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Research Article

Permanence of Periodic Predator-Prey System with Functional Responses and Stage Structure for Prey

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A stage-structured three-species predator-prey model with Beddington-DeAngelis and Holling II functional response is introduced. Based on the comparison theorem, sufficient and necessary conditions which guarantee the predator and the prey species to be permanent are obtained. An example is also presented to illustrate our main results.

1. Introduction

The aim of this paper is to investigate the permanence of the following periodic stage-structure predator-prey system with Beddington-DeAngelis and Holling II functional response:

$$\begin{aligned}
x_1'(t) &= a(t)x_2(t) - b(t)x_1(t) - d(t)x_1^2(t) - \frac{p_1(t)x_1(t)}{k_1(t)+m(t)y_1(t)+n(t)x_1(t)}y_1(t),\\
x_2'(t) &= c(t)x_1(t) - f(t)x_2^2(t) - \frac{p_2(t)x_2(t)}{k_2(t)+x_2(t)}y_2(t),\\
y_1'(t) &= y_1(t)\left[-g_1(t) + \frac{h_1(t)x_1(t)}{k_1(t)+m(t)y_1(t)+n(t)x_1(t)} - q_1(t)y_1(t)\right],\\
y_2'(t) &= y_2(t)\left[-g_2(t) + \frac{h_2(t)x_2(t)}{k_2(t)+x_2(t)} - q_2(t)y_2(t)\right],
\end{aligned} \tag{1.1}$$

where $a(t)$, $b(t)$, $c(t)$, $d(t)$, $f(t)$, $g_i(t)$, $h_i(t)$, $k_i(t)$, $m(t)$, $n(t)$, $p_i(t)$, and $q_i(t)$ $(i = 1,2)$ are all continuous positive ω-periodic functions. Here, $x_1(t)$ and $x_2(t)$ denote the density of immature and mature prey species at time t, respectively, $y_1(t)$ represents the density of the predator that preys on immature prey, and $y_2(t)$ represents the density of the other predator that preys on mature prey at time t.

The birth rate into the immature population is given by $a(t)x_2(t)$, that is, it is assumed to be proportional to the existing mature population, with a proportionality coefficient $a(t)$. The death rate of the immature population is proportional to the existing immature population and to its square with coefficients $b(t)$ and $d(t)$, respectively. The death rate of the mature population is of a logistic nature, that is, it is proportional to the square of the population with a proportionality $f(t)$. The transition rate from the immature individuals to the mature individuals is assumed to be proportional to the existing immature population, with a proportionality coefficient $c(t)$. Similarly, $-g_1(t)y_1(t) - q_1(t)y_1^2(t)$ and $-g_2(t)y_2(t) - q_2(t)y_2^2(t)$ give the density dependent death rate of the two predators, respectively. $p_1(t)$ and $p_2(t)$ are the capturing rate of the two predators, respectively. $h_1(t)/p_1(t)$ and $h_2(t)/p_2(t)$ are the rate of conversion of nutrients into the reproduction of the two mature predators, respectively.

The functional response of predator species $y_1(t)$ to immature prey species takes the Beddington-DeAngelis form, that is, $x_1(t)/(k_1(t) + m(t)y_1(t) + n(t)x_1(t))$. It was introduced by Beddington [1] and DeAngelis et al. [2] independently in 1975. It is similar to the well-known Holling type II functional response but has an extra term $m(t)y_1(t)$ in the denominator which models mutual interference between predators. The Beddington-DeAngelis form of functional response has some of the same qualitative features as the ratio-dependent models form but avoids some of the same behaviors of ratio-dependent models at low densities which have been the source of controversy. The function $x_2(t)/(k_2(t) + x_2(t))$ represents the functional response of predator $y_2(t)$ to mature prey, which is called Holling type II function or Michaelis-Menten function. Holling type II is the second function that Holling proposed three kinds of functional response of the predator to prey based on numerous experiments for different species. The Holling type form of functional response is intituled prey-dependent model form. It is applied to almost invertebrate that is one of the most extensive applied functional responses.

Cui and Song [3] proposed the following predator-prey model with stage structure for prey:

$$
\begin{aligned}
x_1'(t) &= a(t)x_2(t) - b(t)x_1(t) - d(t)x_1^2(t) - p(t)x_1(t)y(t), \\
x_2'(t) &= c(t)x_1(t) - f(t)x_2^2(t), \\
y'(t) &= y(t)\big(- g(t) + h(t)x_1(t) - q(t)y(t)\big).
\end{aligned}
\tag{1.2}
$$

They obtained a set of sufficient and necessary conditions which guarantee the permanence of the system. For more back ground and the relevant work on system (1.2), one could refer to [3–6] and the references cited therein. Recently, Chen [7, 8] and Yang [9] consider the functional response of the predator to immature prey species. Lin and Hong [10] consider a biological model for two predators and one prey with periodic delays.

In reality, mature prey was also consumed by some predators. Different predator usually consumes prey in different stage structure. Some predators only prey on immature prey, and some predators only prey on mature prey. There is different functional response in different predator. So, we add a predator species which consumes mature prey to the model (1.2). By assuming that one predator consumes immature prey according to the Beddington-DeAngelis functional response while the other predator consumes mature prey according to Holling II functional response, we get model (1.1). In the resource limited environment, could the wild animals be coexistence for long-term under the animals' law of the jungle? To keep the biology's variety of the nature, the permanence of biotic population is a significant and comprehensive problem in biomathematics. So, it is meaningful to investigate the permanence of the model (1.1).

The aim of this paper is, by further developing the analysis technique of Cui [3, 11], to derive a set of sufficient and necessary conditions which ensure the permanence of the system (1.1). The rest of the paper is arranged as follows. In Section 2, we introduce some lemmas and then state the main result of this paper. The result is proved in Section 3. In Section 4, we give an example which shows the feasibility of our result. The last section is devoted to make some explanation on the biological meaning of our result.

Throughout this paper, for a continuous ω-periodic function $f(t)$, we set

$$A_\omega(f) = \frac{1}{\omega}\int_0^\omega f(t)dt. \tag{1.3}$$

2. Main result

In this section, we introduce a definition and some lemmas which will be useful in subsequent sections and state the main result.

Definition 2.1. System (1.1) is said to be permanent if there exist positive constants m, M, and T_0, such that each positive solution $(x_1(t), x_2(t), y_1(t), y_2(t))$ of the system (1.1) with any positive initial value φ fulfills $m \le x_i(t) \le M$, $m \le y_i(t) \le M$, $i = 1, 2$ for all $t \ge T_0$, where T_0 may depend on φ.

Lemma 2.2 (see [12]). *If $a(t)$, $b(t)$, $c(t)$, $d(t)$, and $f(t)$ are all ω-periodic, then system*

$$\begin{aligned} x_1'(t) &= a(t)x_2(t) - b(t)x_1(t) - d(t)x_1^2(t), \\ x_2'(t) &= c(t)x_1(t) - f(t)x_2^2(t) \end{aligned} \tag{2.1}$$

has a positive ω-periodic solution $(x_1^*(t), x_2^*(t))$ which is globally asymptotically stable with respect to $R_+^2 = \{(x_1, x_2) : x_1 > 0,\ x_2 > 0\}$.

Lemma 2.3 (see [13]). *If $b(t)$ and $a(t)$ are all ω-periodic, and if $A_\omega(b) > 0$ and $A_\omega(a) > 0$ for all $t \in R$, then the system*

$$x' = x\big(b(t) - a(t)x\big) \tag{2.2}$$

has a positive ω-periodic solution which is globally asymptotically stable.

Now, we state the main result of this paper.

Theorem 2.4. *System* (1.1) *is permanent if and only if*

$$A_\omega\left(-g_1(t) + \frac{h_1(t)x_1^*(t)}{k_1(t) + n(t)x_1^*(t)}\right) > 0, \qquad A_\omega\left(-g_2(t) + \frac{h_2(t)x_2^*(t)}{k_2(t) + x_2^*(t)}\right) > 0, \tag{2.3}$$

where $(x_1^*(t), x_2^*(t))$ is the unique positive periodic solution of system (2.1) *given by Lemma 2.2.*

3. Proof of the main result

We need the following propositions to prove Theorem 2.4. The hypothesis of the lemmas and theorem of the preceding section is assumed to hold in what follows.

Proposition 3.1. *There exist positive constants M_x and M_y such that*

$$\lim_{t\to+\infty}\sup x_i(t) \le M_x, \quad \lim_{t\to+\infty}\sup y_i(t) \le M_y, \quad i = 1,2 \tag{3.1}$$

for all solution of system (1.1) *with positive initial values.*

Proof. Obviously, $R_+^4 = \{(x_1, x_2, y_1, y_2) \mid x_i > 0,\ y_i > 0\}$ is a positively invariant set of system (1.1). Given any solution (x_1, x_2, y_1, y_2) of system (1.1), we have

$$\begin{aligned} x_1'(t) &\le a(t)x_2(t) - b(t)x_1(t) - d(t)x_1^2(t), \\ x_2'(t) &\le c(t)x_1(t) - f(t)x_2^2(t). \end{aligned} \tag{3.2}$$

By Lemma 2.2, the following auxiliary equation:

$$\begin{aligned} u_1'(t) &= a(t)u_2(t) - b(t)u_1(t) - d(t)u_1^2(t), \\ u_2'(t) &= c(t)u_1(t) - f(t)u_2^2(t) \end{aligned} \tag{3.3}$$

has a globally asymptotically stable positive ω-periodic solution $(x_1^*(t), x_2^*(t))$. Let $(u_1(t), u_2(t))$ be the solution of (3.3) with $(u_1(0), u_2(0)) = (x_1(0), x_2(0))$. By comparison theorem, we then have

$$x_i(t) \le u_i(t), \quad i = 1,2, \tag{3.4}$$

for $t \geqslant 0$. By (2.3), we can choose positive $\varepsilon > 0$ small enough such that

$$A_\omega\left(- g_i(t) + \frac{h_i(t)\left(x_i^*(t) + \varepsilon\right)}{k_i(t)} \right) > 0. \tag{3.5}$$

Thus, from the global attractivity of $(x_1^*(t), x_2^*(t))$, for the above given $\varepsilon > 0$, there exists a $T_0 > 0$, such that

$$\left| u_i(t) - x_i^*(t) \right| < \varepsilon, \quad t \ge T_0. \tag{3.6}$$

Inequality (3.4) combined with (3.6) leads to

$$x_i(t) < x_i^*(t) + \varepsilon, \quad t > T_0. \tag{3.7}$$

Let $M_x = \max_{0\le t\le\omega}\{x_i^*(t) + \varepsilon,\ i = 1,2\}$, we have

$$\lim_{t\to+\infty}\sup x_i(t) \le M_x. \tag{3.8}$$

In addition, for $t \ge T_0$, from the third and fourth equations of (1.1) and (3.7) we get

$$\begin{aligned} y_i'(t) &\le y_i(t)\left[-g_i(t) + \frac{h_i(t)x_i(t)}{k_i(t)} - q_i(t)y_i(t)\right] \\ &\le y_i(t)\left[-g_i(t) + \frac{h_i(t)\left(x_i^*(t)+\varepsilon\right)}{k_i(t)} - q_i(t)y_i(t)\right]. \end{aligned} \tag{3.9}$$

Consider the following auxiliary equation:

$$v_i'(t) = v_i(t)\left[-g_i(t) + \frac{h_i(t)\left(x_i^*(t)+\varepsilon\right)}{k_i(t)} - q_i(t)v_i(t)\right]. \tag{3.10}$$

It follows from (3.5) and Lemma 2.3 that (3.10) has a unique positive ω-periodic solution $y_i^*(t) > 0$ which is globally asymptotically stable. Similar to the above analysis, there exists a $T_1 > T_0$ such that for the above ε, one has

$$y_i(t) < y_i^*(t) + \varepsilon, \quad t \ge T_1. \tag{3.11}$$

Let $M_y = \max_{0\le t\le\omega}\{y_i^*(t) + \varepsilon,\ i = 1,2\}$, then we have

$$\lim_{t\to+\infty}\sup y_i(t) \le M_y, \quad i = 1,2. \tag{3.12}$$

This completes the proof of Proposition 3.1. □

Proposition 3.2. *There exist positive constants* $\delta_{ix} < M_x$, $i = 1,2$, *such that*

$$\lim_{t\to+\infty}\inf x_i(t) \ge \delta_{ix}, \quad i = 1,2. \tag{3.13}$$

Proof. By Proposition 3.1, there exists $T_1 > 0$ such that

$$0 < x_i(t) \le M_x, \quad 0 < y_i(t) \le M_y, \quad t \ge T_1. \tag{3.14}$$

Hence, from the first and second equations of system (1.1), we have

$$\begin{aligned} x_1'(t) &\ge a(t)x_2(t) - \left(b(t) + \frac{p_1(t)}{k_1(t)}M_y\right)x_1(t) - d(t)x_1^2(t), \\ x_2'(t) &\ge c(t)x_1(t) - \left(f(t) + \frac{p_2(t)}{k_2(t)}M_y\right)x_2^2(t), \end{aligned} \tag{3.15}$$

for $t \geq T_1$. By Lemma 2.2, the following auxiliary equation:

$$\begin{aligned} u_1'(t) &= a(t)u_2(t) - \left(b(t) + \frac{p_1(t)}{k_1(t)}M_y\right)u_1(t) - d(t)u_1^2(t), \\ u_2'(t) &= c(t)u_1(t) - \left(f(t) + \frac{p_2(t)}{k_2(t)}M_y\right)u_2^2(t) \end{aligned} \tag{3.16}$$

has a globally asymptotically stable positive ω-periodic solution $(\widetilde{x}_1^*(t), \widetilde{x}_2^*(t))$. Let $(u_1(t), u_2(t))$ be the solution of (3.16) with $(u_1(T_1), u_2(T_2)) = (x_1(T_1), x_2(T_2))$, by comparison theorem, we have

$$x_i(t) \geq u_i(t) \quad (i = 1, 2),\ t > T_{1.} \tag{3.17}$$

Thus, from the global attractivity of $(\widetilde{x}_1^*(t), \widetilde{x}_2^*(t))$, there exists a $T_2 > T_1$, such that

$$\left|u_i(t) - \widetilde{x}_i^*(t)\right| < \frac{\widetilde{x}_i^*(t)}{2} \quad (i = 1, 2),\ t > T_2. \tag{3.18}$$

Inequality (3.18) combined with (3.17) leads to

$$x_i(t) > \delta_{ix} = \min_{0 \leq t \leq \omega}\left\{\frac{\widetilde{x}_i^*(t)}{2}\right\} \quad (i = 1, 2),\ t > T_2. \tag{3.19}$$

And so

$$\lim_{t \to +\infty} \inf x_i(t) \geq \delta_{ix}, \quad i = 1, 2. \tag{3.20}$$

The proof of Proposition 3.2 is complete. □

Proposition 3.3. *Suppose that* (2.3) *holds, then there exist positive constants* δ_{iy}, $i = 1, 2$, *such that any solution* $(x_1(t), x_2(t), y_1(t), y_2(t))$ *of system* (1.1) *with positive initial value satisfies*

$$\lim_{t \to +\infty} \sup y_i(t) \geq \delta_{iy}, \quad i = 1, 2. \tag{3.21}$$

Proof. By Assumption (2.3), we can choose constant $\varepsilon_0 > 0$ (without loss of generality, we may assume that $\varepsilon_0 < (1/2)\min_{0 \leq t \leq \omega}\{x_i^*(t)\}$, where $(x_1^*(t), x_2^*(t))$ is the unique positive periodic solution of system (2.1)) such that

$$A_\omega(\varphi_{\varepsilon_0}(t)) > 0, \qquad A_\omega(\psi_{\varepsilon_0}(t)) > 0, \tag{3.22}$$

where

$$\begin{aligned} \varphi_{\varepsilon_0}(t) &= -g_1(t) + \frac{h_1(t)\left(x_1^*(t) - \varepsilon_0\right)}{k_1(t) + m(t)\varepsilon_0 + n(t)\left(x_1^*(t) - \varepsilon_0\right)} - q_1(t)\varepsilon_0, \\ \psi_{\varepsilon_0}(t) &= -g_2(t) + \frac{h_2(t)\left(x_2^*(t) - \varepsilon_0\right)}{k_2(t) + \left(x_2^*(t) - \varepsilon_0\right)} - q_2(t)\varepsilon_0. \end{aligned} \tag{3.23}$$

Consider the following equations with a parameter $\beta > 0$:

$$\begin{aligned} x_1'(t) &= a(t)x_2(t) - \left(b(t) + 2\beta\frac{p_1(t)}{k_1(t)}\right)x_1(t) - d(t)x_1^2(t), \\ x_2'(t) &= c(t)x_1(t) - \left(f(t) + 2\beta\frac{p_2(t)}{k_2(t)}\right)x_2^2(t). \end{aligned} \tag{3.24}$$

By Lemma 2.2, the system (3.24) has a unique positive ω-periodic solution $(x_{1\beta}^*(t), x_{2\beta}^*(t))$, which is globally attractive. Let $(\overline{x}_{1\beta}(t), \overline{x}_{2\beta}(t))$ be the solution of (3.24) with initial condition $\overline{x}_{i\beta}(0) = x_i^*(0)$, $i = 1, 2$. Hence, for above ε_0, there exists a sufficiently large $T_3 > T_2$ such that

$$\left|\overline{x}_{i\beta}(t) - x_{i\beta}^*(t)\right| < \frac{\varepsilon_0}{4} \quad (i = 1, 2),\ t > T_3. \tag{3.25}$$

By the continuity of the solution in the parameter, we have $\overline{x}_{i\beta}(t) \to x_i^*(t)$ uniformly in $[T_3, T_3 + \omega]$ as $\beta \to 0$. Hence, for $\varepsilon_0 > 0$, there exists a $\beta_0 = \beta_0(\varepsilon_0) > 0$ such that

$$\left|\overline{x}_{i\beta}(t) - x_i^*(t)\right| < \frac{\varepsilon_0}{4} \quad (i = 1, 2),\ t \in [T_3, T_3 + \omega],\ 0 < \beta < \beta_0. \tag{3.26}$$

So, we have

$$\left|x_{i\beta}^*(t) - x_i^*(t)\right| \le \left|\overline{x}_{i\beta}(t) - x_{i\beta}^*(t)\right| + \left|\overline{x}_{i\beta}(t) - x_i^*(t)\right| < \frac{\varepsilon_0}{2}, \quad t \in [T_3, T_3 + \omega]. \tag{3.27}$$

Since $x_{i\beta}^*(t)$ and $x_i^*(t)$ are all ω-periodic, we have

$$\left|x_{i\beta}^*(t) - x_i^*(t)\right| < \frac{\varepsilon_0}{2} \quad (i = 1, 2),\ t \ge 0,\ 0 < \beta < \beta_0. \tag{3.28}$$

Choosing a constant β_1 $(0 < \beta_1 < \beta_0,\ 2\beta_1 < \varepsilon_0)$, we have

$$x_{i\beta_1}^*(t) \ge x_i^*(t) - \frac{\varepsilon_0}{2} \quad (i = 1, 2),\ t \ge 0. \tag{3.29}$$

Suppose that Conclusion (3.21) is not true. Then, there exists $F \in R_+^4$ such that, for the positive solution $(x_1(t), x_2(t), y_1(t), y_2(t))$ of (1.1) with an initial condition $(x_1(0), x_2(0), y_1(0), y_2(0)) = F$, we have

$$\lim_{t\to+\infty}\sup y_i(t) < \beta_1, \quad i = 1, 2. \tag{3.30}$$

So, there exists $T_4 > T_3$ such that

$$y_i(t) < 2\beta_1 < \varepsilon_0, \quad t \geq T_4. \tag{3.31}$$

By applying (3.31), from the first and second equations of system (1.1) it follows that for all $t \geq T_4$,

$$\begin{aligned} x_1'(t) &\geq a(t)x_2(t) - \left(b(t) + 2\beta_1 \frac{p_1(t)}{k_1(t)}\right)x_1(t) - d(t)x_1^2(t), \\ x_2'(t) &\geq c(t)x_1(t) - \left(f(t) + 2\beta_1 \frac{p_2(t)}{k_2(t)}\right)x_2^2(t). \end{aligned} \tag{3.32}$$

Let $(u_1(t), u_2(t))$ be the solution of (3.24) with $\beta = \beta_1$ and $u_i(T_4) = x_i(T_4)$, $i = 1, 2$, then

$$x_i(t) \geq u_i(t) \quad (i = 1, 2),\ t \geq T_4. \tag{3.33}$$

By the global asymptotic stability of $(x_{1\beta_1}^*(t), x_{2\beta_2}^*(t))$, for the given $\varepsilon = \varepsilon_0/2$, there exists $T_5 \geq T_4$, such that

$$\left|u_i(t) - x_{i\beta_1}^*(t)\right| < \frac{\varepsilon_0}{2} \quad (i = 1, 2),\ t \geq T_5. \tag{3.34}$$

So,

$$x_i(t) \geq u_i(t) > x_{i\beta_1}^*(t) - \frac{\varepsilon_0}{2} \quad (i = 1, 2),\ t \geq T_5, \tag{3.35}$$

and hence, by using (3.29), we get

$$x_i(t) > x_i^*(t) - \varepsilon_0 \quad (i = 1, 2),\ t \geq T_5. \tag{3.36}$$

Therefore, by (3.31) and (3.36), we have

$$\begin{aligned} y_1'(t) &\geq y_1(t)\left(-g_1(t) + \frac{h_1(t)\left(x_1^*(t) - \varepsilon_0\right)}{k_1(t) + m(t)\varepsilon_0 + n(t)\left(x_1^*(t) - \varepsilon_0\right)} - q_1(t)\varepsilon_0\right) = \varphi_{\varepsilon_0}(t)y_1(t), \\ y_2'(t) &\geq y_2(t)\left(-g_2(t) + \frac{h_2(t)\left(x_2^*(t) - \varepsilon_0\right)}{k_2(t) + \left(x_2^*(t) - \varepsilon_0\right)} - q_2(t)\varepsilon_0\right) = \varphi_{\varepsilon_0}(t)y_2(t), \end{aligned} \tag{3.37}$$

for $t \geq T_5$. Integrating (3.37) from T_5 to t yields

$$
\begin{aligned}
y_1(t) &\geq y_1(T_5)\exp\left\{\int_{T_5}^{t}\varphi_{\varepsilon_0}(t)dt\right\},\\
y_2(t) &\geq y_2(T_5)\exp\left\{\int_{T_5}^{t}\varphi_{\varepsilon_0}(t)dt\right\}.
\end{aligned}
\tag{3.38}
$$

Thus, from (3.22) we know that $\varphi_{\varepsilon_0}(t) > 0$, $\varphi_{\varepsilon_0}(t) > 0$. It follows that $y_1(t) \to +\infty$, $y_2(t) \to +\infty$ as $t \to +\infty$. It is a contradiction. This completes the proof. □

Proposition 3.4. *Suppose that* (2.3) *holds, then there exist positive constants* η_{iy}, $i = 1, 2$, *such that any solution* $(x_1(t), x_2(t), y_1(t), y_2(t))$ *of system* (1.1) *with positive initial value satisfies*

$$
\lim_{t\to+\infty}\inf y_i(t) > \eta_{iy}, \quad i = 1, 2. \tag{3.39}
$$

Proof. Suppose that (3.39) is not true, then there exists a sequence $\{\xi_m\} \in R_+^4$, such that

$$
\lim_{t\to+\infty}\inf y_i(t,\xi_m) < \frac{\delta_{iy}}{(m+1)^2}, \quad m = 1, 2, \ldots. \tag{3.40}
$$

On the other hand, by Proposition 3.3, we have

$$
\lim_{t\to+\infty}\sup y_i(t,\xi_m) > \delta_{iy}, \quad m = 1, 2, \ldots. \tag{3.41}
$$

Hence, there are time sequences $\{s_q^{(m)}\}$ and $\{t_q^{(m)}\}$ satisfying

$$
\begin{gathered}
0 < s_1^{(m)} < t_1^{(m)} < s_2^{(m)} < t_2^{(m)} < \cdots < s_q^{(m)} < t_q^{(m)} < \cdots,\\
s_q^{(m)} \longrightarrow +\infty, \quad t_q^{(m)} \longrightarrow +\infty \quad \text{as } q \longrightarrow +\infty,\\
y_i(s_q^{(m)},\xi_m) = \frac{\delta_{iy}}{m+1}, \qquad y_i(t_q^{(m)},\xi_m) = \frac{\delta_{iy}}{(m+1)^2},\\
\frac{\delta_{iy}}{(m+1)^2} < y_i(t,\xi_m) < \frac{\delta_{iy}}{m+1}, \quad t \in (s_q^{(m)}, t_q^{(m)}).
\end{gathered}
\tag{3.42}
$$

By Proposition 3.1, for a given positive integer m, there is a $T_1^{(m)} > 0$, such that for all $t > T_1^{(m)}$

$$
x_i(t,\xi_m) < M_x, \quad y_i(t,\xi_m) < M_y, \quad i = 1, 2. \tag{3.43}
$$

Because of $s_q^{(m)} \to +\infty$ as $q \to +\infty$, there is a positive integer $Z^{(m)}$, such that $s_q^{(m)} > T_1^{(m)}$ as $q \geq Z^{(m)}$, hence

$$
y_i'(t,\xi_m) \geq y_i(t,\xi_m)\left(-g_i(t) - q_i(t)M_y\right) \tag{3.44}
$$

for $t \in [s_q^{(m)}, t_q^{(m)}]$, $q \geq Z^{(m)}$. Integrating (3.44) from $s_q^{(m)}$ to $t_q^{(m)}$ yields

$$y_i(t_q^{(m)}, \xi_m) \geq y_i(s_q^{(m)}, \xi_m) \exp\left\{\int_{s_q^{(m)}}^{t_q^{(m)}} (-g_i(t) - q_i(t)M_y)dt\right\}, \tag{3.45}$$

or

$$\int_{s_q^{(m)}}^{t_q^{(m)}} (g_i(t) + q_i(t)M_y)dt \geq \ln(m+1) \quad \text{for } q \geq Z^{(m)}. \tag{3.46}$$

Thus, from the boundedness of $g_i(t) + q_i(t)M_y$, we have

$$t_q^{(m)} - s_q^{(m)} \longrightarrow +\infty \quad \text{as } m \longrightarrow +\infty,\ q \geq Z^{(m)}. \tag{3.47}$$

By (3.22) and (3.47), there are constants $P > 0$ and $N_0 > 0$, such that

$$\frac{\delta_{iy}}{m+1} < \beta_1 < \varepsilon_0, \qquad t_q^{(m)} - s_q^{(m)} > 2P, \tag{3.48}$$

$$\int_0^a \varphi_{\varepsilon_0}(t)dt > 0, \qquad \int_0^a \psi_{\varepsilon_0}(t)dt > 0, \tag{3.49}$$

for $m \geq N_0$, $q \geq Z^{(m)}$, and $a \geq P$. Inequality (3.48) implies that

$$y_i(t, \xi_m) < \beta_1 < \varepsilon_0, \quad t \in [s_q^{(m)}, t_q^{(m)}], \tag{3.50}$$

for $m \geq N_0$, $q \geq Z^{(m)}$. In addition, from (3.43) and (3.50) we have

$$\begin{aligned} x_1'(t, \xi_m) &\geq a(t)x_2(t, \xi_m) - \left(b(t) + \frac{2p_1(t)\beta_1}{k_1(t)}\right)x_1(t, \xi_m) - d(t)x_1^2(t, \xi_m), \\ x_2'(t, \xi_m) &\geq c(t)x_1(t, \xi_m) - \left(f(t) + \frac{2p_2(t)\beta_1}{k_2(t)}\right)x_2^2(t, \xi_m), \end{aligned} \tag{3.51}$$

for $t \in [s_q^{(m)}, t_q^{(m)}]$. Let $(u_1(t), u_2(t))$ be the solution of (3.24) with $\beta = \beta_1$ and $u_i(s_q^{(m)}) = x_i(s_q^{(m)}, \xi_m)$, then by applying comparison theorem, we have

$$x_i(t, \xi_m) \geq u_i(t), \quad t \in [s_q^{(m)}, t_q^{(m)}]. \tag{3.52}$$

Further, by using Propositions 3.1 and 3.2, there exists an enough large $Z_1^{(m)} > Z^{(m)}$ such that

$$\eta_{ix} < x_i(s_q^{(m)}, \xi_m) < M_x, \tag{3.53}$$

for $q \geq Z_1^{(m)}$. For $\beta = \beta_1$, (3.24) has a unique positive ω-periodic solution $(x_{1\beta_1}^*(t), x_{2\beta_1}^*(t))$ which is globally asymptotically stable. In addition, by the periodicity of (3.24), the periodic solution $(x_{1\beta_1}^*(t), x_{2\beta_1}^*(t))$ is uniformly asymptotically stable with respect to the compact set $\Omega = \{x \mid \eta_{ix} < x < M_x\}$. Hence, for given ε_0 in Proposition 3.3, there exists $T_0 > P$, which is independent of m and q, such that

$$u_i(t) > x_{i\beta_1}^*(t) - \frac{\varepsilon_0}{2}, \quad i = 1, 2 \text{ as } t > T_0 + s_q^{(m)}. \tag{3.54}$$

Thus, by using (3.29), we get

$$u_i(t) > x_i^*(t) - \varepsilon_0, \quad i = 1, 2 \text{ as } t > T_0 + s_q^{(m)}. \tag{3.55}$$

By (3.47), there exists a positive integer $N_1 \geq N_0$ such that $t_q^{(m)} > s_q^{(m)} + 2T_0 > s_q^{(m)} + 2P$ for $m \geq N_1$ and $q \geq Z_1^{(m)}$. So, we have

$$x_i(t, \xi_m) \geq x_i^*(t) - \varepsilon_0, \quad i = 1, 2 \text{ as } t \in \left[T_0 + s_q^{(m)}, t_q^{(m)}\right], \tag{3.56}$$

where $m \geq N_1$ and $q \geq Z_1^{(m)}$. Hence, by using (3.50) and (3.56), from the third and fourth equations of system (1.1), we have

$$y_1'(t, \xi_m) \geq \varphi_{\varepsilon_0}(t) y_1(t, \xi_m), \quad y_2'(t, \xi_m) \geq \psi_{\varepsilon_0}(t) y_2(t, \xi_m), \quad t \in \left[T_0 + s_q^{(m)}, t_q^{(m)}\right]. \tag{3.57}$$

Integrating the above inequalities from $T_0 + s_q^{(m)}$ to $t_q^{(m)}$, we have

$$\begin{aligned} y_1\left(t_q^{(m)}, \xi_m\right) &\geq y_1\left(T_0 + s_q^{(m)}, \xi_m\right) \exp\left\{\int_{T_0 + s_q^{(m)}}^{t_q^{(m)}} \varphi_{\varepsilon_0}(t) dt\right\}, \\ y_2\left(t_q^{(m)}, \xi_m\right) &\geq y_2\left(T_0 + s_q^{(m)}, \xi_m\right) \exp\left\{\int_{T_0 + s_q^{(m)}}^{t_q^{(m)}} \psi_{\varepsilon_0}(t) dt\right\}, \end{aligned} \tag{3.58}$$

that is

$$\begin{aligned} \frac{\delta_{1y}}{(m+1)^2} &\geq \frac{\delta_{1y}}{(m+1)^2} \exp\left\{\int_{T_0 + s_q^{(m)}}^{t_q^{(m)}} \varphi_{\varepsilon_0}(t) dt\right\} > \frac{\delta_{1y}}{(m+1)^2}, \\ \frac{\delta_{2y}}{(m+1)^2} &\geq \frac{\delta_{2y}}{(m+1)^2} \exp\left\{\int_{T_0 + s_q^{(m)}}^{t_q^{(m)}} \psi_{\varepsilon_0}(t) dt\right\} > \frac{\delta_{2y}}{(m+1)^2}. \end{aligned} \tag{3.59}$$

These are contradictions. This completes the proof of Proposition 3.4. □

Proof of Theorem 2.4. The sufficiency of Theorem 2.4 now follows from Propositions 3.1–3.4. We thus only need to prove the necessity of Theorem 2.4. Suppose that

$$A_\omega\left(-g_1(t)+\frac{h_1(t)x_1^*(t)}{k_1(t)+n(t)x_1^*(t)}\right)\le 0,\qquad A_\omega\left(-g_2(t)+\frac{h_2(t)x_2^*(t)}{k_2(t)+x_2^*(t)}\right)\le 0. \tag{3.60}$$

We will show that

$$\lim_{t\to+\infty} y_i(t)=0,\quad i=1,2. \tag{3.61}$$

In fact, by (3.60), we know that, for any given positive constant $0<\varepsilon<1$, there exist $\varepsilon_1>0$, $(0<\varepsilon_1<\varepsilon)$, $\varepsilon_0>0$ such that

$$\begin{aligned}A_\omega\left(-g_1(t)+\frac{h_1(t)\left(x_1^*(t)+\varepsilon_1\right)}{k_1(t)+n(t)\left(x_1^*(t)+\varepsilon_1\right)}-q_1(t)\varepsilon\right)&\le-\frac{\varepsilon}{2}A_\omega(q_1(t))<-\varepsilon_0,\\ A_\omega\left(-g_2(t)+\frac{h_2(t)\left(x_2^*(t)+\varepsilon_1\right)}{k_2(t)+\left(x_2^*(t)+\varepsilon_1\right)}-q_2(t)\varepsilon\right)&\le-\frac{\varepsilon}{2}A_\omega(q_2(t))<-\varepsilon_0.\end{aligned} \tag{3.62}$$

Since

$$\begin{aligned}x_1'(t)&\le a(t)x_2(t)-b(t)x_1(t)-d(t)x_1^2(t),\\ x_2'(t)&\le c(t)x_1(t)-f(t)x_2^2(t).\end{aligned} \tag{3.63}$$

We know that, for above ε_1 there exists a $T^{(1)}>0$ such that

$$x_i(t)<x_i^*(t)+\varepsilon,\quad t\ge T^{(1)}. \tag{3.64}$$

It follows from (3.62) and (3.64) that for $t\ge T^{(1)}$,

$$\begin{aligned}A_\omega\left(-g_1(t)+\frac{h_1(t)x_1(t)}{k_1(t)+n(t)x_1(t)}-q_1(t)\varepsilon\right)&<-\varepsilon_0,\\ A_\omega\left(-g_2(t)+\frac{h_2(t)x_2(t)}{k_2(t)+x_2(t)}-q_2(t)\varepsilon\right)&<-\varepsilon_0.\end{aligned} \tag{3.65}$$

First, we show that there exists a $T^{(2)}>T^{(1)}$ such that $y_i(T^{(2)})<\varepsilon$, $i=1,2$. Otherwise, by (3.65), we have

$$\begin{aligned}\varepsilon&\le y_1(t)\\ &\le y_1\left(T^{(1)}\right)\exp\left\{\int_{T^{(1)}}^{t}\left(-g_1(s)+\frac{h_1(s)x_1(s)}{k_1(s)+n(s)x_1(s)}-q_1(s)\varepsilon\right)ds\right\}\\ &\le y_1\left(T^{(1)}\right)\exp\left\{-\varepsilon_0\left(t-T^{(1)}\right)\right\}\longrightarrow 0\end{aligned} \tag{3.66}$$

as $t \longrightarrow +\infty$. Similarly, we have

$$\varepsilon \le y_2(t) \le y_2(T^{(1)}) \exp\{-\varepsilon_0(t - T^{(1)})\} \longrightarrow 0, \quad t \longrightarrow +\infty, \tag{3.67}$$

which are contradictions.

Second, we now show that

$$y_i(t) \le \varepsilon \exp\{M(\varepsilon)\omega\}, \quad i = 1,2, \text{ for } t \ge T^{(2)}, \tag{3.68}$$

where

$$M(\varepsilon) = \max_{0\le t\le\omega}\left\{ g_1(t) + \frac{h_1(t)(x_1^*(t)+\varepsilon)}{k_1(t)+n(t)(x_1^*(t)+\varepsilon)} + q_1(t)\varepsilon,\ g_2(t) + \frac{h_2(t)(x_2^*(t)+\varepsilon)}{k_2(t)+(x_2^*(t)+\varepsilon)} + q_2(t)\varepsilon \right\} \tag{3.69}$$

is a bounded constant for $0 < \varepsilon < 1$. Otherwise, there exists a $T^{(3)} > T^{(2)}$ such that

$$y_i(T^{(3)}) > \varepsilon \exp\{M(\varepsilon)\omega\}, \quad i = 1,2. \tag{3.70}$$

By the continuity of $y_i(t)$, there must exist $T^{(4)} \in (T^{(2)}, T^{(3)})$ such that $y_i(T^{(4)}) = \varepsilon$ and $y_i(t) > \varepsilon$ for $t \in (T^{(4)}, T^{(3)}]$. Let P_1 be the nonnegative integer such that $T^{(3)} \in (T^{(4)}+P_1\omega, T^{(4)}+(P_1+1)\omega]$. By the first inequality of (3.65), we have

$$\begin{aligned}
\varepsilon \exp\{M(\varepsilon)\omega\} &< y_1(T^{(3)}) \\
&< y_1(T^{(4)}) \exp\left\{\int_{T^{(4)}}^{T^{(3)}} \left(-g_1(t) + \frac{h_1(t)x_1(t)}{k_1(t)+n(t)x_1(t)} - q_1(t)\varepsilon\right)dt\right\} \\
&= \varepsilon \exp\left\{\int_{T^{(4)}}^{T^{(4)}+P_1\omega} + \int_{T^{(4)}+P_1\omega}^{T^{(3)}}\right\}\left(-g_1(t) + \frac{h_1(t)x_1(t)}{k_1(t)+n(t)x_1(t)} - q_1(t)\varepsilon\right)dt \\
&< \varepsilon \exp\left\{\int_{T^{(4)}+P_1\omega}^{T^{(3)}} \left(g_1(t) + \frac{h_1(t)x_1(t)}{k_1(t)+n(t)x_1(t)} + q_1(t)\varepsilon\right)dt\right\} \\
&< \varepsilon \exp\left\{\int_{T^{(4)}+P_1\omega}^{T^{(3)}} \left(g_1(t) + \frac{h_1(t)(x_1^*(t)+\varepsilon)}{k_1(t)+n(t)(x_1^*(t)+\varepsilon)} + q_1(t)\varepsilon\right)dt\right\} \\
&\le \varepsilon \exp\{M(\varepsilon)\omega\}.
\end{aligned} \tag{3.71}$$

Similarly, by the second inequality of (3.65), we have

$$\varepsilon \exp\{M(\varepsilon)\omega\} < y_2(T^{(3)}) \le \varepsilon \exp\{M(\varepsilon)\omega\}, \tag{3.72}$$

which are contradictions. These imply that (3.68) holds. By the arbitrariness of ε, it immediately follows that $y_i(t) \to 0$ as $t \to +\infty$. This completes the proof of Theorem 2.4. □

4. Example

Consider the following predator-prey system:

$$\begin{aligned}
x_1'(t) &= 5x_2(t) - 2x_1(t) - x_1^2(t) - \frac{(2 + \sin(t)/200)x_1(t)}{5 + y_1(t) + x_1(t)} y_1(t), \\
x_2'(t) &= 3x_1(t) - x_2^2(t) - \frac{(2 + \sin(t)/100)x_2(t)}{4 + x_2(t)}, \\
y_1'(t) &= y_1(t)\left[-\frac{1}{3} - \frac{\sin(t)}{100} + \frac{(2 + \sin(t)/200)x_1(t)}{5 + y_1(t) + x_1(t)} - (4 + \cos(t))y_1(t)\right], \\
y_2'(t) &= y_2(t)\left[-\frac{1}{2} - \frac{\sin(t)}{100} + \frac{(2 + \sin(t)/100)x_2(t)}{4 + x_2(t)} - (3 + \cos(t))y_2(t)\right].
\end{aligned} \tag{4.1}$$

In this case, corresponding to system (1.1), one has $a(t) = 5$, $b(t) = 2$, $c(t) = 3$, $d(t) = 1$, $f(t) = 1$, $g_1(t) = 1/3 + \sin(t)/100$, $g_2(t) = 1/2 + \sin(t)/100$, $h_1(t) = p_1(t) = 2 + \sin(t)/200$, $h_2(t) = p_2(t) = 2 + \sin(t)/100$, $k_1(t) = 5$, $k_2(t) = 4$, $m(t) = n(t) = 1$, $q_1(t) = 4 + \cos(t)$, $q_2(t) = 3 + \cos(t)$.

One could easily see that

$$\begin{aligned}
x_1'(t) &= 5x_2(t) - 2x_1(t) - x_1^2(t), \\
x_2'(t) &= 3x_1(t) - x_2^2(t)
\end{aligned} \tag{4.2}$$

has a unique positive periodic solution $(x_1^*(t), x_2^*(t)) = (3, 3)$, that is, in this case, the positive periodic solution is the positive equilibrium. By simple computation, one has

$$\begin{aligned}
A_\omega\left(-g_1(t) + \frac{h_1(t)x_1^*(t)}{k_1(t) + n(t)x_1^*(t)}\right) &= \frac{5}{12} > 0, \\
A_\omega\left(-g_2(t) + \frac{h_2(t)x_2^*(t)}{k_2(t) + x_2^*(t)}\right) &= \frac{5}{14} > 0.
\end{aligned} \tag{4.3}$$

Hence, corresponding to Theorem 2.4, we know that system (4.1) is permanent.

5. Conclusion

In this paper, a model which describes the nonautonomous periodic predator-prey system with Beddington-DeAngelis and Holling II functional response and stage structure for prey is proposed. Under Assumption (2.3), sufficient and necessary conditions which guarantee the predator and the prey species to be permanent are obtained.

The results of this paper suggest the following biological implication. Note that $(x_1^*(t), x_2^*(t))$ is the globally asymptotically stable periodic solution of system (1.1) without predation, which, as showed by Lemma 2.2, always exists. Hence, condition (2.3) implies that if the death rate of the two predator species is all small enough and the growth by foraging minus the death for the second predator is sufficiently high, the system is permanent.

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