

*Research Article*

## **Recovery of Time-Dependent Parameters of a Black-Scholes-Type Equation: An Inverse Stieltjes Moment Approach**

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We show that the problem of recovering the time-dependent parameters of an equation of Black-Scholes type can be formulated as an inverse Stieltjes moment problem. An application to the problem of implied volatility calculation in the case when the model parameters are time varying is provided and results of numerical simulations are presented.

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### **1. Introduction**

Suppose that we are given a function  $u = u(x, \tau)$  ( $x \geq 0$ ,  $\tau \geq 0$ ) obtained from interpolation of experimental data. We wish to fit  $u$  so as to satisfy the following equation of Black-Scholes type:

$$u_\tau = \alpha(\tau)x^2u_{xx} + \beta(\tau)xu_x + \gamma(\tau)u, \quad (1.1)$$

where  $\alpha$ ,  $\beta$ , and  $\gamma$  are unknown functions. We are interested in recovering these time-dependent model parameters. Finding a method to solve this inverse problem is of prime importance in several areas of application. For instance, it is a central concern in quantitative finance to recover the implied volatility embedded in  $\alpha$  through (1.1) [1–3].

In this paper, we show that this problem can be formulated as an inverse Stieltjes moment problem and demonstrate how to apply our method in the computation of the implied volatility. We also present some results of numerical simulations illustrating our method.

## 2. The inverse Stieltjes moment problem

For each nonnegative integer  $n$  and for each  $\tau \geq 0$ , define the  $n$ th moment  $m_n$  of  $u$  as

$$m_n(\tau) := \int_0^\infty x^n u(x, \tau) dx. \quad (2.1)$$

We assume that  $u$  is such that for all  $\tau \geq 0$ ,  $u(0, \tau)$  and  $u_x(0, \tau)$  are bounded, and that

$$\lim_{x \rightarrow \infty} x^n u(x, \tau) = \lim_{x \rightarrow \infty} x^n u_x(x, \tau) = 0 \quad (2.2)$$

for all  $n$ . In applications involving the Black-Scholes equation [4], the function  $u$  is also taken to be nonnegative.

Using integration by parts and the assumptions on  $u$ , we obtain

$$\begin{aligned} \int_0^\infty x^{n+1} u_x(x, \tau) dx &= -(n+1)m_n(\tau), \\ \int_0^\infty x^{n+2} u_{xx}(x, \tau) dx &= -(n+2) \int_0^\infty x^{n+1} u_x(x, \tau) dx = (n+1)(n+2)m_n(\tau), \\ \int_0^\infty x^n u_\tau(x, \tau) dx &= \frac{d}{d\tau} \int_0^\infty x^n u(x, \tau) dx = m'_n(\tau). \end{aligned} \quad (2.3)$$

Multiplying (1.1) by  $x^n$  and using the above integrals, we see that  $m_n$  satisfies the ordinary differential equation

$$m'_n(\tau) = p_n(\tau)m_n(\tau), \quad p_n(\tau) := (n+1)(n+2)\alpha(\tau) - (n+1)\beta(\tau) + \gamma(\tau). \quad (2.4)$$

The corresponding initial condition is  $m_n(0) = a_n := \int_0^\infty x^n u(x, 0) dx$ . Thus, the moments of  $u$  are expressible as

$$m_n(\tau) = \int_0^\infty x^n u(x, \tau) dx = a_n \exp \left[ \int_0^\tau p_n(s) ds \right]. \quad (2.5)$$

In the classical Stieltjes moment problem [5], we are given the moments  $m_n$  and we seek the function  $u$ . In this case, however, we are given  $u$  and we want to find  $\alpha$ ,  $\beta$ , and  $\gamma$  which are implicit in  $m_n$ . Hence, the problem of the recovery of the time-dependent model parameters of (1.1) is equivalent to an inverse Stieltjes moment problem.

Taking any three consecutive moments of  $u$ , we see from (2.4) that

$$\begin{aligned} (n+1)(n+2)\alpha(\tau) - (n+1)\beta(\tau) + \gamma(\tau) &= \frac{m'_n(\tau)}{m_n(\tau)}, \\ (n+2)(n+3)\alpha(\tau) - (n+2)\beta(\tau) + \gamma(\tau) &= \frac{m'_{n+1}(\tau)}{m_{n+1}(\tau)}, \\ (n+3)(n+4)\alpha(\tau) - (n+3)\beta(\tau) + \gamma(\tau) &= \frac{m'_{n+2}(\tau)}{m_{n+2}(\tau)}. \end{aligned} \quad (2.6)$$

The solution of this linear system is

$$\begin{aligned}
 \alpha(\tau) &= \frac{1}{2} \frac{m_n'(\tau)}{m_n(\tau)} - \frac{m_{n+1}'(\tau)}{m_{n+1}(\tau)} + \frac{1}{2} \frac{m_{n+2}'(\tau)}{m_{n+2}(\tau)}, \\
 \beta(\tau) &= (n+3) \frac{m_n'(\tau)}{m_n(\tau)} - (2n+5) \frac{m_{n+1}'(\tau)}{m_{n+1}(\tau)} + (n+2) \frac{1}{2} \frac{m_{n+2}'(\tau)}{m_{n+2}(\tau)}, \\
 \gamma(\tau) &= \frac{1}{2} (n+2)(n+3) \frac{m_n'(\tau)}{m_n(\tau)} - (n+1)(n+3) \frac{m_{n+1}'(\tau)}{m_{n+1}(\tau)} \\
 &\quad + \frac{1}{2} (n+1)(n+2) \frac{m_{n+2}'(\tau)}{m_{n+2}(\tau)}.
 \end{aligned} \tag{2.7}$$

Since  $\alpha$ ,  $\beta$ , and  $\gamma$  should be independent of  $n$ , we can take  $n = 0$  for simplicity.

In some applications, not all three functions are needed to be determined since some are given. For example, suppose that  $\beta$  and  $\gamma$  are known and so only  $\alpha$  is required. In this case, only one moment is utilised in (2.4):

$$\alpha(\tau) = \frac{1}{(n+1)(n+2)} \left[ \frac{m_n'(\tau)}{m_n(\tau)} + (n+1)\beta(\tau) - \gamma(\tau) \right]. \tag{2.8}$$

Again, we can take  $n = 0$  for simplicity. Alternatively, we can choose to express  $\alpha$  in terms of three consecutive moments using the first equation of (2.7), which does not make explicit use of the given data  $\beta$  and  $\gamma$  but are “encapsulated” in the moments.

In the special case when  $\alpha$ ,  $\beta$ , and  $\gamma$  are all constants, then (2.5) simplifies to

$$m_n(\tau) = a_n \exp(p_n \tau). \tag{2.9}$$

Isolating  $\alpha$  yields

$$\alpha = \frac{1}{(n+1)(n+2)} \left[ \frac{1}{\tau} \log \frac{m_n(\tau)}{a_n} + (n+1)\beta - \gamma \right]. \tag{2.10}$$

We can also eliminate  $\beta$  and  $\gamma$  by solving for  $p_n$  in (2.9) and considering three consecutive moments, thus obtaining the linear system

$$\begin{aligned}
 (n+1)(n+2)\alpha - (n+1)\beta + \gamma &= \frac{1}{\tau} \log \frac{m_n(\tau)}{a_n}, \\
 (n+2)(n+3)\alpha - (n+2)\beta + \gamma &= \frac{1}{\tau} \log \frac{m_{n+1}(\tau)}{a_{n+1}}, \\
 (n+3)(n+4)\alpha - (n+3)\beta + \gamma &= \frac{1}{\tau} \log \frac{m_{n+2}(\tau)}{a_{n+2}}.
 \end{aligned} \tag{2.11}$$

Solving this system for  $\alpha$  gives

$$\alpha = \frac{1}{2\tau} \log \frac{m_n(\tau)}{a_n} - \frac{1}{\tau} \log \frac{m_{n+1}(\tau)}{a_{n+1}} + \frac{1}{2\tau} \log \frac{m_{n+2}(\tau)}{a_{n+2}}. \tag{2.12}$$

Note that we cannot use (2.8) nor the first equation of (2.7) directly to get (2.10) and (2.12), respectively, since the quotient of a moment's derivative and the moment is a constant, so that the moment disappears from the expression and we obtain an identity.

### 3. An application to implied volatility calculation

Our result can be applied in a straightforward manner to address the problem of implied volatility calculation [6] arising in option pricing theory. Specifically, we wish to recover the volatility  $\sigma$  from (1.1) given that the other parameters are known. In this context, (1.1) is called Dupire's equation and  $u(x, \tau)$  is the value of a European call option at the strike price  $x$  and at time to maturity  $\tau$ . The model parameters are

$$\alpha(\tau) = \frac{1}{2}\sigma(\tau)^2, \quad \beta(\tau) = q(\tau) - r(\tau), \quad \gamma(\tau) = -q(\tau), \quad (3.1)$$

where  $r$  is the riskless interest rate,  $q$  is the dividend yield (both known a priori), and  $\sigma$  is the unknown volatility. The initial condition is  $u(x, 0) = \max(S - x, 0)$ , where  $S$  is the asset price, which implies that

$$a_n = m_n(0) = \int_0^\infty x^n u(x, 0) dx = \frac{S^{n+2}}{(n+1)(n+2)}. \quad (3.2)$$

Using (2.8), we obtain

$$\sigma(\tau)^2 = \frac{2}{(n+1)(n+2)} \left[ \frac{m_n'(\tau)}{m_n(\tau)} + (n+2)q(\tau) - (n+1)r(\tau) \right]. \quad (3.3)$$

Equation (3.3) gives a closed-form representation of the volatility parameter estimated from market prices of options and when the values of  $r$  and  $q$  are available. An alternative expression is obtained by using the first equation of (2.7):

$$\sigma(\tau)^2 = \frac{m_n'(\tau)}{m_n(\tau)} - 2 \frac{m_{n+1}'(\tau)}{m_{n+1}(\tau)} + \frac{m_{n+2}'(\tau)}{m_{n+2}(\tau)}. \quad (3.4)$$

When  $r$ ,  $q$ , and  $\sigma$  are all constants, then (2.10) gives

$$\sigma^2 = \frac{2}{(n+1)(n+2)} \left[ \frac{1}{\tau} \log \frac{m_n(\tau)}{a_n} + (n+2)q - (n+1)r \right], \quad (3.5)$$

whilst from (2.12) we have another representation in terms of three consecutive moments:

$$\sigma^2 = \frac{1}{\tau} \left[ \log \frac{a_{n+1}^2}{a_n a_{n+2}} + \log \frac{m_n(\tau) m_{n+2}(\tau)}{m_{n+1}(\tau)^2} \right], \quad (3.6)$$

where  $a_n$  is given in (3.2).

### 4. Numerical results

For the numerical simulations, we assume that  $r$  and  $\sigma$  are constants and that  $q = 0$ . This allows us to compare the numerical and theoretical values for the volatility. To obtain a

numerical estimate of the implied volatility, we need to input a table of observed values for the call price  $u$  versus the strike price  $x$ , that is, a table of the form

$x$	$x_1$	$x_2$	$\cdots$	$x_M$
$u$	$u_1$	$u_2$	$\cdots$	$u_M$

where the time to maturity  $\tau$ , the riskless rate  $r$ , and the asset price  $S$  are assumed to be given if we are to use the formula in (3.5). However, if we decide to utilise the alternative formula (3.6), all we need is the time to maturity  $\tau$  and the given table.

Next, we generate an interpolating function which passes through all the given data points  $(x_1, u_1), (x_2, u_2), \dots, (x_M, u_M)$  and satisfies the conditions  $u(0, \tau) = S$  (see the Black-Scholes formula (4.1) below) and  $\lim_{x \rightarrow \infty} u(x, \tau) = 0$ . For numerical calculations, in the left-hand condition the strike price is replaced by a value very close to zero but positive with the call price equal to  $S$ , whilst in the right-hand condition the strike price is replaced by a sufficiently large positive value with the call price equal to a very small but positive value. After we obtain the interpolating function, we compute the desired moments and substitute into either (3.5) or (3.6) to estimate the implied volatility.

To generate the input table, we use the theoretical values obtained from the Black-Scholes formula

$$\begin{aligned}
 u &= SN(d_1) - xe^{-r\tau}N(d_2), \\
 d_1 &= \frac{\log(S/x) + (r + \sigma^2/2)\tau}{\sigma\sqrt{\tau}}, \\
 d_2 &= d_1 - \sigma\sqrt{\tau}.
 \end{aligned} \tag{4.1}$$

Here,  $N$  is the cumulative distribution function of a standard normal variable. We assume the parameter values  $r = 0.03$ ,  $S = 150$ ,  $\tau = 0.3$ , and  $\sigma = 0.3$ . We take 12 values for  $x$  (i.e.,  $M = 10$ ), choose  $x_0 = 1.0 \times 10^{-6}$  and  $x_{M+1} = x_{11} = 2S$ , and compute the call price for each such  $x$ . Figure 4.1 shows the calculated data points and the interpolating function generated by MATLAB. The estimated volatilities using (3.5) for  $n = 0, 1, 2, 3$  are also shown for comparison. We can see a very good agreement in the estimated values and the actual value used. In Figure 4.2, we use the same parameter values and data points as in Figure 4.1 but estimate the volatility using (3.6) for  $n = 0, 1$ . Note that here we do not need the values of  $r$  and  $S$ , only that of  $\tau$ , to calculate  $\sigma$ . This fact may be useful in practice when, for example, only the call prices and the time to maturity are available. Again, we see a very good agreement in the estimated values and the actual value used.

The usual method to compute the implied volatility from the Black-Scholes formula is to use an iterative Newton-Raphson algorithm [7]. More precisely, assuming that  $r$ ,  $S$ ,  $x$ ,  $\tau$ , and  $u$  are given, then (4.1) can be viewed as a nonlinear equation in  $\sigma$ , for which the iterative root-finding algorithm of Newton and Raphson can be applied. The approximate solution thus obtained is implicit. By contrast, the expressions for  $\sigma$  given in (3.5) or (3.6) are explicit. Another difference between these two methods is that the Newton-Raphson solution may be quite inaccurate if the option is far in the money or far out of the money because then the option price is essentially independent of the volatility. The formulae

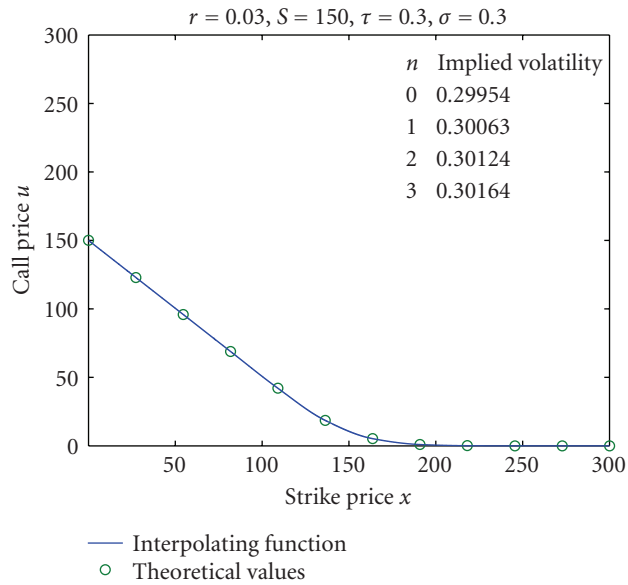


FIGURE 4.1. Theoretical call prices and implied volatility using (3.5).

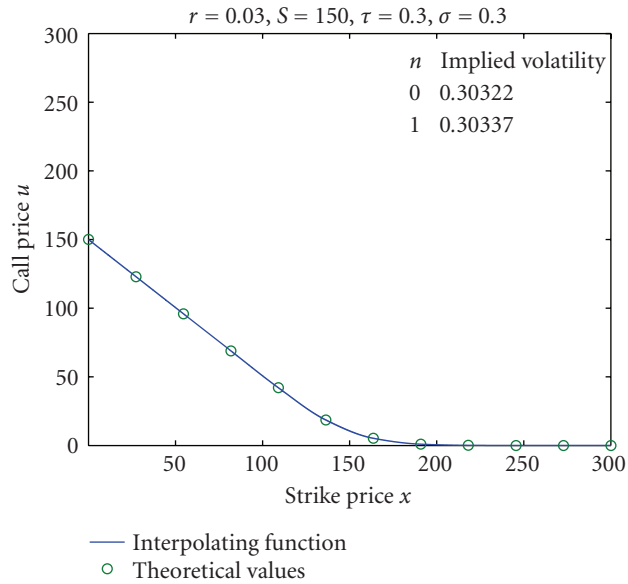


FIGURE 4.2. Theoretical call prices and implied volatility using (3.6).

given in (3.5) or (3.6) can be advantageous in this case since the moneyness is not crucial because the moments are essentially the average of the option prices.

On the other hand, as can be seen in the numerical results above, our method requires as input more than one option price to be able to construct the interpolating function.

This is not a real issue since in practice a list of observed values for the call price versus the strike price is available anyway. Of course, the ideal scenario is to have as many data points as possible to get a better fit for the interpolating function.

An effect of having to input a table of option prices with varying strikes is that the observed prices will most likely have a skew or smile, and would therefore be incompatible with the Black-Scholes model we are assuming. The question is whether the implied volatility obtained is meaningful in the smile/skew case. For example, for a one-year option with forward price 100 and the following observed skew:

Strike	80	100	120
Observed market vol	20%	30%	40%

(with linear interpolation between the strikes), taking  $n = 0$  in formula (3.5) roughly gives the volatility as 37%. This result may not be easy to interpret; however, such a value could serve as a good estimate or benchmark for the current volatility of the price of the underlying asset or variable as it “summarises” the informational content of all available market option prices. Such benchmark for the current volatility is important for other financial modelling endeavours such as the calculation of value-at-risk, where only a single estimate of the current volatility of a financial variable is needed.

## 5. Concluding remarks

In this paper, we showed that the problem of recovering the time-dependent parameters of a Black-Scholes-type equation can be viewed as an inverse Stieltjes moment problem. We showed how this can be applied to the problem of implied volatility calculation in the time-varying case by giving an explicit analytical formula for the volatility. Numerical results were also presented to illustrate the accuracy of our method.

A related problem is on recovering the local volatility surface (see, e.g., [8] and the survey in [9]). In this case, Dupire’s equation takes the form of

$$u_\tau = \frac{1}{2}\sigma(x, \tau)^2 x^2 u_{xx} + [q(\tau) - r(\tau)]xu_x + q(\tau)u, \quad (5.1)$$

where now the volatility  $\sigma$  is a function of both the strike price and the time to maturity. If we solve for  $\sigma$  in this equation, then we can obtain an expression for the volatility surface in terms of the option price and its derivatives. The power of the Dupire formulation lies in the fact that the function  $\sigma$  can fit every given smile or skew. Like the method we proposed in this paper, Dupire’s method also requires an interpolation process. The drawback is that numerical differentiation is very unstable, so a further regularisation procedure has to be implemented. Although the problem we considered here is a special case of the volatility surface problem, the method via moments we proposed relies on integration, which is a smoothening process and does not need regularisation.

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## References

- [1] L. Andersen and R. Brotherton-Ratcliffe, "The equity option volatility smile: an implicit finite-difference approach," *Journal of Computational Finance*, vol. 1, no. 2, pp. 5–37, 1998.
- [2] P. P. Boyle and D. Thangaraj, "Volatility estimation from observed option prices," *Decisions in Economics and Finance*, vol. 23, no. 1, pp. 31–52, 2000.
- [3] B. Dupire, "Pricing with a smile," *Risk*, vol. 7, no. 1, pp. 18–20, 1994.
- [4] F. Black and M. Scholes, "The pricing of options and corporate liabilities," *Journal of Political Economy*, vol. 81, no. 3, pp. 637–654, 1973.
- [5] N. Akhiezer, *The Classical Moment Problem and Some Related Questions in Analysis*, Hafner, New York, NY, USA, 1965.
- [6] E. Derman and I. Kani, "Riding on a smile," *Risk*, vol. 7, no. 2, pp. 32–39, 1994.
- [7] D. Chance, "Leap into the unknown," in *Over the Rainbow: New Developments in Exotic Options and Complex Swaps*, R. Jarrow, Ed., pp. 251–256, Risk, London, UK, 1995.
- [8] M. R. Rodrigo and R. S. Mamon, "A new representation of the local volatility surface," CARISMA Technical Report, Brunel University, London, UK, 2007.
- [9] R. Hafner, *Stochastic Implied Volatility*, vol. 545 of *Lecture Notes in Economics and Mathematical Systems*, Springer, Berlin, Germany, 2004.

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