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Research Article

Slow Rotation of Concentric Spheres with Source at Their Centre in a Viscous Fluid

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The problem of concentric pervious spheres carrying a fluid source at their centre and rotating slowly with different uniform angular velocities Ω_1, Ω_2 about a diameter has been studied. The analysis reveals that only azimuthal component of velocity exists, and the couple, rate of dissipated energy is found analytically in the present situation. The expression of couple on inner sphere rotating slowly with uniform angular velocity Ω_1, while outer sphere also rotates slowly with uniform angular velocity Ω_2, is evaluated. The special cases, like (i) inner sphere is fixed (i.e., $\Omega_1 = 0$), while outer sphere rotates with uniform angular velocity Ω_2, (ii) outer sphere is fixed (i.e., $\Omega_2 = 0$), while inner sphere rotates with uniform angular velocity Ω_1, and (iii) inner sphere rotates with uniform angular velocity Ω_1, while outer sphere rotates at infinity with angular velocity Ω_2, have been deduced.

1. Introduction

Stokes flow is becoming increasingly important due to the miniaturization of fluid mechanical parts for example, in micromechanics as well as in nanomechanics. Slow rotation of spheroids (including the disc) in an infinite fluid was first solved by Jeffrey [1] using curvilinear coordinates. His approach was later extended to the spherical lens, torus, and other axisymmetric shapes. Proudman [2] and Stewartson [3] analyzed the dynamical properties of a fluid occupying the space between two concentric rotating spheres when the angular velocities of the spheres are slightly different, in other words, when the motion relative to a reference frame rotating with one of the spheres is due to an imposed azimuthal velocity which is symmetric about the equator. Rubinow and Keller [4] have considered the force on a spinning sphere which is moving through an incompressible viscous fluid by employing the method of matched asymptotic expansions to describe the asymmetric flow. Brenner [5] also obtained some general results for the drag and couple on an obstacle which is moving through the fluid. Childress [6] has investigated the motion of a sphere moving through a rotating fluid and calculated a correction to the drag coefficient. Wakiya [7] numerically evaluated the drag and angular velocity experienced by freely rotating spheres and compared with calculated from corresponding approximate formulae known before. Barrett [8] has tackled the problem of impulsively started sphere rotating with angular velocity Ω about a diameter. He modified the standard time-dependent boundary layer equation to give series solutions satisfying all the boundary conditions and gave solutions that are applicable at small times for non-zero Reynolds numbers. He found that the velocity components decay algebraically rather than exponentially at large distances. Pearson [9] has presented the numerical solution for the time-dependent viscous flow between two concentric rotating spheres. He governed the motion of a pair of coupled nonlinear partial differential equations in three independent variables, with singular end conditions. He also described the computational process for cases in which one (or both) of the spheres is given an impulsive change in angular velocity-starting from a state of either rest or uniform rotation. Majumdar [10] has solved, by using bispherical coordinates, the nonaxisymmetrical Stokes flow of an incompressible homogeneous viscous liquid in space between two eccentric spheres. It was proved that the resultant force acting upon the spheres is at right angles to the axis of rotation and the line of centres. The effect of the stationary sphere on the force and couple exerted by the liquid on the rotating sphere has been discussed, and the results are compared with those of the axisymmetrical case of Jeffrey [1]. Kanwal [11] has considered a disk performing simple harmonic rotary oscillations about its axis of symmetry in a nonconducting viscous fluid which is at rest at infinity. O'Neill and Majumdar [12, 13] have discussed the problem of asymmetrical slow viscous fluid motions caused by the translation or rotation of two spheres. The exact solutions for any values of the ratio of radii and separation parameters are found by them.

Ranger [14] tackled the problem of axially symmetric flow past a rotating sphere due to a uniform stream of infinity. He has shown that leading terms for the flow consist of a linear superposition of a primary Stokes flow past a nonrotating sphere together with an anti symmetric secondary flow in the azimuthal plane induced by the spinning sphere. Philander [15] presented a note on the flow properties of a fluid between concentric spheres. This note concerns the flow properties of a spherical shell of fluid when motion is forced across the equator. The fluid under consideration is contained between two concentric spheres which rotate about a diameter with angular velocity Ω. The consequences of the forcing motion across the equator are explored in his work. Cooley [16] has investigated the problem of fluid motion generated by a sphere rotating close to a fixed sphere about a diameter perpendicular to the line of centres in the case when the motion is sufficiently slow to permit the linearization of the Navier-Stokes equations by neglecting the inertia terms. He used a method of matched asymptotic expansions to find asymptotic expressions for the forces and couples acting on the spheres as the minimum clearance between them tends to zero. In his paper, the forces and couples are shown to have the form $a_0 \ln \varepsilon + a_1 + o(\varepsilon \ln \varepsilon)$, where ε is the ratio of the minimum clearance between the spheres and the radius of the rotating sphere and where a_0 and a_1 are found explicitly. Munson and Joseph [17, 18] have obtained the high-order analytic perturbation solution for the viscous incompressible flow between concentric rotating spheres. In the second part of their analysis, they have applied the energy theory of hydrodynamic stability to the viscous incompressible flow of a fluid contained between two concentric spheres which rotate about a common axis with prescribed angular velocities. Riley and Mack [19] has discussed the thermal effects on slow viscous flow between rotating concentric spheres. Takagi [20] has considered the flow around a spinning sphere moving in a viscous fluid. He solved the Navier-Stokes equations, using the method of matched asymptotic expansions for small values of the Reynolds number. With the solution, the force and torque on the sphere are computed, and he found that the sphere experiences a force orthogonal to its direction of motion and that the drag is increased in proportion to the square of the spin velocity. Takagi [21] has studied the Stokes flow for the case in which two solid spheres in contact are steadily rotating with different angular velocities about their line of centres. For the case of two equal spheres, he found that, one of which is kept rotating with angular velocity ω while the other is left free, the latter will rotate with angular velocity $\omega/7$. Munson and Menguturk [22] have studied the stability of flow of a viscous incompressible fluid between a stationary outer sphere and rotating inner sphere theoretically and experimentally. Wimmer [23] has provided some experimental results on incompressible viscous fluid flow in the gap between two concentric rotating spheres. Takagi [24] further studied the problem of steady flow which is induced by the slow rotation of a solid sphere immersed in an infinite incompressible viscous fluid, on the basis of Navier-Stokes equations. He obtained the solution in the form of power series with respect to Reynolds number. Drew [25] has found the force on a small sphere translating relative to a slow viscous flow to order of the $1/2$ power of Re for two different fluid flows far from the sphere, namely, pure rotation and pure shear. For pure rotation, the correction of this order to the Stokes drag consists of an increase in the drag. Kim [26] has calculated the torque and frictional force exerted by a viscous fluid on a sphere rotating on the axis of a circular cone of arbitrary vertex angle about an axis perpendicular to the cone axis in the Stokes approximation. Dennis et al. [28] have investigated the problem of viscous incompressible, rotationally symmetric flow due to the rotation of a sphere with a constant angular velocity about a diameter. The solutions of the finite-difference equations are presented for Reynolds number ranging from 1.0 to 5000. Davis and Brenner [29] have used the matched asymptotic expansion methods to solve the problem of steady rotation of a tethered sphere at small, nonzero Reynolds numbers. They obtained first-order Taylor number correction to both the Stokes-law drag and Kirchhoff's law couple on the sphere for Rossby numbers of order unity. Gagliardi [30] has developed the boundary conditions for the equations of motion for a viscous incompressible fluid in a rotating spherical annulus. The solutions of the stream and circumferential functions were obtained in the form of a series of powers of the Reynolds number. Transient profiles were obtained for the dimensional torque, dimensionless angular velocity of the rotating sphere, and the dimensionless angular momentum of the fluid. Marcus and Tuckerman [31, 32] have computed numerically the steady and translation simulation of flow between concentric rotating spheres. O'Neill and Yano [33] derived the boundary condition at the surfactant and substrate fluids caused by the slow rotation of a solid sphere which is partially submerged in the substrate fluid. Yang et al. [34] have provided the numerical schemes for the problem of the axially symmetric motion of an incompressible viscous fluid in an annulus between two concentric rotating spheres. Gagliardi et al. [35] reported the study of the steady state and transient motion of a system consisting of an incompressible, Newtonian fluid in an annulus between two concentric, rotating, rigid spheres. They solved the governing equations for the variable coefficients by separation of variables and Laplace Transform methods. They presented the results for the stream function, circumferential function, angular velocity of the spheres, and torque coefficient as a function of time for various values of the dimensionless system parameters. Ranger [36] has found an exact solution of the Navier-Stokes equations for the axi-symmetric motion (with swirl) representing exponentially time-dependent decay of a solid sphere translating and rotating in a viscous fluid relative to a uniform stream whose speed also decays exponentially with time. He also described a similar solution for the two-dimensional analogue where the sphere is replaced by a circular cylinder of infinite length. Tekasakul et al. [37] have studied the problem of the rotatory oscillation of an axi-symmetric body in an axi-symmetric viscous flow at low Reynolds numbers. They evaluated numerically the local stresses and torques on a selection of free, oscillating, axi-symmetric bodies in the continuum regime in an axi-symmetric viscous incompressible flow. Datta and Srivastava [38] have tackled the problem of slow rotation of a sphere with fluid source at its centre in a viscous fluid. In their investigation, it was found that the effect of fluid source at the centre is to reduce the couple on slowly rotating sphere about its diameter. Kim and Choi [39] conducted the numerical simulations for laminar flow past a sphere rotating in the streamwise direction, in order to investigate the effect of the rotation on the characteristics of flow over the sphere.

Tekasakul and Loyalka [41] have investigated the rotary oscillations of several axi-symmetric bodies in axi-symmetric viscous flows with slip. A numerical method based on Green's function technique is used, and analytic solutions for local stress and torque on spheres and spheroids as function of the frequency parameter and the slip coefficients are obtained. They have analysed that, in all cases, slip reduces stress and torque, and increasingly so with the increasing frequency parameter. Liu et al. [42] have developed a very efficient numerical method based on the finite difference technique for solving time-dependent nonlinear flow problems. They have applied this method to study the unsteady axisymmetric isotherm flow of an incompressible viscous fluid in a spherical shell with a stationary inner sphere and a rotating outer sphere. Ifidon [43] numerically investigated the problem of determining the induced steady axially symmetric motion of an incompressible viscous fluid confined between two concentric spheres, with the outer sphere rotating with constant angular velocity and the inner sphere fixed for large Reynolds numbers. Davis [44] obtained the expression for force and torque on a rotating sphere close to and within a fluid-filled rotating sphere. Romano [45] has introduced new exact analytic solutions for the rotational motion of a axially symmetric rigid body having two equal principal moments of inertia and subjected to an external torque which is constant in magnitude.

In the present paper, the problem of slow rotation of concentric spheres, both assumed to be pervious, with a source at their centre has been tackled. If the strength Q of the source was of the same order as the angular velocity Ω of rotating spheres, the inertia terms could still be neglected and the total flow then consists of only the source solution superimposed on the Stokes solution. Therefore, in this case the Stokes drag and couple are not affected by the source. Also, if Q is large enough so that $Q\Omega$ is not negligible, the inertia terms, being nonlinear, cannot be altogether omitted. The Navier-Stokes equation can still be linearized by assuming that the velocity perturbation in the source flow on account of the Stokes flow is small so that the terms containing square of angular velocity (i.e., of order Ω^2) can be neglected. This assumption is justifiable at least in the vicinity of the spheres where the Stokes approximation is valid. The present problem corresponds to the problem of Stokes flow past a sphere with source at its centre investigated by Datta [46] and slow rotation of sphere with source at its centre in a viscous fluid investigated by Datta and Srivastava [38], the results of which have found application in investigation of the diffusiophoresis target efficiency for an evaporating or condensing drop [47].

2. Formulation of the Problem

Let us consider two pervious spheres of radius "a" and "b" (where $b > a$) with source of strength "Q" at its centre generating radial flow around it in an infinite expense of incompressible fluid of density ρ and kinematic viscosity ν. The spheres are also made to rotate with small steady angular velocities Ω_1 and Ω_2 so that terms of an $o(\Omega^2)$ may be neglected but terms of $o(Q\Omega)$ retained. The motivation of this formulation has been taken from the author's previous work [38] due to the fact that body geometry has not been changed, although the two concentric spheres are rotating slowly with different angular velocities instead of only one. The governed equations of motion will remain same and provide the new solutions under the defined boundary conditions.

The motion is governed by Navier-Stokes equations

$$\mathbf{u}\cdot\operatorname{grad}\mathbf{u} = -\left(\frac{1}{\rho}\right)\operatorname{grad}p + \nu\nabla^2\mathbf{u} \tag{2.1}$$

and continuity equation

$$\operatorname{div}\ \mathbf{u} = 0, \tag{2.2}$$

together with no-slip boundary condition

$$\mathbf{u} = a\Omega\ \widehat{\mathbf{e}}_x \times \widehat{\mathbf{e}}_r, \quad \text{on the inner sphere } r = a, \tag{2.3a}$$

$$\mathbf{u} = b\Omega\ \widehat{\mathbf{e}}_x \times \widehat{\mathbf{e}}_r, \quad \text{on the outer sphere } r = b, \tag{2.3b}$$

and the condition of vanishing of velocity at far off points

$$\mathbf{u} = 0 \quad \text{as } r \longrightarrow \infty. \tag{2.4}$$

1In the above equations symbols $\mathbf{u}, p, \rho, \nu$ stand for velocity, pressure, density, and kinematic viscosity, and unit vectors $\widehat{e}_x$ and $\widehat{e}_r$ are along x-axis and radial direction. It will be convenient to work in spherical polar coordinates (r, θ, φ) with x-axis as the polar axis. We nondimensionalize the space variables by a, velocity by a Ω, and pressure by $\rho\nu\Omega$. Moreover, the symmetry of the problem and the boundary conditions ensure that velocity components are $v_r = v_\theta = 0$ and then we may express the nondimensional velocity vector $\mathbf{u}$ as

$$\mathbf{u} = \frac{Q}{a^2 r^2}\widehat{\mathbf{e}}_r + v_\varphi(r,\theta)\widehat{\mathbf{e}}_\varphi \tag{2.5}$$

and pressure as

$$p = \rho\nu\Omega\ \left[p_0(r) + p_1(r,\theta)\right]. \tag{2.6}$$

By introducing the expressions (2.5) and (2.6) in (2.1), the azimuthal component v_φ is seen to satisfy the equation

$$\nabla^2 v_\varphi - \frac{v_\varphi}{r^2\sin\theta} = \frac{s}{r^3}\frac{\partial}{\partial r}(r v_\varphi), \tag{2.7}$$

where $s = Q/(\nu a)$ is the source parameter.

The above equation is to be solved under the boundary conditions

$$\begin{gathered}
v_\varphi = \sin\theta \quad \text{at } r = 1 \text{ (nondimensional equation of spheres)}, \\
v_\varphi \longrightarrow 0 \quad \text{as } r \longrightarrow \infty.
\end{gathered} \tag{2.8}$$

3. Solution

We take the trial solution as

$$v_\varphi = r\omega(r)\sin\theta; \tag{3.1}$$

substituting this value of v_φ in (2.7), we get, after some calculation and adjustment,

$$\frac{d}{dr}\left[r^4\frac{d\omega}{dr} - sr^2\omega\right] = 0, \tag{3.2}$$

and the boundary conditions (2.8) in nondimensional form become

$$\begin{gathered}
\omega = 1 \quad \text{at } r = 1 \text{ (i.e., on the surface)}, \\
\omega \longrightarrow 0 \quad \text{as} \quad r \longrightarrow \infty.
\end{gathered} \tag{3.3}$$

The above boundary conditions may also be express in dimensional form as

$$\begin{gathered}
\omega = \Omega_1 \quad \text{at } r = a \ \text{(i.e., on the inner sphere)}, \\
\omega \longrightarrow 0 \quad \text{as} \quad r \longrightarrow \infty.
\end{gathered} \tag{3.4}$$

On integration of (3.2), we get the solution in nondimensional form as

$$\omega(r) = -\frac{A}{s^3}\left[\frac{s^3}{r^2} - 2\frac{s}{r} + 2\right] + B\,e^{-s/r} \tag{3.5}$$

and in dimensional form as

$$\omega(r) = -\frac{A}{s^3}\left[\frac{s^2a^2}{r^2} - 2\frac{sa}{r} + 2\right] + B\,e^{-sa/r}, \tag{3.6}$$

where A and B are constants of integration which can be obtained by applying boundary conditions (3.4) as

$$\begin{gathered}
-\frac{A}{s^3} = \Omega_1\left[s^2 - 2s + 2 - 2e^{-s}\right]^{-1}, \\
B = -2\Omega_1\left[s^2 - 2s + 2 - 2e^{-s}\right]^{-1}.
\end{gathered} \tag{3.7}$$

Substituting the values of constants A and B in (3.6), we get the expression of angular velocity $\omega(r)$ in dimensional form as

$$\omega(r) = -\frac{A}{s^3}\left[\frac{s^2a^2}{r^2} - 2\frac{sa}{r} + 2\right] + B\, e^{-sa/r} \tag{3.8}$$

or

$$\omega(r) = \Omega_1\left[\frac{s^2a^2}{r^2} - 2\frac{sa}{r} + 2 - e^{-sa/r}\right]\left[s^2 - 2s + 2 - 2e^{-s}\right]^{-1} \tag{3.9}$$

and consequently, with the help of (3.1), the expression for azimuthal component of velocity v_φ comes out to be in dimensional form as

$$v_\varphi = r\omega(r)\sin\theta = \Omega r\sin\theta\left[\frac{s^2a^2}{r^2} - 2\frac{sa}{r} + 2 - e^{-(sa/r)}\right]\left[s^2 - 2s + 2 - 2e^{-s}\right]. \tag{3.10}$$

3.1. Couple on Inner Sphere Rotating with Uniform Angular Velocity Ω_1 (When Outer Sphere Is Also Rotating with Different Uniform Angular Velocity Ω_2)

If there exists an external concentric pervious sphere of radius b $(b > a)$, rotating with small angular velocity Ω_2, that is, the boundary conditions for this situation will be

$$\begin{aligned} &\omega = \Omega_2 \quad \text{at } r = b \text{ (at outer surface)},\\ &\omega = \Omega_1 \quad \text{at } r = a \text{ (at inner surface)}, \end{aligned} \tag{3.11}$$

by using the above boundary conditions, in (3.6), the constant of integration A and B comes out to be

$$\begin{aligned} \frac{A}{s^3} &= \frac{\Omega_1 - \Omega_2 e^{-s(1-a/b)}}{\left[(s^2\ a^2/b^2 - 2(sa/b) + 2)e^{-s(1-a/b)} + (-s^2 + 2s - 2)\right]},\\ B &= e^{sa/b}\left[\frac{\Omega_1\{s^2a^2/b^2 - 2(sa/b) + 2\} + \Omega_2\{-s^2 + 2s - 2\}}{e^{-s(1-a/b)}\{s^2a^2/b^2 - 2(sa/b) + 2\} + \{-s^2 + 2s - 2\}}\right]. \end{aligned} \tag{3.12}$$

The expression of angular velocity $\omega(r)$ can be written with the help of (3.6)

$$\omega(r) = -\frac{A}{s^3}\left[\frac{s^2a^2}{r^2} - 2\frac{sa}{r} + 2\right] + B\, e^{-sa/r}, \tag{3.13}$$

where A and B are given in (3.12). On differentiating the function $\omega(r)$, we have

$$\frac{d\omega}{dr} = -\frac{A}{s^3}\left[-\frac{2s^2a^2}{r^3} + \frac{2sa}{r^2}\right] + B\,e^{-sa/r}\left(\frac{sa}{r^2}\right), \tag{3.14}$$

and the value of $d\omega/dr$ at $r = a$ can be written as

$$\left(\frac{d\omega}{dr}\right)_{r=a} = \frac{1}{a}\left[\frac{2A}{s^3}\left(s^2 - s\right) + B\,e^{-s}\right]. \tag{3.15}$$

The moment of force p_φ is $p_\varphi \cdot r\ \sin\theta$, where $p_\varphi = \mu \cdot r\ \sin\theta \cdot (d\omega/dr)$ is the only nonvanishing component of force p. If N is the couple on the sphere of radius a, then by using (3.15), we have

$$\begin{aligned}
N &= \int_0^\pi \left(p_\varphi \cdot r\ \sin\theta\right)_{r=a} dS \\
&= \int_0^\pi \left(\mu r \sin\theta \frac{d\omega}{dr} r \sin\theta\right)_{r=a} \cdot \left(2\pi a\ \sin\theta \cdot a d\theta\right) \\
&= \frac{8}{3}\pi a^3 \mu \left[\frac{2A}{s^3}\left(s^2 - s\right) + B\,e^{-s}\right] \\
&= \frac{8}{3}\pi a^3 \mu \Bigg[2\left(s^2 - s\right)\left\{\Omega_1 - \Omega_2 e^{-s(1-a/b)}\right\} + e^{-s(1-a/b)} \\
&\qquad \times \left\{\Omega_1\left(\frac{s^2a^2}{b^2} - 2\frac{sa}{b} + 2\right) + \Omega_2\left(-s^2 + 2s - 2\right)\right\}\Bigg] \\
&\quad \times \left[\left\{e^{-s(1-a/b)}\left(\frac{s^2a^2}{b^2} - 2\frac{sa}{b} + 2\right) + \left(-s^2 + 2s - 2\right)\right\}\right]^{-1}.
\end{aligned} \tag{3.16}$$

The rate of dissipated energy is given by $N\Omega_1$, where the value of N is given in (3.16).

3.2. Couple on Outer Sphere Rotating with Uniform Angular Velocity Ω_2 *(When Inner Sphere is Fixed, that is,* $\Omega_1 = 0$*)*

The expression for angular velocity $\omega(r)$ is given by (3.6):

$$\omega(r) = -\frac{A}{s^3}\left[\frac{s^2a^2}{r^2} - 2\frac{sa}{r} + 2\right] + B\,e^{-sa/r}. \tag{3.17}$$

Now we use the following boundary conditions:

$$\begin{aligned}
\omega(r) &= \Omega_2 \quad \text{on surface } r = b, \\
\omega(r) &\longrightarrow 0 \quad \text{as } r \longrightarrow \infty.
\end{aligned} \tag{3.18}$$

Under these boundary conditions, the values of constant A and B can be obtained as follows:

$$\begin{aligned}\frac{A}{s^3} &= \Omega_2\left[\frac{s^2a^2}{b^2}+\frac{2sa}{b}+2e^{-sa/b}-2\right]^{-1},\\ B &= 2\Omega_2\left[\frac{s^2a^2}{b^2}+\frac{2sa}{b}+2e^{-sa/b}-2\right]^{-1}.\end{aligned} \tag{3.19}$$

Now, the expression for derivative of angular velocity at $r = b$ comes out to be

$$\begin{aligned}\left[\frac{d}{dr}\omega(r)\right]_{r=b} &= \left[-\frac{A}{s^3}\left(-\frac{2s^2a^2}{r^3}+\frac{2sa}{r^2}\right)+B\left(\frac{sa}{r^2}\right)e^{-sa/r}\right]_{r=b}\\ &= \frac{1}{b}\left[\frac{2A}{s^3}\left(\frac{s^2a^2}{b^2}-\frac{sa}{b}\right)+B\frac{sa}{b}e^{-sa/b}\right],\end{aligned} \tag{3.20}$$

which reduces in final form by (3.19):

$$= \frac{2\Omega_2}{b}\left[\frac{s^2a^2}{b^2}-\frac{sa}{b}+\frac{as}{b}e^{-sa/b}\right]\left[\frac{s^2a^2}{b^2}+\frac{2sa}{b}+2e^{-sa/b}-2\right]^{-1}. \tag{3.21}$$

Hence, the couple N on the outer sphere in the presence of inner sphere is

$$\begin{aligned}N &= \int_0^{\pi}\left(\mu\cdot r\sin\theta\cdot\frac{d\omega}{dr}\cdot r\sin\theta\right)_{r=b}(2\pi b\sin\theta\cdot b\,d\theta)\\ &= \frac{8}{3}\pi b^3\mu\left(\frac{d}{dr}\omega(r)\right)_{r=b},\end{aligned} \tag{3.22}$$

and by using (3.21), it reduces to

$$= \frac{16}{3}\pi b^3\mu\Omega_2\left[\frac{s^2a^2}{b^2}-\frac{sa}{b}+\frac{sa}{b}e^{-sa/b}\right]\left[\frac{s^2a^2}{b^2}+2\frac{sa}{b}+2e^{-sa/b}-2\right]^{-1}. \tag{3.23}$$

The expression for rate of dissipated energy will be $N\Omega_2$, where N is given in (3.23).

4. Particular Cases

Case 1. We consider the outer spherical surface to be fixed, that is, $\Omega_2 = 0$; then in this case, by (3.16), we have couple on the inner sphere rotating with angular velocity Ω_1 as

$$N = \frac{8}{3}\pi a^3 \mu \Omega_1 \left[2\left(s^2 - s\right) + \left(\frac{s^2 a^2}{b^2} - 2\frac{sa}{b} + 2\right)e^{-s(1-a/b)}\right] \times \left[\left(-s^2 + 2s - 2\right) + \left(\frac{s^2 a^2}{b^2} - 2\frac{sa}{b} + 2\right)e^{-s(1-a/b)}\right]^{-1}. \tag{4.1}$$

Now, on shifting the solid outer spherical body having radius b ($b > a$) to infinity that is, $b \to \infty$, then $e^{-s(1-a/b)} \to e^{-s}$, and by (4.1), we can have the expression for couple on slowly rotating sphere of radius "a" alone and given by (4.1) as

$$N = \frac{16}{3}\pi \mu a^3 \Omega_1 \frac{\left[s^2 - s + se^{-s}\right]}{\left[-s^2 + 2s - 2 + 2e^{-s}\right]}, \tag{4.2}$$

which matches with the expression of couple obtained by Datta and Srivastava [38] for slowly rotating pervious sphere of radius "a" rotating with slow uniform angular velocity Ω_1 and further reduces to classical one $8\pi\mu a^3\Omega_1$ for $s = 0$ (i.e., in the absence of source at the center).

Case 2. We consider the inner spherical surface to be fixed, that is, $\Omega_1 = 0$; then in this case, by (3.16), we have the couple on the outer sphere rotating with angular velocity Ω_2 as

$$N = \frac{8}{3}\pi \mu a^3 \Omega_2 \frac{\left[-3s^2 + 4s - 2\right]}{\left[(s^2a^2/b^2 - 2(sa/b) + 2) + (-s^2 + 2s - 2)e^{s(1-a/b)}\right]}. \tag{4.3}$$

Case 3. If the inner sphere rotates with uniform angular velocity Ω_1, while outer rotates with uniform angular velocity Ω_2 at infinity, that is, $b \to \infty$, then by expression (3.16), we have the couple on inner sphere as

$$N = \frac{8}{3}\pi \mu a^3 \frac{\left[2\Omega_1\left(s^2 - s + e^{-s}\right) + \Omega_2 e^{-s}\left(-3s^2 + 4s - 2\right)\right]}{\left[-s^2 + 2s - 2 + 2e^{-s}\right]}. \tag{4.4}$$

For large value of source parameter "s", (4.4) reduces in to the form

$$N \approx \frac{16}{3}\pi \mu a^3 \Omega_1 \left[1 + \left(\frac{1}{s} - \frac{6}{s^3}\right)\right], \tag{4.5}$$

which gives 2/3 M_0, where $M_0 = 8\pi\mu a^3\Omega_1$, in the limit as $s \to \alpha$, again in good agreement with the result of couple on rotating sphere with uniform angular velocity for large source parameter obtained by Datta and Srivastava [38].

Case 4. If we consider the limiting situation as $b \rightarrow a$ and $\Omega_2 \rightarrow \Omega_1$, then we have the expression for couple on slowly rotating sphere having radius "a" by (3.23)

$$N = \frac{16}{3}\pi a^3 \mu \Omega_1 \left[s^2 - s + se^{-s}\right]\left[-s^2 + 2s + 2e^{-s} - 2\right]^{-1}, \tag{4.6}$$

which matches with the result existed in the paper of Datta and Srivastava [38] and further reduces to the classical one $8\pi\mu a^3\Omega_1$ for $s = 0$ (i.e., in the absence of source at the center).

The expressions for couple in general case ((3.16), (3.23)) and in cases ((4.1) to (4.6)) are expected to be new and never seen in literature. It was concluded there that the effect of fluid source at the center of sphere is to reduce the couple.

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