \tag{2.51}$$

Then combining (2.40), (2.48), and (2.51), we obtain

$$u(x, y) \le v(x, y) \le z(x, y) \le \sqrt{C} + \int_{y_0}^{y} \int_{x_0}^{x} f(s, t)\Delta s\, \Delta t, \tag{2.52}$$

and the proof is complete. □

Remark 2.12. If we take $\mathbb{T} = \mathbb{R}$, then Theorem 2.11 becomes the extension of the known Ou-Iang's inequality [13] to the 2D case.

The following theorem provides a more general result than Theorem 2.11.

Theorem 2.13. *Suppose p is a positive integer, and $p \geq 2$. Under the conditions of Theorem 2.11, if $u(x,y)$ satisfies*

$$u^p(x,y) \leq C + \int_{y_0}^{y}\int_{x_0}^{x} f(s,t)\sum_{l=0}^{p-1}\left\{u^l[(\tau_1(s),\tau_2(t))]u^{p-1-l}[(\tau_1(s),\sigma(\tau_2(t)))]\right\}\Delta s\,\Delta t, \quad (x,y) \in \left(\mathbb{T}_0 \times \widetilde{\mathbb{T}}_0\right), \tag{2.53}$$

with the initial condition

$$\begin{gathered} u(x,y) = \phi(x,y), \quad \text{if } x \in [\alpha, x_0]\bigcap \mathbb{T} \text{ or } y \in [\beta, y_0]\bigcap \mathbb{T}, \\ \phi(\tau_1(x),\tau_2(y)) \leq C^{1/p}, \quad \forall (x,y) \in \mathbb{T}_0 \times \widetilde{\mathbb{T}}_0, \text{ if } \tau_1(x) \leq x_0 \text{ or } \tau_2(y) \leq y_0, \end{gathered} \tag{2.54}$$

then

$$u(x,y) \leq C^{1/p} + \int_{y_0}^{y}\int_{x_0}^{x} f(s,t)\Delta s\,\Delta t, \quad x \in \mathbb{T}_0,\ y \in \widetilde{\mathbb{T}}_0. \tag{2.55}$$

The proof of Theorem 2.13 is similar to Theorem 2.11. As long as we notice a *delta d ifferentiable* function $z(x,y)$, the following formula [26, Equation (6.2)] holds:

$$(z^p(x,y))_y^{\Delta} = (z(x,y))_y^{\Delta}\sum_{l=0}^{p-1}\left[z^l(x,y)z^{p-1-l}(x,\sigma(y))\right]. \tag{2.56}$$

Then following a similar manner as in Theorem 2.11, we can deduce the desired result.

Theorem 2.14. *Suppose $u, f, \tau_1, \tau_2, \phi, \alpha, \beta$ are the same as in Theorem 2.5, $\omega \in C(\mathbb{R}_+, \mathbb{R}_+)$ is nondecreasing, and p, C are constants with $p \geq 1, C > 0$. Furthermore, define a bijective function G such that $[G(z(x,y))]_y^{\Delta} = (z(x,y))_y^{\Delta}/\omega(z^{1/p}(x,y))$. If for $(x,y) \in \mathbb{T}_0 \times \widetilde{\mathbb{T}}_0$, $u(x,y)$ satisfies the following inequality:*

$$u^p(x,y) \leq C + \int_{y_0}^{y}\int_{x_0}^{x}\left[f(s,t)\omega(u(\tau_1(s),\tau_2(t)))\right]\Delta s\,\Delta t, \tag{2.57}$$

with the initial condition

$$\begin{gathered} u(x,y) = \phi(x,y), \quad \text{if } x \in [\alpha, x_0]\bigcap \mathbb{T} \text{ or } y \in [\beta, y_0]\bigcap \mathbb{T}, \\ \phi(\tau_1(x),\tau_2(y)) \leq C^{1/p}, \quad \forall (x,y) \in \mathbb{T}_0 \times \widetilde{\mathbb{T}}_0, \text{ if } \tau_1(x) \leq x_0 \text{or } \tau_2(y) \leq y_0, \end{gathered} \tag{2.58}$$

then

$$u(x,y) \leq \left\{G^{-1}\left[G(C) + \int_{y_0}^{y}\eta_1(x,t)\Delta t\right]\right\}^{1/p}, \quad (x,y) \in \left(\mathbb{T}_0 \times \widetilde{\mathbb{T}}_0\right), \tag{2.59}$$

where $\eta_1(x,y) = \int_{x_0}^{x} f(s,y)\Delta s$.

Proof. Fix $X \in \mathbb{T}_0$, and $x \in [x_0, X] \bigcap \mathbb{T}$, $y \in \widetilde{\mathbb{T}}_0$. Let

$$v(x,y) = C + \int_{y_0}^{y} \int_{x_0}^{x} [f(s,t)] \Delta s \, \Delta t, \quad x \in [x_0, X] \bigcap \mathbb{T}, \ y \in \widetilde{\mathbb{T}}_0. \tag{2.60}$$

Then

$$u(x,y) \le v^{1/p}(x,y) \le v^{1/p}(X,y), \quad \forall x \in [x_0, X] \bigcap \mathbb{T}, \ y \in \widetilde{\mathbb{T}}_0. \tag{2.61}$$

Similar to (2.14)–(2.16), we obtain

$$u(\tau_1(x), \tau_2(y)) \le v^{1/p}(x,y), \quad x \in [x_0, X] \bigcap \mathbb{T}, \ y \in \widetilde{\mathbb{T}}_0. \tag{2.62}$$

Moreover,

$$\begin{aligned} v_y^{\Delta}(X,y) &= \int_{x_0}^{X} [f(s,y)\omega(u(\tau_1(s), \tau_2(y)))] \Delta s \\ &\le \int_{x_0}^{X} \left[f(s,y)\omega\left(v^{1/p}(s,y)\right)\right] \Delta s \\ &\le \left(\int_{x_0}^{X} f(s,y) \Delta s\right) \omega\left(v^{1/p}(X,y)\right) = \eta_1(X,y)\omega\left(v^{1/p}(X,y)\right). \end{aligned} \tag{2.63}$$

Let $\overline{v}(X,y)$ be the solution of the following problem:

$$\overline{v}_y^{\Delta}(X,y) = \eta_1(X,y)\omega\left(\overline{v}^{1/p}(X,y)\right), \quad \overline{v}(X,y_0) = C. \tag{2.64}$$

Considering $v(X,y_0) = C$ and ω is nondecreasing and continuous, then from (2.63), (2.64), and Lemma 2.4, we have

$$v(X,y) \le \overline{v}(X,y), \quad y \in \widetilde{\mathbb{T}}_0. \tag{2.65}$$

On the other hand, from the definition of G we have $(G(\overline{v}(X,y)))_y^{\Delta} = \overline{v}_y^{\Delta}(X,y)/\omega(\overline{v}(X,y)) = \eta_1(X,y)$. Then an integration with respect to y from y_0 to y yields

$$G(\overline{v}(X,y)) - G(\overline{v}(X,y_0)) = \int_{y_0}^{y} \eta_1(X,t)\Delta t, \tag{2.66}$$

that is,

$$\overline{v}(X,y) \le G^{-1}\left[G(C) + \int_{y_0}^{y} \eta_1(X,t)\Delta t\right], \quad y \in \widetilde{\mathbb{T}}_0. \tag{2.67}$$

Combining (2.61), (2.65), and (2.67), we have

$$u(x,y) \le \left\{ G^{-1}\left[G(C) + \int_{y_0}^{y} \eta_1(X,t)\Delta t \right] \right\}^{1/p}, \quad x \in [x_0, X] \bigcap \mathbb{T},\ y \in \widetilde{\mathbb{T}}_0. \tag{2.68}$$

Setting $x = X$ in (2.68), we get the desired result. □

Remark 2.15. If we take $\mathbb{T} = \mathbb{R}$, then Theorem 2.14 reduces to [14, Theorem 2.1], while Theorem 2.14 reduces to [15, Theorem 2.1] if we take $\mathbb{T} = \mathbb{Z}$.

Theorem 2.16. *Suppose $u, f, \tau_1, \tau_2, \phi, \alpha, \beta$ are the same as in Theorem 2.5, and furthermore, u is delta differential on $\widetilde{\mathbb{T}}_0$ with respect to y, $g \in C_{rd}(\mathbb{T}_0 \times \widetilde{\mathbb{T}}_0, \mathbb{R}_+)$. $\omega \in C(\mathbb{R}_+, \mathbb{R}_+)$ is nondecreasing, and ω is submultiplicative, that is, $\omega(\alpha\beta) \le \omega(\alpha)\omega(\beta)$, for all $\alpha, \beta \in \mathbb{R}_+$. $C > 0$ is a constant. $\widetilde{G}$ is a bijective function such that $[\widetilde{G}(z(x,y))]_y^{\Delta} = (z(x,y))_y^{\Delta} / \omega(z(x,y))$. If for $(x,y) \in \mathbb{T}_0 \times \widetilde{\mathbb{T}}_0$, $u(x,y)$ satisfies the following inequality:*

$$u(x,y) \le C + \int_{y_0}^{y} \int_{x_0}^{x} \left[f(s,t)\omega(u(\tau_1(s), \tau_2(t))) + g(s,t)u(\tau_1(s), \tau_2(t)) \right] \Delta s\, \Delta t, \tag{2.69}$$

with the initial condition

$$\begin{gathered} u(x,y) = \phi(x,y), \quad \text{if } x \in [\alpha, x_0] \bigcap \mathbb{T} \text{ or } y \in [\beta, y_0] \bigcap \mathbb{T}, \\ \phi(\tau_1(x), \tau_2(y)) \le C, \quad \forall (x,y) \in \left(\mathbb{T}_0 \times \widetilde{\mathbb{T}}_0 \right), \text{ if } \tau_1(x) \le x_0 \text{ or } \tau_2(y) \le y_0, \end{gathered} \tag{2.70}$$

then

$$u(x,y) \le \widetilde{G}^{-1}\left[\widetilde{G}(C) + \int_{y_0}^{y} \eta_2(x,t)\Delta t \right] e_{B_1}(y, y_0), \quad (x,y) \in \left(\mathbb{T}_0 \times \widetilde{\mathbb{T}}_0 \right), \tag{2.71}$$

where $B_1(x,y) = \int_{x_0}^{x} g(s,y)\Delta s$, $\eta_2(x,y) = \omega(e_{B_1}(y,y_0)) \int_{x_0}^{x} f(s,y)\Delta s$ and $e_{B_1}(y,y_0)$ is the unique solution of the following equation:

$$z_y^{\Delta}(x,y) = B_1(x,y)z(x,y), \quad z(x,y_0) = 1. \tag{2.72}$$

Proof. Fix $X \in \mathbb{T}_0$, and $x \in [x_0, X] \bigcap \mathbb{T}$, $y \in \widetilde{\mathbb{T}}_0$. Let

$$v(x,y) = C + \int_{y_0}^{y} \int_{x_0}^{x} \left[f(s,t)\omega(u(\tau_1(s), \tau_2(t))) + g(s,t)u(\tau_1(s), \tau_2(t)) \right] \Delta s\, \Delta t. \tag{2.73}$$

Then

$$u(x,y) \le v(x,y) \le v(X,y), \quad \forall x \in [x_0, X] \bigcap \mathbb{T},\ y \in \widetilde{\mathbb{T}}_0. \tag{2.74}$$

Similar to (2.14)–(2.16), we can obtain

$$u(\tau_1(x),\tau_2(y)) \le v(x,y), \quad x \in [x_0, X]\bigcap \mathbb{T},\ y \in \widetilde{\mathbb{T}}_0. \tag{2.75}$$

Furthermore we have

$$\begin{aligned} v(X,y) &= C + \int_{y_0}^{y}\int_{x_0}^{X}\left[f(s,t)\omega(u(\tau_1(s),\tau_2(t))) + g(s,t)u(\tau_1(s),\tau_2(t))\right]\Delta s\,\Delta t \\ &\le C + \int_{y_0}^{y}\int_{x_0}^{X}\left[f(s,t)\omega(v(s,t)) + g(s,t)v(s,t)\right]\Delta s\,\Delta t \\ &\le C + \int_{y_0}^{y}\int_{x_0}^{X} f(s,t)\omega(v(s,t))\Delta s\,\Delta t + \int_{y_0}^{y}\left(\int_{x_0}^{X} g(s,t)\Delta s\right)v(X,t)\Delta t, \quad y \in \widetilde{\mathbb{T}}_0. \end{aligned} \tag{2.76}$$

Let $B_2(X,y) = C + \int_{y_0}^{y}\int_{x_0}^{X} f(s,t)\omega(v(s,t))\Delta s\,\Delta t$. Then from (2.76) it follows that

$$v(X,y) \le B_2(X,y) + \int_{y_0}^{y} B_1(X,t)v(X,t)\Delta t, \quad y \in \widetilde{\mathbb{T}}_0. \tag{2.77}$$

Considering $B_2(X,y)$ is nondecreasing in y, by applying Lemma 2.2 to (2.77), we obtain

$$v(X,y) \le B_2(X,y)e_{B_1}(y,y_0), \quad y \in \widetilde{\mathbb{T}}_0. \tag{2.78}$$

On the other hand,

$$\begin{aligned} \left[B_2(X,y)\right]_y^{\Delta} &= \int_{x_0}^{X}\left[f(s,y)\omega(v(s,y))\right]\Delta s \le \left[\int_{x_0}^{X} f(s,y)\Delta s\right]\omega(v(X,y)) \\ &\le \left[\int_{x_0}^{X} f(s,y)\Delta s\right]\omega\left[B_2(X,y)e_{B_1}(y,y_0)\right] \\ &\le \left[\int_{x_0}^{X} f(s,y)\Delta s\right]\omega(B_2(X,y))\omega(e_{B_1}(y,y_0)) \\ &= \omega(B_2(X,y))\eta_2(X,y). \end{aligned} \tag{2.79}$$

Let $\overline{v}(X,y)$ be the solution of the following equation:

$$\overline{v}_y^{\Delta}(X,y) = \eta_2(X,y)\omega(\overline{v}(X,y)), \quad \overline{v}(X,y_0) = C. \tag{2.80}$$

Considering $B_2(X,y_0) = C$ and ω is nondecreasing and continuous, then from (2.79), (2.80), and Lemma 2.4, we have

$$B_2(X,y) \le \overline{v}(X,y), \quad y \in \widetilde{\mathbb{T}}_0. \tag{2.81}$$

From the definition of $\widetilde{G}$ and (2.80), we have $(\widetilde{G}(\overline{v}(X,y)))_y^{\Delta} = \overline{v}_y^{\Delta}(X,y)/\omega(\overline{v}(X,y)) = \eta_2(X,y)$. Then similar to (2.66) and (2.67), we obtain

$$B_2(X,y) \le \overline{v}(X,y) \le \widetilde{G}^{-1}\left[\widetilde{G}(C) + \int_{y_0}^{y} \eta_2(X,t)\Delta t\right], \quad y \in \widetilde{\mathbb{T}}_0. \tag{2.82}$$

Combining (2.74), (2.78), and (2.82), we have

$$u(x,y) \le \widetilde{G}^{-1}\left[\widetilde{G}(C) + \int_{y_0}^{y} \eta_2(X,t)\Delta t\right] e_{B_1}(y,y_0), \quad x \in [x_0, X] \bigcap \mathbb{T}, \ y \in \widetilde{\mathbb{T}}_0. \tag{2.83}$$

Setting $x = X$ in (2.83), we obtain

$$u(X,y) \le \widetilde{G}^{-1}\left[\widetilde{G}(C) + \int_{y_0}^{y} \eta_2(X,t)\Delta t\right] e_{B_1}(y,y_0), \quad y \in \widetilde{\mathbb{T}}_0. \tag{2.84}$$

Replacing X with x in (2.84) yields the desired inequality (2.71). □

Theorem 2.17. *Under the conditions of Theorem 2.16, if p, C are constants with $p > 0, C > 0$, and for $(x,y) \in \mathbb{T}_0 \times \widetilde{\mathbb{T}}_0$, $u(x,y)$ satisfies the following inequality:*

$$u^p(x,y) \le C + \int_{y_0}^{y}\int_{x_0}^{x} \left[f(s,t)\omega(u(\tau_1(s),\tau_2(t))) + g(s,t)u^p(\tau_1(s),\tau_2(t))\right]\Delta s\, \Delta t, \tag{2.85}$$

with the initial condition (2.58)*, then*

$$u(x,y) \le \left\{G^{-1}\left[G(C) + \int_{y_0}^{y} \eta_3(x,t)\Delta t\right] e_{J_1}(y,y_0)\right\}^{1/p}, \quad (x,y) \in \mathbb{T}_0 \times \widetilde{\mathbb{T}}_0, \tag{2.86}$$

where G is defined as in Theorem 2.14, $J_1(x,y) = \int_{x_0}^{x} g(s,y)\Delta s$, $\eta_3(x,y) = \omega((e_{J_1}(y,y_0))^{1/p}) \int_{x_0}^{x} f(s,y)\Delta s$, and $e_{J_1}(y,y_0)$ is the unique solution of the following equation:

$$z_y^{\Delta}(x,y) = J_1(x,y)z(x,y), \quad z(x,y_0) = 1. \tag{2.87}$$

The proof of Theorem 2.17 is similar to that of Theorem 2.16, and we omit it here.

3. Some Simple Applications

In this section, we will present some examples to illustrate the validity of our results in deriving explicit bounds of solutions of certain delay dynamic equations on time scales.

Example 3.1. Consider the following delay dynamic integral equation:

$$u^p(x,y) = C + \int_{y_0}^{y}\int_{x_0}^{x} M[s,t,u(\tau_1(s),\tau_2(t))]\Delta s\,\Delta t, \quad (x,y)\in \mathbb{T}_0\times\widetilde{\mathbb{T}}_0, \tag{3.1}$$

with the initial condition

$$\begin{aligned} u(x,y) &= \phi(x,y), \quad \text{if } x\in[\alpha,x_0]\bigcap\mathbb{T} \text{ or } y\in[\beta,y_0]\bigcap\mathbb{T}, \\ \phi(\tau_1(x),\tau_2(y)) &\le |C|^{1/p}, \quad \forall(x,y)\in\left(\mathbb{T}_0,\widetilde{\mathbb{T}}_0\right), \text{ if } \tau_1(x)\le x_0 \text{ or } \tau_2(y)\le y_0, \end{aligned} \tag{3.2}$$

where $u\in C_{\mathrm{rd}}(\mathbb{T}_0\times\widetilde{\mathbb{T}}_0,\mathbb{R})$, $\phi,\alpha,\beta,\tau_1,\tau_2$ are the same as in Theorem 2.8, and $M\in C_{\mathrm{rd}}(\mathbb{T}_0\times\widetilde{\mathbb{T}}_0\times\mathbb{R},\mathbb{R})$. Furthermore, assume $|M(s,t,u)|\le f(s,t)|u|^q+g(s,t)|u|^r$, where $f,g\in C_{\mathrm{rd}}(\mathbb{T}_0\times\widetilde{\mathbb{T}}_0,\mathbb{R}_+)$, and p,q,r are the same as in Theorem 2.8.

From (3.1) we have

$$\begin{aligned} |u^p(x,y)| &\le |C| + \int_{y_0}^{y}\int_{x_0}^{x} |M[s,t,u(\tau_1(s),\tau_2(t))]|\Delta s\,\Delta t \\ &\le |C| + \int_{y_0}^{y}\int_{x_0}^{x}\left[f(s,t)|u(\tau_1(s),\tau_2(t))|^q + g(s,t)|u(\tau_1(s),\tau_2(t))|^r\right]\Delta s\,\Delta t. \end{aligned} \tag{3.3}$$

Then according to Theorem 2.8, we can obtain the following estimate:

$$|u(x,y)| \le \left[\widetilde{H}_1(x,y) + \int_{y_0}^{y} e_{\widetilde{H}_2}(y,\sigma(t))\widetilde{H}_2(x,t)\widetilde{H}_1(x,t)\Delta t\right]^{1/p}, \quad (x,y)\in\mathbb{T}_0\times\widetilde{\mathbb{T}}_0, \tag{3.4}$$

where

$$\begin{aligned} \widetilde{H}_1(x,y) &= |C| + \int_{y_0}^{y}\int_{x_0}^{x}\left[f(s,t)\frac{p-q}{p}K^{q/p} + g(s,t)\frac{p-r}{p}K^{r/p}\right]\Delta s\,\Delta t, \quad \forall K>0, \\ \widetilde{H}_2(x,y) &= \int_{x_0}^{x}\left[f(s,y)\frac{q}{p}K^{(q-p)/p} + g(s,y)\frac{r}{p}K^{(r-p)/p}\right]\Delta s, \quad \forall K>0. \end{aligned} \tag{3.5}$$

Example 3.2. Considering the following delay dynamic integral equation:

$$u^3(x,y) = C + \int_{y_0}^{y}\int_{x_0}^{x} N[s,t,u(\tau_1(s),\tau_2(t)),u(\tau_1(s),\sigma(\tau_2(t)))]\Delta s\,\Delta t, \quad (x,y)\in\mathbb{T}_0\times\widetilde{\mathbb{T}}_0, \tag{3.6}$$

with the initial condition

$$\begin{aligned} u(x,y) &= \phi(x,y), \quad \text{if } x\in[\alpha,x_0]\bigcap\mathbb{T} \text{ or } y\in[\beta,y_0]\bigcap\mathbb{T}; \\ \phi(\tau_1(x),\tau_2(y)) &\le |C|^{1/3}, \quad \forall x\in\mathbb{T}_0,\ y\in\widetilde{\mathbb{T}}_0, \text{ if } \tau_1(x)\le x_0 \text{ or } \tau_2(y)\le y_0, \end{aligned} \tag{3.7}$$

where $u\in C_{\mathrm{rd}}(\mathbb{T}_0\times\widetilde{\mathbb{T}}_0,\mathbb{R})$, and $N\in C_{\mathrm{rd}}(\mathbb{T}_0\times\widetilde{\mathbb{T}}_0\times\mathbb{R}^2,\mathbb{R})$.

Assume $|N(x,y,u,v)| \le f(x,y)(|u|^2+|v|^2)$, where $f \in C_{\mathrm{rd}}(\mathbb{T}_0 \times \widetilde{\mathbb{T}}_0, \mathbb{R}_+)$, then from (3.6) we have

$$\begin{aligned}\left|u^3(x,y)\right| &\le |C| + \int_{y_0}^{y}\int_{x_0}^{x} |N[s,t,u(\tau_1(s),\tau_2(t)),u(\tau_1(s),\sigma(\tau_2(t)))]|\Delta s\,\Delta t \\ &\le |C| + \int_{y_0}^{y}\int_{x_0}^{x} f(s,t)\left[|u(\tau_1(s),\tau_2(t))|^2 + |u(\tau_1(s),\sigma(\tau_2(t)))|^2\right]\Delta s\,\Delta t \\ &\le |C| + \int_{y_0}^{y}\int_{x_0}^{x} f(s,t)\Big[|u(\tau_1(s),\tau_2(t))|^2 + |u(\tau_1(s),\sigma(\tau_2(t)))|^2 \\ &\quad +|u(\tau_1(s),\tau_2(t))||u(\tau_1(s),\sigma(\tau_2(t)))|\Big]\Delta s\,\Delta t.\end{aligned} \tag{3.8}$$

According to Theorem 2.13 (p = 3), we can reach the following estimate:

$$\left|u(x,y)\right| \le |C|^{1/3} + \int_{y_0}^{y}\int_{x_0}^{x} f(s,t)\Delta s\,\Delta t, \quad (x,y) \in \mathbb{T}_0 \times \widetilde{\mathbb{T}}_0. \tag{3.9}$$

4. Conclusions

In this paper, we established some new Gronwall-Bellman-type delay integral inequalities in two independent variables on time scales. As one can see, the presented results provide a handy tool for deriving bounds for solutions of certain delay dynamic equations on time scales. Furthermore, the process of constructing Theorems 2.5, 2.8, 2.14, 2.16 and 2.17 can be applied to the situation with n independent variables.

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