Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2011, Article ID 346745, 9 pages
doi:10.1155/2011/346745

Research Article

Existence of Nonoscillatory Solutions of First-Order Neutral Differential Equations

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Received 4 January 2011; Revised 4 March 2011; Accepted 27 April 2011

Academic Editor: Josef Diblík

This paper contains some sufficient conditions for the existence of positive solutions which are bounded below and above by positive functions for the first-order nonlinear neutral differential equations. These equations can also support the existence of positive solutions approaching zero at infinity

1. Introduction

This paper is concerned with the existence of a positive solution of the neutral differential equations of the form

$$\frac{d}{dt}[x(t) - a(t)x(t-\tau)] = p(t)f(x(t-\sigma)), \quad t \geq t_0, \tag{1.1}$$

where $\tau > 0$, $\sigma \geq 0$, $a \in C([t_0, \infty), (0, \infty))$, $p \in C(R, (0, \infty))$, $f \in C(R, R)$, f is nondecreasing function, and $xf(x) > 0$, $x \neq 0$.

By a solution of (1.1) we mean a function $x \in C([t_1 - m, \infty), R)$, $m = \max\{\tau, \sigma\}$, for some $t_1 \geq t_0$, such that $x(t) - a(t)x(t-\tau)$ is continuously differentiable on $[t_1, \infty)$ and such that (1.1) is satisfied for $t \geq t_1$.

The problem of the existence of solutions of neutral differential equations has been studied by several authors in the recent years. For related results we refer the reader to [1–11] and the references cited therein. However there is no conception which guarantees the existence of positive solutions which are bounded below and above by positive functions. In this paper we have presented some conception. The method also supports the existence of positive solutions approaching zero at infinity.

As much as we know, for (1.1) in the literature, there is no result for the existence of solutions which are bounded by positive functions. Only the existence of solutions which are bounded by constants is treated, for example, in [6, 10, 11]. It seems that conditions of theorems are rather complicated, but cannot be simpler due to Corollaries 2.3, 2.6, and 3.2.

The following fixed point theorem will be used to prove the main results in the next section.

Lemma 1.1 ([see [6, 10] Krasnoselskii's fixed point theorem]). *Let X be a Banach space, let Ω be a bounded closed convex subset of X, and let S_1, S_2 be maps of Ω into X such that $S_1x + S_2y \in \Omega$ for every pair $x, y \in \Omega$. If S_1 is contractive and S_2 is completely continuous, then the equation*

$$S_1x + S_2x = x \tag{1.2}$$

has a solution in Ω.

2. The Existence of Positive Solution

In this section we will consider the existence of a positive solution for (1.1). The next theorem gives us the sufficient conditions for the existence of a positive solution which is bounded by two positive functions.

Theorem 2.1. *Suppose that there exist bounded functions $u, v \in C^1([t_0, \infty), (0, \infty))$, constant $c > 0$ and $t_1 \geq t_0 + m$ such that*

$$u(t) \leq v(t), \quad t \geq t_0, \tag{2.1}$$

$$v(t) - v(t_1) - u(t) + u(t_1) \geq 0, \quad t_0 \leq t \leq t_1, \tag{2.2}$$

$$\begin{aligned} &\frac{1}{u(t-\tau)}\left(u(t) + \int_t^{\infty} p(s) f(v(s-\sigma))ds\right) \leq a(t) \\ &\leq \frac{1}{v(t-\tau)}\left(v(t) + \int_t^{\infty} p(s) f(u(s-\sigma))ds\right) \leq c < 1, \quad t \geq t_1. \end{aligned} \tag{2.3}$$

Then (1.1) *has a positive solution which is bounded by functions u, v.*

Proof. Let $C([t_0, \infty), R)$ be the set of all continuous bounded functions with the norm $\|x\| = \sup_{t \geq t_0} |x(t)|$. Then $C([t_0, \infty), R)$ is a Banach space. We define a closed, bounded, and convex subset Ω of $C([t_0, \infty), R)$ as follows:

$$\Omega = \{x = x(t) \in C([t_0, \infty), R) : u(t) \leq x(t) \leq v(t), \ t \geq t_0\}. \tag{2.4}$$

We now define two maps S_1 and $S_2 : \Omega \rightarrow C([t_0, \infty), R)$ as follows:

$$(S_1x)(t) = \begin{cases} a(t)x(t-\tau), & t \geq t_1, \\ (S_1x)(t_1), & t_0 \leq t \leq t_1, \end{cases}$$
$$(S_2x)(t) = \begin{cases} -\displaystyle\int_t^\infty p(s)f(x(s-\sigma))ds, & t \geq t_1, \\ (S_2x)(t_1) + v(t) - v(t_1), & t_0 \leq t \leq t_1. \end{cases} \tag{2.5}$$

We will show that for any $x, y \in \Omega$ we have $S_1x + S_2y \in \Omega$. For every $x, y \in \Omega$ and $t \geq t_1$, we obtain

$$(S_1x)(t) + (S_2y)(t) \leq a(t)v(t-\tau) - \int_t^\infty p(s)f(u(s-\sigma))ds \leq v(t). \tag{2.6}$$

For $t \in [t_0, t_1]$, we have

$$\begin{aligned}(S_1x)(t) + (S_2y)(t) &= (S_1x)(t_1) + (S_2y)(t_1) + v(t) - v(t_1) \\ &\leq v(t_1) + v(t) - v(t_1) = v(t).\end{aligned} \tag{2.7}$$

Furthermore, for $t \geq t_1$, we get

$$(S_1x)(t) + (S_2y)(t) \geq a(t)u(t-\tau) - \int_t^\infty p(s)f(v(s-\sigma))ds \geq u(t). \tag{2.8}$$

Let $t \in [t_0, t_1]$. With regard to (2.2), we get

$$v(t) - v(t_1) + u(t_1) \geq u(t), \quad t_0 \leq t \leq t_1. \tag{2.9}$$

Then for $t \in [t_0, t_1]$ and any $x, y \in \Omega$, we obtain

$$\begin{aligned}(S_1x)(t) + (S_2y)(t) &= (S_1x)(t_1) + (S_2y)(t_1) + v(t) - v(t_1) \\ &\geq u(t_1) + v(t) - v(t_1) \geq u(t).\end{aligned} \tag{2.10}$$

Thus we have proved that $S_1x + S_2y \in \Omega$ for any $x, y \in \Omega$.

We will show that S_1 is a contraction mapping on Ω. For $x, y \in \Omega$ and $t \geq t_1$ we have

$$|(S_1x)(t) - (S_1y)(t)| = |a(t)||x(t-\tau) - y(t-\tau)| \leq c\|x - y\|. \tag{2.11}$$

This implies that

$$\|S_1x - S_1y\| \leq c\|x - y\|. \tag{2.12}$$

Also for $t \in [t_0, t_1]$, the previous inequality is valid. We conclude that S_1 is a contraction mapping on Ω.

We now show that S_2 is completely continuous. First we will show that S_2 is continuous. Let $x_k = x_k(t) \in \Omega$ be such that $x_k(t) \to x(t)$ as $k \to \infty$. Because Ω is closed, $x = x(t) \in \Omega$. For $t \geq t_1$ we have

$$\begin{aligned} |(S_2x_k)(t) - (S_2x)(t)| &\leq \left|\int_t^{\infty} p(s)\left[f(x_k(s-\sigma)) - f(x(s-\sigma))\right]ds\right| \\ &\leq \int_{t_1}^{\infty} p(s)\left|f(x_k(s-\sigma)) - f(x(s-\sigma))\right|ds. \end{aligned} \tag{2.13}$$

According to (2.8), we get

$$\int_{t_1}^{\infty} p(s)f(v(s-\sigma))ds < \infty. \tag{2.14}$$

Since $|f(x_k(s-\sigma)) - f(x(s-\sigma))| \to 0$ as $k \to \infty$, by applying the Lebesgue dominated convergence theorem, we obtain

$$\lim_{k\to\infty} \|(S_2x_k)(t) - (S_2x)(t)\| = 0. \tag{2.15}$$

This means that S_2 is continuous.

We now show that $S_2\Omega$ is relatively compact. It is sufficient to show by the Arzela-Ascoli theorem that the family of functions $\{S_2x : x \in \Omega\}$ is uniformly bounded and equicontinuous on $[t_0, \infty)$. The uniform boundedness follows from the definition of Ω. For the equicontinuity we only need to show, according to Levitans result [7], that for any given $\varepsilon > 0$ the interval $[t_0, \infty)$ can be decomposed into finite subintervals in such a way that on each subinterval all functions of the family have a change of amplitude less than ε. Then with regard to condition (2.14), for $x \in \Omega$ and any $\varepsilon > 0$, we take $t^* \geq t_1$ large enough so that

$$\int_{t^*}^{\infty} p(s)f(x(s-\sigma))ds < \frac{\varepsilon}{2}. \tag{2.16}$$

Then, for $x \in \Omega$, $T_2 > T_1 \geq t^*$, we have

$$\begin{aligned} |(S_2x)(T_2) - (S_2x)(T_1)| &\leq \int_{T_2}^{\infty} p(s)f(x(s-\sigma))ds \\ &\quad + \int_{T_1}^{\infty} p(s)f(x(s-\sigma))ds < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned} \tag{2.17}$$

For $x \in \Omega$ and $t_1 \le T_1 < T_2 \le t^*$, we get

$$\begin{aligned} |(S_2x)(T_2) - (S_2x)(T_1)| &\le \int_{T_1}^{T_2} p(s) f(x(s-\sigma))ds \\ &\le \max_{t_1 \le s \le t^*} \{p(s) f(x(s-\sigma))\}(T_2 - T_1). \end{aligned} \tag{2.18}$$

Thus there exists $\delta_1 = \varepsilon / M$, where $M = \max_{t_1 \le s \le t^*}\{p(s) f(x(s-\sigma))\}$, such that

$$|(S_2x)(T_2) - (S_2x)(T_1)| < \varepsilon \quad \text{if } 0 < T_2 - T_1 < \delta_1. \tag{2.19}$$

Finally for any $x \in \Omega,\ t_0 \le T_1 < T_2 \le t_1$, there exists a $\delta_2 > 0$ such that

$$\begin{aligned} |(S_2x)(T_2) - (S_2x)(T_1)| = |v(T_1) - v(T_2)| &= \left| \int_{T_1}^{T_2} v'(s)ds \right| \\ &\le \max_{t_0 \le s \le t_1} \{ |v'(s)| \} (T_2 - T_1) < \varepsilon \quad \text{if } 0 < T_2 - T_1 < \delta_2. \end{aligned} \tag{2.20}$$

Then $\{S_2x : x \in \Omega\}$ is uniformly bounded and equicontinuous on $[t_0, \infty)$, and hence $S_2\Omega$ is relatively compact subset of $C([t_0, \infty), R)$. By Lemma 1.1 there is an $x_0 \in \Omega$ such that $S_1x_0 + S_2x_0 = x_0$. We conclude that $x_0(t)$ is a positive solution of (1.1). The proof is complete. □

Corollary 2.2. *Suppose that there exist functions $u, v \in C^1([t_0, \infty), (0, \infty))$, constant $c > 0$ and $t_1 \ge t_0 + m$ such that* (2.1), (2.3) *hold and*

$$v'(t) - u'(t) \le 0, \quad t_0 \le t \le t_1. \tag{2.21}$$

Then (1.1) *has a positive solution which is bounded by the functions u, v.*

Proof. We only need to prove that condition (2.21) implies (2.2). Let $t \in [t_0, t_1]$ and set

$$H(t) = v(t) - v(t_1) - u(t) + u(t_1). \tag{2.22}$$

Then with regard to (2.21), it follows that

$$H'(t) = v'(t) - u'(t) \le 0, \quad t_0 \le t \le t_1. \tag{2.23}$$

Since $H(t_1) = 0$ and $H'(t) \le 0$ for $t \in [t_0, t_1]$, this implies that

$$H(t) = v(t) - v(t_1) - u(t) + u(t_1) \ge 0, \quad t_0 \le t \le t_1. \tag{2.24}$$

Thus all conditions of Theorem 2.1 are satisfied. □

Corollary 2.3. *Suppose that there exists a function* $v \in C^1([t_0,\infty),(0,\infty))$, *constant* $c > 0$ *and* $t_1 \geq t_0 + m$ *such that*

$$a(t) = \frac{1}{v(t-\tau)}\left(v(t) + \int_t^\infty p(s)f(v(s-\sigma))ds\right) \leq c < 1, \quad t \geq t_1. \tag{2.25}$$

Then (1.1) *has a solution* $x(t) = v(t)$, $t \geq t_1$.

Proof. We put $u(t) = v(t)$ and apply Theorem 2.1. □

Theorem 2.4. *Suppose that there exist functions* $u, v \in C^1([t_0,\infty),(0,\infty))$, *constant* $c > 0$ *and* $t_1 \geq t_0 + m$ *such that* (2.1), (2.2), *and* (2.3) *hold and*

$$\lim_{t\to\infty} v(t) = 0. \tag{2.26}$$

Then (1.1) *has a positive solution which is bounded by the functions* u, v *and tends to zero.*

Proof. The proof is similar to that of Theorem 2.1 and we omit it. □

Corollary 2.5. *Suppose that there exist functions* $u, v \in C^1([t_0,\infty),(0,\infty))$, *constant* $c > 0$ *and* $t_1 \geq t_0 + m$ *such that* (2.1), (2.3), (2.21), *and* (2.26) *hold. Then* (1.1) *has a positive solution which is bounded by the functions* u, v *and tends to zero.*

Proof. The proof is similar to that of Corollary 2.2, and we omitted it. □

Corollary 2.6. *Suppose that there exists a function* $v \in C^1([t_0,\infty),(0,\infty))$, *constant* $c > 0$ *and* $t_1 \geq t_0 + m$ *such that* (2.25), (2.26) *hold. Then* (1.1) *has a solution* $x(t) = v(t)$, $t \geq t_1$ *which tends to zero.*

Proof. We put $u(t) = v(t)$ and apply Theorem 2.4. □

3. Applications and Examples

In this section we give some applications of the theorems above.

Theorem 3.1. *Suppose that*

$$\int_{t_0}^\infty p(t)dt = \infty, \tag{3.1}$$

$0 < k_1 \le k_2$ and there exist constants $c > 0$, $\gamma \ge 0$, $t_1 \ge t_0 + m$ such that

$$\frac{k_1}{k_2}\exp\left((k_2 - k_1)\int_{t_0-\gamma}^{t_0} p(t)dt\right) \ge 1, \tag{3.2}$$

$$\begin{aligned}
&\exp\left(-k_2\int_{t-\tau}^{t} p(s)ds\right) + \exp\left(k_2\int_{t_0-\gamma}^{t-\tau} p(s)ds\right)\\
&\quad\times\int_t^\infty p(s)f\left(\exp\left(-k_1\int_{t_0-\gamma}^{s-\sigma} p(\xi)d\xi\right)\right)ds \le a(t)\\
&\le \exp\left(-k_1\int_{t-\tau}^{t} p(s)ds\right) + \exp\left(k_1\int_{t_0-\gamma}^{t-\tau} p(s)ds\right)\\
&\quad\times\int_t^\infty p(s)f\left(\exp\left(-k_2\int_{t_0-\gamma}^{s-\sigma} p(\xi)d\xi\right)\right)ds \le c < 1, \quad t \ge t_1.
\end{aligned} \tag{3.3}$$

Then (1.1) *has a positive solution which tends to zero.*

Proof. We set

$$u(t) = \exp\left(-k_2\int_{t_0-\gamma}^{t} p(s)ds\right), \quad v(t) = \exp\left(-k_1\int_{t_0-\gamma}^{t} p(s)ds\right), \quad t \ge t_0. \tag{3.4}$$

We will show that the conditions of Corollary 2.5 are satisfied. With regard to (2.21), for $t \in [t_0, t_1]$, we get

$$\begin{aligned}
v'(t) - u'(t) &= -k_1p(t)v(t) + k_2p(t)u(t)\\
&= p(t)v(t)\left[-k_1 + k_2u(t)\exp\left(k_1\int_{t_0-\gamma}^{t} p(s)ds\right)\right]\\
&= p(t)v(t)\left[-k_1 + k_2\exp\left((k_1-k_2)\int_{t_0-\gamma}^{t} p(s)ds\right)\right]\\
&\le p(t)v(t)\left[-k_1 + k_2\exp\left((k_1-k_2)\int_{t_0-\gamma}^{t_0} p(s)ds\right)\right] \le 0.
\end{aligned} \tag{3.5}$$

Other conditions of Corollary 2.5 are also satisfied. The proof is complete. □

Corollary 3.2. *Suppose that* $k > 0,\ c > 0,\ t_1 \geq t_0 + m$, (3.1) *holds, and*

$$a(t) = \exp\left(-k\int_{t-\tau}^{t} p(s)ds\right) + \exp\left(k\int_{t_0}^{t-\tau} p(s)ds\right) \times \int_{t}^{\infty} p(s)f\left(\exp\left(-k\int_{t_0}^{s-\sigma} p(\xi)d\xi\right)\right)ds \leq c < 1, \quad t \geq t_1. \tag{3.6}$$

Then (1.1) *has a solution*

$$x(t) = \exp\left(-k\int_{t_0}^{t} p(s)ds\right), \quad t \geq t_1, \tag{3.7}$$

which tends to zero.

Proof. We put $k_1 = k_2 = k,\ \gamma = 0$ and apply Theorem 3.1. □

Example 3.3. Consider the nonlinear neutral differential equation

$$[x(t) - a(t)x(t-2)]' = px^3(t-1), \quad t \geq t_0, \tag{3.8}$$

where $p \in (0,\infty)$. We will show that the conditions of Theorem 3.1 are satisfied. Condition (3.1) obviously holds and (3.2) has a form

$$\frac{k_1}{k_2}\exp((k_2 - k_1)p\gamma) \geq 1, \tag{3.9}$$

$0 < k_1 \leq k_2,\ \gamma \geq 0$. For function $a(t)$, we obtain

$$\begin{aligned}
&\exp(-2pk_2) + \frac{1}{3k_1}\exp(p[k_2(\gamma - t_0 - 2) - 3k_1(\gamma - t_0 - 1) + (k_2 - 3k_1)t]) \\
&\quad \leq a(t) \leq \exp(-2pk_1) \\
&\qquad + \frac{1}{3k_2}\exp(p[k_1(\gamma - t_0 - 2) - 3k_2(\gamma - t_0 - 1) + (k_1 - 3k_2)t]), \quad t \geq t_0.
\end{aligned} \tag{3.10}$$

For $p = 1,\ k_1 = 1,\ k_2 = 2,\ \gamma = 1,\ t_0 = 1$, condition (3.9) is satisfied and

$$e^{-4} + \frac{1}{3e}e^{-t} \leq a(t) \leq e^{-2} + \frac{e^4}{6}e^{-5t}, \quad t \geq t_1 \geq 3. \tag{3.11}$$

If the function $a(t)$ satisfies (3.11), then (3.8) has a solution which is bounded by the functions $u(t) = \exp(-2t),\ v(t) = \exp(-t),\ t \geq 3$.

For example if $p = 1, k_1 = k_2 = 1.5, \gamma = 1, t_0 = 1$, from (3.11) we obtain

$$a(t) = e^{-3} + \frac{e^{1.5}}{4.5} e^{-3t}, \tag{3.12}$$

and the equation

$$\left[x(t) - \left(e^{-3} + \frac{e^{1.5}}{4.5} e^{-3t}\right) x(t-2)\right]' = x^3(t-1), \quad t \geq 3, \tag{3.13}$$

has the solution $x(t) = \exp(-1.5t)$ which is bounded by the function $u(t)$ and $v(t)$.

Acknowledgments

The research was supported by the Grant 1/0090/09 and the Grant 1/1260/12 of the Scientific Grant Agency of the Ministry of Education of the Slovak Republic and Project APVV-0700-07 of the Slovak Research and Development Agency.

References

[1] A. Boichuk, J. Diblík, D. Khusainov, and M. Růžičková, "Fredholm's boundary-value problems for differential systems with a single delay," *Nonlinear Analysis. Theory, Methods & Applications*, vol. 72, no. 5, pp. 2251–2258, 2010.

[2] J. Diblík, "Positive and oscillating solutions of differential equations with delay in critical case," *Journal of Computational and Applied Mathematics*, vol. 88, no. 1, pp. 185–202, 1998, Positive solutions of nonlinear problem.

[3] J. Diblík and M. Kúdelčíková, "Two classes of asymptotically different positive solutions of the equation $\dot{y}(t) = -f(t, y_t)$," *Nonlinear Analysis. Theory, Methods & Applications*, vol. 70, no. 10, pp. 3702–3714, 2009.

[4] J. Diblík, Z. Svoboda, and Z. Šmarda, "Retract principle for neutral functional differential equations," *Nonlinear Analysis. Theory, Methods & Applications*, vol. 71, no. 12, pp. e1393–e1400, 2009.

[5] J. Diblík and M. Růžičková, "Existence of positive solutions of a singular initial problem for a nonlinear system of differential equations," *The Rocky Mountain Journal of Mathematics*, vol. 3, pp. 923–944, 2004.

[6] L. H. Erbe, Q. Kong, and B. G. Zhang, *Oscillation Theory for Functional-Differential Equations*, vol. 190 of *Monographs and Textbooks in Pure and Applied Mathematics*, Marcel Dekker, New York, NY, USA, 1995.

[7] B. M. Levitan, "Some questions of the theory of almost periodic functions. I," *Uspekhi Matematicheskikh Nauk*, vol. 2, no. 5, pp. 133–192, 1947.

[8] X. Lin, "Oscillation of second-order nonlinear neutral differential equations," *Journal of Mathematical Analysis and Applications*, vol. 309, no. 2, pp. 442–452, 2005.

[9] X. Wang and L. Liao, "Asymptotic behavior of solutions of neutral differential equations with positive and negative coefficients," *Journal of Mathematical Analysis and Applications*, vol. 279, no. 1, pp. 326–338, 2003.

[10] Y. Zhou, "Existence for nonoscillatory solutions of second-order nonlinear differential equations," *Journal of Mathematical Analysis and Applications*, vol. 331, no. 1, pp. 91–96, 2007.

[11] Y. Zhou and B. G. Zhang, "Existence of nonoscillatory solutions of higher-order neutral differential equations with positive and negative coefficients," *Applied Mathematics Letters*, vol. 15, no. 7, pp. 867–874, 2002.