Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2008, Article ID 178534, 19 pages
doi:10.1155/2008/178534

Research Article

Harnack Inequalities and ABP Estimates for Nonlinear Second-Order Elliptic Equations in Unbounded Domains

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Received 17 July 2007; Revised 31 January 2008; Accepted 4 April 2008

Recommended by Y. Giga

We are concerned with fully nonlinear uniformly elliptic operators with a superlinear gradient term. We look for local estimates, such as weak Harnack inequality and local maximum principle, and their extension up to the boundary. As applications, we deduce ABP-type estimates and weak maximum principles in general unbounded domains, a strong maximum principle, and a Liouville-type theorem.

1. Introduction

The qualitative theory of second-order elliptic equations received a strong effort from Harnack inequalities. Here, we will make use of this powerful technique to study continuous viscosity solutions u of fully nonlinear elliptic equations ($F = f$):

$$F\big(x, u(x), Du(x), D^2u(x)\big) = f(x), \quad x \in \Omega, \tag{1.1}$$

in unbounded domains Ω of $\mathbb{R}^n$, where F is a real function of $x \in \Omega$, $t \in \mathbb{R}$, $p \in \mathbb{R}^n$ and X in the set $\mathcal{S}^n$ of $n \times n$ real symmetric matrices.

We recall that F is (degenerate) elliptic if F is nondecreasing in X and uniformly elliptic if there exist (ellipticity) constants λ and Λ such that $0 < \lambda \le \Lambda$ and

$$\lambda \operatorname{tr}(Y) \le F(x, t, p, X + Y) - F(x, t, p, X) \le \Lambda \operatorname{tr}(Y), \tag{1.2}$$

for $Y \ge 0$, that is Y is semidefinite positive, where $\operatorname{tr}(Y)$ denotes the trace of the matrix Y.

In the class of uniformly elliptic operators, there are two extremal ones, well known as Pucci maximal and minimal operators, respectively:

$$\mathcal{P}^{+}_{\lambda,\Lambda}(X) = \Lambda \operatorname{tr}(X^{+}) - \lambda \operatorname{tr}(X^{-}),$$
$$\mathcal{P}^{-}_{\lambda,\Lambda}(X) = \lambda \operatorname{tr}(X^{+}) - \Lambda \operatorname{tr}(X^{-}), \tag{1.3}$$

where $X^{\pm}$ are the positive and negative parts of X, which can be decomposed in a unique way as $X = X^{+} - X^{-}$ with $X^{\pm} \geq 0$ and $X^{+}X^{-} = 0$. Other examples of fully nonlinear uniformly elliptic operators can be found in [1–3].

Throughout this paper, we will consider elliptic operators with the structure conditions

$$F(x,t,p,X) \geq \mathcal{P}^{-}_{\lambda,\Lambda}(X) - b(x)|p|^{q}, \tag{1.4}$$
$$F(x,t,p,X) \leq \mathcal{P}^{+}_{\lambda,\Lambda}(X) + b(x)|p|^{q}, \tag{1.5}$$

where $\mathcal{P}^{\pm}$ are the extremal Pucci operators, $b(x)$ is a continuous function and the exponent $q \in [1,2]$, so that the gradient term can have a superlinear, at most quadratic growth.

Remark 1.1. The above structure conditions are exactly equivalent to the uniform ellipticity when F is linear in the variable $X \in \mathcal{S}^{n}$. In the nonlinear case they allow a slight generalization. Let us consider, for $0 < \lambda < \Lambda$ and $t \geq 0$, the function

$$h(t) = \begin{cases} \Lambda t, & \text{if } 0 \leq t \leq \dfrac{1}{\Lambda}, \\ 1, & \text{if } \dfrac{1}{\Lambda} < t \leq \dfrac{1}{\lambda}, \\ \lambda t, & \text{if } t > \dfrac{1}{\lambda}, \end{cases} \tag{1.6}$$

then the operator $F = h(\operatorname{tr}(X^{+})) - h(\operatorname{tr}(X^{-}))$ is elliptic and satisfies both the conditions (1.4) and (1.5), even that it is not uniformly elliptic.

However, if (1.5) (resp., (1.4)) holds, then subsolutions (resp., supersolutions) of the equation $F = f$ are subsolutions (resp., supersolutions) of uniformly elliptic equations, and this is needed to prove our results.

We will be concerned principally with the following topics in unbounded domains; see [4–6] for classical results.

[MP] maximum principle for u.s.c. subsolutions w of $F = 0$ in the viscosity sense (v.s.), in the form

$$F \geq 0 \text{ in } \Omega,\ w \leq 0 \text{ on } \partial\Omega,\ \sup_{\Omega} w < +\infty \Longrightarrow w \leq 0 \text{ in } \Omega; \tag{1.7}$$

[LT] Liouville theorem for continuous solutions of $F = 0$ v.s., in the form

$$F = 0 \text{ in } \mathbb{R}^{n},\ \sup_{\mathbb{R}^{n}} w < +\infty \Longrightarrow w = \text{ constantin } \mathbb{R}^{n}. \tag{1.8}$$

Concerning MP, it is worth to note that the condition from above on the size of w can be weakened in the framework of the Phragmén-Lindelöf theory (see, e.g., [7–9]) but not omitted at all, even for classical subsolutions (see, e.g., [4, 10]). It is also well known that MP fails to hold in general in exterior domains. In fact, due to the boundedness of the fundamental solution $u(x) = |x|^{2-n}$ of the Laplace equation $\Delta u = 0$, the function $w = 1 - u$ provides a counterexample to MP in $\Omega = \mathbb{R}^n \setminus \overline{B}_1(0)$. Thus we introduce a local measure-geometric condition $\mathbf{G}_\sigma$ in Ω at $y \in \mathbb{R}^n$, which depends on the real parameter $\sigma \in (0,1)$: *there exists a ball* $B = B(y)$ *such that*

$$y \in B, \quad \left|B \setminus \Omega_y\right| \geq \sigma |B|, \tag{1.9}$$

where Ω_y *is the connected component of* $B \setminus \partial\Omega$ *containing* y.

If $\mathbf{G}_\sigma$ is satisfied in Ω at all $y \in \Omega$, we simply say that Ω is a **wG** domain (with parameter σ). This is a generalization of condition **G** of Cabré [10], which ultimately goes back to Berestycki et al. [11].

Let $R(y)$ denote the radius of the ball $B = B(y)$ provided by condition **wG**. We will call *domains of cylindrical and conical type* the **wG** domains such that $R(y) = O(1)$ and $R(y) = O(|y|)$ as $|y| \to +\infty$, respectively. Examples of the first kind are domains with finite measure, cylinders, slabs, complements of a periodic lattice of balls, whereas cones, and complements, in the plane, of logarithmic spirals, are examples of the second kind.

In [12], it is shown that MP holds true in a **wG** domain for strong solutions of a linear second-order uniformly elliptic operator $F = \operatorname{tr} A(x)X$; see also [13, 14] for earlier results and [15, 16] for viscosity solutions of a fully nonlinear operator with linear and quadratic growth in the gradient (i.e., in the case of the structure condition (1.5) with $q = 1$ and $q = 2$) provided that $b(x) = O(1/|x|)$ and $b(x) = O(1)$ as $|x| \to \infty$, respectively.

With the aim to find conditions on the coefficient $b(x)$ such that MP holds in **wG** domains when $1 \leq q \leq 2$, our result is the following.

Theorem 1.2 (MP). *Let* $0 < \sigma < 1$ *and* $1 \leq q \leq 2$. *Let* Ω *be a domain of* $\mathbb{R}^n$ *satisfying condition* **wG** *or alternatively such that, for a closed subset* H *of* $\mathbb{R}^n$,

(i) MP *holds in each connected component of* $\Omega \setminus H$;

(ii) *condition* $\mathbf{G}_\sigma$ *is satisfied in* Ω *at each* $y \in \Omega \cap H$.

Suppose that $w \in \mathrm{USC}(\overline{\Omega})$ *is a viscosity solution of* $F(x, w, Dw, D^2w) \geq 0$ *and structure condition* (1.5) *holds with* $b \in C(\overline{\Omega})$, *such that* $b(x) = O(1/|x|^{2-q})$.

If $w \leq 0$ *on* $\partial\Omega$ *and* $\sup w < +\infty$ *in* Ω, *then* $w \leq 0$ *in* Ω.

This yields indeed MP in a wider class of domains than **wG**, for example, the cut plane and more generally the complement of continuous semi-infinite curves in $\mathbb{R}^2$ and their generalizations to hypersurfaces in $\mathbb{R}^n$.

We also outline that the limit cases $q = 1$ and $q = 2$ of the above mentioned papers are obtained by continuity from the intermediate cases $1 < q < 2$, as it follows from Theorem 1.2. Nonetheless, there are technical improvements with respect to the previous works even in the limit cases.

Consider in particular a *parabolic shaped* domain Ω, satisfying condition **wG** with $R(y) = O(|y|^\alpha)$, $0 < \alpha < 1$; the limit cases $\alpha = 0$ and $\alpha = 1$ correspond to domains of cylindrical and conical types, respectively.

Based on an argument of [12], eventually passing to a smaller $r_y \leq R(y)$, we can suppose that condition $\mathbf{G}_\sigma$ is satisfied with $|B \setminus \Omega_y| = \sigma|B|$ exactly. We get the new following variant of ABP estimate.

Theorem 1.3 (ABP). *Let $0 < \sigma, \tau < 1$, $\tau' > 1$, $R_0, \beta \geq 0$, $1 \leq q \leq 2$, and $N > 0$. Let Ω be a* **wG** *domain, such that condition $\mathbf{G}_\sigma$ in Ω is fulfilled at each $y \in \Omega$ with $R(y) \leq R_0 + \beta|y|^\alpha$, $0 \leq \alpha \leq 1$. Assume that F satisfies the structure condition* (1.5)*, with $b, f \in C(\overline{\Omega})$ and $b_0 := \sup_\Omega |b(x)|(1 + |x|^{\alpha(2-q)}) < +\infty$.*

If $w \in \mathrm{USC}(\overline{\Omega})$ is a viscosity solution of $F(x, w, Dw, D^2w) \geq f$ such that $w \leq N$ in Ω and $w \leq 0$ on $\partial\Omega$, then

$$\sup_\Omega w \leq C \lim_{\varepsilon \to 0^+} \sup_{y \in \Omega; |y| \geq \varepsilon r_y} r_y \left\| f^- \right\|_{L^n(\Omega \cap B_{\tau\varepsilon r_y, \tau' r_y})}, \tag{1.10}$$

where C is a positive constant depending on n, q, λ, Λ, $b_0 N^{q-1}$, σ, τ, τ', R_0, β.

Note that in the case of a domain of cylindrical type ($\alpha = 0$), it is sufficient to have $b(x) = O(1)$, for all $q \in [1, 2]$, as well as in the case of a quadratic growth in the gradient variable ($q = 2$), for all $\alpha \in [0, 1]$.

This result extends the previous ones contained in [10, 14] for the linear case, and [8, 16], dealing with fully nonlinear equations, in the limit situations of cylindrical/conical domains and linear/quadratic gradient terms.

Remark 1.4. In general, unless $q = 1$, the above ABP type estimate is different from the so-called ABP maximum principles since C depends on the upper bound N of w if $b_0 > 0$ and $q > 1$. For ABP-type estimates of this kind in bounded domains we refer to [17]. Counterexamples to the ABP maximum principle can been found in [17–19].

Consider now $\Omega = \mathbb{R}^n$. The classical Liouville theorem says that harmonic functions in the entire $\mathbb{R}^n$, which are bounded either above or below, are constant. This result continues to hold for strong solutions of quasilinear uniformly elliptic equations; see [20]. For viscosity solutions of fully nonlinear uniformly elliptic equations with an additive gradient term having linear growth, we refer to [21, 22]. Our result is the following.

Theorem 1.5 (Liouville theorem). *Let $w \in C(\mathbb{R}^n)$ be such that $F(x, w, Dw, D^2w) = 0$ in the viscosity sense, and assume structure conditions* (1.4) *and* (1.5)*, with $b \in C(\mathbb{R}^n)$ such that $b(x) = O(1/|x|^{2-q})$ as $|x| \to +\infty$. If w is bounded either above or below, then w is constant.*

Remark 1.6. Under some additional assumptions, Liouville-type results also hold in unbounded domains of $\mathbb{R}^n$ containing balls of arbitrary large radius; see [23].

Our main tools are Krylov-Safonov Harnack inequalities and local MP; see [20] for strong solutions of quasilinear uniformly elliptic equations. For viscosity solutions and F satisfying the structure condition (1.4), they can be found in [3] if $b = 0$ and in [24] if $q = 2$; see also [25]. In the case of linear or superlinear, almost quadratic, growth in the gradient ($1 \leq q < 2$), weak Harnack (wH) inequality and local MP can be deduced using arguments of [17], in which a (full) Harnack inequality has been established for L^p viscosity solutions; see also [26].

Nevertheless, for convenience of the reader we believed that it is worth to report systematically on this kind of inequalities in Section 3.

As the previous ones, our approach follows the lines of [3], based on the methods of [27, 28] and on the ABP maximum principle for viscosity solutions in bounded domains, due to Caffarelli [29].

Remark 1.7. In deriving wH inequality and local MP, we only need the Alexandroff-Bakelman-Pucci (ABP) estimate with $q = 1$ and f continuous, so [30, Proposition 2.12] and also [17, Theorem 4.1] in the case of linear growth in the gradient term, are sufficient to our purpose. But we notice that new ABP-type estimates have been established for L^p-viscosity solutions of equations with discontinuous coefficients by Koike and Święch [19, 27] for $q \in [1, 2]$ and $f \in L^p$.

Remark 1.8. In the case of a superlinear first-order term, wH inequality and local MP are obtained by interpolation between the linear and quadratic cases, eliminating the square gradient term by means of an exponential transformation used before by Trudinger [24], see Lemmas 3.1 and 3.2 below. This kind of ideas have been also considered by Sirakov in [31].

What we definitely need are, for MP, the scaled boundary wH inequality (3.16), derived in Section 3 by means of typical viscosity methods, and, for technical reasons, its version in annular regions (3.24), and, for LT, the scaled Harnack inequality (3.11). Moreover, using the interior wH inequality (3.7) and assuming the structure condition (1.5), we also state a strong MP theorem, according to which a subsolution u of equation $F = 0$ cannot achieve a positive maximum inside any domain (open connected set) of $\mathbb{R}^n$ unless it is constant; see Theorem 5.1 below. For a different approach, based on Hopf lemma, and more general versions see [32].

The paper is organized as follows. In Section 2, we recall some basic results of elliptic theory for viscosity solutions of second-order fully nonlinear equations with a linear gradient term; in Section 3, we extend local maximum principle and weak Harnack inequality, even up to the boundary, to the case of a superlinear gradient term; these results are applied in Section 4 to get Alexandroff-Bakelman-Pucci-type estimates and maximum principles, with the proof of Theorems 1.2 and 1.3; finally, a strong maximum principle is derived and the proof of Liouville theorem (Theorem 1.5) is given in Section 5. In the appendix, for the sake of completeness, we show the basic weak Harnack inequality and local MP for a uniformly elliptic operator with an additive first-order term having linear growth in the gradient.

2. Basic estimates (linear gradient term)

Let Ω be a domain of $\mathbb{R}^n$, and denote by $\mathrm{USC}(\Omega)$ and $\mathrm{LSC}(\Omega)$, respectively, the sets of the upper and lower semicontinuous functions in Ω. The function $u \in \mathrm{USC}(\Omega)$ is said to be a viscosity subsolution of $F = f$ if

$$F\left(x, u(x), D\varphi(x), D^2\varphi(x)\right) \geq f(x) \tag{2.1}$$

at any point $x \in \Omega$ and for all $\varphi \in C^2(\Omega)$ such that $\varphi - u$ has a local minimum in x. Similarly, a viscosity supersolution $u \in \mathrm{LSC}(\Omega)$ of $F = f$ satisfies

$$F\left(x, u(x), D\varphi(x), D^2\varphi(x)\right) \leq f(x) \tag{2.2}$$

at any point $x \in \Omega$ and for all $\varphi \in C^2(\Omega)$ such that $u - \varphi$ has a local minimum in x.

We may also assume that $\varphi(x) = u(x)$ in the above definition, that is the graph of the test function φ touches that one of u from above for subsolutions and from below for supersolutions [3]. Moreover, if F is continuous in the matrix-variable, as for uniformly elliptic operators, then we may assume that $\varphi(x)$ is a paraboloid, that is a quadratic polynomial.

We will make use of the following version of the ABP estimate, in which Γ_u^+ denotes the upper contact set

$$\Gamma_u^+ = \left\{x \in \Omega \;/\; \exists p \in \mathbb{R}^n \text{ s.t. } u(y) \le u(x) + p\cdot(y-x) \text{ for } x \in \Omega\right\} \tag{2.3}$$

of the graph of the function u. Using [30, Proposition 2.12] or [17, Theorem 4.1], we have the following.

Lemma 2.1 (ABP estimate). *Let $u \in \mathrm{LSC}(\overline{B})$ be a viscosity supersolution of the equation*

$$\mathcal{P}^-_{\lambda,\Lambda}(D^2u) - b_0|Du| = f \tag{2.4}$$

in a ball B of unit radius, such that $u \ge 0$ on ∂B, where $f \in L^n(B) \cap C(B)$, for some constant $b_0 \ge 0$. Then

$$\sup_B u^- \le C\|f^+\|_{L^n(\Gamma^+_{u^-})}, \tag{2.5}$$

for a positive constant $C = C(n, \lambda, \Lambda, b_0)$. Similarly, if $u \in \mathrm{USC}(\overline{B})$ is a viscosity subsolution of the equation

$$\mathcal{P}^+_{\lambda,\Lambda}(D^2u) + b_0|Du| = f \tag{2.6}$$

such that $u \le 0$ on ∂B, then

$$\sup_B u^+ \le C\|f^-\|_{L^n(\Gamma^+_{u^+})}. \tag{2.7}$$

From Lemma 2.1, we obtain the following results, see the appendix, which extend [3, Theorem 4.8, (1) and (2)]; see also [15].

Here we denote by B_r a ball centered at $x_0 \in \mathbb{R}^n$ of radius $r > 0$.

Lemma 2.2 (wH inequality). *Let $b_0 \ge 0$ and $0 < \tau < 1$. Suppose that $u \in \mathrm{LSC}(B_{1/\tau})$ is a viscosity supersolution of* (2.4)*, with $f \in C(\overline{B}_{1/\tau})$, and $u \ge 0$ in $B_{1/\tau}$. Then*

$$\left(\frac{1}{|B_1|}\int_{B_1} u^{p_0}\right)^{1/p_0} \le C\left(\inf_{B_1} u + \|f^+\|_{L^n(B_{1/\tau})}\right), \tag{2.8}$$

where C and p_0 are positive numbers, depending on n, λ, Λ, b_0, and τ.

Lemma 2.3 (local MP). *Let $b_0 \ge 0$ and $0 < \tau < 1$. Suppose that $u \in \mathrm{USC}(B_1)$ is a viscosity subsolution of* (2.6) *with $f \in C(\overline{B}_1)$. Then for all $p > 0$,*

$$\sup_{B_\tau} u \le C\left(\left(\frac{1}{|B_1|}\int_{B_1} (u^+)^p\right)^{1/p} + \|f^-\|_{L^n(B_1)}\right), \tag{2.9}$$

where C is a positive constant, depending on $n, \lambda, \Lambda, b_0, \tau$, and p.

3. Interior and boundary Harnack estimates and local MP (superlinear gradient term)

Firstly, we extend interior estimates (2.8) and (2.9) to fully nonlinear operators F with a superlinear first-order term, such that, respectively, (1.4) and (1.5) hold.

Lemma 3.1 (wH inequality). *Let $b_0 \geq 0$, $0 < \tau < 1$, and $1 \leq q \leq 2$. Suppose that $u \in \mathrm{LSC}(B_{1/\tau})$ is a viscosity solution of $F(x, u, Du, D^2u) \leq f$, under structure condition (1.4) with $b \leq b_0$, $f \in C(\overline{B}_{1/\tau})$, and $0 \leq u \leq 1$ in $B_{1/\tau}$. Then (2.8) holds with positive constants C and p_0, depending on $n, \lambda, \Lambda, b_0, \tau$, and q.*

Lemma 3.2 (local MP). *Let $b_0 \geq 0$, $0 < \tau < 1$, and $1 \leq q \leq 2$. Suppose that $u \in \mathrm{USC}(B_1)$ is a viscosity solution of $F(x, u, Du, D^2u) \geq f$, under structure condition (1.5), with $b \leq b_0$, $f \in C(\overline{B}_1)$, and $u \leq 1$. Then (2.9) holds for all $p > 0$ with a positive constant C, depending on $n, \lambda, \Lambda, b_0, q, \tau$, and p.*

Proof of Lemmas 3.1 and 3.2. We only show the proof of Lemma 3.2, since that one of Lemma 3.1 is similar. By the structure condition (1.5), we have

$$\mathcal{P}^+_{\lambda,\Lambda}(D^2u) + b_0|Du|^q \geq f(x) \tag{3.1}$$

and also, in the viscosity sense,

$$\mathcal{P}^+_{\lambda,\Lambda}(D^2u^+) + b_0\left|Du^+\right|^q \geq -f^-(x). \tag{3.2}$$

From this, by Young's inequality, it follows that

$$\mathcal{P}^+_{\lambda,\Lambda}(D^2u^+) + b_1\left|Du^+\right| + b_2\left|Du^+\right|^2 \geq -f^-(x) \tag{3.3}$$

with

$$\begin{aligned} b_1 &= (2-q)b_0^{1/q}, \\ b_2 &= (q-1)b_0^{2/q}. \end{aligned} \tag{3.4}$$

Using the transformation $u^+ = (\lambda/b_2)\ \log(1 + (b_2/\lambda)v)$, then the USC function $v = (\lambda/b_2)(\exp((b_2/\lambda)u^+) - 1)$ satisfies the differential inequality

$$\mathcal{P}^+_{\lambda,\Lambda}(D^2v) + b_1|Dv| \geq -f^-(x)\left(1 + \frac{b_2}{\lambda}v(x)\right) \tag{3.5}$$

in $B_{1/\tau}$. Therefore, we can apply Lemma 2.3 to the subsolution v. To conclude the proof of Lemma 3.2, it is sufficient to observe that

$$u^+ \leq v \leq \frac{\lambda}{b_2}\left(\exp\left(\frac{b_2}{\lambda}\right) - 1\right)u^+. \tag{3.6}$$

□

Rescaling variables and functions, we highlight the dependence on geometric parameters.

Theorem 3.3 (scaled wH inequality). *Let $b_0 \geq 0$, $0 < \tau < 1$, $N > 0$, and $1 \leq q \leq 2$. Suppose that $u \in \mathrm{LSC}(B_{R/\tau})$ is a viscosity solution of $F(x, u, Du, D^2u) \leq f$, under structure condition* (1.4)*, with $b \leq b_0$, $f \in C(\overline{B}_{R/\tau})$, and $0 \leq u \leq N$ in $B_{R/\tau}$. Then*

$$\left(\frac{1}{|B_R|}\int_{B_R} u^{p_0}\right)^{1/p_0} \leq C\Big(\inf_{B_R} u + R\|f\|_{L^n(B_{R/\tau})}\Big), \tag{3.7}$$

with positive constants C and p_0, depending on $n, \lambda, \Lambda, q, \tau$, and $b_0 N^{q-1} R^{2-q}$.

Proof. Considering, for $y \in B_{1/\tau}$, the function $v(y)$, defined by $u(x) = Nv(x/R)$, we have

$$\mathcal{P}^-_{\lambda,\Lambda}(D^2 v) - b_0 N^{q-1} R^{2-q} |Dv|^q \leq R^2 N^{-1} f^+(Ry). \tag{3.8}$$

Thus, applying Lemma 3.1, we get

$$\left(\frac{1}{|B_1|}\int_{B_1} v^{p_0}\right)^{1/p_0} \leq C\Big(\inf_{B_1} v + R^2 N^{-1} \left\|f^+(Ry)\right\|_{L^n_y(B_{1/\tau})}\Big), \tag{3.9}$$

with $C = C(n, \lambda, \Lambda, q, \tau, b_0 N^{q-1} R^{2-q})$, from which the assert follows. □

Note that constants p_0 and C of the above wH inequality depend in general on the upper bound N for the supersolution and on the radius R of the ball, but in the case $q = 1$ there is no dependence on N and in the case $q = 2$ no dependence on R.

In the same manner as in Theorem 3.3 for wH inequality, we make the dependence on the geometric constants explicit in the following local MP.

Theorem 3.4 (scaled local MP). *Let $b_0 \geq 0$, $0 < \tau < 1$, $N > 0$, and $1 \leq q \leq 2$. Suppose that $u \in \mathrm{USC}(B_R)$ is a viscosity solution of $F(x, u, Du, D^2u) \geq f$, under structure condition* (1.5)*, with $b \leq b_0$, $f \in C(\overline{B}_R)$, and $u \leq N$. Then for all $p > 0$,*

$$\sup_{B_{\tau R}} u \leq C\left(\left(\frac{1}{|B_R|}\int_{B_R} (u^+)^p\right)^{1/p} + R\left\|f^-\right\|_{L^n(B_R)}\right), \tag{3.10}$$

with a positive constant C, depending on $n, \lambda, \Lambda, q, \tau, b_0 N^{q-1} R^{2-q}$, and p.

Combining Theorems 3.3 and 3.4, we get the full Harnack inequality for solutions.

Theorem 3.5 (Harnack inequality). *Let $b_0 \geq 0$, $0 < \tau < 1$, $N > 0$, and $1 \leq q \leq 2$. Suppose that $u \in C(B_{R/\tau})$ is a viscosity solution of $F(x, u, Du, D^2u) = f$ in $B_{R/\tau}$, under structure conditions* (1.4) *and* (1.5)*, with $b \leq b_0$, $f \in C(\overline{B}_{R/\tau})$, and $0 \leq u \leq N$. Then*

$$\sup_{B_R} u \leq C\Big(\inf_{B_R} u + R\|f\|_{L^n(B_{R/\tau})}\Big), \tag{3.11}$$

with a positive constant $C = C(n, \lambda, \Lambda, q, \tau, b_0 N^{q-1} R^{2-q})$.

We wish to extend the above estimates up to the boundary, that is, to balls intersecting the boundary of the domain $A \subset \mathbb{R}^n$, where the solutions are defined. For this purpose we will need suitable extensions of such solutions outside A. Precisely, take concentric balls $B_{\tau R} \subset B_R \subset B_{R/\tau}$ such that $B_{\tau R} \cap A \neq \varnothing$ and $B_{R/\tau} \setminus A \neq \varnothing$. For a nonnegative viscosity supersolution $u \in \mathrm{LSC}(\overline{A})$ of equation $F(x, u, Du, D^2u) = f$ in A, we put

$$m = \inf_{B_{R/\tau}\cap\partial A} u; \quad u_m^-(x) = \begin{cases} \min\left(u(x), m\right), & \text{if } x \in A, \\ m, & \text{if } x \notin A, \end{cases} \tag{3.12}$$

where $0 < \tau < 1$. Similarly, for a viscosity subsolution $u \in \mathrm{USC}(\overline{A})$, we put

$$M = \sup_{B_R\cap\partial A} u^+; \quad u_M^+(x) = \begin{cases} \max\left(u^+(x), M\right), & \text{if } x \in A, \\ M, & \text{if } x \notin A. \end{cases} \tag{3.13}$$

Denote also by f_0^+ and f_0^- the continuations of f^+ and f^- vanishing outside A, respectively. Following [3, Proposition 2.8] and using the structure conditions (1.4) and (1.5), we have

$$\mathcal{P}^-_{\lambda,\Lambda}(D^2u_m^-) - b_0\left|Du_m^-\right|^q \le f_0^+ \tag{3.14}$$

in $B_{R/\tau}$ for a viscosity supersolution $u \in \mathrm{LSC}(\overline{A})$, and

$$\mathcal{P}^+_{\lambda,\Lambda}(D^2u_\tau^+) + b_0\left|Du_\tau^+\right|^q \ge -f_0^- \tag{3.15}$$

in B_R for a viscosity subsolution $u \in \mathrm{USC}(\overline{A})$.

Observe that, if $f^+ = 0$ on ∂A, then f_0^+ is continuous, and then we can apply Theorem 3.3 to get a boundary wH inequality. Similarly, if $f^- = 0$ on ∂A, we can use Theorem 3.4 to deduce a boundary local MP.

Nevertheless, even in the general case, when f_0^+ and f_0^- are not necessarily continuous, we can get boundary estimates by means of an approximation process, as shown here below, where we use the notations defined just above.

Theorem 3.6 (boundary wH inequality). *Let $b_0 \ge 0$, $0 < \tau < 1$, $N > 0$, and $1 \le q \le 2$. Suppose that $u \in \mathrm{LSC}(\overline{A})$ is a viscosity solution of $F(x, u, Du, D^2u) \le f$, under structure condition* (1.4)*, with $b(x) \le b_0$, $f \in C(\overline{A})$, and $0 \le u \le N$ in A. Then*

$$\left(\frac{1}{|B_R|}\int_{B_R}(u_m^-)^{p_0}\right)^{1/p_0} \le C\left(\inf_{B_R\cap A} u + R\|f\|_{L^n(B_{R/\tau}\cap A)}\right), \tag{3.16}$$

with positive constants C and p_0, depending on $n, \lambda, \Lambda, q, \tau$, and $b_0N^{q-1}R^{2-q}$.

Proof. For $\varepsilon > 0$, set

$$m_\varepsilon = \inf_{I_\varepsilon(\partial A)} u, \quad I_\varepsilon(\partial A) = \left\{x \in B_{R/\tau} \cap \overline{A} \,:\, \mathrm{dist}(x, \partial A) \le \varepsilon\right\}, \tag{3.17}$$

and, for $x \in \overline{B}_{R/\tau}$,

$$u^-_{m_\varepsilon}(x) = \begin{cases} \min(u(x), m_\varepsilon), & \text{if } x \in A, \\ m_\varepsilon, & \text{if } x \notin A, \end{cases}$$
$$f_\varepsilon(x) = f^+(x)\rho\left(\frac{\operatorname{dist}(x, \mathbb{R}^n \setminus A)}{\varepsilon}\right), \tag{3.18}$$

where

$$\rho(t) = \begin{cases} t, & \text{if } 0 \le t < 1, \\ 1, & \text{if } t \ge 1. \end{cases} \tag{3.19}$$

It is easy to check that $u^-_{m_\varepsilon} \in \mathrm{LSC}(B_{R/\tau})$, $0 \le u^-_{m_\varepsilon} \le N$, $f_\varepsilon \in C(\overline{B}_{R/\tau})$, and

$$\mathcal{P}^-_{\lambda,\Lambda}(D^2 u^-_{m_\varepsilon}) - b_0|Du^-_{m_\varepsilon}|^q \le f_\varepsilon(x) \tag{3.20}$$

in $B_{R/\tau}$. Therefore, we can apply Theorem 3.3 with $u^-_{m_\varepsilon}$ instead of u and f_ε instead of f to get

$$\left(\frac{1}{|B_R|}\int_{B_R}(u^-_{m_\varepsilon})^{p_0}\right)^{1/p_0} \le C\left(\inf_{B_R} u^-_{m_\varepsilon} + R\|f_\varepsilon\|_{L^n(B_{R/\tau})}\right). \tag{3.21}$$

Note that $\inf_{B_R} u^-_{m_\varepsilon} \le \inf_{B_R\cap A} u$ and $0 \le f_\varepsilon \le f^+$ in A, $f_\varepsilon = 0$ outside A. Also observing that, by lower semicontinuity,

$$m \le \liminf_{\varepsilon\to 0} m_\varepsilon \tag{3.22}$$

and therefore, by Fatou's lemma,

$$\int_{B_R}(u^-_m)^{p_0} \le \liminf_{\varepsilon\to 0}\int_{B_R}(u^-_{m_\varepsilon})^{p_0}, \tag{3.23}$$

from inequality (3.21) we get the assert. □

In the sequel, we will make also use of a version of boundary wH inequality for annular regions $B_{\varepsilon R,R} = B_R \setminus B_{\varepsilon R}(0)$, $0 < \varepsilon < 1$, which can be deduced by Theorem 3.6 reasoning as in [10, Theorem 3.1].

In this case $m = \inf_{\partial A\cap B_{\varepsilon\tau R,\tau' R}} u$, where $0 < \varepsilon \le 1/2$, $0 < \tau < 1$, $\tau' > 1$.

Corollary 3.7 (boundary wH inequality). *Let $0 < \tau < 1$, $\tau' > 1$, $N > 0$, and $1 \le q \le 2$. Suppose that $u \in \mathrm{LSC}(\overline{A})$ is a viscosity solution of $F(x,u,Du,D^2u) \le f$, under structure condition* (1.4)*, with $f \in C(\overline{A})$, and $0 \le u \le N$ in A. Then*

$$\left(\frac{1}{|B_{\varepsilon R,R}|}\int_{B_{\varepsilon R,R}}(u^-_m)^{p_0}\right)^{1/p_0} \le C\left(\inf_{A\cap B_{\varepsilon R,R}} u + R\|f\|_{L^n(A\cap B_{\varepsilon\tau R,\tau' R})}\right), \tag{3.24}$$

with positive constants C and p_0, depending on $n, \lambda, \Lambda, q, \tau, \tau'$, and $N^{q-1}R^{2-q}\|b\|_{L^\infty(A\cap B_{\varepsilon\tau R,R/\tau})}$.

In a similar manner, we extend the local MP up to the boundary.

Theorem 3.8 (boundary local MP). *Let $b_0 \geq 0$, $0 < \tau < 1$, $N > 0$, and $1 \leq q \leq 2$. Suppose that $u \in \mathrm{USC}(\overline{A})$ is a viscosity solution of $F(x, u, Du, D^2u) \geq f$, under structure condition* (1.5), *with $f \in C(\overline{A})$ and $u \leq N$ in A. Then for all $p > 0$,*

$$\sup_{B_{\tau R} \cap A} u \leq C\left(\left(\frac{1}{|B_R|}\int_{B_R}(u_M^+)^p\right)^{1/p} + R\|f^-\|_{L^n(B_R \cap A)}\right), \tag{3.25}$$

with a positive constant C, depending on $n, \lambda, \Lambda, q, \tau, b_0 N^{q-1} R^{2-q}$, and p.

4. ABP-type estimates and maximum principles

Here we use boundary estimates of previous section to obtain MP in unbounded domains Ω of $\mathbb{R}^n$ for viscosity subsolutions $u \in \mathrm{USC}(\overline{\Omega})$, bounded above, of equation $F(x, u, Du, D^2u) = 0$ under structure condition (1.5).

We will make use of the measure-geometric condition $\mathbf{G}_\sigma$, $0 < \sigma < 1$, given in the introduction. By a continuity argument, see [12], eventually passing to a smaller R, which we will call r_y, we can assume that condition $\mathbf{G}_\sigma$ is satisfied with $|B \setminus \Omega_y| = \sigma |B|$ exactly.

We also recall that Ω is a **wG** domain (with parameter σ) if each point $y \in \Omega$ satisfies condition $\mathbf{G}_\sigma$ in Ω. In particular, if $R(y)$ is the radius of the ball $B = B(y)$ provided by condition $\mathbf{G}_\sigma$, we define domains of cylindrical and conical type as **wG** domains such that $R(y) = O(1)$ and $R(y) = O(|y|)$, respectively as $|y| \to \infty$.

4.1. Domains of cylindrical type

We start with the condition **G** of Cabré [10]. Let $\sigma < 1$, $\tau < 1$, and R_0 be positive real numbers. We say that an open connected set Ω of $\mathbb{R}^n$ is a **G** domain if to each $y \in \Omega$ we can associate a ball $B = B_R(x_y)$ of radius $R \leq R_0$ such that

$$y \in B_{\tau R}(x_y), \quad |B \setminus \Omega_y| \geq \sigma |B|, \qquad (\mathbf{G}_{\sigma,\tau}) \tag{4.1}$$

where Ω_y is the connected component of $\Omega \cap B$ containing y.

Since $\mathbf{G}_\sigma \equiv \mathbf{G}_{\sigma,1}$, then a **G** domain of $\mathbb{R}^n$ is of cylindrical type, like domains of finite Lebesgue n-dimensional measure, subdomains of $\omega \times \mathbb{R}^{n-k}$, where ω has finite Lebesgue k-dimensional measure, the complement of the spiral of equation $r = \theta$ in polar coordinates of $\mathbb{R}^2$.

Given a differential operator with structure conditions, like (1.4) and (1.5), Ω will be called a *narrow domain* when, for given τ and R_0, condition $\mathbf{G}_{\sigma,\tau}$ is satisfied for σ suitably close to 1, depending on the structure constants and the remaining geometric constants.

A straightforward application of Theorem 3.8 yields MP in narrow domains. Indeed, assume that $u \leq N$ and $F(x, u, Du, D^2u) \geq 0$ in Ω. Then, by (1.5), we have

$$\mathcal{P}^+(D^2u^+) + b_0|Du^+|^q \geq 0. \tag{4.2}$$

Suppose that $u \leq 0$ on $\partial\Omega$ and set $M = \sup_\Omega u^+$. Applying Theorem 3.8 in $A = \Omega_y$ with $p = 1$, we obtain

$$u(y) \leq \sup_{\Omega_y \cap B_{\tau R}(x_y)} u \leq \frac{C}{|B|}\int_{\Omega_y \cap B} u^+ \leq CM\frac{|\Omega_y \cap B|}{|B|} \leq CM(1 - \sigma). \tag{4.3}$$

From this, taking the supremum over $y \in \Omega$, we get $M \leq 0$, that is $u \leq 0$ in Ω, provided $\sigma > 1 - 1/C$, and hence MP holds in this case.

In order to pass from narrow domains to arbitrary cylindrical domains we will use Theorem 3.6, from which the following ABP-type estimate follows.

Theorem 4.1 (ABP estimate). *Let $\sigma, \tau < 1$, let R_0 and N be positive real numbers, and $1 \leq q \leq 2$. Let Ω be a cylindrical domain such that condition $\mathbf{G}_\sigma$ in Ω is satisfied at each $y \in \Omega$ with $R(y) \leq R_0$.*

Suppose that $w \in \mathrm{USC}(\overline{\Omega})$ is a viscosity solution of $F(x, w, Dw, D^2w) \geq f$, under the structure condition (1.5), with $b \leq b_0$ and $f \in C(\overline{\Omega})$.

If $w \leq N$ in Ω and $w \leq 0$ on $\partial\Omega$, then

$$\sup_{\Omega} w \leq C R_0 \sup_{y \in \Omega} \|f\|_{L^n(B_{R(y)/\tau}(x_y) \cap \Omega)}, \tag{4.4}$$

where C depends on $n, \lambda, \Lambda, \sigma, \tau$, and $b_0 R_0^{2-q} N^{q-1}$.

Proof. It is enough to show the result for $\tau \to 1^-$.

Set $M = \sup_\Omega w^+$ and $u = M - w$. Let $y \in \Omega$ and $B = B_R$ of radius R, provided by condition $\mathbf{G}_\sigma$ in y. We choose $R = r_y$ such that $|B \setminus \Omega_y| = \sigma |B|$; see the beginning of this section. We also denote by $B_{\tau R}$ the concentric ball of radius τR.

Now we apply Theorem 3.6 to u in $A = \Omega_y$ with $B_{\tau R}(x_y)$ instead of B_R and τ close enough to 1 in such a way that $|B_{\tau R}(x_y) \setminus \Omega_y| \geq (\sigma/2)|B|$ and $|B_{\tau R}(x_y) \cap \Omega_y| \geq ((1-\sigma)/2)|B|$. Since $w \leq 0$ on $\partial\Omega$, then $m \geq M$, hence we get

$$\begin{aligned}\left(\frac{\sigma}{2}\right)^{1/p_0} M &\leq \left(\frac{1}{|B_{\tau R}(x_y)|} \int_{B_{\tau R}(x_y)} (u_m^-)^{p_0}\right)^{1/p_0} \\ &\leq C\left(M - \sup_{B_{\tau R}(x_y) \cap \Omega_y} w + R\|f\|_{L^n(B \cap \Omega)}\right),\end{aligned} \tag{4.5}$$

from which, for $x \in B_{\tau R}(x_y) \cap \Omega_y$, we obtain the pointwise inequality

$$w(x) \leq \sup_{B_{\tau R}(x_y) \cap \Omega_y} w \leq tM + R\|f\|_{L^n(B \cap \Omega)}, \tag{4.6}$$

with $0 < t < 1$. On the other hand, setting $K = tM + R\|f\|_{L^n(B \cap \Omega)}$ and $\Omega_K = \{x \in \Omega \,/\, w(x) > K\}$, by virtue of (4.6) we have $B \setminus \Omega_K \supset B_{\tau R}(x_y) \cap \Omega_y$ and therefore, by our choice of R and τ,

$$\frac{|B \setminus \Omega_K|}{|B|} \geq \frac{|B_{\tau R}(x_y) \cap \Omega_y|}{|B|} \geq \frac{1-\sigma}{2}. \tag{4.7}$$

A further application of Theorem 3.6 to $u = M - w$ in $A = \Omega_K$ yields

$$\begin{aligned}\left(\frac{1-\sigma}{2}\right)^{1/p_0} (M - K) &\leq \left(\frac{1}{|B|} \int_B (u_m^-)^{p_0}\right)^{1/p_0} \\ &\leq C\left(M - \sup_{B \cap \Omega_K} w + R\|f\|_{L^n(B_{R/\tau}(x_y) \cap \Omega)}\right),\end{aligned} \tag{4.8}$$

since in this case $m \le M - K$. From this we deduce that, for $x \in B \cap \Omega_K$

$$\begin{aligned} w(x) &\le \sup_{B\cap\Omega_K} w \le \left(1-t'\right)M + t'K + R\|f\|_{L^n(B_{R/\tau}(x_y)\cap\Omega)} \\ &\le (1-t'+tt')M + 2R\|f\|_{L^n(B_{R/\tau}(x_y)\cap\Omega)}, \end{aligned} \tag{4.9}$$

with $0 < t' < 1$. From the definition of Ω_K, see also (4.6), it follows that

$$w(x) \le t''M + 2R\|f\|_{L^n(B_{R/\tau}(x_y)\cap\Omega)}, \tag{4.10}$$

with $0 < t'' < 1$, for all $x \in \Omega \cap B$ and hence also for $x = y$.

Finally, passing to the supremum over $y \in \Omega$, we get the result. □

4.2. General domains

Firstly, we consider **wG** domains Ω, such that condition $\mathbf{G}_\sigma$ in Ω holds at each $y \in \Omega$ without bounds for the radii $R(y)$ of the balls provided by $\mathbf{G}_\sigma$.

Note that in general the ABP-type estimate of Theorem 4.1 is useless unless $b_0 = 0$, see [13], since the the constant C of ABP estimate blows up when $R \to +\infty$. This is why we assume $b(x) = O(1/|x|^{2-q})$ as $|x| \to +\infty$ in the structure condition (1.5). Moreover, to take advantage from the decay of $b(x)$, it is convenient to use the boundary wH inequality for annular regions of Corollary 3.7 rather than Theorem 3.6.

Reasoning as in the proof of Theorem 4.1, but quite more carefully with the aid of (3.24) instead of (3.16), see [16], we get the following ABP-type estimate.

Theorem 4.2 (ABP). *Let σ and N be positive real numbers and $1 \le q \le 2$. Let Ω be a **wG** domain (with parameter $\sigma < 1$).*

Suppose that $w \in \mathrm{USC}(\overline{\Omega})$ is a viscosity solution of $F(x, w, Dw, D^2w) \ge f$, under the structure condition (1.5)*, with $b, f \in C(\overline{\Omega})$ such that*

$$b_q := \sup_{y\in\Omega;|y|\ge\varepsilon r_y} r_y^{2-q}\|b\|_{L^\infty(\Omega\cap B_{\tau\varepsilon r_y,\tau' r_y})} < +\infty, \tag{4.11}$$

for all $\varepsilon > 0$ small enough, all $\tau < 1$ sufficiently close to 1, and some $\tau' > 1$.

If $w \le N$ in Ω and $w \le 0$ on $\partial\Omega$, then

$$\sup_\Omega w \le C \sup_{y\in\Omega;|y|\ge\varepsilon r_y} r_y\left\|f^-\right\|_{L^n(\Omega\cap B_{\tau\varepsilon r_y,\tau' r_y})} + \sup_{y\in\Omega;|y|\le\varepsilon r_y} C_y r_y\left\|f^-\right\|_{L^n(\Omega\cap B_{\varepsilon r_y})} \tag{4.12}$$

for possibly smaller $\varepsilon > 0$ and larger $\tau < 1$, depending on n and σ.

Here C and C_y are positive constants depending on n, q, λ, Λ, b_qN^{q-1}, σ, ε, τ, τ', while C_y also depends on $N^{q-1}r_y^{2-q}\|b\|_{L^\infty(\Omega\cap B_{\varepsilon r_y})}$.

Proof of Theorem 1.2. In the case of **wG** domains, Theorem 1.2 follows at once letting $f = 0$ in Theorem 4.2. Suppose now that Ω can be split by a closed set $H \subset \mathbb{R}^n$ in components where MP holds and each $y \in H$ satisfies condition $\mathbf{G}_\sigma$ in Ω. By MP in the components, since we assume that $w \le 0$ on $\partial\Omega$, then for $x \in \Omega$ we have

$$w(x) \le \sup_{(\Omega\cap H)\cup\partial\Omega} w^+ = \sup_{\Omega\cap H} w^+. \tag{4.13}$$

Reasoning as above for (4.10), but using Corollary 3.7 instead of Theorem 3.6 as before to obtain Theorem 4.2, from condition $\mathbf{G}_\sigma$ we deduce for $y \in \Omega \cap H$ that

$$w(y) \le t \sup_{\Omega} w^+, \tag{4.14}$$

where $t \in]0,1[$ is independent of y. Inserting this inequality in the former one, and taking the supremum over Ω, we get the result. □

Examples

Provided that $b(x) = O(1/|x|^{2-q})$ as $|x| \to \infty$, this last result yields MP in very general domains such as, for instance:

(i) **wG** domains, like a proper cone Ω such that $\overline{\Omega} \ne \mathbb{R}^n$ and in general a domain of conical type, like the complement Ω in $\mathbb{R}^n$ of $\Gamma \times \mathbb{R}^{n-2}$, where Γ is a logarithmic spiral of equation $r = e^\theta$ in polar coordinates, or also complement of a larger spiral of equation $r = s(\theta)$, with s a positive increasing function.

(ii) Domains which can be split in **wG** subdomains by a suitable closed set H of $\mathbb{R}^n$, like the cut plane in $\mathbb{R}^2$ or in general the complement in $\mathbb{R}^n$ of a graph $\{(x,y) \in \mathbb{R}^{n-1} \times \mathbb{R} \mid x_i \ge 0,\ i = 1,\dots,n-1,\ y = f(x)\}$ such that $|f(x)| \le h + k|x|$ for positive constants h and k.

As a further example, we show a repeated application of Theorem 1.2. Look at the complement Ω in $\mathbb{R}^2$ of a sequence of balls $B_r(k)$, $k = (k_x, 0), k_x \in \mathbb{N}$, with $0 < r < 1/3$. Consider the nonnegative x -axis as H, then $\Omega_H = \Omega \setminus H$ is connected. If K is the half-line of equations $y = (1/2)x$, $x \ge 0$, then we have the following:

(i) $\Omega_H \setminus K$ has two components, which are domains of conical type, where MP holds;

(ii) each point of $\Omega_H \cap K$ satisfies condition $\mathbf{G}_{1/2}$ in Ω_H.

Thus MP holds in $\Omega \setminus H$ by Theorem 1.2. Also, each point of H satisfies condition $\mathbf{G}_\sigma$ in Ω for some $\sigma \in\ ]0,1[$ depending on r. Therefore, again by Theorem 1.2, we conclude that MP holds in Ω.

4.3. Parabolic shaped domains

For a parabolic shaped **wG** domain, condition $\mathbf{G}_\sigma$ at $y \in \Omega$ holds with $R(y) = O(|y|^\alpha)$ as $|y| \to \infty$, for some $0 \le \alpha \le 1$, the limit cases $\alpha = 0$ and $\alpha = 1$ representing, respectively, the cylindrical and the conical cases. Hence $r_y \le R(y) \le R_0 + \beta|y|^\alpha$ for all $y \in \Omega$ with positive constants R_0 and β. Then, choosing ε sufficiently small in Theorem 4.2, if $|y| \le \varepsilon r_y$, then

$$|y| \le \varepsilon R_0 + \varepsilon\beta|y|^\alpha \le \varepsilon\left(R_0 + \beta\right) + \varepsilon\beta|y|, \tag{4.15}$$

so that the supremum in the second term of the right-hand side of (4.12) is taken over a bounded subset of $y \in \Omega$, in which $r_y \le R_1$ for some positive constant R_1. Thus

$$\sup_{y \in \Omega; |y| \le \varepsilon r_y} r_y \left\| f^- \right\|_{L^n(\Omega \cap B_{\varepsilon r_y})} \le R_1 \sup_{y \in \Omega; r_y \le R_1} \left\| f^- \right\|_{L^n(\Omega \cap B_{\varepsilon r_y})}. \tag{4.16}$$

Proof of Theorem 1.3. Since condition **wG** holds with $r_y = O(|y|^\alpha)$, $0 \le \alpha \le 1$, the assumption $b(x) = O(1/|x|^{\alpha(2-q)})$ as $|x| \to +\infty$ implies the finiteness of b_q in (4.11). Taking account of (4.16), by continuity of f the estimate (1.10) follows letting $\varepsilon \to 0$. □

5. Strong maximum principle and Liouville theorem

The weak Harnack inequality of Theorem 3.3 can be used to show the following strong MP.

Theorem 5.1 (strong MP). *Let Ω be a domain of $\mathbb{R}^n$. Let $w \in C(\Omega)$ be such that $F(x, w, Dw, D^2w) \geq 0$ in the viscosity sense, and assume structure condition* (1.5)*, with $b \in C(\overline{\Omega})$. If $x_0 \in \Omega$ and $M := w(x_0) \geq w(x)$, for all $x \in \Omega$, then $w \equiv M$ in Ω.*

Proof. Following [33], set $\Omega_1 = w^{-1}(\{M\})$ and $\Omega_2 = \Omega \backslash \Omega_1$. By assumption $\Omega_1 \neq \varnothing$. By continuity of w, it turns out that $\Omega_2 = w^{-1}(]-\infty, M[)$ is an open subset of $\mathbb{R}^n$. Moreover, plainly, $\Omega = \Omega_1 \cup \Omega_2$ and $\Omega_2 \cap \Omega_1 = \varnothing$.

Recall that Ω is an open connected set. Thus it is sufficient to show that Ω_1 is in turn an open subset to have $\Omega = \Omega_1$, as claimed in the statement of the theorem. Indeed, let $x_1 \in \Omega_1$, that is, $w(x_1) = M$, and set $u = M - w$, then u is a nonnegative viscosity solution of $F(x, u, Du, D^2u) \leq 0$. Applying (3.7) in a ball $B_R := B_R(x_1) \subset B_{R/\tau}(x_1) \subset\subset \Omega$, we get

$$\left(\frac{1}{|B_R|}\int_{B_R}(M-w)^{p_0}\right)^{1/p_0} \leq C \inf_{B_R} u = 0, \tag{5.1}$$

from which, by continuity, $u \equiv M$ in $B_R(x_1)$. This shows that Ω_1 is an open subset of $\mathbb{R}^n$ and concludes the proof. □

The Liouville type result of Theorem 1.5 is instead based on Harnack inequality (3.11) of Theorem 3.5. It is convenient to consider its version in annular regions $B_{R,2R} = B_{2R}(0) \setminus \overline{B}_R(0)$ to take advantage of the decay of $b(x)$, obtained in standard way, using inequality (3.11) in a chain of linked balls. This yields, for continuous solutions $u \in C(\overline{B}_{R/2,4R})$ of equation $F = f$, $0 \leq u \leq N$, under the structure conditions (1.4) and (1.5), with $b, f \in C(\overline{B}_{R/2,4R})$, the following inequality:

$$\sup_{B_{R,2R}} u \leq C\Big(\inf_{B_{R,2R}} u + R\|f\|_{L^n(B_{R/2,4R})}\Big), \tag{5.2}$$

with a positive constant $C = C(n, \lambda, \Lambda, q, \tau, \|b\|_{L^\infty(B_{R/2,4R})} N^{q-1} R^{2-q})$.

Proof of Theorem 1.5. By the strong MP of Theorem 5.1, we know that w can achieve neither a maximum nor a minimum at a point of $\mathbb{R}^n$ unless it is constant, in which case we should be done.

Suppose for instance that $w \leq M := \sup w < +\infty$. Let R_k be an increasing sequence of positive numbers such that $\lim_{k\to\infty} R_k = \infty$. Set $M_k = \sup_{\partial B_{R_k}} w$ and $m_k = \inf_{\partial B_{R_k}} w$. By weak maximum principle, M_k is increasing and m_k is decreasing; thus

$$\lim_{k\to\infty} M_k = M, \qquad \lim_{k\to\infty} m_k = m \in [-\infty, +\infty[. \tag{5.3}$$

Then, using Harnack inequality (5.2), with $u = M - w$, we get

$$M - m_k = \sup_{\partial B_{R_k}}(M - w) \leq C \inf_{\partial B_{R_k}} (M - w) = C\big(M - M_k\big), \tag{5.4}$$

from which

$$M \leq C\big(M - M_k\big) + m_k \leq C\big(M - M_k\big) + m_k \tag{5.5}$$

and, letting $k \to \infty$, we get $M = m$, as we wanted to show. □

Appendix

Proof of Lemmas 2.2 and 2.3

Although the proof of Lemma 2.2 is already contained in previous papers also in the case of an almost-quadratic gradient term, see for instance [17], here, for the sake of completeness, we give a sketch of the simple version in the case of linear gradient term, following [3], where the fundamental case of a second-order uniformly elliptic operator is treated, with no lower-order terms.

However, it seems useless to repeat the nice proof of [3], to which we refer for the ideas and details. We only outline the steps which are influenced by the first-order term. For this reason, we keep the same notations of [3].

Also, for the sake of brevity, we will refer to constants depending only on n, λ, Λ, b_0 as to structural constants.

Firstly, we introduce a test function; see [3, Lemma 4.1].

Lemma A.1. *There exist positive structural constants M, C and a function $\varphi \in C^2(\mathbb{R}^n)$ such that*

$$\varphi \leq -2 \quad \text{in } B_{(3/2)\sqrt{n}}, \qquad \varphi \geq 0 \quad \text{in } \mathbb{R}^n \setminus B_{2\sqrt{n}}, \tag{A.1}$$

$$\varphi \geq -M \quad \text{in } \mathbb{R}^n, \tag{A.2}$$

$$\mathcal{P}^+_{\lambda,\Lambda}(D^2\varphi) + b_0|D\varphi| \leq C\xi \quad \text{in } \mathbb{R}^n, \tag{A.3}$$

where $\xi \in C(\mathbb{R}^n)$, $0 \leq \xi \leq 1$, $\operatorname{supp} \xi \subset B_{1/2}$.

Proof. We search for a function of type $\varphi(x) = A_1 - A_2 e^{-\alpha r}$, for $r = |x| \geq 1/4$, where A_1 and A_2 are positive constants to be chosen in order that $\phi((3/2)\sqrt{n}) = -2$ and $\phi(2\sqrt{n}) = 0$. Next, we extend φ to $\mathbb{R}^n$ in such a way that $\varphi \geq -M$. By calculations, choosing $\alpha = 4\Lambda(n-1) + 1/\lambda$ we have

$$\mathcal{P}^+_{\lambda,\Lambda}(D^2\varphi) + b_0|D\varphi| = \alpha A_2 e^{-\alpha r}\left(\frac{\Lambda(n-1)}{r} - \alpha\lambda + 1\right) \leq 0 \tag{A.4}$$

for $r \geq 1/4$. Also, for $r \leq 1/4$,

$$\mathcal{P}^+_{\lambda,\Lambda}(D^2\varphi) + b_0|D\varphi| \leq C \tag{A.5}$$

and therefore (A.3) holds taking a cut-off function $\xi \in C(\mathbb{R}^n)$ such that $\xi = 1$ in $\overline{B}_{1/4}$ and $\xi = 0$ outside $B_{1/2}$. □

Next, we get a lower bound for the size of level sets of supersolutions. Denoting by Q_l a cube of side l, consider a nonnegative viscosity solution $u \in \mathrm{LSC}(Q_{4\sqrt{n}})$ of the differential inequality $\mathcal{P}^-_{\lambda,\Lambda}(D^2u) - b_0|Du| \leq f$.

Setting $w = u + \varphi$ and observing that

$$\mathcal{P}^-_{\lambda,\Lambda}(D^2w) - b_0|Dw| \leq \mathcal{P}^-_{\lambda,\Lambda}(D^2u) + \mathcal{P}^+_{\lambda,\Lambda}(D^2\varphi) - b_0|Du| + b_0|D\varphi| \leq f^+ + C\xi, \tag{A.6}$$

a positive lower bound

$$|\{u \leq M\} \cap Q_1| > \mu, \tag{A.7}$$

with a structural positive constant $\mu < 1$, see (A.2), follows, in the same way as in [3, Lemma 4.5], using ABP estimate (2.5), provided that $\inf_{Q_3} u \leq 1$ and $\|f^+\|_{L^n(Q_{4\sqrt{n}})} \leq \varepsilon_0$ for a positive structural constant ε_0. Moreover, under the same assumptions, [3, Lemma 4.6] says that

$$\left|\{u > M^k\} \cap Q_1\right| \leq (1-\mu)^k \tag{A.8}$$

for all $k \in \mathbb{N}$, which for $k = 1$ agrees with (A.7). Then we point out that (A.8) follows by an induction process, based on the Calderón-Zygmund decomposition of the cube Q_1, centered at the origin, supposing (A.8) to hold for $k-1$.

To perform the induction step it is crucial that, for a supersolution $u(x)$ of (2.4), the rescaled function $\widetilde{u}(y) = u(x)/M^{k-1}$, where $x = x_0 + 2^{-i}y$ runs in the dyadic cube $Q_{2^{-i}4\sqrt{n}}$, centered at x_0, is in turn a supersolution of (2.4) with a correspondingly scaled f, namely

$$\mathcal{P}^-_{\lambda,\Lambda}\left(D^2\widetilde{u}(y)\right) - b_0\left|D\widetilde{u}(y)\right| \leq \frac{f^+(x)}{2^{2i}M^{k-1}} \tag{A.9}$$

for $y \in Q_{4\sqrt{n}}$. From (A.8) it follows that

$$\left|\{u > t\} \cap Q_1\right| \leq dt^{-\varepsilon} \tag{A.10}$$

for all $t > 0$, with d and ε positive structural constants. Then, following the proof of Theorem 4.8(1) of [3], we use (A.8) in the identity

$$\int_{Q_1} u^{p_0} = p_0 \int_0^{+\infty} \left|\{u \geq t\} \cap Q_1\right| dt, \tag{A.11}$$

see [1], with $p_0 = \varepsilon/2$ and, by rescaling, remove the normalization conditions $\inf_{Q_3} u \leq 1$ and $\|f^+\|_{L^n(Q_{4\sqrt{n}})} \leq \varepsilon_0$ to get

$$\|u\|_{L^{p_0}(Q_1)} \leq C\left(\inf_{Q_3} u + \left\|f^+\right\|_{L^n(Q_{4\sqrt{n}})}\right). \tag{A.12}$$

From this, with a covering argument as in [10, Theorem 3.1], we obtain (2.8).

We argue in the same manner for Lemma 2.3. Suppose again that Q_1 is centered at the origin. Following the proof of Theorem 4.8(2) of [3], firstly we consider a subsolution u of (2.6) such that $\|f^+\|_{L^n(Q_{4\sqrt{n}})} \leq \varepsilon_0$ and $\|u^+\|_{L^\varepsilon(Q_1)} \leq d^{1/\varepsilon}$ to get, even in this case, (A.10). Then, arguing as in [3, Lemma 4.7], there exist structural constants $M_0 > 1$ and $\sigma > 0$ such that, for all $j \in \mathbb{N}$ large enough,

$$\left|x_0\right| < \frac{1}{4}, \quad u(x_0) \geq \nu^{j-1} \Longrightarrow Q^j := Q_{l_j}(x_0) \subset Q_1, \ \sup_{Q^j} \geq \nu^j M_0, \tag{A.13}$$

where $\nu = M_0/(M_0 - 1/2)$ and $l_j = \sigma M_0^{-\varepsilon/n}\nu^{-\varepsilon j/n}$. As above, to get this result we use the invariance of equation by scale transformations, namely that the function $v(y) = \nu/(\nu - 1) - u(x)/\nu^{j-1}(\nu-1)M_0$, where $x = x_0 + (4\sqrt{n})^{-1}l_j y$ runs in the small cube Q^j, is in turn a supersolution of (2.4) with a correspondingly scaled f, that is,

$$\mathcal{P}^-_{\lambda,\Lambda}\left(D^2 v(y)\right) - b_0\left|D\widetilde{v}(y)\right| \leq \frac{f^-(x)}{\nu^{j-1}(\nu-1)M_0} \tag{A.14}$$

for $y \in Q_{4\sqrt{n}}$, provided that $j > 1 + \log(2 - 1/M_0)/\log\nu$.

On the base of (A.13), reasoning as in the proof of Lemma 4.4 of [3], we infer that $\sup_{Q_{1/4}} u \le C$, from which, by rescaling to remove normalization conditions $\|f^+\|_{L^n(Q_{4\sqrt{n}})} \le \varepsilon_0$ and $\|u^+\|_{L^\varepsilon(Q_1)} \le d^{1/\varepsilon}$, we get

$$\sup_{Q_{1/4}} u \le C\left(\|u^+\|_{L^\varepsilon(Q_1)} + \|f^+\|_{L^n(Q_{4\sqrt{n}})}\right). \tag{A.15}$$

as in the proof of Theorem 4.8 (2) of [3]. By a covering argument, as above for supersolutions, we get (2.9) for $p = \varepsilon$. Note that (A.10) a fortiori holds replacing ε with $p < \varepsilon$. Thus (2.9) follows for all $0 < p < \varepsilon$. Finally, by Hölder inequality, we obtain (2.9) for all $p > 0$.

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