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Research Article

Permanence of a Semi-Ratio-Dependent Predator-Prey System with Nonmonotonic Functional Response and Time Delay

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Sufficient conditions for permanence of a semi-ratio-dependent predator-prey system with nonmonotonic functional response and time delay $\dot{x}_1(t) = x_1(t)[r_1(t) - a_{11}(t)x_1(t-\tau(t)) - a_{12}(t)x_2(t)/(m^2+x_1^2(t))]$, $\dot{x}_2(t) = x_2(t)[r_2(t)-a_{21}(t)x_2(t)/x_1(t)]$, are obtained, where $x_1(t)$ and $x_2(t)$ stand for the density of the prey and the predator, respectively, and $m \neq 0$ is a constant. $\tau(t) \geq 0$ stands for the time delays due to negative feedback of the prey population.

1. Introduction

The dynamic relationship between predators and their preys has long been and will continue to be one of the dominant themes in both ecology and mathematical ecology due to its universal existence and importance [1]. These problems may appear to be simple mathematically at first sight, but they are, in fact, often very challenging and complicated [2, 3].

Recently, Ding et al. [4] studied dynamics of a semi-ratio-dependent predator-prey system with the nonmonotonic functional response and delay

$$\begin{aligned} \dot{x}_1(t) &= x_1(t)\left[r_1(t) - a_{11}(t)x_1(t-\tau(t)) - \frac{a_{12}(t)x_2(t)}{m^2 + x_1^2(t)}\right], \\ \dot{x}_2(t) &= x_2(t)\left[r_2(t) - \frac{a_{21}(t)x_2(t)}{x_1(t)}\right], \end{aligned} \tag{1.1}$$

with initial conditions

$$x_i(\theta) = \phi_i(\theta), \quad \theta \in [-\tau^u, 0], \quad \phi_i(0) > 0, \quad \phi_i \in C([-\tau^u, 0], R_+), \quad i = 1, 2, \tag{1.2}$$

where $x_1(t)$ and $x_2(t)$ stand for the density of the prey and the predator, respectively, and $m \neq 0$ is a constant. $\tau(t) \geq 0$ stands for the time delays due to negative feedback of the prey population. $r_1(t)$, $r_2(t)$ stand for the intrinsic growth rates of the prey and the predator, respectively. $a_{11}(t)$ is the intraspecific competition rate of the prey. $a_{12}(t)$ is the capturing rate of the predator. The predator grows with the carrying capacity $x(t)/a_{21}(t)$ proportional to the population size of the prey or prey abundance. $a_{21}(t)$ is a measure of the food quality that the prey provided for conversion into predator birth. Assumed that $r_i(t)$, $a_{ij}(t)$, $i, j = 1, 2$, are continuously positive periodic functions with period ω, by using the continuation theorem of coincidence degree theory, the existence of a positive periodic solution for the semi-ratio-dependent predator-prey system with nonmonotonic functional responses and time delay is established. For the ecological sense of the system (1.1) we refer to [5–8] and the references cited therein.

As we know, permanence is one of the most important topics on the study of population dynamics. One of the most interesting questions in mathematical biology concerns the survival of species in ecological models. Biologically, when a system of interacting species is persistent in a suitable sense, it means that all the species survive in the long term. It is reasonable to ask for conditions under which the system is permanent. However, Ding et al. [4] did not investigate this property of the system (1.1).

Motivated by the above question, we will consider the permanence of the system (1.1). Unlike the assumptions of Ding et al. [4], we argue that a general nonautonomous nonperiodic system is more appropriate, and thus, we assume that the coefficients of system (1.1) satisfy: (A) $r_i(t)$, $a_{ij}(t)$, $\tau(t)$, $i, j = 1, 2$, are nonnegative functions bounded above and below by positive constants.

Throughout this paper, for a continuous function $g(t)$, we set

$$g^l = \inf_{t \in R} g(t), \qquad g^u = \sup_{t \in R} g(t). \tag{1.3}$$

It is easy to verify that solutions of system (1.1) corresponding to initial conditions (1.2) are defined on $[0, +\infty)$ and remain positive for all $t \geq 0$. In this paper, the solution of system (1.1) satisfying initial conditions (1.2) is said to be positive.

The aim of this paper is, by using the differential inequality theory, to obtain a set of sufficient conditions to ensure the permanence of the system (1.1).

2. Permanence

In this section, we establish a permanence result for system (1.1).

Lemma 2.1 (see [9]). *If $a > 0$, $b > 0$ and $\dot{x} \geq x(b - ax)$, when $t \geq 0$ and $x(0) > 0$, one has:*

$$\lim_{t \to +\infty} \inf x(t) \geq \frac{b}{a}. \tag{2.1}$$

If $a > 0$, $b > 0$ and $\dot{x} \leq x(b - ax)$, when $t \geq 0$ and $x(0) > 0$, one has:

$$\lim_{t \to +\infty} \sup x(t) \leq \frac{b}{a}. \tag{2.2}$$

Proposition 2.2. *Let* $(x_1(t), x_2(t))$ *be any positive solution of system* (1.1), *then*

$$\lim_{n\to+\infty} \sup x_i(n) \le M_i, \quad i = 1, 2, \tag{2.3}$$

where

$$M_1 = \frac{r_1^u}{a_{11}^l \exp\{-r_1^u \tau^u\}}, \qquad M_2 = \frac{r_2^u M_1}{a_{21}^l}. \tag{2.4}$$

Proof. Let $(x_1(t), x_2(t))$ be any positive solution of system (1.1), from the first equation of system (1.1) one has

$$\dot{x}_1(t) \le r_1(t) x_1(t). \tag{2.5}$$

By integrating both sides of the above inequality from $t - \tau(t)$ to t with respect to t, we obtain

$$x_1(t - \tau(t)) \ge x_1(t) \exp\left\{\int_{t-\tau(t)}^{t} -r_1(s) ds\right\} \ge x_1(t) \exp\{-r_1^u \tau^u\}. \tag{2.6}$$

By substituting (2.6) into the first equation of system (1.1), one has

$$\dot{x}_1(t) \le x_1(t)\left[r_1(t) - a_{11}(t) \exp\{-r_1^u \tau^u\} x_1(t)\right] \le x_1(t)\left[r_1^u - a_{11}^l \exp\{-r_1^u \tau^u\} x_1(t)\right]. \tag{2.7}$$

By Lemma 2.1, according to (2.7), it immediately follows that

$$\lim_{t\to+\infty} \sup x_1(t) \le \frac{r_1^u}{a_{11}^l \exp\{-r_1^u \tau^u\}} := M_1. \tag{2.8}$$

It follows that for any small positive constant $\varepsilon > 0$, there exists a $T_1 > 0$ such that

$$x_1(t) \le M_1 + \varepsilon, \quad t > T_1. \tag{2.9}$$

By substituting (2.9) into the second equation of system (1.1), one has

$$\dot{x}_2(t) \le x_2(t)\left[r_2(t) - \frac{a_{21}(t) x_2(t)}{M_1 + \varepsilon}\right] \le x_2(t)\left[r_2^u - \frac{a_{21}^l}{M_1 + \varepsilon} x_2(t)\right]. \tag{2.10}$$

By Lemma 2.1, according to (2.10), we get

$$\lim_{t\to+\infty} \sup x_2(t) \le \frac{r_2^u (M_1 + \varepsilon)}{a_{21}^l}. \tag{2.11}$$

Setting $\varepsilon \longrightarrow 0$ yields that

$$\lim_{t\to+\infty} \sup x_2(t) \leq \frac{r_2^u M_1}{a_{21}^l} := M_2. \tag{2.12}$$

This completes the proof of Proposition 2.2. □

Now we are in the position of stating the permanence of the system (1.1).

Theorem 2.3. *Assume that $r_1^l - a_{12}^u M_2/m^2 > 0$ hold, then system* (1.1) *is permanent, that is, there exist positive constants m_i, M_i, $i = 1, 2$, which are independent of the solutions of system* (1.1)*, such that for any positive solution $(x_1(t), x_2(t))$ of system* (1.1) *with initial condition* (1.2)*, one has*

$$m_i \leq \lim_{t\to+\infty} \inf x_i(t) \leq \lim_{t\to+\infty} \sup x_i(t) \leq M_i, \quad i = 1, 2. \tag{2.13}$$

Proof. By applying Proposition 2.2, we see that to end the proof of Theorem 2.3, it is enough to show that under the conditions of Theorem 2.3,

$$\lim_{t\to+\infty} \inf x_i(t) \geq m_i. \tag{2.14}$$

From Proposition 2.2, for all $\varepsilon > 0$, there exists a $T_2 > 0$, for all $t > T_2$,

$$x_i(t) \leq M_i + \varepsilon. \tag{2.15}$$

By substituting (2.15) into the first equation of system (1.1), it follows that

$$\dot{x}_1(t) \geq x_1(t)\left[r_1(t) - a_{11}(t)(M_1 + \varepsilon) - \frac{a_{12}(t)(M_2 + \varepsilon)}{m^2}\right]. \tag{2.16}$$

By integrating both sides of the above inequality from $t - \tau(t)$ to t with respect to t, we obtain

$$\begin{aligned} x_1(t - \tau(t)) &\leq x_1(t) \exp\left\{\int_{t-\tau(t)}^{t} -\left[r_1(s) - a_{11}(s)(M_1 + \varepsilon) - \frac{a_{12}(s)(M_2 + \varepsilon)}{m^2}\right] ds\right\} \\ &\leq x_1(t) \exp\left\{\left[-r_1^l \tau^l + a_{11}^u (M_1 + \varepsilon)\tau^u + \frac{a_{12}^u (M_2 + \varepsilon)\tau^u}{m^2}\right]\right\}. \end{aligned} \tag{2.17}$$

By substituting the above inequality into the first equation of system (1.1), one has

$$\begin{aligned}
\dot{x}_1(t) &\geq x_1(t)\Bigg\{r_1(t) - \frac{a_{12}(t)(M_2+\varepsilon)}{m^2} \\
&\qquad -a_{11}(t)\exp\left\{\left[-r_1^l\tau^l + a_{11}^u(M_1+\varepsilon)\tau^u + \frac{a_{12}^u(M_2+\varepsilon)\tau^u}{m^2}\right]\right\}x_1(t)\Bigg\} \\
&\geq x_1(t)\Bigg\{r_1^l - \frac{a_{12}^u(M_2+\varepsilon)}{m^2} \\
&\qquad -a_{11}^u\exp\left\{\left[-r_1^l\tau^l + a_{11}^u(M_1+\varepsilon)\tau^u + \frac{a_{12}^u(M_2+\varepsilon)\tau^u}{m^2}\right]\right\}x_1(t)\Bigg\}.
\end{aligned} \tag{2.18}$$

By Lemma 2.1, under the conditions of Theorem 2.3, it immediately follows that

$$\liminf_{t\to+\infty} x_1(t) \geq \frac{r_1^l - a_{12}^u(M_2+\varepsilon)/m^2}{a_{11}^u\exp\left\{\left[-r_1^l\tau^l + a_{11}^u(M_1+\varepsilon)\tau^u + a_{12}^u(M_2+\varepsilon)\tau^u/m^2\right]\right\}}. \tag{2.19}$$

Setting $\varepsilon \to 0$, yields that

$$\liminf_{t\to+\infty} x_1(t) \geq \frac{r_1^l - a_{12}^u M_2/m^2}{a_{11}^u\exp\left\{\left[-r_1^l\tau^l + a_{11}^u M_1\tau^u + a_{12}^u M_2\tau^u/m^2\right]\right\}} := m_1. \tag{2.20}$$

It follows that for the above positive constant $\varepsilon > 0$, there exists a $T_2 > 0$ such that

$$x_1(t) \geq m_1 - \varepsilon, \quad t > T_2. \tag{2.21}$$

By substituting (2.21) into the second equation of system (1.1), one has

$$\dot{x}_2(t) \geq x_2(t)\left[r_2(t) - \frac{a_{21}(t)x_2(t)}{m_1-\varepsilon}\right] \geq x_2(t)\left[r_2^l - \frac{a_{21}^u}{m_1-\varepsilon}x_2(t)\right]. \tag{2.22}$$

By Lemma 2.1, according to (2.22), it immediately follows that

$$\liminf_{t\to+\infty} x_2(t) \geq \frac{r_2^l(m_1-\varepsilon)}{a_{21}^u}. \tag{2.23}$$

Setting $\varepsilon \to 0$ yields that

$$\liminf_{t\to+\infty} x_2(t) \geq \frac{r_2^l m_1}{a_{21}^u} := m_2. \tag{2.24}$$

This completes the proof of Theorem 2.3. □

References

[1] A. A. Berryman, "The origins and evolution of predator-prey theory," *Ecology*, vol. 73, no. 5, pp. 1530–1535, 1992.

[2] S.-B. Hsu, T.-W. Hwang, and Y. Kuang, "Global analysis of the Michaelis-Menten-type ratio-dependent predator-prey system," *Journal of Mathematical Biology*, vol. 42, no. 6, pp. 489–506, 2001.

[3] Y. Kuang and E. Beretta, "Global qualitative analysis of a ratio-dependent predator-prey system," *Journal of Mathematical Biology*, vol. 36, no. 4, pp. 389–406, 1998.

[4] X. Ding, C. Lu, and M. Liu, "Periodic solutions for a semi-ratio-dependent predator-prey system with nonmonotonic functional response and time delay," *Nonlinear Analysis: Real World Applications*, vol. 9, no. 3, pp. 762–775, 2008.

[5] Y.-H. Fan, W.-T. Li, and L.-L. Wang, "Periodic solutions of delayed ratio-dependent predator-prey models with monotonic or nonmonotonic functional responses," *Nonlinear Analysis: Real World Applications*, vol. 5, no. 2, pp. 247–263, 2004.

[6] S. Ruan and D. Xiao, "Global analysis in a predator-prey system with nonmonotonic functional response," *SIAM Journal on Applied Mathematics*, vol. 61, no. 4, pp. 1445–1472, 2001.

[7] Q. Wang, M. Fan, and K. Wang, "Dynamics of a class of nonautonomous semi-ratio-dependent predator-prey systems with functional responses," *Journal of Mathematical Analysis and Applications*, vol. 278, no. 2, pp. 443–471, 2003.

[8] D. Xiao and S. Ruan, "Multiple bifurcations in a delayed predator-prey system with nonmonotonic functional response," *Journal of Differential Equations*, vol. 176, no. 2, pp. 494–510, 2001.

[9] F. Chen and M. You, "Permanence for an integrodifferential model of mutualism," *Applied Mathematics and Computation*, vol. 186, no. 1, pp. 30–34, 2007.