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Research Article

On Penalty and Gap Function Methods for Bilevel Equilibrium Problems

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We consider bilevel pseudomonotone equilibrium problems. We use a penalty function to convert a bilevel problem into one-level ones. We generalize a pseudo-∇-monotonicity concept from ∇-monotonicity and prove that under pseudo-∇-monotonicity property any stationary point of a regularized gap function is a solution of the penalized equilibrium problem. As an application, we discuss a special case that arises from the Tikhonov regularization method for pseudomonotone equilibrium problems.

1. Introduction

Let C be a nonempty closed-convex subset in $\mathbb{R}^n$, and let $f, g : C \times C \to \mathbb{R}$ be two bifunctions satisfying $f(x,x) = g(x,x) = 0$ for every $x \in C$. Such a bifunction is called an equilibrium bifunction. We consider the following bilevel equilibrium problem (BEP for short):

$$\text{find } \overline{x} \in S_g \text{ such that } f(\overline{x}, y) \geq 0, \quad \forall y \in S_g, \tag{1.1}$$

where $S_g = \{u \in C : g(u,y) \geq 0, \ \ \forall y \in C\}$, that is, S_g is the solution set of the equilibrium problems

$$\text{find } u \in C \text{ such that } g(u,y) \geq 0, \ \forall y \in C. \tag{1.2}$$

As usual, we call problem (1.1) the upper problem and (1.2) the lower one. BEPs are special cases of mathematical programs with equilibrium constraints. Sources for such problems can be found in [1–3]. Bilevel monotone variational inequality, which is a special case of problem

(1.1), was considered in [4, 5]. Moudafi in [6] suggested the use of the proximal point method for monotone BEPs. Recently, Ding in [7] used the auxiliary problem principle to BEPs. In both papers, the bifunctions f and g are required to be monotone on C. It should be noticed that under the pseudomonotonicity assumption on g the solution-set S_g of the lower problem (1.2) is a closed-convex set whenever $g(x,\cdot)$ is lower semicontinuous and convex on C for each x. However, the main difficulty is that, even the constrained set S_g is convex, it is not given explicitly as in a standard mathematical programming problem, and therefore the available methods (see, e.g., [8–14] and the references therein) cannot be applied directly.

In this paper, first, we propose a penalty function method for problem (1.1). Next, we use a regularized gap function for solving the penalized problems. Under certain pseudo-∇-monotonicity properties of the regularized bifunction, we show that any stationary point of the gap function on the convex set C is a solution to the penalized subproblem. Finally, we apply the proposed method to the Tikhonov regularization method for pseudomonotone equilibrium problems.

2. A Penalty Function Method

Penalty function method is a fundamental tool widely used in optimization to convert a constrained problem into unconstrained (or easier constrained) ones. This method was used to monotone variational inequalities in [5] and equilibrium problems in [15]. In this section, we use the penalty function method in the bilevel problem (1.1). First, let us recall some well-known concepts on monotonicity and continuity (see, e.g., [16]) that will be used in the sequel.

Definition 2.1. The bifunction $\phi : C \times C \to \mathbb{R}$ is said to be as follows:

(a) strongly monotone on C with modulus $\beta > 0$ if

$$\phi(x,y) + \phi(y,x) \le -\beta\|x-y\|^2, \quad \forall x,y \in C, \tag{2.1}$$

(b) monotone on C if

$$\phi(x,y) + \phi(y,x) \le 0, \quad \forall x,y \in C, \tag{2.2}$$

(c) pseudomonotone on C if

$$\forall x,y \in C: \quad \phi(x,y) \ge 0 \Longrightarrow \phi(y,x) \le 0, \tag{2.3}$$

(d) upper semicontinuous at x with respect to the first argument on C if

$$\overline{\lim_{z\to x}}\, \phi(z,y) \le \phi(x,y), \quad \forall y \in C, \tag{2.4}$$

(e) lower semicontinuous at y with respect to the second argument on C if

$$\varliminf_{w \to y} \phi(x, w) \geq \phi(x, y), \quad \forall x \in C. \tag{2.5}$$

Clearly, (a)⇒(b)⇒(c).

Definition 2.2 (see [17]). The bifunction $\phi : C \times C \to \mathbb{R}$ is said to be *coercive* on C if there exists a compact subset $B \subset \mathbb{R}^n$ and a vector $y_0 \in B \cap C$ such that

$$\phi(x, y_0) < 0, \quad \forall x \in C \setminus B. \tag{2.6}$$

Theorem 2.3 (see [18, Proposition 2.1.14]). *Let $\phi : C \times C \to \mathbb{R}$ be a equilibrium bifunction such that $\phi(\cdot, y)$ is upper semicontinuous on C for each $y \in C$ and $\phi(x, \cdot)$ is lower semicontnous, convex on C for each $x \in C$. Suppose that C is compact or ϕ is coercive on C, then there exists at least one $x^* \in C$ such that $\phi(x^*, y) \geq 0$ for every $y \in C$.*

The following proposition tells us about a relationship between the coercivity and the strong monotonicity.

Proposition 2.4. *Suppose that the equilibrium bifunction ϕ is strongly monotone on C, and $\phi(x, \cdot)$ is convex, lower semicontinuous with respect to the second argument for all $x \in C$, then for each $y \in C$, there exists a compact set B such that $y \in B$ and $\phi(x, y) < 0$ for all $x \in C \setminus B$.*

Proof. Suppose by contradiction that the conclusion does not hold, then there exists an element $y_0 \in C$ such that for every compact set B there is an element $x_B \in C \setminus B$ such that $\phi(x_B, y_0) \geq 0$. Take $B := B_r$ as the closed ball centered at y_0 with radius $r > 1$. Then there exists $x_r \in C \setminus B_r$ such that $\phi(x^r, y_0) \geq 0$. Let x be the intersection of the line segment $[y_0, x_r]$ with the unit sphere $S(y_0; 1)$ centered at y_0 and radius 1. Hence, $x_r = y_0 + t(r)(x - y_0)$, where $t(r) > r$. By the strong monotonicity of ϕ, we have

$$\phi(y_0, x_r) \leq -\phi(x_r, y_0) - \beta \|x_r - y_0\|^2 \leq -\phi(x_r, y_0) - \beta t(r)^2 \|x - y_0\|^2. \tag{2.7}$$

Since $\phi(y_0, \cdot)$ is convex on C, it follows that

$$\phi(y_0, x) \leq \frac{1}{t(r)} \phi(y_0, x_r) + \frac{t(r) - 1}{t(r)} \phi(y_0, y_0), \tag{2.8}$$

which implies that $\phi(y_0, x) \leq -\beta t(r) \|x - y_0\|^2 \leq -\beta r$. Thus,

$$\phi(y_0, x) \longrightarrow -\infty \quad \text{as } r \longrightarrow \infty. \tag{2.9}$$

However, since $\phi(y_0, \cdot)$ is lower semicontinuous on C, by the well-known Weierstrass Theorem, $\phi(y_0, \cdot)$ attains its minimum on the compact set $S(y_0; 1) \cap C$. This fact contradicts (2.9). □

From this proposition, we can derive the following corollaries.

Corollary 2.5 (see [18]). *If the bifunction ϕ is strongly monotone on C, and $\phi(x,\cdot)$ is convex, lower semicontinuous with respect to the second argument for all $x \in C$, then ϕ is coercive on C.*

Corollary 2.6. *Suppose that the bifunction f is strongly monotone on C, and $f(x,\cdot)$ is convex, lower semicontinuous with respect to the second argument for all $x \in C$. If the bifunction g is coercive on C then, for every $\epsilon > 0$, the bifunction $g + \epsilon f$ is uniformly coercive on C, for example, there exists a point $y_0 \in C$ and a compact set B both independent of ϵ such that*

$$g(x, y_0) + \epsilon f(x, y_0) < 0, \quad \forall x \in C \setminus B. \tag{2.10}$$

Proof. From the coercivity of g, we conclude that there exists a compact B_1 and $y_0 \in C$ such that $g(x, y_0) < 0$ for all $x \in C \setminus B_1$. Since f is strongly monotone, convex, lower semicontinuous on C, by choosing $y = y_0$, from Proposition 2.4, there exists a compact B_2 such that $f(x, y_0) < 0$ for all $x \in C \setminus B_2$. Set $B = B_1 \cup B_2$, then B is compact and $g(x, y_0) + \epsilon f(x, y_0) < 0$ for all $x \in C \setminus B$. □

Remark 2.7. It is worth to note that if both f, g are coercive and pseudomonotone on C, then the function $f + g$ is not necessary coercive or pseudomonotone on C.

To see this, let us consider the following bifunctions.

Example 2.8. Let $f(x,y) := (x_1y_2 - x_2y_1)e^{x_1}$, $g(x,y) := (x_2y_1 - x_1y_2)e^{x_2}$, and $C = \{(x_1,x_2) : x_1 \geq -1, (1/10)(x_1 - 9) \leq x_2 \leq 10x_1 + 9\}$ then we have

(i) $f(x,y), g(x,y)$ are pseudomonotone and coercive on C,

(ii) for all $\epsilon > 0$ the bifunctions $f_\epsilon(x,y) = g(x,y) + \epsilon f(x,y)$ are neither pseudomonotone nor coercive on C.

Indeed,

(i) if $f(x,y) \leq 0$, then $f(y,x) \geq 0$, thus f is pseudomonotone on C. By choosing $y^0 = (y_1^0, 0), (0 < y_1^0 \leq 1)$ and $B = \{(x_1,x_2) : x_1^2 + x_2^2 \leq r\}$ $(r > 1)$, we have $f(x, y^0) = -x_2y_1^0e^{x_1} < 0$ for all $y \in C \setminus B$, which means that f is coercive on C. Similarly, we can see that g is coercive on C,

(ii) by definition of f, we have that

$$f_\epsilon(x,y) = (x_2y_1 - x_1y_2)(e^{x_2} - \epsilon e^{x_1}), \quad \forall \epsilon > 0. \tag{2.11}$$

Take $x(t) = (t, 2t)$, for all $y(t) = (2t, t)$, then $f_\epsilon(x(t), y(t)) = 3t^2(e^{2t} - \epsilon e^t) > 0$, whereas $f_\epsilon(y(t), x(t)) = -3t^2(e^t - \epsilon e^{2t}) > 0$ for t is sufficiently large. So f_ϵ is not pseudomonotone on C.

Now, we show that the bifunction $f_\epsilon(x,y) = (x_2y_1 - x_1y_2)(e^{x_2} - \epsilon e^{x_1})$ is not coercive on C. Suppose, by contradiction, that there exist a compact set B and $y^0 = (y_1^0, y_2^0) \in B \cap C$ such that $f_\epsilon(x, y^0) < 0$ for all $x \in C \setminus B$, then, by coercivity of f_ϵ, it follows, $y_1^0, y_2^0 > 0$ and $y_1^0 \neq y_0^2$. With $x(t) = (t, kt), (t > 0)$, we have $f_\epsilon(x(t), y^0) = t(ky_1^0 - y_2^0)(e^{kt} - \epsilon e^t)$. However

(i) if $y_1^0 > y_2^0$, then from $1 < k < 10$ it follows that $x(t) \in C$ and $f_\epsilon(x(t), y^0) > 0$ for t is sufficiently large, which contradicts with coercivity,

(ii) if $y_1^0 < y_2^0$, then, by choosing $1/10 < k < 1$, we obtain $x(t) \in C$ and $f_\epsilon(x(t), y^0) > 0$ for t is large enough. But this cannot happen because of the coercivity of f_ϵ.

Now, for each fixed $\epsilon > 0$, we consider the penalized equilibrium problem PEP(C, f_ϵ) defined as

$$\text{find } \overline{x}_\epsilon \in C \text{ such that } f_\epsilon(\overline{x}_\epsilon, y) := g(\overline{x}_\epsilon, y) + \epsilon f(\overline{x}_\epsilon, y) \geq 0, \quad \forall y \in C. \tag{2.12}$$

By SOL(C, f_ϵ), we denote the solution set of PEP(C, f_ϵ).

Theorem 2.9. *Suppose that the equilibrium bifunctions f, g are pseudomonotone, upper semicontinuous with respect to the first argument and lower semicontinuous, convex with respect to the second argument on C, then any cluster point of the sequence $\{x_k\}$ with $x_k \in SOL(C, f_{\epsilon_k})$, $\epsilon_k \to 0$ is a solution to the original bilevel problem* (1.1). *In addition, if f is strongly monotone and g is coercive on C, then for each $\epsilon_k > 0$ the penalized problem $PEP(C, f_{\epsilon_k})$ is solvable, and any sequence $\{x_k\}$ with $x_k \in SOL(C, f_{\epsilon_k})$ converges to the unique solution of the bilevel problem* (1.1) *as $k \to \infty$.*

Proof. Let $\{x_k\}$ be any sequence with $x_k \in$ SOL(C, f_{ϵ_k}), and let $\overline{x}$ be any of its cluster points. Without lost of generality, we may assume that $x_k \to \overline{x}$ as $k \to \infty$. Since $x_k \in$ SOL(C, f_{ϵ_k}), one has

$$g(x_k, y) + \epsilon_k f(x_k, y) \geq 0, \quad \forall y \in C. \tag{2.13}$$

For any $z \in S_g$, we have $g(z, y) \geq 0$, for all $y \in C$ and in particular, $g(z, x_k) \geq 0$. Then, by the pseudomonotonicity of g, we have $g(x_k, z) \leq 0$. Replacing y by z in (2.13), we obtain

$$g(x_k, z) + \epsilon_k f(x_k, z) \geq 0, \tag{2.14}$$

which implies that

$$\epsilon_k f(x_k, z) \geq -g(x_k, z) \geq 0 \Longrightarrow f(x_k, z) \geq 0. \tag{2.15}$$

Let $k \to \infty$, by upper semicontinuity of f, we have $f(\overline{x}, z) \geq 0$ for all $z \in S_g$.

To complete the proof, we need only to show that $\overline{x} \in S_g$. Indeed, for any $y \in C$, we have

$$g(x_k, y) + \epsilon_k f(x_k, y) \geq 0, \quad \forall y \in C. \tag{2.16}$$

Again, by upper semicontinuity of f and g, we obtain in the limit, as $\epsilon_k \to 0$, that $g(\overline{x}, y) \geq 0$ for all $y \in C$. Hence, $\overline{x} \in S_g$.

Now suppose, in addition, that f is strongly monotone on C. By Corollary 2.6, f_{ϵ_k} is uniformly coercive on C. Thus, problem PEP(C, f_{ϵ_k}) is solvable and, for all $\epsilon_k > 0$, the solution sets of these problems are contained in a compact set B. So any infinite sequence $\{x_k\}$ of the solutions has a cluster point, say, $\overline{x}$. By the first part, $\overline{x}$ is a solution of (1.1). Note that, from the assumption on g, the solution set S_g of the lower equilibrium (EP(C, g)) is a closed, convex, compact set. Since f is lower semicontinuous and convex with respect to the second argument and is strongly monotone on C, the upper equilibrium problem $\mathrm{EP}(S_g, f)$ has a unique solution. Using again the first part of the theorem, we can see that $x_k \to \overline{x}$ as $k \to \infty$ □

Remark 2.10. In a special case considered in [6], where both f and g are monotone, the penalized problem (PEP) is monotone too. In this case, (PEP) can be solved by some existing methods (see, e.g., [6, 11–14, 19]) and the references therein. However, when one of these two bifunctions is pseudomonotone, the penalized problem (PEP), in general, does not inherit any monotonicity property from f and g. In this case, problem (PEP) cannot be solved by the above-mentioned existing methods.

3. Gap Function and Descent Direction

A well-known tool for solving equilibrium problem is the gap function. The regularized gap function has been introduced by Taji and Fukushima in [20] for variational inequalities, and extended by Mastroeni in [11] to equilibrium problems. In this section, we use the regularized gap function for the penalized equilibrium problem (PEP). As we have mentioned above, this problem, even when g is pseudomonotone and f is strongly monotone, is still difficult to solve.

Throughout this section, we suppose that both f and g are lower semicontinuous, convex on C with respect to the second argument. First, we recall (see, e.g., [11]) the definition of a gap function for the equilibrium problem.

Definition 3.1. A function $\varphi : C \to \mathbb{R} \cup \{+\infty\}$ is said to be a gap function for (PEP) if

(i) $\varphi(x) \geq 0$, for all $x \in C$,

(ii) $\varphi(\overline{x}) = 0$ if and only if $\overline{x}$ is a solution for (PEP).

A gap function for (PEP) is $\varphi(x) = -\min_{y\in C} f_\epsilon(x, y)$. This gap function may not be finite and, in general, is not differentiable. To obtain a finite, differentiable gap function, we use the regularized gap function introduced in [20] and recently used by Mastroeni in [11] to equilibrium problems. From Proposition 2.2 and Theorem 2.1 in [11], the following proposition is immediate.

Proposition 3.2. *Suppose that $l : C \times C \to \mathbb{R}$ is a nonnegative differentiable, strongly convex bifunction on C with respect to the second argument and satisfies*

(a) $l(x, x) = 0$ *for all* $x \in C$,

(b) $\nabla_y l(x, x) = 0$ *for all* $x \in C$.

Then the function

$$\varphi_\epsilon(x) = -\min_{y\in C}\left[g(x, y) + \epsilon\left[f(x, y) + l(x, y)\right]\right] \tag{3.1}$$

is a finite gap function for (PEP). In addition, if f and g are differentiable with respect to the first argument and $\nabla_x f(x, y), \nabla_x g(x, y)$ are continuous on C, then $\varphi_\epsilon(x)$ is continuously differentiable on C and

$$\nabla\varphi_\epsilon(x) = -\nabla_x g(x, y_\epsilon(x)) - \epsilon\nabla_x\left[f(x, y_\epsilon(x)) + l(x, y_\epsilon(x))\right] = -\nabla_x g_\epsilon(x, y_\epsilon(x)), \tag{3.2}$$

where

$$g_\epsilon(x,y) = g(x,y) + \epsilon\left[f(x,y) + l(x,y)\right],$$
$$y_\epsilon(x) = \arg\min_{y\in C}\left\{g_\epsilon(x,y)\right\}. \tag{3.3}$$

Note that the function $l(x,y) := (1/2)\langle M(y-x), y-x\rangle$, where M is a symmetric positive definite matrix of order n that satisfies the assumptions on l.

We need some definitions on ∇-monotonicity.

Definition 3.3. A differentiable bifunction $h : C\times C \to \mathbb{R}$ is called as follows:

(a) strongly ∇-monotone on C if there exists a constant $\tau > 0$ such that,

$$\left\langle \nabla_x h(x,y) + \nabla_y h(x,y), y-x\right\rangle \geq \tau\left\|y-x\right\|^2, \quad \forall x,y\in C, \tag{3.4}$$

(b) strictly ∇-monotone on C if

$$\left\langle \nabla_x h(x,y) + \nabla_y h(\mathrm{x},y), y-x\right\rangle > 0, \quad \forall x,y\in C,\ x\neq y, \tag{3.5}$$

(c) ∇-monotone on C if

$$\langle \nabla_x h(x,y) + \nabla_y h(x,y), y-x\rangle \geq 0, \quad \forall x,y\in C, \tag{3.6}$$

(d) strictly pseudo-∇-monotone on C if

$$\left\langle \nabla_x h(x,y), y-x\right\rangle \leq 0 \Longrightarrow \left\langle \nabla_y h(x,y), y-x\right\rangle > 0,\ \forall x,y\in C, \quad x\neq y, \tag{3.7}$$

(e) pseudo-∇-monotone on C if

$$\left\langle \nabla_x h(x,y), y-x\right\rangle \leq 0 \Longrightarrow \left\langle \nabla_y h(x,y), y-x\right\rangle \geq 0, \quad \forall x,y\in C. \tag{3.8}$$

Remark 3.4. The definitions (a), (b), and (c) can be found, for example, in [8, 11]. The definitions (d) and (e), to our best knowledge, are not used before. From the definitions, we have

$$\text{(a)} \Longrightarrow \text{(b)} \Longrightarrow \text{(c)} \Longrightarrow \text{(e)}, \qquad \text{(a)} \Longrightarrow \text{(b)} \Longrightarrow \text{(d)} \Longrightarrow \text{(e)}. \tag{3.9}$$

However, (c) may not imply (d) and vice versa as shown by the following simple examples.

Example 3.5. Consider the bifunction $h(x,y) = e^{x^2}(y^2 - x^2)$ defined on $C\times C$ *with* $C = \mathbb{R}$. This bifunction is not ∇-monotone on C, because

$$\left\langle \nabla_x h(x,y) + \nabla_y h(x,y), y-x\right\rangle = 2e^{x^2}(y-x)^2\left(x^2 + xy + 1\right) \tag{3.10}$$

is negative for $x = -1, \quad y = 3$. However, $h(x,y)$ is strictly pseudo- ∇-monotone. Indeed, we have

$$\begin{gathered}\langle \nabla_x h(x,y), y-x\rangle = 2xe^{x^2}\left(y^2 - x^2 - 1\right)(y-x) \le 0 \Longleftrightarrow x\left(y^2 - x^2 - 1\right)(y-x) \le 0, \\ \langle \nabla_y h(x,y), y-x\rangle = 2ye^{x^2}(y-x) > 0 \Longleftrightarrow y(y-x) > 0.\end{gathered} \tag{3.11}$$

It is not difficult to verify that

$$x\left(y^2 - x^2 - 1\right)(y-x) \le 0 \Longrightarrow y(y-x) > 0, \quad \text{as } x \ne y. \tag{3.12}$$

Hence this function is strictly pseudo- ∇-monotone but is not ∇-monotone.

Vice versa, considering the bifunction $h(x,y) = (y-x)^T M(y-x)$ defined on $\mathbb{R}^n \times \mathbb{R}^n$, where M is a matrix of order $n \times n$, we have the following:

(i) h is ∇-monotone, because

$$\begin{aligned}&\langle \nabla_x h(x,y) + \nabla_y h(x,y), y-x\rangle \\ &\quad = \left\langle -(y-x)^T\left(M + M^T\right) + (y-x)^T\left(M + M^T\right), y-x\right\rangle = 0, \quad \forall x,y.\end{aligned} \tag{3.13}$$

Clearly, h is not strictly-∇-monotone,

(ii) h is strictly pseudo ∇-monotone if and only if

$$\langle \nabla_x h(x,y), y-x\rangle = -\left\langle (y-x)^T\left(M + M^T\right), y-x\right\rangle \le 0 \tag{3.14}$$

implies

$$\langle \nabla_y h(x,y), y-x\rangle = (y-x)^T\left(M + M^T\right), y-x\rangle > 0, \ \forall x,y, \quad x \ne y. \tag{3.15}$$

The latter inequality equivalent to $M + M^T$ is a positive definite matrix of order $n \times n$.

Remark 3.6. As shown in [8] when $h(x,y) = \langle T(x), y-x\rangle$ with T a differentiable monotone operator on C, h is monotone on C if and only if T is monotone on C, and in this case, monotonicity of h on C coincides with ∇-monotonicity of h on C.

The following example shows that pseudomonotonicity may not imply pseudo-∇-monotonicity.

Example 3.7. Let $h(x,y) = -ax(y-x)$, defined on $\mathbb{R}_+ \times \mathbb{R}_+$, $(a > 0)$. It is easy to see that

$$h(x,y) \ge 0 \Longrightarrow h(y,x) \le 0, \quad \forall x,y \ge 0. \tag{3.16}$$

Thus, h is pseudomonotone on $\mathbb{R}_+$.

We have

$$\langle \nabla_x h(x,y), y-x \rangle = -a(y-x)(y-2x) < 0, \quad \forall y > 2x > 0. \tag{3.17}$$

But

$$\langle \nabla_y h(x,y), y-x \rangle = -ax(y-x) < 0, \quad \forall y > 2x > 0. \tag{3.18}$$

So h is not pseudo-∇-monotone on $\mathbb{R}_+$.

From the definition of the gap function φ_ϵ, a global minimal point of this function over C is a solution to problem (PEP). Since φ_ϵ is not convex, its global minimum is extremely difficult to compute. In [8], the authors have shown that under the strict ∇-monotonicity a stationary point is also a global minimum of gap function. By a counterexample, the authors in [8] also pointed out that the strict ∇-monotonicity assumption cannot be relaxed to ∇-monotonicity. The following theorem shows that the stationary property is still guaranteed under the strict pseudo-∇-monotonicity.

Theorem 3.8. *Suppose that g_ϵ is strictly pseudo- ∇-monotone on C. If $\overline{x}$ is a stationary point of φ_ϵ over C, that is,*

$$\langle \nabla\varphi_\epsilon(\overline{x}), y-\overline{x} \rangle \geq 0, \quad \forall y \in C. \tag{3.19}$$

then $\overline{x}$ solves (PEP).

Proof. Suppose that $\overline{x}$ does not solve (PEP), then $y_\epsilon(\overline{x}) \neq \overline{x}$.

Since $\overline{x}$ is a stationary point of φ_ϵ on C, from the definition of φ_ϵ, we have

$$\langle \nabla\varphi_\epsilon(\overline{x}), y-\overline{x} \rangle = -\langle \nabla_x g_\epsilon(x, y_\epsilon(x)), y_\epsilon(x)-x \rangle \geq 0. \tag{3.20}$$

By strict pseudo-∇-monotonicity of g_ϵ, it follows that

$$\langle \nabla_y g_\epsilon(\overline{x}, y_\epsilon(\overline{x})), y_\epsilon(\overline{x})-\overline{x} \rangle > 0. \tag{3.21}$$

On the other hand, since $y_\epsilon(\overline{x})$ minimizes $g_\epsilon(x,\cdot)$ over C, we have

$$\langle \nabla_y g_\epsilon(\overline{x}, y_\epsilon(\overline{x})), y_\epsilon(\overline{x})-\overline{x} \rangle \leq 0, \tag{3.22}$$

which is in contradiction with (3.21). □

To compute a stationary point of a differentiable function over a closed-convex set, we can use the existing descent direction algorithms in mathematical programming (see, e.g., [8, 21]). The next proposition shows that if $y(x)$ is a solution of the problem $\min_{y\in C} g_\epsilon(x,y)$, then $y(x)-x$ is a descent direction on C of φ_ϵ at x. Namely, we have the following proposition.

Proposition 3.9. *Suppose that g_ϵ is strictly pseudo-∇-monotone on C and x is not a solution to Problem (PEP), then*

$$\langle \nabla\varphi_\epsilon(x), y_\epsilon(x) - x\rangle < 0. \tag{3.23}$$

Proof. Let $d_\epsilon(x) = y_\epsilon(x) - x$. Since x is not a solution to (PEP), then $d_\epsilon(x) \neq 0$. Suppose that, by contradiction, $d_\epsilon(x)$ is not a descent direction on C of φ_ϵ at x, then

$$\langle \nabla\varphi_\epsilon(x), y_\epsilon(x) - x\rangle \geq 0 \Longleftrightarrow -\langle \nabla_x g_\epsilon(x, y_\epsilon(x)), y_\epsilon(x) - x\rangle \geq 0, \tag{3.24}$$

which, by strict pseudo-∇-monotonicity of g_ϵ, implies

$$\langle \nabla_y g_\epsilon(x, y_\epsilon(x)), y_\epsilon(x) - x\rangle > 0. \tag{3.25}$$

On the other hand, since $y_\epsilon(x)$ minimizes $g_\epsilon(x,\cdot)$ over C, by the well-known optimality condition, we have

$$\langle \nabla_y g_\epsilon(x, y_\epsilon(x)), y_\epsilon(x) - x\rangle \leq 0, \tag{3.26}$$

which contradicts (3.25). □

Proposition 3.10. *Suppose that $g(x,\cdot)$ is strictly convex on C for every $x \in C$ and g is strictly pseudo-∇-monotone on C. If $x \in C$ is not a solution of (PEP), then there exists $\overline{\epsilon} > 0$ such that $y_\epsilon(x) - x$ is a descent direction of φ_ϵ on C at x for all $0 < \epsilon \leq \overline{\epsilon}$.*

Proof. By contradiction, suppose that the statement of the proposition does not hold, then there exist $\epsilon_k \searrow 0$ and $x \in C$ such that

$$\langle \nabla\varphi_{\epsilon_k}(x), y_{\epsilon_k}(x) - x\rangle \geq 0 \Longleftrightarrow -\langle \nabla_x g_{\epsilon_k}(x, y_{\epsilon_k}(x)), y_{\epsilon_k}(x) - x\rangle \geq 0. \tag{3.27}$$

Since $g_\epsilon(x,\cdot)$ is strictly convex differentiable on C, by Theorem 2.1 in [9], the function $\epsilon \mapsto y_\epsilon(x)$ is continuous with respect to ϵ, thus $y_{\epsilon_k}(x)$ tends to $y_0(x)$ as $\epsilon_k \to 0$, where $y_0(x) = \arg\min_{y\in C} g(x,y)$. Since $g_{\epsilon_k}(x,y) = g(x,y) + \epsilon_k f(x,y)$ is continuously differentiable, letting $\epsilon_k \to 0$ in (3.27), we obtain

$$-\langle \nabla_x g(x, y_0(x)), y_0(x) - x\rangle \geq 0. \tag{3.28}$$

By strict pseudo-∇-monotonicity of g, it follows that

$$\langle \nabla_y g(x, y_0(x)), y_0(x) - x\rangle > 0. \tag{3.29}$$

On the other hand, since $y_{\epsilon_k}(x)$ minimizes $g_{\epsilon_k}(x,\cdot)$ over C, we have

$$\langle \nabla_y g_{\epsilon_k}(x, y_{\epsilon_k}(x)),\ y_{\epsilon_k}(x) - x\rangle \leq 0. \tag{3.30}$$

Taking the limit, we obtain

$$\langle \nabla_y g(x, y_0(x)), y_0(x) - x \rangle \leq 0, \tag{3.31}$$

which contradicts (3.29). □

To illustrate Theorem 3.8, let us consider the following examples.

Example 3.11. Consider the bifunctions $g(x, y) = e^{x^2}(y^2 - x^2)$ and $f(x, y) = 10^{x^2}(y^2 - x^2)$ defined on $\mathbb{R} \times \mathbb{R}$. It is not hard to verify that,

(i) $g(x, y), f(x, y)$ are monotone, strictly pseudo-∇-monotone on $\mathbb{R}$,

(ii) for all $\epsilon > 0$ the bifunction $g(x, y) + \epsilon f(x, y)$ is monotone and strictly pseudo-∇-monotone on $\mathbb{R}$ and satisfying all of the assumptions of Theorem 3.8.

Example 3.12. Let $f(x, y) = -x^2 - xy + 2y^2$ and $g(x, y) = -3x^2y + xy^2 + 2y^3$ defined on $\mathbb{R}_+ \times \mathbb{R}_+$ it is easy to see that,

(i) g, f are pseudomonotone, strictly ∇-monotone on $\mathbb{R}_+$,

(ii) for all $\epsilon > 0$ the bifunction $g(x, y) + \epsilon f(x, y)$ is pseudomonotone and strictly ∇-monotosne on $\mathbb{R}_+$ and satisfying all of the assumptions of Theorem 3.8.

4. Application to the Tikhonov Regularization Method

The Tikhonov method [22] is commonly used for handling ill-posed problems. Recently, in [23] the Tikhonov method has been extended to the pseudomonotone equilibrium problem

$$\text{Find } x^* \in C \text{ such that } g(x^*, y) \geq 0, \quad \forall y \in C, \tag{EP(C, g)}$$

where, as before, C is a closed-convex set in $\mathbb{R}^n$ and $g : C \to \mathbb{R}$ is a pseudomonotone bifunction satisfying $g(x, x) = 0$ for every $x \in C$.

In the Tikhonov regularization method considered in [23], problem (EP(C, g)) is regularized by the problems

$$\text{find } x^* \in C \text{ such that } g_\epsilon(x^*, y) := g(x^*, y) + \epsilon f(x^*, y) \geq 0, \quad \forall y \in C, \tag{EP(C, g_\epsilon)}$$

where f is an equilibrium bifunction on C and $\epsilon > 0$ and play the role of the regularization bifunction and regularization parameter, respectively.

In [23], the following theorem has been proved.

Theorem 4.1. *Suppose that $f(\cdot, y)$, $g(\cdot, y)$ are upper semicontinuous and $f(x, \cdot)$, $g(x, \cdot)$ are lower semicontinuous convex on C for each $x, y \in C$ and that g is pseudomonotone on C. Suppose further that f is strongly monotone on C satisfying the condition*

$$\exists \delta > 0: \quad |f(x, y)| \leq \delta \|x - x^g\| \|y - x\|, \quad \forall x, y \in C, \tag{4.1}$$

where $x^g \in C$ (plays the role of a guess solution) is given.

Then the following three statements are equivalent:

(a) *the solution set of* $(\mathrm{EP}(C, g_\epsilon))$ *is nonempty for each* $\varepsilon > 0$ *and* $\lim_{\varepsilon \to 0^+} x(\varepsilon)$ *exists, where* $x(\varepsilon)$ *is arbitrarily chosen in the solution set of* $(\mathrm{EP}(C, g_\epsilon))$*,*

(b) *the solution set of* $(\mathrm{EP}(C, g_\epsilon))$ *is nonempty for each* $\varepsilon > 0$ *and* $\lim_{\varepsilon \to 0^+} \sup \|x(\varepsilon)\| < \infty$*, where* $x(\varepsilon)$ *is arbitrarily chosen in the solution set of* $(\mathrm{EP}(C, g_\epsilon))$*,*

(c) *the solution set of* $(\mathrm{EP}(C, g))$ *is nonempty.*

Moreover, if any one of these statements holds, then $\lim_{\varepsilon \to 0^+} x(\varepsilon)$ *is equal to the unique solution of the strongly monotone equilibrium problem* $\mathrm{EP}(S_g, f)$*, where* S_g *denotes the solution set of the original problem* $(\mathrm{EP}(C, g))$*.*

Note that, when g is monotone on C, the regularized subproblems are strongly monotone and therefore, they can be solved by some existing methods. When g is pseudomonotone, the subproblems, in general, are no longer strongly monotone, even not pseudomonotone. So solving them becomes a difficult task. However, the problem of finding the limit point of the sequences of iterates leads to the unique solution of problem $\mathrm{EP}(S_g, f)$.

In order to apply the penalty and gap function methods described in the preceding sections, let us take, for instant,

$$f(x, y) = \langle x - x^g, y - x \rangle. \tag{4.2}$$

Clearly, f is both strongly monotone and strongly ∇-monotone with the same modulus 1. Moreover, f satisfies the condition (4.1). Therefore, the problem of finding the limit point in the above Tikhonov regularization method can be formulated as the bilevel equilibrium problem

$$\text{find } x \in S_g \text{ such that } f(x^*, y) \geq 0, \quad \forall y \in S_g, \tag{4.3}$$

which is of the form (1.1). Now, for each fixed $\epsilon_k > 0$, we consider the penalized equilibrium problem $\mathrm{PEP}(C, f_{\epsilon_k})$ defined as

$$\text{find } \overline{x}_k \in C \text{ such that } f_{\epsilon_k}(\overline{x}_k, y) := g(\overline{x}_k, y) + \epsilon_k f(\overline{x}_k, y) \geq 0, \quad \forall y \in C. \tag{4.4}$$

As before, by $\mathrm{SOL}(C, f_{\epsilon_k})$, we denote the solution set of $\mathrm{PEP}(C, f_{\epsilon_k})$.

Applying Theorems 2.9 and 3.8, we obtain the following result.

Theorem 4.2. *Suppose that the bifunction* g *satisfies the following conditions:*

(i) $g(x, \cdot)$ *is convex, lower semicontinuous for all* $x \in C$*,*

(ii) g *is pseudomonotone and coercive on* C*.*
Then for any $\epsilon_k > 0$*, the penalized problem* $PEP(C, f_{\epsilon_k})$ *is solvable, and any sequence* $\{x_k\}$ *with* $x_k \in SOL(C, f_{\epsilon_k})$ *for all* k *converges to the unique solution of the problem* (4.3) *as* $k \to \infty$*.*

(iii) *In addition, if* $g(x, y) + \epsilon_k f(x, y)$ *is strictly pseudo-* ∇*-monotone on* C *(in particular,* $g(x, y)$ *is* ∇*-monotone), and* $\overline{x}_k$ *is any stationary point of the mathematical program* $\min_{x\in C}\varphi_k(x)$ *with*

$$\varphi_k(x) := \min_{y\in C}\{g(x, y) + \epsilon_k f(x, y)\}, \tag{4.5}$$

then $\{\overline{x}_k\}$ *converges to the unique solution of the problem* (4.3) *as* $k \to \infty$.

5. Conclusion

We have considered a class of bilevel pseudomonotone equilibrium problems. The main difficulty of this problem is that its feasible domain is not given explicitly as in a standard mathematical programming problem. We have proposed a penalty function method to convert the bilevel problem into one-level ones. Then we have applied the regularized gap function method to solve the penalized equilibrium subproblems. We have generalized the pseudo-∇-monotonicity concept from ∇-monotonicity. Under the pseudo-∇-monotonicity property, we have proved that any stationary point of the gap function is a solution to the original bilevel problem. As an application, we have shown how to apply the proposed method to the Tikhonov regularization method for pseudomonotone equilibrium problems.

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