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Research Article

Propagation of Elastic Waves in Prestressed Media

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3D solutions of the dynamical equations in the presence of external forces are derived for a homogeneous, prestressed medium. 2D plane waves solutions are obtained from general solutions and show that there exist two types of plane waves, namely, quasi-P waves and quasi-SV waves. Expressions for slowness surfaces and apparent velocities for these waves are derived analytically as well as numerically and represented graphically.

1. Introduction

In fact, the Earth is prestressed medium, due to many physical causes, that is, gravity variation, slow process of creep and variation of temperature, and so forth. Therefore, these problems are of much interest to seismologists due to its application in mineral prospecting and prediction of earthquakes.

For studying the propagation of elastic waves in prestressed solids of infinite extent, Sidhu and Singh [1, 2], Norris [3], and Day et al. [4] used a much simpler form of equation of motion of Biot [5]. The medium considered by earlier investigators is a homogenous prestressed with incremental elastic coefficients possessing orthotropic anisotropy due to normal components of initial stresses.

In the present paper, 3D problem of propagation of P, SV and SH waves for a homogeneous prestressed medium is discussed. The model of prestressed medium used here is much more general than that used by earlier investigators.

The 2D plane waves solutions are obtained from general solutions for different conditions of initial stresses with or without external forces. Graphs of slowness surfaces are derived.

2. Basic Equations

Consider a homogenous prestressed solid. The state of prestresses is, therefore, defined by six components, that is, $S_{11,}S_{22,}S_{33,}S_{12} = S_{21}, S_{31} = S_{13}$, and $S_{23} = S_{32}$. Let all stress components be functions of (x, y, z). The state of initial stress introduces anisotropy so that even for an initially isotropic solid defined by two Lame's constants λ, μ, the number of the incremental elastic coefficients $(B_{ij,}Q_i)$ will be always larger than 2. Let (X, Y, Z) be components of body forces along coordinate axes, respectively, where X, Y, Z are all constant. The general form of dynamical equation for prestressed solids in the presence of external forces is given by Biot [5, page 52]:

$$\begin{aligned}
&\frac{\partial s_{11}}{\partial x} + \frac{\partial s_{12}}{\partial y} + \frac{\partial s_{31}}{\partial z} + \rho \Delta x + \rho(\omega_z Y - \omega_y Z) - \rho e X - e_{xx}\frac{\partial S_{11}}{\partial x} - e_{yy}\frac{\partial S_{12}}{\partial y} - e_{zz}\frac{\partial S_{31}}{\partial z} \\
&\quad - e_{yz}\left(\frac{\partial S_{31}}{\partial y} + \frac{\partial S_{12}}{\partial z}\right) - e_{zx}\left(\frac{\partial S_{11}}{\partial z} + \frac{\partial S_{31}}{\partial x}\right) - e_{xy}\left(\frac{\partial S_{12}}{\partial x} + \frac{\partial S_{11}}{\partial y}\right) + (S_{11} - S_{22})\frac{\partial W_z}{\partial y} \\
&\quad - 2S_{12}\frac{\partial W_z}{\partial x} + S_{12}\frac{\partial W_x}{\partial z} + (S_{33} - S_{11})\frac{\partial W_y}{\partial z} + 2S_{12}\frac{\partial W_y}{\partial x} - S_{31}\frac{\partial W_x}{\partial y} + S_{23}\left(\frac{\partial W_y}{\partial y} - \frac{\partial W_z}{\partial z}\right) \\
&= \rho\frac{\partial^2 u_1}{\partial t^2}.
\end{aligned} \tag{2.1}$$

The two other equations are obtained by cyclic permutation of $x, y, z, 1, 2, 3$ and X, Y, Z. ρ is the density and (u_1, u_2, u_3) the displacement components along the axes. ΔX_i are the components of incremental body force, which are assumed to satisfy the equation

$$\begin{gathered}
\Delta X_i = u_j X_{i,j} = 0, \\
e = e_{xx} + e_{yy} + e_{zz}.
\end{gathered} \tag{2.2}$$

The S_{ij} are the components of prestress, which are assumed to satisfy the equilibrium equation.

$$S_{ij,j} + \rho X_i = 0, \tag{2.3}$$

and are related to the initial strain ϵ_{ij} by Hooke's law

$$S_{ij} = \lambda \epsilon_{ii}\delta_{ij} + 2\mu\epsilon_{ij}, \tag{2.4}$$

The incremental stresses s_{ij} are supposed to be linearly related to the incremental strains e_{ij} through the incremental elastic coefficients B_{ij} and Q_i

$$\begin{aligned}
s_{11} &= B_{11}e_{xx} + B_{12}e_{yy} + B_{13}e_{zz,} \\
s_{22} &= B_{21}e_{xx} + B_{22}e_{yy} + B_{23}e_{zz,} \\
s_{33} &= B_{31}e_{xx} + B_{32}e_{yy} + B_{33}e_{zz,} \\
s_{23} &= 2Q_1 e_{yz,} \\
s_{31} &= 2Q_2 e_{zx,} \\
s_{23} &= 2Q_3 e_{xy.}
\end{aligned} \tag{2.5}$$

We assume that

$$\begin{gathered}
(u_1, u_2, u_3) = (u, v, w), \qquad (X_1, X_2, X_3) = (X, Y, Z), \\
e_{xx} = \frac{\partial u}{\partial x}, \qquad e_{yy} = \frac{\partial v}{\partial y}, \qquad e_{zz} = \frac{\partial w}{\partial z}, \\
e_{xy} = \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right), \qquad e_{yz} = \frac{1}{2}\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right), \qquad e_{zx} = \frac{1}{2}\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right), \\
W_x = \frac{1}{2}\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right), \qquad W_y = \frac{1}{2}\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right), \qquad W_z = \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right).
\end{gathered} \tag{2.6}$$

Substituting (2.2), (2.5), and (2.6) in (2.1), we have

$$\begin{aligned}
&B_{11}\frac{\partial^2 u}{\partial x^2} + \left(Q_3 - \frac{P_1}{2}\right)\frac{\partial^2 u}{\partial y^2} + \left(Q_2 + \frac{P_3}{2}\right)\frac{\partial^2 u}{\partial z^2} - S_{12}\frac{\partial^2 v}{\partial x^2} - \frac{1}{2}S_{12}\frac{\partial^2 v}{\partial z^2} - S_{31}\frac{\partial^2 w}{\partial x^2} \\
&\quad - \frac{1}{2}S_{31}\frac{\partial^2 w}{\partial y^2} + S_{12}\frac{\partial^2 u}{\partial x \partial y} + S_{23}\frac{\partial^2 u}{\partial y \partial z} + S_{31}\frac{\partial^2 u}{\partial x \partial z} + \left(B_{12} + Q_3 + \frac{P_1}{2}\right)\frac{\partial^2 v}{\partial x \partial y} \\
&\quad + \frac{1}{2}S_{31}\frac{\partial^2 v}{\partial y \partial z} - \frac{1}{2}S_{23}\frac{\partial^2 v}{\partial x \partial z} + \left(B_{13} + Q_2 - \frac{P_3}{2}\right)\frac{\partial^2 w}{\partial x \partial z} \\
&\quad + \frac{1}{2}S_{12}\frac{\partial^2 w}{\partial y \partial z} - \frac{1}{2}S_{23}\frac{\partial^2 w}{\partial x \partial y} - \left(\frac{\partial S_{11}}{\partial x} + \rho X\right)\frac{\partial u}{\partial x} \\
&\quad - \frac{1}{2}\left(\frac{\partial S_{12}}{\partial x} + \frac{\partial S_{11}}{\partial y}\right)\frac{\partial u}{\partial y} - \frac{1}{2}\left(\frac{\partial S_{31}}{\partial x} + \frac{\partial S_{11}}{\partial z} + \rho Z\right)\frac{\partial u}{\partial z}
\end{aligned}$$

$$
\begin{aligned}
&-\frac{1}{2}\left(\frac{\partial S_{12}}{\partial x}+\frac{\partial S_{11}}{\partial y}-\rho Y\right)\frac{\partial v}{\partial x}+\left(\frac{\partial S_{12}}{\partial y}+\rho X\right)\frac{\partial v}{\partial y}\\
&-\frac{1}{2}\left(\frac{\partial S_{31}}{\partial y}+\frac{\partial S_{12}}{\partial z}\right)\frac{\partial v}{\partial z}-\frac{1}{2}\left(\frac{\partial S_{31}}{\partial x}+\frac{\partial S_{11}}{\partial z}-\rho Z\right)\frac{\partial w}{\partial x}\\
&-\frac{1}{2}\left(\frac{\partial S_{31}}{\partial y}+\frac{\partial S_{12}}{\partial z}\right)\frac{\partial w}{\partial y}-\left(\frac{\partial S_{31}}{\partial z}+\rho X\right)\frac{\partial w}{\partial z}\\
=\,&\rho\frac{\partial^2 u}{\partial t^2}.
\end{aligned}
\tag{2.7}
$$

The two other equations are obtained from (2.7) by cycle permutation of $x, y, z, 1, 2, 3, X, Y, Z$ and u, v, w.

Here

$$
\begin{aligned}
P_1 &= S_{11} - S_{22},\\
P_2 &= S_{22} - S_{33},\\
P_3 &= S_{33} - S_{11}.
\end{aligned}
\tag{2.8}
$$

3. Propagation of Waves

For elastic waves propagating in a direction specified by direction cosines (l, m, n) along the axes, we take

$$
u = U_i \exp(iP), \qquad v = V_i \exp(iP), \qquad w = W_i \exp(iP). \tag{3.1}
$$

(U_i, V_i, W_i) are amplitude factors along the axes and P is phase factor:

$$
P = k\{ct - (lx + my + nz)\}, \tag{3.2}
$$

where c is phase velocity and k is wave number.

Putting (3.1) and (3.2) in (2.7), we get

$$
\begin{aligned}
&\left(\Omega_1+\Omega_1'+\rho c^2\right)U_i+(\Im_1+\Im_1')V_i+(\mathcal{K}_1+\mathcal{K}_1')W_i=0,\\
&(\mathcal{K}_2+\mathcal{K}_2')U_i+\left(\Omega_2+\Omega_2'+\rho c^2\right)V_i+(\Im_2+\Im_2')W_i=0,\\
&(\Im_3+\Im_3')U_i+(\mathcal{K}_3+\mathcal{K}_3')V_i+\left(\Omega_3+\Omega_3'+\rho c^2\right)W_i=0,
\end{aligned}
\tag{3.3}
$$

where

$$\Omega_1 = -\left\{ B_{11} l^2 + \left(Q_3 - \frac{P_1}{2} \right) m^2 + \left(Q_2 + \frac{P_3}{2} \right) n^2 + S_{12} lm + S_{23} mn + S_{31} nl \right\}, \tag{3.4}$$

$$\Im_1 = S_{12} l^2 + \left(\frac{S_{12}}{2} \right) n^2 - \left(B_{12} + Q_3 + \frac{P_1}{2} \right) lm - \left(\frac{S_{31}}{2} \right) mn + \left(\frac{S_{23}}{2} \right) ln, \tag{3.5}$$

$$\mathcal{K}_1 = S_{31} l^2 + \left(\frac{S_{31}}{2} \right) m^2 - \left(B_{13} + Q_2 - \frac{P_3}{2} \right) ln - \left(\frac{S_{12}}{2} \right) mn + \left(\frac{S_{23}}{2} \right) ln, \tag{3.6}$$

$$\Omega_1' = \frac{i}{k} \left\{ \left(\frac{\partial S_{11}}{\partial x} + \rho X \right) l + \frac{1}{2} \left(\frac{\partial S_{12}}{\partial x} + \frac{\partial S_{11}}{\partial y} + \rho Y \right) m + \frac{1}{2} \left(\frac{\partial S_{31}}{\partial x} + \frac{\partial S_{11}}{\partial z} + \rho Z \right) n \right\}, \tag{3.7}$$

$$\Im_1' = \frac{i}{k} \left\{ \frac{1}{2} \left(\frac{\partial S_{12}}{\partial x} + \frac{\partial S_{11}}{\partial y} - \rho Y \right) l - \left(\frac{\partial S_{12}}{\partial y} + \rho X \right) m + \frac{1}{2} \left(\frac{\partial S_{31}}{\partial y} + \frac{\partial S_{12}}{\partial z} \right) n \right\}, \tag{3.8}$$

$$\mathcal{K}_1' = \frac{i}{k} \left\{ \frac{1}{2} \left(\frac{\partial S_{31}}{\partial x} + \frac{\partial S_{11}}{\partial z} - \rho Z \right) l + \frac{1}{2} \left(\frac{\partial S_{31}}{\partial y} + \frac{\partial S_{12}}{\partial z} \right) m + \frac{1}{2} \left(\frac{\partial S_{31}}{\partial z} + \rho X \right) n \right\}. \tag{3.9}$$

The explicit expressions for $(\Omega_2, \Omega_3), (\Im_2, \Im_3), (\mathcal{K}_2, \mathcal{K}_3), (\Omega_2', \Omega_3'), (\Im_2', \Im_3')$, and $(\mathcal{K}_2', \mathcal{K}_3')$ are obtained from (3.4)–(3.9), by cyclic permutation of $x, y, z, 1, 2, 3, l, m, n$ and X, Y, Z, respectively.

Setting the determinant of the coefficients of U_i, V_i and W_i of (3.3) equal to zero, on simplification, we get the cubic equation in ρc^2:

$$\begin{aligned} &\rho^3 c^6 + \rho^2 c^4 (A + B + D) + \rho c^2 (AB + BD + DA - FJ - EI - HG) \\ &\quad + (ABD + EFG + HIJ - AFJ - EID - HGB) = 0, \end{aligned} \tag{3.10}$$

where

$$\begin{aligned} A &= \Omega_1 + \Omega_1', & E &= \Im_1 + \Im_1', & H &= \mathcal{K}_1 + \mathcal{K}_1', \\ B &= \Omega_2 + \Omega_2', & F &= \Im_2 + \Im_2', & I &= \mathcal{K}_2 + \mathcal{K}_2', \\ D &= \Omega_3 + \Omega_3', & G &= \Im_3 + \Im_3', & J &= \mathcal{K}_3 + \mathcal{K}_3'. \end{aligned} \tag{3.11}$$

4. Special Cases

Two special cases may be dealt with immediately.

4.1. Three Dimensional Prestressed Medium

4.1.1. Propagation of Waves along the Unique Axis

Putting $n = 1$, $l = m = 0$ in (3.3)–(3.8) and in the expressions of (Ω_2, Ω_3), $(\Im_2, \Im_3)$, $(\mathcal{K}_2, \mathcal{K}_3)$, (Ω'_2, Ω'_3), $(\Im'_2, \Im'_3)$, and $(\mathcal{K}'_2, \mathcal{K}'_3)$, we get

$$\begin{gathered}\Omega_{11} = -\left(Q_2 + \frac{P_3}{2}\right), \qquad \Omega_{21} = -\left(Q_1 - \frac{P_2}{2}\right), \qquad \Omega_{31} = -B_{33},\\ \Im_{11} = \frac{S_{12}}{2}, \qquad \Im_{21} = 0, \qquad \Im_{31} = S_{31},\\ \mathcal{K}_{11} = 0, \qquad \mathcal{K}_{21} = \frac{S_{12}}{2}, \qquad \mathcal{K}_{31} = S_{23},\end{gathered} \tag{4.1}$$

$$\begin{gathered}\Omega'_{11} = \frac{i}{2k}\left(\frac{\partial S_{31}}{\partial x} + \frac{\partial S_{11}}{\partial z} + \rho z\right), \qquad \Omega'_{21} = \frac{i}{2k}\left(\frac{\partial S_{23}}{\partial y} + \frac{\partial S_{22}}{\partial z} + \rho z\right), \qquad \Omega'_{31} = \frac{i}{k}\left(\frac{\partial S_{33}}{\partial x} + \rho z\right),\\ \Im'_{11} = \frac{i}{2k}\left(\frac{\partial S_{31}}{\partial y} + \frac{\partial S_{12}}{\partial z}\right), \qquad \Im'_{21} = -\frac{i}{k}\left(\frac{\partial S_{23}}{\partial z} + \rho Y\right), \qquad \Im'_{31} = \frac{i}{k}\left(\frac{\partial S_{33}}{\partial x} + \frac{\partial S_{31}}{\partial z} - \rho X\right),\\ \mathcal{K}'_{11} = \frac{i}{k}\left(\frac{\partial S_{31}}{\partial z} + \rho X\right), \qquad \mathcal{K}'_{21} = \frac{i}{2k}\left(\frac{\partial S_{23}}{\partial x} + \frac{\partial S_{12}}{\partial z}\right), \qquad \mathcal{K}'_{31} = \frac{i}{2k}\left(\frac{\partial S_{33}}{\partial y} + \frac{\partial S_{23}}{\partial z} + \rho Y\right).\end{gathered} \tag{4.2}$$

Equation (3.10) takes the form

$$\begin{aligned}&\rho^3 c^6 + \rho^2 c^4 (A_1 + B_1 + D_1) + \rho c^2 (A_1 B_1 + B_1 D_1 + D_1 A_1 - F_1 J_1 - E_1 I_1 - H_1 G_1)\\ &\quad + (A_1 B_1 D_1 + E_1 F_1 G_1 + H_1 I_1 J_1 - A_1 F_1 J_1 - E_1 I_1 D_1 - H_1 G_1 B_1) = 0,\end{aligned} \tag{4.3}$$

where

$$\begin{gathered}A_1 = \Omega_{11} + \Omega'_{11}, \qquad B_1 = \Omega_{21} + \Omega'_{21}, \qquad D_1 = \Omega_{31} + \Omega'_{31},\\ E_1 = \Im_{11} + \Im'_{11}, \qquad F_1 = \Im'_{21}, \qquad G_1 = \Im_{31} + \Im'_{31},\\ H_1 = \mathcal{K}'_{11}, \qquad I_1 = \mathcal{K}_{21} + \mathcal{K}'_{21}, \qquad J_1 = \mathcal{K}_{31} + \mathcal{K}'_{31}.\end{gathered} \tag{4.4}$$

4.1.2. In the Absence of Body Forces

When $X = Y = Z = 0$ and $S_{11}, S_{22}, S_{33}, S_{12}$ and so forth are all constant, then (4.3) becomes, with the help of (4.2) and (4.4),

$$\begin{aligned}&\rho^3 c^6 + \rho^2 c^4 (\Omega_{11} + \Omega_{21} + \Omega_{31}) + \rho c^2 (\Omega_{11}\Omega_{21} + \Omega_{21}\Omega_{31} + \Omega_{31}\Omega_{11} - \Im_{11}\mathcal{K}_{21})\\ &\quad + (\Omega_{11}\Omega_{21}\Omega_{31} - \Im_{11}\mathcal{K}_{21}\Omega_{31}) = 0.\end{aligned} \tag{4.5}$$

4.1.3. Prestressed Is Defined by Normal Components

There are only normal components of prestress present in the medium, that is putting $S_{12} = S_{31} = S_{23} = 0$ in (4.1), (4.5) takes the form

$$\left(\Omega_{11} + \rho c^2\right)\left(\Omega_{21} + \rho c^2\right)\left(\Omega_{31} + \rho c^2\right) = 0; \tag{4.6}$$

on simplification, we get three values of c^2:

$$c^2 = \frac{Q_2 + P_3/2}{\rho}, \qquad c^2 = \frac{Q_1 - P_2/2}{\rho}, \qquad c^2 = \frac{B_{33}}{\rho}. \tag{4.7}$$

4.2. Two Dimensional Prestressed Medium

4.2.1. Plane Waves Solution

Here, we consider the behaviour of plane waves in xy-plane perpendicular to the z-axis; putting $n = 0$ in (3.4)–(3.9) and in the expressions of (Ω_2, Ω_3) and so forth, we get the set of equations from (3.3):

$$\begin{aligned} &\left(\Omega_{111} + \Omega'_{111} + \rho c^2\right)U_i + \left(\Im_{111} + \Im'_{111}\right)V_i = 0, \\ &\left(\mathcal{K}_{211} + \mathcal{K}'_{211}\right)U_i + \left(\Omega_{211} + \Omega'_{211} + \rho c^2\right)V_i = 0, \end{aligned} \tag{4.8}$$

where

$$\begin{aligned} \Omega_{111} &= -\left\{B_{11}l^2 + \left(Q_3 - \frac{P_1}{2}\right)m^2 + S_{12}ml\right\}, \\ \Im_{111} &= \left\{S_{12}l^2 - \left(B_{12} + Q_3 + \frac{P_1}{2}\right)ml\right\}, \\ \Omega_{211} &= -\left\{\left(Q_3 + \frac{P_1}{2}\right)l^2 + B_{22}m^2 + S_{12}ml\right\}, \\ \mathcal{K}_{211} &= \left\{S_{12}m^2 - \left(B_{12} + Q_3 - \frac{P_1}{2}\right)nl\right\}, \\ \Omega'_{111} &= \frac{i}{k}\left\{\left(\frac{\partial S_{11}}{\partial x} + \rho X\right)l + \frac{1}{2}\left(\frac{\partial S_{12}}{\partial x} + \frac{\partial S_{11}}{\partial y} + \rho Y\right)m\right\}, \\ \Omega'_{211} &= \frac{i}{k}\left\{\frac{1}{2}\left(\frac{\partial S_{22}}{\partial x} + \frac{\partial S_{12}}{\partial y} + \rho X\right)l + \left(\frac{\partial S_{22}}{\partial y} + \rho Y\right)m\right\}, \\ \Im'_{111} &= \frac{i}{k}\left\{\frac{1}{2}\left(\frac{\partial S_{12}}{\partial x} + \frac{\partial S_{11}}{\partial y} - \rho Y\right)l - \left(\frac{\partial S_{12}}{\partial y} + \rho X\right)m\right\}, \\ \mathcal{K}'_{211} &= \frac{i}{k}\left\{\left(\frac{\partial S_{12}}{\partial x} + \rho Y\right)l + \frac{1}{2}\left(\frac{\partial S_{22}}{\partial x} + \frac{\partial S_{12}}{\partial y} + \rho X\right)m\right\}. \end{aligned} \tag{4.9}$$

Equation (4.8) has nontrivial solution when

$$\rho^2 c^4 + \left(\Omega_{111} + \Omega_{211} + \Omega'_{111} + \Omega'_{211}\right)\rho c^2 - \left(\mathfrak{I}_{111} + \mathfrak{I}'_{111}\right)\left(\mathcal{K}_{211} + \mathcal{K}'_{211}\right) = 0. \tag{4.10}$$

It is quadratic equation in ρc^2, and it has two values of c^2 corresponding to quasi-SV waves and quasi-P waves.

In the absence of body forces and when S_{11}, S_{22}, and so forth are constants, then from (4.8) and (4.9), we get

$$\begin{gathered}\left(\Omega_{111} + \rho c^2\right)U_i + \mathfrak{I}_{111} V_i = 0, \\ \mathcal{K}_{211} U_i + \left(\Omega_{211} + \rho c^2\right) V_i = 0.\end{gathered} \tag{4.11}$$

If $S_{12} = S_{23} = S_{31} = 0$, then (4.11) takes the form

$$\begin{gathered}\left(\Omega_{1111} + \rho c^2\right)U_i + \mathfrak{I}_{1111} V_i = 0, \\ \mathcal{K}_{2111} U_i + \left(\Omega_{2111} + \rho c^2\right) V_i = 0,\end{gathered} \tag{4.12}$$

where

$$\begin{gathered}\Omega_{1111} = -\left\{B_{11} l^2 + \left(Q_3 - \frac{P_1}{2}\right)m^2\right\}, \\ \mathfrak{I}_{1111} = -\left(B_{12} + Q_3 + \frac{P_1}{2}\right)ml, \\ \Omega_{2111} = -\left\{\left(Q_3 + \frac{P_1}{2}\right)l^2 + B_{22} m^2\right\}, \\ \mathcal{K}_{2111} = -\left(B_{21} + Q_3 - \frac{P_1}{2}\right)lm.\end{gathered} \tag{4.13}$$

The set of homogeneous (4.12) in U_i, V_i has a nontrivial solution when

$$\begin{vmatrix}\left(\Omega_{1111} + \rho c^2\right) & \mathfrak{I}_{1111} \\ \mathcal{K}_{2111} & \left(\Omega_{2111} + \rho c^2\right)\end{vmatrix} = 0. \tag{4.14}$$

This quadratic equation in ρc^2 may be solved to obtain

$$
\begin{aligned}
&2\rho c^2 \\
&= \left\{ \left(B_{11}+Q_3 + \frac{P_1}{2} \right) l^2 + \left(B_{21}+Q_3 - \frac{P_1}{2} \right) m^2 \right\} \\
&\pm \sqrt{ \left\{ \left(B_{11} - Q_3 - \frac{P_1}{2} \right) l^2 + \left(B_{22} - Q_3 + \frac{P_1}{2} \right) m^2 \right\}^2 + 4\left(B_{21} + Q_3 - \frac{P_1}{2} \right) \left(B_{12} + Q_3 + \frac{P_1}{2} \right) l^2 m^2 }.
\end{aligned}
\tag{4.15}
$$

Thus, in general, in this two-dimensional model of the prestressed medium, there exist two types of plane waves, namely, quasi-P waves and quasi-SV waves whose phase velocities correspond to upper and lower signs of (4.15).

4.2.2. *Propagation of Plane Waves in Orthotropic Medium*

Consider a homogenous prestressed elastic solid. The material is either isotropic in finite strain or anisotropic with orthotropic symmetry. The principal directions of initial stress are chosen to coincide with the directions of elastic symmetry and the coordinate axes. Let the state of uniform initial stresses have principal stresses S_{11}, S_{22}, and S_{33}. We further assume that $S_{22} = S_{33}$ and S_{11} and S_{22} are constant. The principal stress S_{33} does not enter explicitly into the equations of motion. Its influence is, however, included indirectly in the values of the incremental elastic coefficients. We put $l = \sin\theta$ and $m = \cos\theta$; (4.15) can be written as

$$
\begin{aligned}
2\rho c^2(\theta) = {}& \left\{ \left(B_{11}+Q_3 + \frac{P_1}{2} \right) \sin^2\theta + \left(B_{21}+Q_3 - \frac{P_1}{2} \right) \cos^2\theta \right\} \\
&\pm \sqrt{ \left\{ \left(B_{11} - Q_3 - \frac{P_1}{2} \right) \sin^2\theta + \left(B_{22} - Q_3 + \frac{P_1}{2} \right) \cos^2\theta \right\}^2 + \mathbb{A} },
\end{aligned}
\tag{4.16}
$$

where $\mathbb{A} = 4(B_{12} + Q_3 + P_1/2)(B_{12} + Q_3 + P_1/2)\sin^2\theta\cos^2\theta$, where

$$
B_{21} - P_1 = B_{12}. \tag{4.17}
$$

Let $C_P(\theta)$ and $C_{\mathrm{SV}}(\theta)$ be the values of c associated with upper and lower signs in (4.16), corresponding to the velocities for quasi-P waves and quasi-SV waves, respectively. Hence these expressions coincide with the expressions obtained by Sidhu and Singh [1] for velocities of quasi-P waves and quasi-SV waves.

5. Numerical Calculation and Discussions

Here we consider the model for an initially stressed medium:

$$
B_{11} = \lambda + 2\mu + P_1, \qquad B_{12} = \lambda + P_1, \qquad B_{22} = \lambda + 2\mu, \qquad Q_3 = \mu. \tag{5.1}
$$

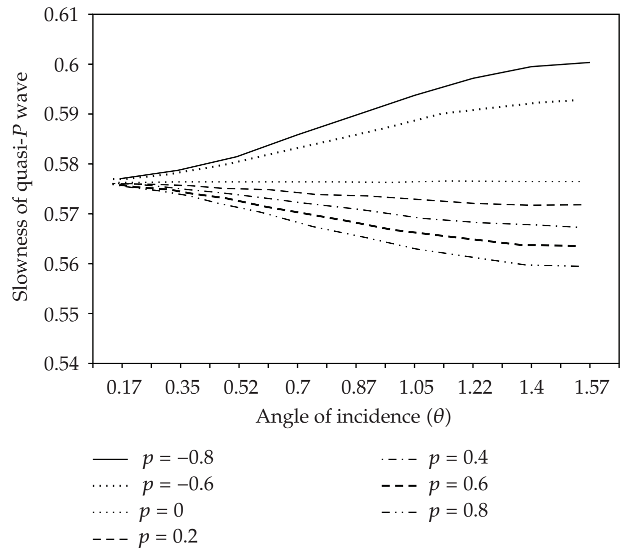

Figure 1

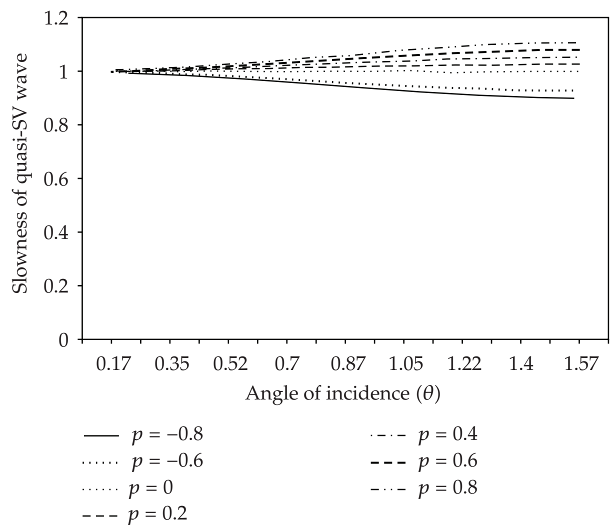

Figure 2

Using (5.1) in (4.17), we get a nondimensional form of velocity equations as

$$\begin{aligned}
\widehat{C}_P(\theta) &= \sqrt{\frac{1}{2}\left[(\delta+3+p)+\left\{(\delta+1+p)^2+\delta p(\delta+1+p)\sin^2\theta\right\}^{1/2}\right]}, \\
\widehat{C}_{\mathrm{SV}}(\theta) &= \sqrt{\frac{1}{2}\left[(\delta+3+p)-\left\{(\delta+1+p)^2+\delta p(\delta+1+p)\sin^2\theta\right\}^{1/2}\right]},
\end{aligned} \tag{5.2}$$

where

$$\delta = \frac{\lambda}{\mu}, \qquad p = \frac{P_1}{2\mu}, \qquad \beta^2 = \frac{\mu}{\rho}. \tag{5.3}$$

The apparent velocities for quasi-P waves and quasi-SV waves can be obtained from (5.2) as

$$\begin{aligned} C_{Pa}(\theta) &= \frac{\widehat{C}_P(\theta)}{\sin\theta}, \\ C_{\mathrm{SV}a}(\theta) &= \frac{\widehat{C}_{\mathrm{SV}}(\theta)}{\sin\theta}. \end{aligned} \tag{5.4}$$

The numerical values of the dimensionless slowness $(1/\widehat{C}_P(\theta), 1/\widehat{C}_{\mathrm{SV}}(\theta))$ have been calculated from (5.2) assuming that $\delta = 1$ for different values of p vary from −0.8 to 0.8 and for different values of θ vary from 0° to 90°. Figures 1 and 2 show the variation of the dimensionless slowness for the quasi-P and quasi-SV waves with the angle of incidence −0.8, −0.6, 0.2, 0.4, 0.6, and 0.8.

6. Conclusion

The study shows that the phase velocities of quasi-P waves and quasi-SV waves are highly affected by the initial stresses present in the medium and also the direction of propagation.

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