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Research Article

Homomorphisms and Derivations in C^*-Ternary Algebras

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In 2006, C. Park proved the stability of homomorphisms in C^*-ternary algebras and of derivations on C^*-ternary algebras for the following generalized Cauchy-Jensen additive mapping: $2f((\sum_{j=1}^{p} x_j/2) + \sum_{j=1}^{d} y_j) = \sum_{j=1}^{p} f(x_j) + 2\sum_{j=1}^{d} f(y_j)$. In this note, we improve and generalize some results concerning this functional equation.

1. Introduction and Preliminaries

The stability problem of functional equations is originated from a question of Ulam [1] concerning the stability of group homomorphisms. Hyers [2] gave a first affirmative partial answer to the question of Ulam for Banach spaces. Hyers' theorem was generalized by Aoki [3] for additive mappings and by Rassias [4] for linear mappings by considering an unbounded Cauchy difference.

Theorem 1.1 (Th. M. Rassias). *Let $f : E \to E'$ be a mapping from a normed vector space E into a Banach space E' subject to the inequality*

$$\| f(x+y) - f(x) - f(y) \| \le \varepsilon(\|x\|^p + \|y\|^p) \tag{1.1}$$

for all $x, y \in E$, where ε and p are constants with $\varepsilon > 0$ and $p < 1$. Then the limit

$$L(x) = \lim_{n\to\infty} \frac{f(2^n x)}{2^n} \tag{1.2}$$

exists for all $x \in E$, *and* $L : E \to E'$ *is the unique additive mapping which satisfies*

$$\|f(x) - L(x)\| \leq \frac{2\varepsilon}{2 - 2^p}\|x\|^p \tag{1.3}$$

for all $x \in E$. *If* $p < 0$, *then inequality* (1.1) *holds for* $x, y \neq 0$ *and* (1.3) *for* $x \neq 0$. *Also, if for each* $x \in E$ *the mapping* $f(tx)$ *is continuous in* $t \in \mathbb{R}$, *then* L *is linear.*

It was shown by Gajda [5] as well as by Rassias and Šemrl [6] that one cannot prove a Rassias's type theorem when $p = 1$. The counter examples of Gajda [5] as well as of Rassias and Šemrl [6] have stimulated several mathematicians to invent new definitions of *approximately additive* or *approximately linear* mappings; compare Găvruţa [7] and Jung [8], who among others studied the stability of functional equations. Theorem 1.1 provided a lot of influence in the development of a generalization of the Hyers-Ulam stability concept. This new concept is known as *Hyers-Ulam-Rassias stability* of functional equations (cf. the books of Czerwik [9], Hyers et al. [10]).

Theorem 1.2 (Rassias [11–13]). *Let* X *be a real normed linear space and* Y *a real Banach space. Assume that* $f : X \to Y$ *is a mapping for which there exist constants* $\theta \geq 0$ *and* $p, q \in \mathbb{R}$ *such that* $r = p + q \neq 1$ *and* f *satisfies the functional inequality* (*Cauchy-Găvruţa-Rassias inequality*)

$$\|f(x+y) - f(x) - f(y)\| \leq \theta\|x\|^p\|y\|^q \tag{1.4}$$

for all $x, y \in X$. *Then there exists a unique additive mapping* $L : X \to Y$ *satisfying*

$$\|f(x) - L(x)\| \leq \frac{\theta}{|2^r - 2|}\|x\|^r \tag{1.5}$$

for all $x \in X$. *If, in addition,* $f : X \to Y$ *is a mapping such that the transformation* $t \to f(tx)$ *is continuous in* $t \in \mathbb{R}$ *for each fixed* $x \in X$, *then* L *is linear.*

For the case $r = 1$, a counter example has been given by Găvruţa [14]. The stability in Theorem 1.2 involving a product of different powers of norms is called *Ulam-Găvruţa-Rassias stability* (see [15–17]). In 1994, a generalization of Theorems 1.1 and 1.2 was obtained by Găvruţa [7], who replaced the bounds $\varepsilon(\|x\|^p + \|y\|^p)$ and $\theta\|x\|^p\|y\|^q$ by a general control function $\varphi(x, y)$. During past few years several mathematicians have published on various generalizations and applications of generalized Hyers-Ulam stability to a number of functional equations and mappings (see [16–44]).

Following the terminology of [45], a nonempty set G with a ternary operation $[\cdot,\cdot,\cdot] : G \times G \times G \to G$ is called a *ternary groupoid* and is denoted by $(G, [\cdot,\cdot,\cdot])$. The ternary groupoid $(G, [\cdot,\cdot,\cdot])$ is called *commutative* if $[x_1, x_2, x_3] = [x_{\sigma(1)}, x_{\sigma(2)}, x_{\sigma(3)}]$ for all $x_1, x_2, x_3 \in G$ and all permutations σ of $\{1, 2, 3\}$.

If a binary operation $\circ$ is defined on G such that $[x, y, z] = (x \circ y) \circ z$ for all $x, y, z \in G$, then we say that $[\cdot,\cdot,\cdot]$ is derived from $\circ$. We say that $(G, [\cdot,\cdot,\cdot])$ is a *ternary semigroup* if the operation $[\cdot,\cdot,\cdot]$ is *associative*, that is, if $[[x, y, z], u, v] = [x, [y, z, u], v] = [x, y, [z, u, v]]$ holds for all $x, y, z, u, v \in G$ (see [46]).

A C^*-ternary algebra is a complex Banach space A, equipped with a ternary product $(x, y, z) \mapsto [x, y, z]$ of A^3 into A, which are $\mathbb{C}$-linear in the outer variables, conjugate $\mathbb{C}$-linear in the middle variable, and associative in the sense that $[x, y, [z, w, v]] = [x, [w, z, y], v] = [[x, y, z], w, v]$, and satisfies $\|[x, y, z]\| \le \|x\| \cdot \|y\| \cdot \|z\|$ and $\|[x, x, x]\| = \|x\|^3$ (see [45, 47]). Every left Hilbert C^*-module is a C^*-ternary algebra via the ternary product $[x, y, z] := \langle x, y\rangle z$.

If a C^*-ternary algebra $(A, [\cdot, \cdot, \cdot])$ has an identity, that is, an element $e \in A$ such that $x = [x, e, e] = [e, e, x]$ for all $x \in A$, then it is routine to verify that A, endowed with $x \circ y := [x, e, y]$ and $x^* := [e, x, e]$, is a unital C^*-algebra. Conversely, if $(A, \circ)$ is a unital C^*-algebra, then $[x, y, z] := x \circ y^* \circ z$ makes A into a C^*-ternary algebra.

A $\mathbb{C}$-linear mapping $H : A \to B$ is called a *C^*-ternary algebra homomorphism* if

$$H([x, y, z]) = [H(x), H(y), H(z)] \tag{1.6}$$

for all $x, y, z \in A$. If, in addition, the mapping H is bijective, then the mapping $H : A \to B$ is called a *C^*-ternary algebra isomorphism*. A $\mathbb{C}$-linear mapping $\delta : A \to A$ is called a *C^*-ternary derivation* if

$$\delta([x, y, z]) = [\delta(x), y, z] + [x, \delta(y), z] + [x, y, \delta(z)] \tag{1.7}$$

for all $x, y, z \in A$ (see [23, 45, 48]).

Let $(A, \circ)$ be a C^*-algebra and $[x, y, z] := x \circ y^* \circ z$ for all $x, y, z \in A$. The mapping $H : A \to A$ defined by $H(x) = -ix$ is a C^*-ternary algebra isomorphism. Let $a \in A$ with $a^* = a$. The mapping $\delta_a : A \to A$ defined by $\delta_a(x) = i(ax - xa)$ is a C^*-ternary derivation. There are some applications, although still hypothetical, in the fractional quantum Hall effect, the nonstandard statistics, supersymmetric theory, and Yang-Baxter equation (cf. [49–51]).

Throughout this paper, assume that p, d are nonnegative integers with $p + d \ge 3$, and that A and B are C^*-ternary algebras.

2. Stability of Homomorphisms in C^*-Ternary Algebras

The stability of homomorphisms in C^*-ternary algebras has been investigated in [31] (see also [37]). In this note, we improve some results in [31]. For a given mapping $f : A \to B$, we define

$$C_\mu f(x_1, \ldots, x_p, y_1, \ldots, y_d) := 2f\left(\frac{\sum_{j=1}^{p} \mu x_j}{2} + \sum_{j=1}^{d} \mu y_j\right) - \sum_{j=1}^{p} \mu f(x_j) - 2\sum_{j=1}^{d} \mu f(y_j) \tag{2.1}$$

for all $\mu \in \mathbb{T}^1 := \{\lambda \in \mathbb{C} : |\lambda| = 1\}$ and all $x_1, \ldots, x_p, y_1, \ldots, y_d \in A$.

One can easily show that a mapping $f : A \to B$ satisfies

$$C_\mu f(x_1, \ldots, x_p, y_1, \ldots, y_d) = 0 \tag{2.2}$$

for all $\mu \in \mathbb{T}^1$ and all $x_1, \ldots, x_p, y_1, \ldots, y_d \in A$ if and only if

$$f(\mu x + \lambda y) = \mu f(x) + \lambda f(y) \tag{2.3}$$

for all $\mu, \lambda \in \mathbb{T}^1$ and all $x, y \in A$.

We will use the following lemmas in this paper.

Lemma 2.1 (see [30]). *Let $f : A \to B$ be an additive mapping such that $f(\mu x) = \mu f(x)$ for all $x \in A$ and all $\mu \in \mathbb{T}^1$. Then the mapping f is $\mathbb{C}$-linear.*

Lemma 2.2. *Let $\{x_n\}_n, \{y_n\}_n$ and $\{z_n\}_n$ be convergent sequences in A. Then the sequence $\{[x_n, y_n, z_n]\}_n$ is convergent in A.*

Proof. Let $x, y, z \in A$ such that

$$\lim_{n\to\infty} x_n = x, \qquad \lim_{n\to\infty} y_n = y, \qquad \lim_{n\to\infty} z_n = z. \tag{2.4}$$

Since

$$\begin{aligned} [x_n, y_n, z_n] - [x, y, z] &= [x_n - x, y_n - y, z_n - z] + [x_n - x, y_n, z] \\ &\quad + [x, y_n - y, z_n] + [x_n, y, z_n - z] \end{aligned} \tag{2.5}$$

for all n, we get

$$\begin{aligned} \left\| [x_n, y_n, z_n] - [x, y, z] \right\| &\le \|x_n - x\| \|y_n - y\| \|z_n - z\| + \|x_n - x\| \|y_n\| \|z\| \\ &\quad + \|x\| \|y_n - y\| \|z_n\| + \|x_n\| \|y\| \|z_n - z\| \end{aligned} \tag{2.6}$$

for all n. So

$$\lim_{n\to\infty} [x_n, y_n, z_n] = [x, y, z]. \tag{2.7}$$

This completes the proof. □

Theorem 2.3 (see [31]). *Let r and θ be nonnegative real numbers such that $r \notin [1,3]$, and let $f : A \to B$ be a mapping such that*

$$\left\| C_\mu f(x_1, \ldots, x_p, y_1, \ldots, y_d) \right\|_B \le \theta \left(\sum_{j=1}^{p} \|x_j\|_A^r + \sum_{j=1}^{d} \|y_j\|_A^r \right), \tag{2.8}$$

$$\left\| f([x, y, z]) - [f(x), f(y), f(z)] \right\|_B \le \theta \left(\|x\|_A^r + \|y\|_A^r + \|z\|_A^r \right) \tag{2.9}$$

for all $\mu \in \mathbb{T}^1$ and all $x, y, z, x_1, \ldots, x_p, y_1, \ldots, y_d \in A$. Then there exists a unique C^*-ternary algebra homomorphism $H : A \to B$ such that

$$\left\| f(x) - H(x) \right\|_B \leq \frac{2^r (p+d)\theta}{\left| 2(p+2d)^r - (p+2d)2^r \right|} \|x\|_A^r \tag{2.10}$$

for all $x \in A$.

In the following theorem we have an alternative result of Theorem 2.3.

Theorem 2.4. *Let r, s, and θ be nonnegative real numbers such that $0 < r < 1$, $0 < s < 3$ (resp., $r > 1$, $s > 3$), and let $d \geq 2$. Suppose that $f : A \to B$ is a mapping with $f(0) = 0$, satisfying* (2.8) *and*

$$\left\| f([x,y,z]) - [f(x), f(y), f(z)] \right\|_B \leq \theta\left(\|x\|_A^s + \|y\|_A^s + \|z\|_A^s \right) \tag{2.11}$$

for all $\mu \in \mathbb{T}^1$ and all $x, y, z \in A$. Then there exists a unique C^*-ternary algebra homomorphism $H : A \to B$ such that

$$\left\| f(x) - H(x) \right\|_B \leq \frac{d\theta}{2|d - d^r|} \|x\|_A^r \tag{2.12}$$

for all $x \in A$.

Proof. We prove the theorem in two cases.

Case 1. $0 < r < 1$ and $0 < s < 3$.

Letting $\mu = 1$, $x_1 = \cdots = x_p = 0$ and $y_1 = \cdots = y_d = x$ in (2.8), we get

$$\left\| f(dx) - df(x) \right\|_B \leq \frac{d\theta}{2} \|x\|_A^r \tag{2.13}$$

for all $x \in A$. If we replace x by $d^n x$ in (2.13) and divide both sides of (2.13) to d^{n+1}, we get

$$\left\| \frac{1}{d^{n+1}} f\left(d^{n+1}x\right) - \frac{1}{d^n} f(d^n x) \right\|_B \leq \frac{\theta}{2} d^{(r-1)n} \|x\|_A^r \tag{2.14}$$

for all $x \in A$ and all nonnegative integers n. Therefore,

$$\left\| \frac{1}{d^{n+1}} f\left(d^{n+1}x\right) - \frac{1}{d^m} f(d^m x) \right\|_B \leq \frac{\theta}{2} \sum_{i=m}^{n} d^{(r-1)i} \|x\|_A^r \tag{2.15}$$

for all $x \in A$ and all nonnegative integers $n \geq m$. From this it follows that the sequence $\{(1/d^n)f(d^n x)\}$ is Cauchy for all $x \in A$. Since B is complete, the sequence $\{(1/d^n)f(d^n x)\}$ converges. Thus one can define the mapping $H : A \to B$ by

$$H(x) := \lim_{n\to\infty} \frac{1}{d^n} f(d^n x) \tag{2.16}$$

for all $x \in A$. Moreover, letting $m = 0$ and passing the limit $n \to \infty$ in (2.15), we get (2.12). It follows from (2.8) that

$$\begin{aligned}
&\left\| 2H\left(\frac{\sum_{j=1}^{p} \mu x_j}{2} + \sum_{j=1}^{d} \mu y_j \right) - \sum_{j=1}^{p} \mu H(x_j) - 2\sum_{j=1}^{d} \mu H(y_j) \right\|_B \\
&\quad = \lim_{n\to\infty} \frac{1}{d^n} \left\| 2f\left(d^n \frac{\sum_{j=1}^{p} \mu x_j}{2} + d^n \sum_{j=1}^{d} \mu y_j \right) - \sum_{j=1}^{p} \mu f(d^n x_j) - 2\sum_{j=1}^{d} \mu f(d^n y_j) \right\|_B \\
&\quad \leq \lim_{n\to\infty} \frac{d^{nr}}{d^n} \theta \left(\sum_{j=1}^{p} \|x_j\|_A^r + \sum_{j=1}^{d} \|y_j\|_A^r \right) = 0
\end{aligned} \tag{2.17}$$

for all $\mu \in \mathbb{T}^1$ and all $x_1, \ldots, x_p, y_1, \ldots, y_d \in A$. Hence

$$2H\left(\frac{\sum_{j=1}^{p} \mu x_j}{2} + \sum_{j=1}^{d} \mu y_j \right) = \sum_{j=1}^{p} \mu H(x_j) + 2\sum_{j=1}^{d} \mu H(y_j) \tag{2.18}$$

for all $\mu \in \mathbb{T}^1$ and all $x_1, \ldots, x_p, y_1, \ldots y_d \in A$. So $H(\lambda x + \mu y) = \lambda H(x) + \mu H(y)$ for all $\lambda, \mu \in \mathbb{T}^1$ and all $x, y \in A$. Therefore by Lemma 2.1 the mapping $H : A \to B$ is $\mathbb{C}$-linear.

It follows from Lemma 2.2 and (2.11) that

$$\begin{aligned}
&\|H([x,y,z]) - [H(x), H(y), H(z)]\|_B \\
&\quad = \lim_{n\to\infty} \frac{1}{d^{3n}} \|f([d^n x, d^n y, d^n z]) - [f(d^n x), f(d^n y), f(d^n z)]\|_B \\
&\quad = \theta \lim_{n\to\infty} \frac{d^{ns}}{d^{3n}} \left(\|x\|_A^s + \|y\|_A^s + \|z\|_A^s\right) = 0
\end{aligned} \tag{2.19}$$

for all $x, y, z \in A$. Thus

$$H([x,y,z]) = [H(x), H(y), H(z)] \tag{2.20}$$

for all $x, y, z \in A$. Therefore the mapping H is a C^*-ternary algebra homomorphism.

Now let $T : A \to B$ be another C^*-ternary algebra homomorphism satisfying (2.12). Then we have

$$\|H(x) - T(x)\|_B = \lim_{n\to\infty} \frac{1}{d^n} \left\| f(d^n x) - T(d^n x) \right\|_B \le \frac{d\theta}{2|d - d^r|} \lim_{n\to\infty} \frac{d^{nr}}{d^n} \|x\|_A^r = 0 \tag{2.21}$$

for all $x \in A$. So we can conclude that $H(x) = T(x)$ for all $x \in A$. This proves the uniqueness of H. Thus the mapping $H : A \to B$ is a unique C^*-ternary algebra homomorphism satisfying (2.12), as desired.

Case 2. $r > 1$ and $s > 3$.

Similar to the proof of Case 1, we conclude that the sequence $\{d^n f(d^{-n}x)\}$ is a Cauchy sequence in B. So we can define the mapping $H : A \to B$ by

$$H(x) := \lim_{n\to\infty} d^n f\left(d^{-n}x\right) \tag{2.22}$$

for all $x \in A$. The rest of the proof is similar to the proof of Case 1.

□

Theorem 2.5 (see [31]). *Let r and θ be nonnegative real numbers such that $r \notin [1/(p+d), 1]$, and let $f : A \to B$ be a mapping such that*

$$\left\| C_\mu f(x_1, \dots, x_p, y_1, \dots, y_d) \right\|_B \le \theta \prod_{j=1}^{p} \|x_j\|_A^r \cdot \prod_{j=1}^{d} \|y_j\|_A^r, \tag{2.23}$$

$$\left\| f([x, y, z]) - [f(x), f(y), f(z)] \right\|_B \le \theta \|x\|_A^r \|y\|_A^r \|z\|_A^r \tag{2.24}$$

for all $\mu \in \mathbb{T}^1$ and all $x, y, z, x_1, \dots, x_p, y_1, \dots, y_d \in A$. Then there exists a unique C^*-ternary algebra homomorphism $H : A \to B$ such that

$$\left\| f(x) - H(x) \right\|_B \le \frac{2^{(p+d)r}\theta}{\left| 2(p+2d)^{(p+d)r} - 2^{(p+d)r}(p+2d) \right|} \|x\|_A^{(p+d)r} \tag{2.25}$$

for all $x \in A$.

The following theorem shows that the mapping $f : A \to B$ in Theorem 2.5 is a C^*-ternary algebra homomorphism when $r > 0$.

Theorem 2.6. *Let $r, s, q, r_1, \ldots, r_p, s_1, \ldots, s_d$, and θ be nonnegative real numbers such that $r+s+q \neq 3$ and $r_k > 0$ ($s_k > 0$) for some $1 \leq k \leq p$, $p \geq 2$ ($1 \leq k \leq d$, $d \geq 2$).*

Let $f : A \to B$ be a mapping satisfying

$$\left\| C_\mu f(x_1, \ldots, x_p, y_1, \ldots, y_d) \right\|_B \leq \theta \prod_{j=1}^{p} \|x_j\|_A^{r_j} \cdot \prod_{j=1}^{d} \|y_j\|_A^{s_j}, \tag{2.26}$$

$$\left\| f([x,y,z]) - [f(x), f(y), f(z)] \right\|_B \leq \theta \|x\|_A^r \|y\|_A^s \|z\|_A^q \tag{2.27}$$

for all $\mu \in \mathbb{T}^1$ and all $x, y, z, x_1, \ldots, x_p, y_1, \ldots, y_d \in A$. Then the mapping $f : A \to B$ is a C^*-ternary algebra homomorphism. (We put $\|\cdot\|_A^0 = 1$).

Proof. Let $r_k > 0$ for some $1 \leq k \leq p$ (we have similar proof when $s_k > 0$ for some $1 \leq k \leq d$). We now assume, without loss of generality, that $r_1 > 0$. Letting $x_1 = \cdots = x_p = y_1 = \cdots = y_d = 0$ in (2.26), we get that $f(0) = 0$. Letting $x_2 = 2x$ and $x_1 = x_3 = \cdots = x_p = y_1 = \cdots = y_d = 0$ in (2.26), we get

$$\mu f(2x) = 2f(\mu x) \tag{2.28}$$

for all $\mu \in \mathbb{T}^1$ and all $x \in A$. Setting $\mu = 1$ in (2.28), we get that $f(2x) = 2f(x)$ for all $x \in A$. Therefore,

$$f(\mu x) = \mu f(x), \qquad f(2\mu x) = 2\mu f(x) \tag{2.29}$$

for all $\mu \in \mathbb{T}^1$ and all $x \in A$. If we put $x_2 = 2x$ and $y_1 = y$ and $x_1 = x_3 = \cdots = x_p = y_2 = \cdots = y_d = 0$ in (2.26), we get

$$2f(\mu x + \mu y) = \mu f(2x) + 2\mu f(y) \tag{2.30}$$

for all $\mu \in \mathbb{T}^1$ and all $x \in A$. It follows from (2.29) and (2.30) that

$$f(\mu x + \lambda y) = \mu f(x) + \lambda f(y) \tag{2.31}$$

for all $\lambda, \mu \in \mathbb{T}^1$ and all $x, y \in A$. Therefore, by Lemma 2.1 the mapping $f : A \to B$ is $\mathbb{C}$-linear. Let $r+s+q > 3$. Then it follows from (2.27) that

$$\begin{aligned} &\left\| f([x,y,z]) - [f(x), f(y), f(z)] \right\|_B \\ &\quad = \lim_{n\to\infty} 8^n \left\| f\left(\left[\frac{x}{2^n}, \frac{y}{2^n}, \frac{z}{2^n}\right]\right) - \left[f\left(\frac{x}{2^n}\right), f\left(\frac{y}{2^n}\right), f\left(\frac{z}{2^n}\right)\right] \right\|_B \\ &\quad \leq \theta \|x\|_A^r \|y\|_A^s \|z\|_A^q \lim_{n\to\infty} \left(\frac{8}{2^{r+s+q}}\right)^n = 0 \end{aligned} \tag{2.32}$$

for all $x, y, z \in A$. Therefore,

$$f([x, y, z]) = [f(x), f(y), f(z)] \tag{2.33}$$

for all $x, y, z \in A$. Similarly, for $r + s + q < 3$, we get (2.33). □

In the rest of this section, assume that A is a unital C^*-ternary algebra with norm $\|\cdot\|_A$ and unit e, and that B is a unital C^*-ternary algebra with norm $\|\cdot\|_B$ and unit e'.

We investigate homomorphisms in C^*-ternary algebras associated with the functional equation $C_\mu f(x_1, \ldots, x_p, y_1, \ldots, y_d) = 0$.

Theorem 2.7 (see [31]). *Let $r > 1$ ($r < 1$) and θ be nonnegative real numbers, and let $f : A \to B$ be a bijective mapping satisfying* (2.8) *such that*

$$f([x, y, z]) = [f(x), f(y), f(z)] \tag{2.34}$$

for all $x, y, z \in A$. If $\lim_{n\to\infty}((p + 2d)^n/2^n) f(2^n e/(p + 2d)^n) = e'$ ($\lim_{n\to\infty}(2^n/(p + 2d)^n) f((p + 2d)^n/2^n) e = e'$), then the mapping $f : A \to B$ is a C^*-ternary algebra isomorphism.

In the following theorems we have alternative results of Theorem 2.7.

Theorem 2.8. *Let $r < 1$, $s < 2$ and θ be nonnegative real numbers, and let $f : A \to B$ be a mapping satisfying* (2.8) *and* (2.11). *If there exist a real number $\lambda > 1$ ($0 < \lambda < 1$) and an element $x_0 \in A$ such that $\lim_{n\to\infty}(1/\lambda^n) f(\lambda^n x_0) = e'$ ($\lim_{n\to\infty}\lambda^n f(x_0/\lambda^n) = e'$), then the mapping $f : A \to B$ is a C^*-ternary algebra homomorphism.*

Proof. By using the proof of Theorem 2.4, there exists a unique C^*-ternary algebra homomorphism $H : A \to B$ satisfying (2.12). It follows from (2.12) that

$$H(x) = \lim_{n\to\infty} \frac{1}{\lambda^n} f(\lambda^n x), \quad \left(H(x) = \lim_{n\to\infty} \lambda^n f\left(\frac{x}{\lambda^n}\right)\right) \tag{2.35}$$

for all $x \in A$ and all real numbers $\lambda > 1$ ($0 < \lambda < 1$). Therefore, by the assumption we get that $H(x_0) = e'$. Let $\lambda > 1$ and $\lim_{n\to\infty}(1/\lambda^n) f(\lambda^n x_0) = e'$. It follows from (2.11) that

$$\begin{aligned}
&\left\|[H(x), H(y), H(z)] - [H(x), H(y), f(z)]\right\|_B \\
&\quad = \left\|H[x, y, z] - [H(x), H(y), f(z)]\right\|_B \\
&\quad = \lim_{n\to\infty} \frac{1}{\lambda^{2n}} \left\|f([\lambda^n x, \lambda^n y, z]) - [f(\lambda^n x), f(\lambda^n y), f(z)]\right\|_B \\
&\quad \le \theta \lim_{n\to\infty} \frac{1}{\lambda^{2n}} \left(\lambda^{ns}\|x\|_A^s + \lambda^{ns}\|y\|_A^s + \|z\|_A^s\right) = 0
\end{aligned} \tag{2.36}$$

for all $x \in A$. So $[H(x), H(y), H(z)] = [H(x), H(y), f(z)]$ for all $x, y, z \in A$. Letting $x = y = x_0$ in the last equality, we get $f(z) = H(z)$ for all $z \in A$. Similarly, one can shows that $H(x) = f(x)$ for all $x \in A$ when $0 < \lambda < 1$ and $\lim_{n\to\infty}\lambda^n f(x_0/\lambda^n) = e'$. Therefore, the mapping $f : A \to B$ is a C^*-ternary algebra homomorphism. □

3. Derivations on C^*-Ternary Algebras

Throughout this section, assume that A is a C^*-ternary algebra with norm $\|\cdot\|_A$.

Park [31] proved the Hyers-Ulam-Rassias stability and Ulam-Găvruţa-Rassias stability of derivations on C^*-ternary algebras for the following functional equation:

$$C_\mu f(x_1,\ldots,x_p,y_1,\ldots,y_d) = 0. \tag{3.1}$$

For a given mapping $f : A \to A$, let

$$\mathbf{D}f(x,y,z) = f([x,y,z]) - [f(x),y,z] - [x,f(y),z] - [x,y,f(z)] \tag{3.2}$$

for all $x,y,z \in A$.

Theorem 3.1 (see [31]). *Let r and θ be nonnegative real numbers such that $r \notin [1,3]$, and let $f : A \to A$ a mapping satisfying* (2.8) *and*

$$\|\mathbf{D}f(x,y,z)\|_A \le \theta\left(\|x\|_A^r + \|y\|_A^r + \|z\|_A^r\right) \tag{3.3}$$

for all $x,y,z \in A$. Then there exists a unique C^*-ternary derivation $\delta : A \to A$ such that

$$\|f(x) - \delta(x)\|_A \le \frac{2^r(p+d)}{\left|2(p+2d)^r - (p+2d)2^r\right|}\theta\|x\|_A^r \tag{3.4}$$

for all $x \in A$.

Theorem 3.2 (see [31]). *Let r and θ be nonnegative real numbers such that $r \notin [1/(p+d),1]$, and let $f : A \to A$ be a mapping satisfying* (2.23) *and*

$$\|\mathbf{D}f(x,y,z)\|_A \le \theta\|x\|_A^r\|y\|_A^r\|z\|_A^r \tag{3.5}$$

for all $x,y,z \in A$. Then there exists a unique C^*-ternary derivation $\delta : A \to A$ such that

$$\|f(x) - \delta(x)\|_A \le \frac{2^{(p+d)r}}{\left|2(p+2d)^{(p+d)r} - (p+2d)2^{(p+d)r}\right|}\theta\|x\|_A^{(p+d)r} \tag{3.6}$$

for all $x \in A$.

In the following theorems we generalize and improve the results in Theorems 3.1 and 3.2.

Theorem 3.3. *Let $\varphi : A^{p+d} \to [0,\infty)$ and $\psi : A^3 \to [0,\infty)$ be functions such that*

$$\widetilde{\varphi}(x) := \sum_{n=0}^{\infty} \gamma^{-n} \varphi(\gamma^n x, \ldots, \gamma^n x) < \infty, \tag{3.7}$$

$$\lim_{n\to\infty} \gamma^{-n} \varphi(\gamma^n x_1, \ldots, \gamma^n x_p, \gamma^n y_1, \ldots, \gamma^n y_d) = 0, \tag{3.8}$$

$$\lim_{n\to\infty} \gamma^{-3n} \psi(\gamma^n x, \gamma^n y, \gamma^n z) = 0, \qquad \lim_{n\to\infty} \gamma^{-2n} \psi(\gamma^n x, \gamma^n y, z) = 0 \tag{3.9}$$

for all $x, y, z, x_1, \ldots, x_p, y_1, \ldots, y_d \in A$ where $\gamma = (p+2d)/2$. Suppose that $f : A \to A$ is a mapping satisfying

$$\left\| C_\mu f(x_1, \ldots, x_p, y_1, \ldots, y_d) \right\|_A \le \varphi(x_1, \ldots, x_p, y_1, \ldots, y_d), \tag{3.10}$$

$$\left\| \mathbf{D} f(x, y, z) \right\|_A \le \psi(x, y, z) \tag{3.11}$$

for all $\mu \in \mathbb{T}^1$ and all $x, y, z, x_1, \ldots, x_p, y_1, \ldots, y_d \in A$. Then the mapping $f : A \to A$ is a C^*-ternary derivation.

Proof. Let us assume $\mu = 1$ and $x_1 = \cdots = x_p = y_1 = \cdots = y_d = x$ in (3.10). Then we get

$$\left\| 2f\left(\frac{p+2d}{2}x\right) - (p+2d)f(x) \right\|_A \le \varphi(x, \ldots, x) \tag{3.12}$$

for all $x \in A$. If we replace x in (3.12) by $\gamma^n x$ and divide both sides of (3.12) to γ^{n+1}, then we get

$$\left\| \frac{1}{\gamma^{n+1}} f(\gamma^{n+1}x) - \frac{1}{\gamma^n} f(\gamma^n x) \right\|_A \le \frac{1}{2\gamma^{n+1}} \varphi(\gamma^n x, \ldots, \gamma^n x) \tag{3.13}$$

for all $x \in A$ and all integers $n \ge 0$. Hence

$$\left\| \frac{1}{\gamma^{n+1}} f(\gamma^{n+1}x) - \frac{1}{\gamma^m} f(\gamma^m x) \right\|_A \le \frac{1}{2\gamma} \sum_{i=m}^{n} \frac{1}{\gamma^i} \varphi\left(\gamma^i x, \ldots, \gamma^i x\right) \tag{3.14}$$

for all $x \in A$ and all integers $n \ge m \ge 0$. From this it follows that the sequence $\{(1/\gamma^n) f(\gamma^n x)\}$ is Cauchy for all $x \in A$. Since A is complete, the sequence $\{(1/\gamma^n) f(\gamma^n x)\}$ converges. Thus we can define the mapping $\delta : A \to A$ by

$$\delta(x) := \lim_{n\to\infty} \frac{1}{\gamma^n} f(\gamma^n x) \tag{3.15}$$

for all $x \in A$. Moreover, letting $m = 0$ and passing the limit $n \to \infty$ in (3.14), we get

$$\left\| \delta(x) - f(x) \right\|_A \le \frac{1}{2\gamma} \widetilde{\varphi}(x) \tag{3.16}$$

for all $x \in A$. It follows from (3.8) and (3.10) that

$$\begin{aligned}
&\left\|C_\mu \delta(x_1, \ldots, x_p, y_1, \ldots, y_d)\right\|_A \\
&\quad = \lim_{n\to\infty} \frac{1}{\gamma^n} \left\|C_\mu f(\gamma^n x_1, \ldots, \gamma^n x_p, \gamma^n y_1, \ldots, \gamma^n y_d)\right\|_A \\
&\quad \le \lim_{n\to\infty} \frac{1}{\gamma^n} \varphi(\gamma^n x_1, \ldots, \gamma^n x_p, \gamma^n y_1, \ldots, \gamma^n y_d) = 0
\end{aligned} \tag{3.17}$$

for all $\mu \in \mathbb{T}^1$ and all $x, y, z, x_1, \ldots, x_p, y_1, \ldots, y_d \in A$. Hence

$$2\delta\left(\frac{\sum_{j=1}^{p} \mu x_j}{2} + \sum_{j=1}^{d} \mu y_j\right) = \sum_{j=1}^{p} \mu \delta(x_j) + 2\sum_{j=1}^{d} \mu \delta(y_j) \tag{3.18}$$

for all $\mu \in \mathbb{T}^1$ and all $x_1, \ldots, x_p, y_1, \ldots, y_d \in A$. So $\delta(\lambda x + \mu y) = \lambda \delta(x) + \mu \delta(y)$ for all $\lambda, \mu \in \mathbb{T}^1$ and all $x, y \in A$. Therefore, by Lemma 2.1 the mapping $\delta : A \to A$ is $\mathbb{C}$-linear.

It follows from (3.9) and (3.11) that

$$\left\|\mathbf{D}\delta(x, y, z)\right\|_A = \lim_{n\to\infty} \frac{1}{\gamma^{3n}} \left\|\mathbf{D} f(\gamma^n x, \gamma^n y, \gamma^n z)\right\|_A \le \lim_{n\to\infty} \frac{1}{\gamma^{3n}} \varphi(\gamma^n x, \gamma^n y, \gamma^n z) = 0 \tag{3.19}$$

for all $x, y, z \in A$. Hence

$$\delta([x, y, z]) = [\delta(x), y, z] + [x, \delta(y), z] + [x, y, \delta(z)] \tag{3.20}$$

for all $x, y, z \in A$. So the mapping $\delta : A \to A$ is a C^*-ternary derivation.

It follows from (3.9) and (3.11)

$$\begin{aligned}
&\left\|\delta[x, y, z] - [\delta(x), y, z] - [x, \delta(y), z] - [x, y, f(z)]\right\|_A \\
&\quad = \lim_{n\to\infty} \frac{1}{\gamma^{2n}} \| f[\gamma^n x, \gamma^n y, z] - [f(\gamma^n x), \gamma^n y, z] \\
&\qquad\qquad - [\gamma^n x, f(\gamma^n y), z] - [\gamma^n x, \gamma^n y, f(z)] \|_A \\
&\quad \le \lim_{n\to\infty} \frac{1}{\gamma^{2n}} \varphi(\gamma^n x, \gamma^n y, z) = 0
\end{aligned} \tag{3.21}$$

for all $x, y, z \in A$. Thus

$$\delta[x, y, z] = [\delta(x), y, z] + [x, \delta(y), z] + [x, y, f(z)] \tag{3.22}$$

for all $x, y, z \in A$. Hence we get from (3.20) and (3.22) that

$$[x, y, \delta(z)] = [x, y, f(z)] \tag{3.23}$$

for all $x, y, z \in A$. Letting $x = y = f(z) - \delta(z)$ in (3.23), we get

$$\|f(z) - \delta(z)\|_A^3 = \|[f(z) - \delta(z), f(z) - \delta(z), f(z) - \delta(z)]\|_A = 0 \tag{3.24}$$

for all $z \in A$. Hence $f(z) = \delta(z)$ for all $z \in A$. So the mapping $f : A \to A$ is a C^*-ternary derivation, as desired. □

Corollary 3.4. *Let $r < 1$, $s < 2$, and θ be nonnegative real numbers, and let $f : A \to A$ be a mapping satisfying* (2.8) *and*

$$\|\mathbf{D}f(x, y, z)\|_A \leq \theta\left(\|x\|_A^s + \|y\|_A^s + \|z\|_A^s\right) \tag{3.25}$$

for all $x, y, z \in A$. Then the mapping $f : A \to A$ is a C^*-ternary derivation.

Proof. Define

$$\varphi(x_1, \ldots, x_p, y_1, \ldots, y_d) = \theta\left(\sum_{j=1}^{p}\|x_j\|_A^r + \sum_{j=1}^{d}\|y_j\|_A^r\right),$$
$$\psi(x, y, z) = \theta\left(\|x\|_A^s + \|y\|_A^s + \|z\|_A^s\right) \tag{3.26}$$

for all $x, y, z, x_1, \ldots, x_p, y_1, \ldots, y_d \in A$, and apply Theorem 3.3. □

Corollary 3.5. *Let r, s, and θ be nonnegative real numbers such that $s, r(p + d) < 1$, and let $f : A \to A$ be a mapping satisfying* (2.23) *and*

$$\|\mathbf{D}f(x, y, z)\|_A \leq \theta\|x\|_A^s\|y\|_A^s\|z\|_A^s \tag{3.27}$$

for all $x, y, z \in A$. Then the mapping $f : A \to A$ is a C^*-ternary derivation.

Proof. Define

$$\varphi(x_1, \ldots, x_p, y_1, \ldots, y_d) = \theta\prod_{j=1}^{p}\|x_j\|_A^r\prod_{j=1}^{d}\|y_j\|_A^r,$$
$$\psi(x, y, z) = \theta\|x\|_A^s\|y\|_A^s\|z\|_A^s \tag{3.28}$$

for all $x, y, z, x_1, \ldots, x_p, y_1, \ldots, y_d \in A$, and apply Theorem 3.3. □

Theorem 3.6. *Let $\varphi : A^{p+d} \to [0,\infty)$ and $\psi : A^3 \to [0,\infty)$ be functions such that*

$$\widetilde{\varphi}(x) := \sum_{n=1}^{\infty} \gamma^n \varphi\left(\frac{x}{\gamma^n}, \dots, \frac{x}{\gamma^n}\right) < \infty,$$
$$\lim_{n\to\infty} \gamma^n \varphi\left(\frac{x_1}{\gamma^n}, \dots, \frac{x_p}{\gamma^n}, \frac{y_1}{\gamma^n}, \dots, \frac{y_d}{\gamma^n}\right) = 0, \tag{3.29}$$
$$\lim_{n\to\infty} \gamma^{3n} \psi\left(\frac{x}{\gamma^n}, \frac{y}{\gamma^n}, \frac{z}{\gamma^n}\right) = 0, \qquad \lim_{n\to\infty} \gamma^{2n} \psi\left(\frac{x}{\gamma^n}, \frac{y}{\gamma^n}, z\right) = 0$$

for all $x, y, z, x_1, \dots, x_p, y_1, \dots, y_d \in A$ where $\gamma = (p + 2d)/2$. Suppose that $f : A \to A$ is a mapping satisfying (3.10) *and* (3.11). *Then the mapping $f : A \to A$ is a C^*-ternary derivation.*

Proof. If we replace x in (3.12) by x/γ^{n+1} and multiply both sides of (3.12) by γ^n, then we get

$$\left\| \gamma^{n+1} f\left(\frac{x}{\gamma^{n+1}}\right) - \gamma^n f\left(\frac{x}{\gamma^n}\right) \right\|_A \le \frac{\gamma^n}{2} \varphi\left(\frac{x}{\gamma^{n+1}}, \dots, \frac{x}{\gamma^{n+1}}\right) \tag{3.30}$$

for all $x \in A$ and all integers $n \ge 0$. Hence

$$\left\| \gamma^{n+1} f\left(\frac{x}{\gamma^{n+1}}\right) - \gamma^m f\left(\frac{x}{\gamma^m}\right) \right\|_A \le \frac{1}{2\gamma} \sum_{i=m+1}^{n+1} \gamma^i \varphi\left(\frac{x}{\gamma^i}, \dots, \frac{x}{\gamma^i}\right) \tag{3.31}$$

for all $x \in A$ and all integers $n \ge m \ge 0$. From this it follows that the sequence $\{\gamma^n f(x/\gamma^n)\}$ is Cauchy for all $x \in A$. Since A is complete, the sequence $\{\gamma^n f(x/\gamma^n)\}$ converges. Thus we can define the mapping $\delta : A \to A$ by

$$\delta(x) := \lim_{n\to\infty} \gamma^n f\left(\frac{x}{\gamma^n}\right) \tag{3.32}$$

for all $x \in A$. The rest of the proof is similar to the proof of Theorem 3.3, and we omit it. □

Corollary 3.7. *Let r, s, and θ be nonnegative real numbers such that $s, r(p + d) > 1$, and let $f : A \to A$ be a mapping satisfying* (2.23) *and* (3.27). *Then the mapping $f : A \to A$ is a C^*-ternary derivation.*

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