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Research Article

Oscillation Criteria for Second-Order Superlinear Neutral Differential Equations

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Some oscillation criteria are established for the second-order superlinear neutral differential equations $(r(t)|z'(t)|^{\alpha-1}z'(t))' + f(t, x(\sigma(t))) = 0$, $t \geq t_0$, where $z(t) = x(t) + p(t)x(\tau(t))$, $\tau(t) \geq t$, $\sigma(t) \geq t$, $p \in C([t_0, \infty), [0, p_0])$, and $\alpha \geq 1$. Our results are based on the cases $\int_{t_0}^{\infty} 1/r^{1/\alpha}(t)\mathrm{d}t = \infty$ or $\int_{t_0}^{\infty} 1/r^{1/\alpha}(t)\mathrm{d}t < \infty$. Two examples are also provided to illustrate these results.

1. Introduction

This paper is concerned with the oscillatory behavior of the second-order superlinear differential equation

$$\left(r(t)\left|z'(t)\right|^{\alpha-1}z'(t)\right)' + f(t, x(\sigma(t))) = 0, \quad t \geq t_0, \tag{1.1}$$

where $\alpha \geq 1$ is a constant, $z(t) = x(t) + p(t)x(\tau(t))$.

Throughout this paper, we will assume the following hypotheses:

(A_1) $r \in C^1([t_0, \infty), \mathbb{R})$, $r(t) > 0$ for $t \geq t_0$;

(A_2) $p \in C([t_0, \infty), [0, p_0])$, where p_0 is a constant;

(A_3) $\tau \in C^1([t_0, \infty), \mathbb{R})$, $\tau'(t) \geq \tau_0 > 0$, $\tau(t) \geq t$;

(A_4) $\sigma \in C([t_0, \infty), \mathbb{R})$, $\sigma(t) \geq t$, $\tau \circ \sigma = \sigma \circ \tau$;

(A_5) $f(t,u) \in C([t_0,\infty)\times\mathbb{R},\mathbb{R})$, and there exists a function $q \in C([t_0,\infty),[0,\infty))$ such that

$$f(t,u)\operatorname{sign} u \geq q(t)|u|^{\alpha}, \quad \text{for } u \neq 0,\ t \geq t_0. \tag{1.2}$$

By a solution of (1.1), we mean a function $x \in C([T_x,\infty),\mathbb{R})$ for some $T_x \geq t_0$ which has the property that $r(t)|z'(t)|^{\alpha-1}z'(t) \in C^1([T_x,\infty),\mathbb{R})$ and satisfies (1.1) on $[T_x,\infty)$. We consider only those solutions x which satisfy $\sup\{|x(t)| : t \geq T\} > 0$ for all $T \geq T_x$. As is customary, a solution of (1.1) is called oscillatory if it has arbitrarily large zeros on $[t_0,\infty)$, otherwise, it is called nonoscillatory. Equation (1.1) is said to be oscillatory if all its solutions are oscillatory.

We note that neutral differential equations find numerous applications in electric networks. For instance, they are frequently used for the study of distributed networks containing lossless transmission lines which rise in high-speed computers where the lossless transmission lines are used to interconnect switching circuits; see [1].

In the last few years, there are many studies that have been made on the oscillation and asymptotic behavior of solutions of discrete and continuous equations; see, for example, [2–28]. Agarwal et al. [5], Chern et al. [6], Džurina and Stavroulakis [7], Kusano and Yoshida [8], Kusano and Naito [9], Mirzov [10], and Sun and Meng [11] observed some similar properties between

$$\left(r(t)\left|x'(t)\right|^{\alpha-1}x'(t)\right)' + q(t)|x(\sigma(t))|^{\alpha-1}x(\sigma(t)) = 0 \tag{1.3}$$

and the corresponding linear equations

$$\left(r(t)x'(t)\right)' + q(t)x(t) = 0. \tag{1.4}$$

Baculíková [12] established some new oscillation results for (1.3) when $\alpha = 1$. In 1989, Philos [13] obtained some Philos-type oscillation criteria for (1.4).

Recently, many results have been obtained on oscillation and nonoscillation of neutral differential equations, and we refer the reader to [14–35] and the references cited therein. Liu and Bai [16], Xu and Meng [17, 18], Dong [19], Baculíková and Lacková [20], and Jiang and Li [21] established some oscillation criteria for (1.3) with neutral term under the assumptions $\tau(t) \leq t$, $\sigma(t) \leq t$,

$$R(t) = \int_{t_0}^{t} \frac{1}{r^{1/\alpha}(s)}\mathrm{d}s \longrightarrow \infty \text{ as } t \longrightarrow \infty, \tag{1.5}$$

$$\int_{t_0}^{\infty} \frac{1}{r^{1/\alpha}(t)}\mathrm{d}t < \infty. \tag{1.6}$$

Saker and O'Regan [24] studied the oscillatory behavior of (1.1) when $0 \leq p(t) < 1$, $\tau(t) \leq t$ and $\sigma(t) > t$.

Han et al. [26] examined the oscillation of second-order nonlinear neutral differential equation

$$\left(r(t)\left[x(t) + p(t)x(\tau(t))\right]'\right)' + q(t)f(x(\sigma(t))) = 0, \quad t \geq t_0, \tag{1.7}$$

where $\tau(t) \le t$, $\sigma(t) \le t$, $\tau'(t) = \tau_0 > 0$, $0 \le p(t) \le p_0 < \infty$, and the authors obtained some oscillation criteria for (1.7).

However, there are few results regarding the oscillatory problem of (1.1) when $\tau(t) \ge t$ and $\sigma(t) \ge t$. Our aim in this paper is to establish some oscillation criteria for (1.1) under the case when $\tau(t) \ge t$ and $\sigma(t) \ge t$.

The paper is organized as follows. In Section 2, we will establish an inequality to prove our results. In Section 3, some oscillation criteria are obtained for (1.1). In Section 4, we give two examples to show the importance of the main results.

All functional inequalities considered in this paper are assumed to hold eventually, that is, they are satisfied for all t large enough.

2. Lemma

In this section, we give the following lemma, which we will use in the proofs of our main results.

Lemma 2.1. *Assume that $\alpha \ge 1$, $a, b \in \mathbb{R}$. If $a \ge 0$, $b \ge 0$, then*

$$a^{\alpha} + b^{\alpha} \ge \frac{1}{2^{\alpha-1}}(a+b)^{\alpha} \tag{2.1}$$

holds.

Proof. (i) Suppose that $a = 0$ or $b = 0$. Obviously, we have (2.1). (ii) Suppose that $a > 0$, $b > 0$. Define the function g by $g(u) = u^{\alpha}$, $u \in (0, \infty)$. Then $g''(u) = \alpha(\alpha-1)u^{\alpha-2} \ge 0$ for $u > 0$. Thus, g is a convex function. By the definition of convex function, for $\lambda = 1/2$, $a, b \in (0, \infty)$, we have

$$g\left(\frac{a+b}{2}\right) \le \frac{g(a)+g(b)}{2}, \tag{2.2}$$

that is,

$$a^{\alpha} + b^{\alpha} \ge \frac{1}{2^{\alpha-1}}(a+b)^{\alpha}. \tag{2.3}$$

This completes the proof. □

3. Main Results

In this section, we will establish some oscillation criteria for (1.1).

First, we establish two comparison theorems which enable us to reduce the problem of the oscillation of (1.1) to the research of the first-order differential inequalities.

Theorem 3.1. *Suppose that* (1.5) *holds. If the first-order neutral differential inequality*

$$\left[u(t)+\frac{(p_0)^{\alpha}}{\tau_0}u(\tau(t))\right]'+\frac{1}{2^{\alpha-1}}Q(t)(R(\sigma(t))-R(t_1))^{\alpha}u(\sigma(t))\leq 0 \tag{3.1}$$

has no positive solution for all sufficiently large t_1, *where* $Q(t)=\min\{q(t),q(\tau(t))\}$, *then every solution of* (1.1) *oscillates.*

Proof. Let x be a nonoscillatory solution of (1.1). Without loss of generality, we assume that there exists $t_1\geq t_0$ such that $x(t)>0$, $x(\tau(t))>0$, and $x(\sigma(t))>0$ for all $t\geq t_1$. Then $z(t)>0$ for $t\geq t_1$. In view of (1.1), we obtain

$$\left(r(t)\left|z'(t)\right|^{\alpha-1}z'(t)\right)'\leq -q(t)x^{\alpha}(\sigma(t))\leq 0,\quad t\geq t_1. \tag{3.2}$$

Thus, $r(t)|z'(t)|^{\alpha-1}z'(t)$ is decreasing function. Now we have two possible cases for $z'(t)$: (i) $z'(t)<0$ eventually, (ii) $z'(t)>0$ eventually.

Suppose that $z'(t)<0$ for $t\geq t_2\geq t_1$. Then, from (3.2), we get

$$r(t)\left|z'(t)\right|^{\alpha-1}z'(t)\leq r(t_2)\left|z'(t_2)\right|^{\alpha-1}z'(t_2),\quad t\geq t_2, \tag{3.3}$$

which implies that

$$z(t)\leq z(t_2)-r^{1/\alpha}(t_2)\left|z'(t_2)\right|\int_{t_2}^{t}r^{-1/\alpha}(s)\mathrm{d}s. \tag{3.4}$$

Letting $t\to\infty$, by (1.5), we find $z(t)\to-\infty$, which is a contradiction. Thus

$$z'(t)>0 \tag{3.5}$$

for $t\geq t_2$.

By applying (1.1), for all sufficiently large t, we obtain

$$\left(r(t)\left(z'(t)\right)^{\alpha}\right)'+q(t)x^{\alpha}(\sigma(t))+(p_0)^{\alpha}q(\tau(t))x^{\alpha}(\sigma(\tau(t)))+\frac{(p_0)^{\alpha}}{\tau'(t)}\left(r(\tau(t))\left(z'(\tau(t))\right)^{\alpha}\right)'\leq 0. \tag{3.6}$$

Using inequality (2.1), (3.2), (3.5), $\tau\circ\sigma=\sigma\circ\tau$, and the definition of z, we conclude that

$$\left(r(t)\left(z'(t)\right)^{\alpha}\right)'+\frac{(p_0)^{\alpha}}{\tau_0}r(\tau(t))\left(z'(\tau(t))^{\alpha}\right)'+\frac{1}{2^{\alpha-1}}Q(t)z^{\alpha}(\sigma(t))\leq 0. \tag{3.7}$$

It follows from (3.2) and (3.5) that $u(t) = r(t)(z'(t))^{\alpha} > 0$ is decreasing and then

$$z(t) \geq \int_{t_2}^{t} \frac{\left(r(s)(z'(s))^{\alpha}\right)^{1/\alpha}}{r^{1/\alpha}(s)} \mathrm{d}s \geq u^{1/\alpha}(t) \int_{t_2}^{t} \frac{1}{r^{1/\alpha}(s)} \mathrm{d}s = u^{1/\alpha}(t)(R(t) - R(t_2)). \tag{3.8}$$

Thus, from (3.7) and the above inequality, we find

$$\left[u(t) + \frac{(p_0)^{\alpha}}{\tau_0} u(\tau(t))\right]' + \frac{1}{2^{\alpha-1}} Q(t)(R(\sigma(t)) - R(t_2))^{\alpha} u(\sigma(t)) \leq 0. \tag{3.9}$$

That is, inequality (3.1) has a positive solution u; this is a contradiction. The proof is complete. □

Theorem 3.2. *Suppose that* (1.5) *holds. If the first-order differential inequality*

$$\eta'(t) + \frac{\tau_0}{2^{\alpha-1}\left(\tau_0 + (p_0)^{\alpha}\right)} Q(t)(R(\sigma(t)) \;\; - R(t_1))^{\alpha} \eta(\sigma(t)) \leq 0 \tag{3.10}$$

has no positive solution for all sufficiently large t_1, where Q is defined as in Theorem 3.1, then every solution of (1.1) *oscillates.*

Proof. Let x be a nonoscillatory solution of (1.1). Without loss of generality, we assume that there exists $t_1 \geq t_0$ such that $x(t) > 0$, $x(\tau(t)) > 0$, and $x(\sigma(t)) > 0$ for all $t \geq t_1$. Then $z(t) > 0$ for $t \geq t_1$. Proceeding as in the proof of Theorem 3.1, we obtain that $u(t) = r(t)(z'(t))^{\alpha}$ is decreasing, and it satisfies inequality (3.1). Set $\eta(t) = u(t) + (p_0)^{\alpha} u(\tau(t))/\tau_0$. From $\tau(t) \geq t$, we get

$$\eta(t) = u(t) + \frac{(p_0)^{\alpha}}{\tau_0} u(\tau(t)) \leq \left(1 + \frac{(p_0)^{\alpha}}{\tau_0}\right) u(t). \tag{3.11}$$

In view of the above inequality and (3.1), we see that

$$\eta'(t) + \frac{\tau_0}{2^{\alpha-1}\left(\tau_0 + (p_0)^{\alpha}\right)} Q(t)(R(\sigma(t)) - R(t_1))^{\alpha} \eta(\sigma(t)) \leq 0. \tag{3.12}$$

That is, inequality (3.10) has a positive solution η; this is a contradiction. The proof is complete. □

Next, using Riccati transformation technique, we obtain the following results.

Theorem 3.3. *Suppose that* (1.5) *holds. Moreover, assume that there exists $\rho \in C^1([t_0, \infty), (0, \infty))$ such that*

$$\limsup_{t \to \infty} \int_{t_0}^{t} \left[\frac{\rho(s)Q(s)}{2^{\alpha-1}} - \frac{1}{(\alpha+1)^{\alpha+1}} \left(1 + \frac{(p_0)^{\alpha}}{\tau_0}\right) \frac{r(s)\left(\rho'_{+}(s)\right)^{\alpha+1}}{(\rho(s))^{\alpha}}\right] \mathrm{d}s = \infty \tag{3.13}$$

holds, where Q is defined as in Theorem 3.1, $\rho'_+(t) = \max\{0, \rho'(t)\}$. Then every solution of (1.1) *oscillates.*

Proof. Let x be a nonoscillatory solution of (1.1). Without loss of generality, we assume that there exists $t_1 \geq t_0$ such that $x(t) > 0$, $x(\tau(t)) > 0$, and $x(\sigma(t)) > 0$ for all $t \geq t_1$. Then $z(t) > 0$ for $t \geq t_1$. Proceeding as in the proof of Theorem 3.1, we obtain (3.2)–(3.7).

Define a Riccati substitution

$$\omega(t) = \rho(t)\frac{r(t)(z'(t))^{\alpha}}{(z(t))^{\alpha}}, \quad t \geq t_2. \tag{3.14}$$

Thus $\omega(t) > 0$ for $t \geq t_2$. Differentiating (3.14) we find that

$$\omega'(t) = \rho'(t)\frac{r(t)(z'(t))^{\alpha}}{(z(t))^{\alpha}} + \rho(t)\frac{\left(r(t)(z'(t))^{\alpha}\right)'}{(z(t))^{\alpha}} - \alpha\rho(t)\frac{r(t)(z'(t))^{\alpha}z^{\alpha-1}(t)z'(t)}{(z(t))^{2\alpha}}. \tag{3.15}$$

Hence, by (3.14) and (3.15), we see that

$$\omega'(t) = \frac{\rho'(t)}{\rho(t)}\omega(t) + \rho(t)\frac{\left(r(t)(z'(t))^{\alpha}\right)'}{(z(t))^{\alpha}} - \frac{\alpha}{\rho^{1/\alpha}(t)r^{1/\alpha}(t)}\omega^{(\alpha+1)/\alpha}(t). \tag{3.16}$$

Similarly, we introduce another Riccati substitution

$$\upsilon(t) = \rho(t)\frac{r(\tau(t))(z'(\tau(t)))^{\alpha}}{(z(t))^{\alpha}}, \quad t \geq t_2. \tag{3.17}$$

Then $\upsilon(t) > 0$ for $t \geq t_2$. From (3.2), (3.5), and $\tau(t) \geq t$, we have

$$z'(t) \geq \left(\frac{r(\tau(t))}{r(t)}\right)^{1/\alpha} z'(\tau(t)). \tag{3.18}$$

Differentiating (3.17), we find

$$\upsilon'(t) = \rho'(t)\frac{r(\tau(t))(z'(\tau(t)))^{\alpha}}{(z(t))^{\alpha}} + \rho(t)\frac{\left(r(\tau(t))(z'(\tau(t)))^{\alpha}\right)'}{(z(t))^{\alpha}} - \alpha\rho(t)\frac{r(\tau(t))(z'(\tau(t)))^{\alpha}z^{\alpha-1}(t)z'(t)}{(z(t))^{2\alpha}}. \tag{3.19}$$

Therefore, by (3.17), (3.18), and (3.19), we see that

$$\upsilon'(t) \leq \frac{\rho'(t)}{\rho(t)}\upsilon(t) + \rho(t)\frac{\left(r(\tau(t))(z'(\tau(t)))^{\alpha}\right)'}{(z(t))^{\alpha}} - \frac{\alpha}{\rho^{1/\alpha}(t)r^{1/\alpha}(t)}\upsilon^{(\alpha+1)/\alpha}(t). \tag{3.20}$$

Thus, from (3.16) and (3.20), we have

$$\omega'(t) + \frac{(p_0)^\alpha}{\tau_0}\upsilon'(t) \le \rho(t)\frac{\left(r(t)(z'(t))^\alpha\right)' + \left((p_0)^\alpha/\tau_0\right)\left(r(\tau(t))(z'(\tau(t)))^\alpha\right)'}{(z(t))^\alpha} + \frac{\rho'(t)}{\rho(t)}\omega(t) - \frac{\alpha}{\rho^{1/\alpha}(t)r^{1/\alpha}(t)}\omega^{(\alpha+1)/\alpha}(t) + \frac{(p_0)^\alpha}{\tau_0}\frac{\rho'(t)}{\rho(t)}\upsilon(t) - \frac{(p_0)^\alpha}{\tau_0}\frac{\alpha}{\rho^{1/\alpha}(t)r^{1/\alpha}(t)}\upsilon^{(\alpha+1)/\alpha}(t). \tag{3.21}$$

It follows from (3.5), (3.7), and $\sigma(t) \ge t$ that

$$\omega'(t) + \frac{(p_0)^\alpha}{\tau_0}\upsilon'(t) \le -\frac{1}{2^{\alpha-1}}\rho(t)Q(t) + \frac{\rho'_+(t)}{\rho(t)}\omega(t) - \frac{\alpha}{\rho^{1/\alpha}(t)r^{1/\alpha}(t)}\omega^{(\alpha+1)/\alpha}(t) + \frac{(p_0)^\alpha}{\tau_0}\frac{\rho'_+(t)}{\rho(t)}\upsilon(t) - \frac{(p_0)^\alpha}{\tau_0}\frac{\alpha}{\rho^{1/\alpha}(t)r^{1/\alpha}(t)}\upsilon^{(\alpha+1)/\alpha}(t). \tag{3.22}$$

Integrating the above inequality from t_2 to t, we obtain

$$\omega(t) - \omega(t_2) + \frac{(p_0)^\alpha}{\tau_0}\upsilon(t) - \frac{(p_0)^\alpha}{\tau_0}\upsilon(t_2) \le -\int_{t_2}^{t}\frac{1}{2^{\alpha-1}}\rho(s)Q(s)\mathrm{d}s + \int_{t_2}^{t}\left[\frac{\rho'_+(s)}{\rho(s)}\omega(s) - \frac{\alpha}{\rho^{1/\alpha}(s)r^{1/\alpha}(s)}\omega^{(\alpha+1)/\alpha}(s)\right]\mathrm{d}s + \int_{t_2}^{t}\frac{(p_0)^\alpha}{\tau_0}\left[\frac{\rho'_+(s)}{\rho(s)}\upsilon(s) - \frac{\alpha}{\rho^{1/\alpha}(s)r^{1/\alpha}(s)}\upsilon^{(\alpha+1)/\alpha}(s)\right]\mathrm{d}s. \tag{3.23}$$

Define

$$A := \left[\frac{\alpha}{\rho^{1/\alpha}(t)r^{1/\alpha}(t)}\right]^{\alpha/(\alpha+1)}\omega(t), \qquad B := \left[\frac{\rho'_+(t)}{\rho(t)}\frac{\alpha}{\alpha+1}\left[\frac{\alpha}{\rho^{1/\alpha}(t)r^{1/\alpha}(t)}\right]^{-\alpha/(\alpha+1)}\right]^{\alpha}. \tag{3.24}$$

Using inequality

$$\frac{\alpha+1}{\alpha}AB^{1/\alpha} - A^{(\alpha+1)/\alpha} \le \frac{1}{\alpha}B^{(\alpha+1)/\alpha}, \quad \text{for } A \ge 0,\ B \ge 0 \ \text{ are constants}, \tag{3.25}$$

we have

$$\frac{\rho'_+(t)}{\rho(t)}\omega(t) - \frac{\alpha}{\rho^{1/\alpha}(t)r^{1/\alpha}(t)}\omega^{(\alpha+1)/\alpha}(t) \le \frac{1}{(\alpha+1)^{\alpha+1}}\frac{r(t)\left(\rho'_+(t)\right)^{\alpha+1}}{\rho(t)^\alpha}. \tag{3.26}$$

Similarly, we obtain

$$\frac{\rho'_+(t)}{\rho(t)}v(t) - \frac{\alpha}{\rho^{1/\alpha}(t)r^{1/\alpha}(t)}v^{(\alpha+1)/\alpha}(t) \leq \frac{1}{(\alpha+1)^{\alpha+1}}\frac{r(t)\left(\rho'_+(t)\right)^{\alpha+1}}{\rho(t)^{\alpha}}. \tag{3.27}$$

Thus, from (3.23), we get

$$\begin{aligned}\omega(t) - \omega(t_2) + \frac{(p_0)^{\alpha}}{\tau_0}v(t) - \frac{(p_0)^{\alpha}}{\tau_0}v(t_2)\\ \leq -\int_{t_2}^{t}\left[\frac{\rho(s)Q(s)}{2^{\alpha-1}} - \frac{1}{(\alpha+1)^{\alpha+1}}\left(1+\frac{(p_0)^{\alpha}}{\tau_0}\right)\frac{r(s)\left(\rho'_+(s)\right)^{\alpha+1}}{\rho(s)^{\alpha}}\right]ds,\end{aligned} \tag{3.28}$$

which contradicts (3.13). This completes the proof. □

As an immediate consequence of Theorem 3.3 we get the following.

Corollary 3.4. *Let assumption* (3.13) *in Theorem 3.3 be replaced by*

$$\begin{aligned}\limsup_{t\to\infty}\int_{t_0}^{t}\rho(s)Q(s)ds = \infty,\\ \limsup_{t\to\infty}\int_{t_0}^{t}\frac{r(s)\left(\rho'_+(s)\right)^{\alpha+1}}{(\rho(s))^{\alpha}}ds < \infty.\end{aligned} \tag{3.29}$$

Then every solution of (1.1) *oscillates.*

From Theorem 3.3 by choosing the function ρ, appropriately, we can obtain different sufficient conditions for oscillation of (1.1), and if we define a function ρ by $\rho(t) = 1$, and $\rho(t) = t$, we have the following oscillation results.

Corollary 3.5. *Suppose that* (1.5) *holds. If*

$$\limsup_{t\to\infty}\int_{t_0}^{t}Q(s)ds = \infty, \tag{3.30}$$

where Q is defined as in Theorem 3.1, then every solution of (1.1) *oscillates.*

Corollary 3.6. *Suppose that* (1.5) *holds. If*

$$\limsup_{t\to\infty}\int_{t_0}^{t}\left[\frac{sQ(s)}{2^{\alpha-1}} - \frac{1}{(\alpha+1)^{\alpha+1}}\left(1+\frac{(p_0)^{\alpha}}{\tau_0}\right)\frac{r(s)}{s^{\alpha}}\right]ds = \infty, \tag{3.31}$$

where Q is defined as in Theorem 3.1, then every solution of (1.1) *oscillates.*

In the following theorem, we present a Philos-type oscillation criterion for (1.1). First, we introduce a class of functions $\mathfrak{R}$. Let

$$\mathbb{D}_0 = \{(t,s) : t > s \geq t_0\}, \qquad \mathbb{D} = \{(t,s) : t \geq s \geq t_0\}. \tag{3.32}$$

The function $H \in C(\mathbb{D}, \mathbb{R})$ is said to belong to the class $\mathfrak{R}$ (defined by $H \in \mathfrak{R}$, for short) if

(i) $H(t,t) = 0$, for $t \geq t_0$, $H(t,s) > 0$, for $(t,s) \in \mathbb{D}_0$;

(ii) H has a continuous and nonpositive partial derivative $\partial H(t,s)/\partial s$ on D_0 with respect to s.

We assume that $\varsigma(t)$ and $\rho(t)$ for $t \geq t_0$ are given continuous functions such that $\rho(t) > 0$ and differentiable and define

$$\theta(t) = \frac{\rho'(t)}{\rho(t)} + (\alpha+1)(\varsigma(t))^{1/\alpha}, \psi(t) = \rho(t)\left\{[r(t)\varsigma(t)]' - r(t)(\varsigma(t))^{(1+\alpha)/\alpha}\right\},$$
$$\phi(t,s) = \frac{r(s)\rho(s)}{(\alpha+1)^{\alpha+1}}\left(\theta(s) + \frac{\partial H(t,s)/\partial s}{H(t,s)}\right)^{\alpha+1}. \tag{3.33}$$

Now, we give the following result.

Theorem 3.7. *Suppose that* (1.5) *holds and α is a quotient of odd positive integers. Moreover, let $H \in \mathfrak{R}$ be such that*

$$\limsup_{t\to\infty} \frac{1}{H(t,t_0)} \int_{t_0}^{t} H(t,s)\left[\frac{\rho(s)Q(s)}{2^{\alpha-1}} - \left(1 + \frac{(p_0)^\alpha}{\tau_0}\right)(\psi(s) + \phi(t,s))\right]ds = \infty \tag{3.34}$$

holds, where Q is defined as in Theorem 3.1. Then every solution of (1.1) *oscillates.*

Proof. Let x be a nonoscillatory solution of (1.1). Without loss of generality, we assume that there exists $t_1 \geq t_0$ such that $x(t) > 0$, $x(\tau(t)) > 0$, and $x(\sigma(t)) > 0$ for all $t \geq t_1$. Then $z(t) > 0$ for $t \geq t_1$. Proceeding as in the proof of Theorem 3.1, we obtain (3.2)–(3.7). Define the Riccati substitution ω by

$$\omega(t) = \rho(t)\left[\frac{r(t)(z'(t))^\alpha}{(z(t))^\alpha} + r(t)\varsigma(t)\right], \quad t \geq t_2 \geq t_1. \tag{3.35}$$

Then, we have

$$\omega'(t) = \rho'(t)\left[\frac{r(t)(z'(t))^\alpha}{(z(t))^\alpha} + r(t)\varsigma(t)\right] + \rho(t)\left[\frac{r(t)(z'(t))^\alpha}{(z(t))^\alpha} + r(t)\varsigma(t)\right]'$$
$$= \frac{\rho'(t)}{\rho(t)}\omega(t) + \rho(t)[r(t)\varsigma(t)]' + \rho(t)\frac{\left(r(t)(z'(t))^\alpha\right)'}{(z(t))^\alpha} - \alpha\rho(t)\frac{r(t)(z'(t))^{\alpha+1}}{(z(t))^{\alpha+1}}. \tag{3.36}$$

Using (3.35), we get

$$\omega'(t) = \frac{\rho'(t)}{\rho(t)}\omega(t) + \rho(t)[r(t)\varsigma(t)]' + \rho(t)\frac{\left(r(t)(z'(t))^{\alpha}\right)'}{(z(t))^{\alpha}} - \frac{\alpha\rho(t)}{r^{1/\alpha}(t)}\left[\frac{\omega(t)}{\rho(t)} - r(t)\varsigma(t)\right]^{(1+\alpha)/\alpha}. \tag{3.37}$$

Let

$$A = \frac{\omega(t)}{\rho(t)}, \qquad B = r(t)\varsigma(t). \tag{3.38}$$

By applying the inequality (see [21, 24])

$$A^{(1+\alpha)/\alpha} - (A-B)^{1+\alpha/\alpha} \le B^{1/\alpha}\left[\left(1+\frac{1}{\alpha}\right)A - \frac{1}{\alpha}B\right], \quad \text{for } \alpha = \frac{\text{odd}}{\text{odd}} \ge 1, \tag{3.39}$$

we see that

$$\left[\frac{\omega(t)}{\rho(t)} - r(t)\varsigma(t)\right]^{(1+\alpha)/\alpha} \ge \frac{\omega^{(1+\alpha)/\alpha}(t)}{\rho^{(1+\alpha)/\alpha}(t)} + \frac{1}{\alpha}(r(t)\varsigma(t))^{(1+\alpha)/\alpha} - \frac{\alpha+1}{\alpha}\frac{(r(t)\varsigma(t))^{1/\alpha}}{\rho(t)}\omega(t). \tag{3.40}$$

Substituting (3.40) into (3.37), we have

$$\begin{aligned}\omega'(t) \le &\left[\frac{\rho'(t)}{\rho(t)} + (\alpha+1)(\varsigma(t))^{1/\alpha}\right]\omega(t) + \rho(t)\left\{[r(t)\varsigma(t)]' - r(t)(\varsigma(t))^{(1+\alpha)/\alpha}\right\} \\ &+ \rho(t)\frac{\left(r(t)(z'(t))^{\alpha}\right)'}{(z(t))^{\alpha}} - \frac{\alpha}{r^{1/\alpha}(t)\rho^{1/\alpha}(t)}\omega^{(1+\alpha)/\alpha}(t).\end{aligned} \tag{3.41}$$

That is,

$$\omega'(t) \le \theta(t)\omega(t) + \psi(t) + \rho(t)\frac{\left(r(t)(z'(t))^{\alpha}\right)'}{(z(t))^{\alpha}} - \frac{\alpha}{r^{1/\alpha}(t)\rho^{1/\alpha}(t)}\omega^{(1+\alpha)/\alpha}(t). \tag{3.42}$$

Next, define another Riccati transformation u by

$$u(t) = \rho(t)\left[\frac{r(\tau(t))(z'(\tau(t)))^{\alpha}}{(z(t))^{\alpha}} + r(t)\varsigma(t)\right], \quad t \ge t_2 \ge t_1. \tag{3.43}$$

Then, we have

$$u'(t) = \rho'(t)\left[\frac{r(\tau(t))(z'(\tau(t)))^{\alpha}}{(z(t))^{\alpha}} + r(t)\varsigma(t)\right] + \rho(t)\left[\frac{r(\tau(t))(z'(\tau(t)))^{\alpha}}{(z(t))^{\alpha}} + r(t)\varsigma(t)\right]'$$
$$= \frac{\rho'(t)}{\rho(t)}u(t) + \rho(t)[r(t)\varsigma(t)]' + \rho(t)\frac{\left(r(\tau(t))(z'(\tau(t)))^{\alpha}\right)'}{(z(t))^{\alpha}} - \alpha\rho(t)\frac{r(\tau(t))(z'(\tau(t)))^{\alpha}z'(t)}{(z(t))^{\alpha+1}}. \tag{3.44}$$

From (3.2), (3.5), and $\tau(t) \geq t$, we have that (3.18) holds. Hence, we obtain

$$u'(t) \leq \frac{\rho'(t)}{\rho(t)}u(t) + \rho(t)[r(t)\varsigma(t)]' + \rho(t)\frac{\left(r(\tau(t))(z'(\tau(t)))^{\alpha}\right)}{(z(t))^{\alpha}} - \alpha\rho(t)\frac{\left(r(\tau(t))(z'(\tau(t)))^{\alpha}\right)^{(1+\alpha)/\alpha}}{r^{1/\alpha}(t)(z(t))^{\alpha+1}}. \tag{3.45}$$

Using (3.43), we get

$$u'(t) \leq \frac{\rho'(t)}{\rho(t)}u(t) + \rho(t)[r(t)\varsigma(t)]' + \rho(t)\frac{\left(r(\tau(t))(z'(\tau(t)))^{\alpha}\right)'}{(z(t))^{\alpha}} - \frac{\alpha\rho(t)}{r^{1/\alpha}(t)}\left[\frac{u(t)}{\rho(t)} - r(t)\varsigma(t)\right]^{(1+\alpha)/\alpha}. \tag{3.46}$$

Let

$$A = \frac{u(t)}{\rho(t)}, \qquad B = r(t)\varsigma(t). \tag{3.47}$$

By applying the inequality (3.39), we see that

$$\left[\frac{u(t)}{\rho(t)} - r(t)\varsigma(t)\right]^{(1+\alpha)/\alpha} \geq \frac{u^{(1+\alpha)/\alpha}(t)}{\rho^{(1+\alpha)/\alpha}(t)} + \frac{1}{\alpha}(r(t)\varsigma(t))^{(1+\alpha)/\alpha} - \frac{\alpha+1}{\alpha}\frac{(r(t)\varsigma(t))^{1/\alpha}}{\rho(t)}u(t). \tag{3.48}$$

Substituting (3.48) into (3.46), we have

$$u'(t) \leq \left[\frac{\rho'(t)}{\rho(t)} + (\alpha+1)(\varsigma(t))^{1/\alpha}\right]u(t) + \rho(t)\left\{[r(t)\varsigma(t)]' - r(t)(\varsigma(t))^{(1+\alpha)/\alpha}\right\}$$
$$+ \rho(t)\frac{\left(r(\tau(t))(z'(\tau(t)))^{\alpha}\right)'}{(z(t))^{\alpha}} - \frac{\alpha}{r^{1/\alpha}(t)\rho^{1/\alpha}(t)}u^{(1+\alpha)/\alpha}(t). \tag{3.49}$$

That is,

$$u'(t) \leq \theta(t)u(t) + \psi(t) + \rho(t)\frac{\left(r(\tau(t))(z'(\tau(t)))^{\alpha}\right)'}{(z(t))^{\alpha}} - \frac{\alpha}{r^{1/\alpha}(t)\rho^{1/\alpha}(t)}u^{(1+\alpha)/\alpha}(t). \tag{3.50}$$

By (3.42) and (3.50), we find

$$\begin{aligned}\omega'(t)+\frac{(p_0)^\alpha}{\tau_0}u'(t)\le{}&\left(1+\frac{(p_0)^\alpha}{\tau_0}\right)\varphi(t)+\rho(t)\frac{\left(r(t)(z'(t))^\alpha\right)'+\left((p_0)^\alpha/\tau_0\right)\left(r(\tau(t))(z'(\tau(t)))^\alpha\right)'}{(z(t))^\alpha}\\&+\theta(t)\omega(t)-\frac{\alpha}{r^{1/\alpha}(t)\rho^{1/\alpha}(t)}\omega^{(1+\alpha)/\alpha}(t)+\frac{(p_0)^\alpha}{\tau_0}\theta(t)u(t)\\&-\frac{(p_0)^\alpha}{\tau_0}\frac{\alpha}{r^{1/\alpha}(t)\rho^{1/\alpha}(t)}u^{(1+\alpha)/\alpha}(t).\end{aligned}\tag{3.51}$$

In view of the above inequality, (3.5), (3.7), and $\sigma(t)\ge t$, we get

$$\begin{aligned}\omega'(t)+\frac{(p_0)^\alpha}{\tau_0}u'(t)\le{}&\left(1+\frac{(p_0)^\alpha}{\tau_0}\right)\varphi(t)-\frac{\rho(t)Q(t)}{2^{\alpha-1}}+\theta(t)\omega(t)-\frac{\alpha}{r^{1/\alpha}(t)\rho^{1/\alpha}(t)}\omega^{(1+\alpha)/\alpha}(t)\\&+\frac{(p_0)^\alpha}{\tau_0}\theta(t)u(t)-\frac{(p_0)^\alpha}{\tau_0}\frac{\alpha}{r^{1/\alpha}(t)\rho^{1/\alpha}(t)}u^{(1+\alpha)/\alpha}(t),\end{aligned}\tag{3.52}$$

which follows that

$$\begin{aligned}&\int_{t_2}^{t}H(t,s)\left[\frac{\rho(s)Q(s)}{2^{\alpha-1}}-\left(1+\frac{(p_0)^\alpha}{\tau_0}\right)\varphi(s)\right]\mathrm{d}s\\&\quad\le-\int_{t_2}^{t}H(t,s)\omega'(s)\mathrm{d}s+\int_{t_2}^{t}H(t,s)\theta(s)\omega(s)\mathrm{d}s\\&\quad-\int_{t_2}^{t}H(t,s)\frac{\alpha\omega^{(1+\alpha)/\alpha}(s)}{r^{1/\alpha}(s)\rho^{1/\alpha}(s)}\mathrm{d}s-\frac{(p_0)^\alpha}{\tau_0}\int_{t_2}^{t}H(t,s)u'(s)\mathrm{d}s\\&\quad+\frac{(p_0)^\alpha}{\tau_0}\int_{t_2}^{t}H(t,s)\theta(s)u(s)\mathrm{d}s-\frac{(p_0)^\alpha}{\tau_0}\int_{t_2}^{t}H(t,s)\frac{\alpha u^{(1+\alpha)/\alpha}(s)}{r^{1/\alpha}(s)\rho^{1/\alpha}(s)}\mathrm{d}s.\end{aligned}\tag{3.53}$$

Using the integration by parts formula and $H(t,t)=0$, we have

$$\begin{aligned}\int_{t_2}^{t}H(t,s)\omega'(s)\mathrm{d}s&=-H(t,t_2)\omega(t_2)-\int_{t_2}^{t}\frac{\partial H(t,s)}{\partial s}\omega(s)\mathrm{d}s,\\\int_{t_2}^{t}H(t,s)u'(s)\mathrm{d}s&=-H(t,t_2)u(t_2)-\int_{t_2}^{t}\frac{\partial H(t,s)}{\partial s}u(s)\mathrm{d}s.\end{aligned}\tag{3.54}$$

So, by (3.53), we obtain

$$\begin{aligned}&\int_{t_2}^{t} H(t,s)\left[\frac{\rho(s)Q(s)}{2^{\alpha-1}}-\left(1+\frac{(p_0)^{\alpha}}{\tau_0}\right)\psi(s)\right]\mathrm{d}s\\&\quad\leq H(t,t_2)\omega(t_2)+\frac{(p_0)^{\alpha}}{\tau_0}H(t,t_2)u(t_2)\\&\qquad+\int_{t_2}^{t}H(t,s)\left[\theta(s)+\frac{\partial H(t,s)/\partial s}{H(t,s)}\right]\omega(s)\mathrm{d}s-\int_{t_2}^{t}H(t,s)\frac{\alpha\omega^{(1+\alpha)/\alpha}(s)}{r^{1/\alpha}(s)\rho^{1/\alpha}(s)}\mathrm{d}s\\&\qquad+\frac{(p_0)^{\alpha}}{\tau_0}\int_{t_2}^{t}H(t,s)\left[\theta(s)+\frac{\partial H(t,s)/\partial s}{H(t,s)}\right]u(s)\mathrm{d}s-\frac{(p_0)^{\alpha}}{\tau_0}\int_{t_2}^{t}H(t,s)\frac{\alpha u^{(1+\alpha)/\alpha}(s)}{r^{1/\alpha}(s)\rho^{1/\alpha}(s)}\mathrm{d}s.\end{aligned} \tag{3.55}$$

Using the inequality

$$By-Ay^{(\alpha+1)/\alpha}\leq\frac{\alpha^{\alpha}}{(\alpha+1)^{\alpha+1}}\frac{B^{\alpha+1}}{A^{\alpha}}, \tag{3.56}$$

where

$$A=\frac{\alpha}{r^{1/\alpha}(s)\rho^{1/\alpha}(s)},\qquad B=\theta(s)+\frac{\partial H(t,s)/\partial s}{H(t,s)}, \tag{3.57}$$

we have

$$\int_{t_2}^{t}H(t,s)\left[\frac{\rho(s)Q(s)}{2^{\alpha-1}}-\left(1+\frac{(p_0)^{\alpha}}{\tau_0}\right)(\psi(s)+\phi(t,s))\right]\mathrm{d}s\leq H(t,t_2)\omega(t_2)+\frac{(p_0)^{\alpha}}{\tau_0}H(t,t_2)u(t_2) \tag{3.58}$$

due to (3.55), which yields that

$$\frac{1}{H(t,t_2)}\int_{t_2}^{t}H(t,s)\left[\frac{\rho(s)Q(s)}{2^{\alpha-1}}-\left(1+\frac{(p_0)^{\alpha}}{\tau_0}\right)(\psi(s)+\phi(t,s))\right]\mathrm{d}s\leq\omega(t_2)+\frac{(p_0)^{\alpha}}{\tau_0}u(t_2), \tag{3.59}$$

which contradicts (3.34). The proof is complete. □

From Theorem 3.7, we can obtain different oscillation conditions for all solutions of (1.1) with different choices of H; the details are left to the reader.

Theorem 3.8. *Assume that* (1.6) *and* (3.30) *hold. Furthermore, assume that* $0\leq p(t)\leq p_1<1$. *If*

$$\int_{t_0}^{\infty}\left[\frac{1}{r(s)}\int_{t_0}^{s}q(u)\mathrm{d}u\right]^{1/\alpha}\mathrm{d}s=\infty, \tag{3.60}$$

then every solution x *of* (1.1) *oscillates or* $\lim_{t\to\infty}x(t)=0$.

Proof. Let x be a nonoscillatory solution of (1.1). Without loss of generality, we assume that there exists $t_1 \geq t_0$ such that $x(t) > 0$, $x(\tau(t)) > 0$, and $x(\sigma(t)) > 0$ for all $t \geq t_1$. Then $z(t) > 0$ for $t \geq t_1$. Proceeding as in the proof of Theorem 3.1, we obtain (3.2). Thus $r(t)|z'(t)|^{\alpha-1}z'(t)$ is decreasing function, and there exists a $t_2 \geq t_1$ such that $z'(t) > 0$, $t \geq t_2$ or $z'(t) < 0$, $t \geq t_2$.

Case 1. Assume that $z'(t) > 0$, for $t \geq t_2$. Proceeding as in the proof of Theorem 3.3 and setting $\rho(t) = t$, we can obtain a contradiction with (3.31).

Case 2. Assume that $z'(t) < 0$, for $t \geq t_2$. Then there exists a finite limit

$$\lim_{t\to\infty} z(t) = l, \tag{3.61}$$

where $l \geq 0$. Next, we claim that $l = 0$. If not, then for any $\epsilon > 0$, we have $l < z(t) < l + \epsilon$, eventually. Take $0 < \epsilon < l(1-p_1)/p_1$. We calculate

$$x(t) = z(t) - p(t)x(\tau(t)) > l - p_1 z(\tau(t)) > l - p_1(l+\epsilon) = m(l+\epsilon) > mz(t), \tag{3.62}$$

where

$$m = \frac{l}{l+\epsilon} - p_1 = \frac{l(1-p_1) - \epsilon p_1}{l+\epsilon} > 0. \tag{3.63}$$

From (3.2) and (3.62), we have

$$\left(r(t)\left(-z'(t)\right)^{\alpha}\right)' \geq q(t)x^{\alpha}(\sigma(t)) \geq (ml)^{\alpha} q(t). \tag{3.64}$$

Integrating the above inequality from t_2 to t, we get

$$r(t)\left(-z'(t)\right)^{\alpha} - r(t_2)\left(-z'(t_2)\right)^{\alpha} \geq (ml)^{\alpha} \int_{t_2}^{t} q(s)\mathrm{d}s, \tag{3.65}$$

which implies

$$z'(t) \leq -ml\left[\frac{1}{r(t)}\int_{t_2}^{t} q(s)\mathrm{d}s\right]^{1/\alpha}. \tag{3.66}$$

Integrating the above inequality from t_2 to t, we have

$$z(t) \leq z(t_2) - ml\int_{t_2}^{t}\left[\frac{1}{r(s)}\int_{t_2}^{s} q(u)\mathrm{d}u\right]^{1/\alpha}\mathrm{d}s, \tag{3.67}$$

which yields $z(t) \to -\infty$; this is a contradiction. Hence, $\lim_{t\to\infty} z(t) = 0$. Note that $0 < x(t) \leq z(t)$. Then, $\lim_{t\to\infty} x(t) = 0$. The proof is complete. □

4. Examples

In this section, we will give two examples to illustrate the main results.

Example 4.1. Consider the following linear neutral equation:

$$(x(t) + 2x(t + (2n - 1)\pi))'' + x(t + (2m - 1)\pi) = 0, \quad \text{for } t \geq t_0, \tag{4.1}$$

where n and m are positive integers.

Let

$$r(t) = 1, \quad p(t) = 2, \quad \tau(t) = t + (2n - 1)\pi, \quad q(t) = 1, \quad \sigma(t) = t + (2m - 1)\pi. \tag{4.2}$$

Hence, $Q(t) = 1$. Obviously, all the conditions of Corollary 3.5 hold. Thus by Corollary 3.5, every solution of (4.1) is oscillatory. It is easy to verify that $x(t) = \sin t$ is a solution of (4.1).

Example 4.2. Consider the following linear neutral equation:

$$\left(e^{2t}\left(x(t) + \frac{1}{2}x(t + 3)\right)'\right)' + \left(e^{2t+1} + \frac{1}{2}e^{2t-2}\right)x(t + 1) = 0, \quad \text{for } t \geq t_0, \tag{4.3}$$

where n and m are positive integers.

Let

$$r(t) = e^{2t}, \qquad p(t) = \frac{1}{2}, \qquad q(t) = e^{2t+1} + e^{2t-2}/2, \quad \alpha = 1. \tag{4.4}$$

Clearly, all the conditions of Theorem 3.8 hold. Thus by Theorem 3.8, every solution of (4.3) is either oscillatory or $\lim_{t\to\infty} x(t) = 0$. It is easy to verify that $x(t) = e^{-t}$ is a solution of (4.3).

Remark 4.3. Recent results cannot be applied to (4.1) and (4.3) since $\tau(t) \geq t$ and $\sigma(t) \geq t$.

Remark 4.4. Using the method given in this paper, we can get other oscillation criteria for (1.1); the details are left to the reader.

Remark 4.5. It would be interesting to find another method to study (1.1) when $\tau \circ \sigma \not\equiv \sigma \circ \tau$.

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