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Universality in several-matrix models via approximate transport maps

by

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Contents

1. Introduction.

Large random matrices appear in many different fields, including quantum mechanics, quantum chaos, telecommunications, finance, and statistics. As such, understanding how the asymptotic properties of the spectrum depend on the fine details of the model, in particular on the distribution of the entries, soon appeared as a central question.

An important model is the one of Wigner matrices, that is Hermitian matrices with independent and identically distributed real or complex entries. We will denote by N the dimension of the matrix, and assume that the entries are renormalized to have covariance N^{-1}. It was shown by Wigner [68] that the macroscopic distribution of the spectrum converges, under very mild assumptions, to the so-called semi-circle law. However, because the spectrum is a complicated function of the entries, its local properties took much longer to be revealed. The first approach to the study of local fluctuations of the spectrum was based on exact models, namely the Gaussian models, where the joint law of the eigenvalues has a simple description as a Coulomb Gas law [52], [63], [64], [31], [19]. There, it was shown that the largest eigenvalue fluctuates around the boundary of the support of the semi-circle law in the scale $N^{-2/3}$, and that the limit distribution of these fluctuations were given by the so-called Tracy–Widom law [63], [64]. On the other hand, inside the bulk the distance between two consecutive eigenvalues is of order N^{-1} and the fluctuations at this scale can be described by the sine-kernel distribution. Although this precise description was first obtained only for the Gaussian models, it was already envisioned by Wigner that these fluctuations should be universal, i.e., independent of the precise distribution of the entries.

Recently, a series of remarkable breakthroughs [23], [25], [26], [29], [27], [61], [60], [59], [58] proved that, under rather general assumptions, the local statistics of a Wigner matrix are independent of the precise distribution of the entries, provided they have enough finite moments, are centered and with the same variance. These results were extended to the case where distribution of the entries depend on the indices, still assuming that their variance is uniformly bounded below [28]. The study of band-matrices is still a challenge when the width of the band approaches the critical order of $\sqrt{N}$, see related works [57], [24]. Such universality results were also extended to non-normal square matrices with independent entries [62].

A related question is to study universality for local fluctuations for the so-called β-models, that are laws of particles in interaction according to a Coulomb-gas potential to the power β and submitted to a potential V. When $\beta=1,2,4$ and V is quadratic, these laws correspond to the joint law of the eigenvalues of Gaussian matrices with real, complex, or symplectic entries. Universality was proven for very general potentials in the case $\beta=2$ [45], [47]. In the case $\beta=1,4$, universality was proved in [21] in the bulk, and [20] at the edge, for monomial potentials V (see [22] for a review). For general one-cut potentials, the first proof of universality was given in [56] in the case $\beta=1$, whereas [41] treated the case $\beta=4$. The local fluctuations of more general β-ensembles were only derived recently [65], [54] in the Gaussian case. Universality in the β-ensembles was first addressed in [13] (in the bulk, $\beta>0$, $V\in C^4$), then in [14] (at the edge, $\beta\geqslant 1$, $V\in C^4$), [43] (at the edge, $\beta>0$, V convex polynomial), and finally in [56] (in the bulk, $\beta>0$, V analytic, multi-cut case included) and in [5] (in the bulk and the edge, V smooth enough). The universality at the edge in the several-cut case is treated in [4]. The case where the interaction is more general than a Coulomb gas, but given by a mean-field interaction $\prod_{i<j}\varphi(x_i-x_j)$ where $\varphi(t)$ behaves as $|t|^\beta$ in a neighborhood of the origin and both $\log|x|^{-\beta}\varphi(x)$ and the potential are real-analytic, was considered in [32] (β=2, universality in the bulk), [66] (β>0, universality in the bulk), and [42] (β=2, universality at the edge).

Despite all these new developments, up to now nothing was known about the universality of the fluctuations of the eigenvalues in several-matrix models, except in very particular situations. The aim of this paper is to provide new universality results for general perturbative several-matrix models, giving a firm mathematical ground to the widely spread belief coming from physics that universality of local fluctuations should hold, at least until some phase transition occurs.

An important application of our results is given by polynomials in Gaussian Wigner matrices and deterministic matrices. More precisely, let $X_1^N, ..., X_d^N$ be independent $N\times N$ matrices in the Gaussian Unitary Ensemble (GUE), i.e. $N\times N$ Hermitian matrices with independent complex Gaussian entries with covariance $1/N$, and let $B_1^N, ..., B_m^N$ be $N\times N$ Hermitian deterministic matrices. Assume that for any choices of $i_1, ..., i_k\in\{1, ..., m\}$ and $k\in\mathbb{N}$,

$$\frac{1}{N}\mathrm{Tr}(B_{i_1}^N \dots B_{i_k}^N) \tag{1.1}$$

converges to some limit $\tau(b_{i_1}\dots b_{i_k})$, where τ is a linear form on the set of polynomials in the variables $\{b_\ell\}_{\ell=1}^m$ that inherits properties of the trace (such as positivity, mass one, and traciality, see (6.2)), and is called a *tracial state* or a *non-commutative distribution* in free probability.

A key result due to Voiculescu [67] shows the existence of a non-commutative distribution σ such that for any polynomial p in $d+m$ self-adjoint non-commutative variables

$$\lim_{N\to\infty}\frac{1}{N}\mathrm{Tr}(p(X_1^N, ..., X_d^N, B_1^N, ..., B_m^N)) = \sigma(p(S_1, ..., S_d, b_1, ..., b_m)) \quad \text{a.s.}$$

where, under σ, $S_1, ..., S_d$ are d free semi-circular variables, free from $b_1, ..., b_m$ with law τ. More recently, Haagerup and Thorbjørnsen [39] (when the matrices $\{B_i^N\}_{i=1}^m$ vanish) and then Male [49] (when the spectral radius of polynomials $p(B_1^N, ..., B_m^N)$ in $\{B_i^N\}_{i=1}^m$ converge to the norm of their limit $p(b_1, ..., b_m)$) showed that this convergence is also true for the operator norms, namely the following convergence holds almost surely:

$$\lim_{N\to\infty}\|p(X_1^N, ..., X_d^N, B_1^N, ..., B_m^N)\|_\infty = \|p(S_1, ..., S_d, b_1, ..., b_m)\|_\infty,$$

where

$$\|p(S_1, ..., S_d, b_1, ..., b_m)\|_\infty = \lim_{r\to\infty}\sigma((p(S_1, ..., S_d, b_1, ..., b_m)p(S_1, ..., S_d, b_1, ..., b_m)^*)^r)^{1/2r}.$$

However, it was not known in general how the eigenvalues of such a polynomial fluctuate locally.

In this paper we show that if p is a perturbation of x_1 then, under some weak additional assumptions on the deterministic matrices $B_1^N, ..., B_m^N$, the eigenvalues of $p(X_1^N, ..., X_d^N, B_1^N, ..., B_m^N)$ fluctuate as the eigenvalues of X_1^N. In particular, if

$$p(X_1, ..., X_d) = X_1 + \varepsilon\, Q(X_1, ..., X_d)$$

with ε small enough and Q self-adjoint, then we can show that, once properly renormalized, the fluctuations of the eigenvalues of $p(X_1^N, ..., X_d^N)$ follow the sine-kernel inside the bulk and the Tracy–Widom law at the edges. In addition, this universality result holds also for (averages with respect to E of) m-point correlation functions around some energy level E in the bulk. Furthermore, all these results extend to the case of matrices in the Gaussian Orthogonal Ensemble (GOE).

Although we shall not investigate this here, our results should extend to non-Gaussian entries at least when the entries have the same first four moments as the Gaussian. This would however be a non-trivial generalization, as it would involve fine analysis such as the local law and rigidity.

To our knowledge this type of result is completely new except in the case of the very specific polynomial $p(S, b) = b + S$, which was recently treated in non-perturbative situations [17], [44] or when p is a product of non-normal random matrices [46], [1]. Notice that although our results hold only in a perturbative setting, it is clear that some assumptions on p are needed and universality cannot hold for any polynomial. Indeed, even if one considers only one matrix, if p is not strictly increasing then the largest eigenvalue of $p(X_1^N)$ could be the image by p of an eigenvalue of X_1^N inside the bulk, and hence it would follow the sine-kernel law instead of the Tracy–Widom law.

Our approach to universality for polynomials in several matrices goes through the universality for unitarily invariant matrices interacting via a potential. Indeed, as shown in §7, the law of the eigenvalues of such polynomials is a special case of the latter models, that we describe now.

Let V be a polynomial in non-commutative variables, $W_1, ..., W_d: \mathbb{R} \to \mathbb{R}$ be smooth functions, and consider the following probability measure on the space of d-tuples of $N \times N$ Hermitian or symmetric matrices (see also §2 for more details):

$$\begin{aligned} d\mathbb{P}_\beta^{N,V}&(dX_1, ..., dX_d) \\ &= \frac{1}{Z_\beta^{N,V}} e^{N \operatorname{Tr} V(X_1, ..., X_d, B_1, ..., B_m)} e^{-N \sum_{k=1}^d \operatorname{Tr} W_k(X_k)} \prod_{i=1}^d \mathbf{1}_{\|X_i\|_\infty \leqslant M}\, dX, \end{aligned}$$

where $dX = dX_1 \dots dX_d$ is the Lebesgue measure on the set of d-tuples of $N\times N$ Hermitian or symmetric matrices (from now on, to simplify the notation, we remove the superscript N on X_i and B_i). Also, $M>0$ is a cut-off which ensures that

$$Z_\beta^{N,V} := \int e^{N\,\mathrm{Tr} V(X_1,\dots,X_d,B_1,\dots,B_m)} e^{-N\sum_{k=1}^d \mathrm{Tr} W_k(X_k)} \prod_{i=1}^d \mathbf{1}_{\|X_i\|_\infty \leqslant M}\, dX$$

is finite despite the fact that V is a polynomial which could go to infinity faster than the W_k's. We assume that V is self-adjoint in the sense that $V(X_1,\dots,X_d,B_1,\dots,B_m)$ is Hermitian (resp. symmetric) for any $N\times N$ Hermitian (resp. symmetric) matrices $X_1,\dots,X_d,B_1,\dots,B_m$. As a consequence, $\mathbb{P}_\beta^{N,V}$ has a real non-negative density. Since we shall later need to assume that V is small, we shall not try to get the best assumptions on the W_k's, and we shall assume that they are uniformly convex. As discussed in Remark 2.2 below, this could be relaxed.

Such multi-matrix models appear in physics, in connection with the enumeration of colored maps [16], [51], [40], [30], and in planar algebras and the Potts model on random graphs [33], [34]. However, despite the introduction of biorthogonal polynomials [8] to compute precisely observables in these models, the local properties of the spectrum in these models could not be studied so far, except in very specific situations [3]. Our proof shows that the limiting spectral measure of the matrix models has a connected support and behaves as a square root at the boundary when a is small enough and the W_k are uniformly convex, see Lemma 3.2. This in particular shows that in great generality the nth moments for the related models, which can be identified with generating functions for planar maps, grow like $C^n n^{-3/2}$, as for the semi-circle law and rooted trees. More interesting exponents could be found at criticality, a case that we can hardly study in this article since we need a to be small. The transport maps between the limiting measures could themselves provide valuable combinatorial information, as a way to analyze the limiting spectral measures, but they would also need to be extended to criticality too. Yet, the extension of our techniques to the non-commutative setting yields interesting isomorphisms of related algebras [38], [53].

In [35], [36] it was shown that there exists $M_0<\infty$ such that the following holds: for $M>M_0$ there exists $a_0>0$ so that, for $a\in[-a_0,a_0]$, there is a non-commutative distribution τ^{aV} satisfying

$$\lim_{N\to\infty} \mathbb{P}_\beta^{N,aV}\left(\frac{1}{N}\mathrm{Tr}\big(p(X_1,\dots,X_d)\big)\right) = \tau^{aV}(p)$$

for any polynomials p in d non-commutative letters. In particular, if $\{\lambda_i^k\}_{i=1}^N$ denote the eigenvalues of X_k, the spectral measure

$$L_k^N := \frac{1}{N}\sum_{i=1}^N \delta_{\lambda_i^k}$$

converges weakly and in moments towards the probability measure μ_k^{aV} defined by

$$\mu_k^{aV}(x^\ell) := \tau^{aV}((X_k)^\ell) \quad \text{for all } \ell \in \mathbb{N}. \tag{1.2}$$

Moreover, one can bound these moments to see that μ_k^{aV} is compactly supported and hence defined by the family of its moments. In addition, it can be proved that μ_k^{aV} does not depend on the cutoff M. Furthermore, a central limit theorem for this problem was studied in [36] where it was proved that, for any polynomial p,

$$\operatorname{Tr}\big(p(X_1, ..., X_d)\big) - N\,\tau^{aV}(p)$$

converges in law towards a Gaussian variable. A higher-order expansion (the "topological expansion") was derived in [50].

In this article we show that, if a is small enough, the local fluctuations of the eigenvalues of each matrix under $\mathbb{P}_\beta^{N,aV}$ are the same as when $a=0$ and the W_k are just quadratic; in other words, up to rescaling, they follow the sine-kernel distribution inside the bulk and the Tracy–Widom law at the edges of the corresponding ensemble (see Corollaries 2.6 and 2.7). In addition, averaged energy universality of the correlation functions holds in our multi-matrix setting (see Corollary 2.8).

The idea to prove these results consists in finding a map from the law of the eigenvalues of independent GUE or GOE matrices to a probability measure that approximates our matrix models (see Theorem 2.5 and Corollary 2.7). This approach is inspired by the method introduced in [5] to study one-matrix models. However, not only are the arguments here much more involved, but we also improve the results in [5]. Indeed, the estimates on the approximate transport map obtained in [5] allowed one to obtain universality results only with bounded test functions, and could not be used to show averaged energy universality even in the single-matrix setting. Here, we are able to show stronger estimates that allow us to deal also with functions that grow polynomially in N (see equation (2.8)), and we exploit this to prove averaged energy universality in multi-matrix models (see Corollary 2.8).

A second key (and highly non-trivial) step in our proof consists in showing a large N-expansion for integrals over the unitary and orthogonal group (see §6). Such integrals arise when one seeks for the joint law of the eigenvalues by simply performing a change of variables and integrating over the eigenvectors. The expansion of such integrals was only know up to the first order [18] in the orthogonal case, and was derived for linear statistics in the case $\beta=2$ in [37]. However, to be able to study the law of the eigenvalues of polynomials in several matrices we need to treat quadratic statistics. Moreover, we need to prove that the expansions are smooth functions of the empirical measures of the matrices. Indeed, such an expansion allows us to express the joint law of the eigenvalues of our matrix models as the distribution of mean field interaction models (more precisely, as the distribution of d β-ensembles interacting via a mean field smooth interaction), and from this representation we are able to apply to this setting the approximate transport argument mentioned above, and prove our universality results.

In the next section we describe in detail our results.

2. Statement of the results

We are interested in the joint law of the eigenvalues under $\mathbb{P}_\beta^{N,V}$. We shall in fact consider a slightly more general model, where the interaction potential may not be linear in the trace, but rather some tensor power of the trace. This is necessary to deal with the law of a polynomial in several matrices. Hence, we consider the probability measure

$$d\mathbb{P}_\beta^{N,V}(X_1,...,X_d) := \frac{1}{Z_\beta^{N,V}} e^{N^{2-r}\mathrm{Tr}^{\otimes r} V(X_1,...,X_d,B_1,...,B_m)} \prod_{k=1}^{d} dR_{\beta,M}^{N,W_k}(X_k)$$

with

$$dR_{\beta,M}^{N,W}(X) := \frac{1}{Z_{\beta,M}^{N,W}} e^{-N\mathrm{Tr}(W(X))} \mathbf{1}_{\|X\|_\infty \leqslant M} dX,$$

where $\mathbf{1}_E$ denotes the indicator function of a set E, and $Z_\beta^{N,V}$ and $Z_{\beta,M}^{N,W}$ are normalizing constants. Here,

- $\beta=2$ (resp. $\beta=1$) corresponds to integration over the Hermitian (resp. symmetric) set $\mathcal{H}_\beta^N$ of $N\times N$ matrices with complex (resp. real) entries. In particular

$$dX = \begin{cases} \prod_{1\leqslant j\leqslant \ell\leqslant N} dX_{\ell j}, & \text{if } \beta=1,\\ \prod_{1\leqslant j\leqslant \ell\leqslant N} d\,\mathrm{Re}(X_{\ell j}) \prod_{1\leqslant j<\ell\leqslant N} d\,\mathrm{Im}(X_{\ell j}), & \text{if } \beta=2.\end{cases}$$

- Tr denotes the trace over $N\times N$ matrices, that is, $\mathrm{Tr}A=\sum_{j=1}^N A_{jj}$.
- $W_k\colon\mathbb{R}\to\mathbb{R}$ are uniformly convex functions, that is

$$W_k''(x) \geqslant c_0 > 0 \quad \text{for all } x\in\mathbb{R},$$

and given a function $W\colon\mathbb{R}\to\mathbb{R}$ and a $N\times N$ Hermitian matrix X, we define $W(X)$ as

$$W(X) := UW(D)U^*,$$

where U is a unitary matrix which diagonalizes X as $X=UDU^*$, and $W(D)$ is the diagonal matrix with entries $(W(D_{11}),...,W(D_{NN}))$.

- $B_1, ..., B_m$ are Hermitian (resp. symmetric) matrices if $\beta=2$ (resp. $\beta=1$).
- $\mathbb{C}\langle x_1, ..., x_d, b_1, ..., b_m\rangle^{\otimes r}$ denotes the space of rth tensor products of polynomials in d non-commutative variables with complex (resp. real) coefficients when $\beta=2$ (resp. $\beta=1$). For $p\in\mathbb{C}\langle x_1, ..., x_d, b_1, ..., b_m\rangle^{\otimes r}$ we denote by

$$p=\sum\langle p, q_1\otimes q_2 ...\otimes q_r\rangle\, q_1\otimes q_2 ...\otimes q_r$$

its decomposition on the monomial basis, and let p^* denote its adjoint given by

$$p^*:=\sum\overline{\langle p, q_1\otimes q_2 ...\otimes q_r\rangle}\, q_1^*\otimes q_2^* ...\otimes q_r^*,$$

where $*$ denotes the involution given by

$$(Y_{i_1} ... Y_{i_\ell})^*=Y_{i_\ell} ... Y_{i_1} \quad \text{for all } i_1, ..., i_\ell\in\{1, ..., d+m\},$$

where $\{Y_i=X_i\}_{i=1}^d$ and $\{Y_{j+d}=B_j\}_{j=1}^m$. We take V to belong to the closure of

$$\mathbb{C}\langle x_1, ..., x_d, b_1, ..., b_m\rangle^{\otimes r}$$

for the norm given, for $\xi>1$ and $\zeta\geqslant1$, by

$$\|p\|_{\xi,\zeta}:=\sum|\langle p, q_1\otimes q_2 ...\otimes q_r\rangle|\xi^{\sum_{i=1}^r \deg_X(q_i)}\zeta^{\sum_{i=1}^r \deg_B(q_i)} \tag{2.1}$$

where $\deg_X(q)$ (resp. $\deg_B(q)$) denotes the number of letters $\{X_i\}_{i=1}^d$ (resp. $\{B_i\}_{i=1}^m$) contained in q. If p only depends on the X_i (resp. the B_i), its norm does not depend on ζ (resp. ξ) and we simply denote it by $\|p\|_\xi$ (resp. $\|p\|_\zeta$). We also assume that V is self-adjoint, that is $V(X_1, ..., X_d, B_1, ..., B_m)^*=V(X_1, ..., X_d, B_1, ..., B_m)$.
- We use $\|\cdot\|_\infty$ to denote the spectral radius norm.

Performing the change of variables $X_k\mapsto U_kD(\lambda^k)U_k^*$, with U_k being unitary and $D(\lambda^k)$ being the diagonal matrix with entries $\lambda^k:=(\lambda_1^k, ..., \lambda_N^k)$, we find that the joint law of the eigenvalues is given by

$$dP_\beta^{N,V}(\lambda^1, ..., \lambda^d)=\frac{1}{\widetilde{Z}_\beta^{N,V}}I_\beta^{N,V}(\lambda^1, ..., \lambda^d)\prod_{k=1}^d dR_{\beta,M}^{N,W_k}(\lambda^k), \tag{2.2}$$

where

$$I_\beta^{N,V}(\lambda^1, ..., \lambda^d):=\int e^{N^{2-r}\mathrm{Tr}^{\otimes r}V(U_1D(\lambda^1)U_1^*, ..., U_dD(\lambda^d)U_d^*, B_1, ..., B_m)}\, dU_1 ... dU_d,$$

dU being the Haar measure on the unitary group when $\beta=2$ (resp. the orthogonal group when $\beta=1$), $\widetilde{Z}_\beta^{N,V}>0$ is a normalization constant, and $R_{\beta,M}^{N,W}$ is the probability measure on $\mathbb{R}^N$ given by

$$dR_{\beta,M}^{N,W}(\lambda):=\frac{1}{Z_{\beta,M}^{N,W}}\prod_{i<j}|\lambda_i-\lambda_j|^\beta e^{-N\sum_{i=1}^N W(\lambda_i)}\prod_{i=1}^N 1_{|\lambda_i|\leqslant M}d\lambda_i, \quad \lambda=(\lambda_1, ..., \lambda_N). \tag{2.3}$$

As we shall prove in §3, if W_k are uniformly convex and V is sufficiently small, for all $k\in\{1,...,d\}$ the empirical measure L_k^N of the eigenvalues of X_k converges to a compactly supported probability measure μ_k^V. In particular, if the cut-off M is chosen sufficiently large so that $[-M,M]\ni\operatorname{supp}(\mu_k^0)$, for V sufficiently small $[-M,M]\ni\operatorname{supp}(\mu_k^V)$ and the limiting measures μ_k^V will be independent of M. Hence, we shall assume that M is a universally large constant (i.e., the largeness depends only on the potentials W_k). More precisely, throughout the whole paper we will suppose that the following holds.

Hypothesis 2.1. Assume that:

- $W_k\colon\mathbb{R}\to\mathbb{R}$ is uniformly convex for any $k\in\{1,...,d\}$, that is, $W_k''(x)\geqslant c_0>0$ for all $x\in\mathbb{R}$. Moreover, $W_k\in C^\sigma(\mathbb{R})$ for some $\sigma\geqslant 36$.
- $M>1$ is a large universal constant.
- V is self-adjoint and $\|V\|_{M\xi,\zeta}<\infty$ for some ξ large enough (the largeness being universal, see Lemma 6.16) and $\zeta\geqslant 1$.
- The spectral radius of each of the Hermitian matrices $B_1,...,B_m$ is bounded by 1.

Remark 2.2. The convexity assumption on the potentials W_k could be relaxed. Indeed, the main reasons for this assumption are:

– To ensure that the equilibrium measures, obtained as limits of the empirical measure of the eigenvalues, enjoy the properties described in §3.

– To guarantee that the operator $\mathbf{\Xi}_t$ appearing in Proposition 4.4 is invertible.

– To prove the concentration inequalities in §4.5.

– To have rigidity estimates on the eigenvalues, needed in the universality proofs in §5.

As shown in the papers [12], [11], [5], the properties above hold under weaker assumptions on the W_k's. However, because the proofs of our results are already very delicate, we decided to introduce the convexity assumptions in order to avoid additional technicality that would obscure the main ideas in the paper.

In order to be able to apply the approximate transport strategy introduced in [5], a key result we will prove is the following large dimension expansion of $I_\beta^{N,V}$.

THEOREM 2.3. *Under Hypothesis 2.1, there exists $a_0>0$ such that, for $a\in[-a_0,a_0]$,*

$$I_\beta^{N,aV}(\lambda^1,...,\lambda^k)=\left(1+O\left(\frac{1}{N}\right)\right)e^{\sum_{l=0}^2 N^{2-l}F_l^a(L_1^N,...,L_d^N,\tau_B^N)},\tag{2.4}$$

where L_k^N are the spectral measures

$$L_k^N:=\frac{1}{N}\sum_{i=1}^N\delta_{\lambda_i^k},$$

$O(1/N)$ depends only on M, τ_B^N denotes the non-commutative distribution of the B_i given by the collection of complex numbers

$$\tau_B^N(p) := \frac{1}{N}\mathrm{Tr}(p(B_1, ..., B_m)), \quad p \in \mathbb{C}\langle b_1, ..., b_m\rangle, \tag{2.5}$$

and $\{F_l^a(\mu_1, ..., \mu_d, \tau)\}_{l=0}^2$ are smooth functions of $(\mu_1, ..., \mu_d, \tau)$ for the weak topology generated on the space of probability measures $\mathcal{P}([-M, +M])$ by

$$\|\mu\|_{\zeta M} := \max_{k\geqslant 1}(M\zeta)^{-k}|\mu(x^k)|$$

and the norm $\sup_{\|p\|_\zeta\leqslant 1}|\tau(p)|$ on linear forms τ on $\mathbb{C}\langle b_1, ..., b_m\rangle$.

This result is proved in §6. We notice that it was already partially proved in [37] in the unitary case. However, only the case where $r=1$ was considered there, and the expansion was shown to hold only in terms of the joint non-commutative distribution of the diagonal matrices $\{D(\lambda^k)\}_{k=1}^d$ rather than the spectral measure of each of them.

From the latter expansion of the density of $P_\beta^{N,aV}$ we can deduce the convergence of the spectral measures by standard large deviation techniques.

Corollary 2.4. *Assume that, for any polynomial $p\in\mathbb{C}\langle b_1, ..., b_m\rangle$,*

$$\lim_{N\to\infty}\tau_B^N(p) = \tau_B(p). \tag{2.6}$$

Then, under Hypothesis 2.1, there exists $a_0>0$ such that, for $a\in[-a_0, a_0]$, the empirical measures $\{L_k^N\}_{k=1}^d$ converge almost surely under $P_\beta^{N,aV}$ towards probability measures $\{\mu_k^{aV}\}_{k=1}^d$ on the real line.

In the case $r=1$ this result is already a consequence of [35] and [18]. The existence and study of the equilibrium measures is performed in §3.

Starting from the representation of the density given in Theorem 2.3 (see §4), we are able to prove the following existence results on approximate transport maps.

Theorem 2.5. *Under Hypothesis 2.1 with $\zeta>1$, suppose additionally that*

$$\tau_B^N(p) = \tau_B^0(p) + \frac{1}{N}\tau_B^1(p) + \frac{1}{N^2}\tau_B^2(p) + O\left(\frac{1}{N^3}\right), \tag{2.7}$$

where the error is uniform on balls for $\|\cdot\|_\zeta$. Then there exists a constant $\alpha>0$ such that, provided $|a|\leqslant\alpha$, we can construct a map

$$T^N = ((T^N)_1^1, ..., (T^N)_N^1, ..., (T^N)_1^d, ..., (T^N)_N^d): \mathbb{R}^{dN} \longrightarrow \mathbb{R}^{dN}$$

satisfying the following property: Let $\chi:\mathbb{R}^{dN}\to\mathbb{R}^+$ *be a non-negative measurable function such that* $\|\chi\|_\infty\leqslant N^k$ *for some* $k\geqslant 0$. *Then, for any* $\eta>0$, *we have*

$$\left|\log\left(1+\int\chi\circ T^N\,dP_\beta^{N,0}\right)-\log\left(1+\int\chi\,dP_\beta^{N,aV}\right)\right|\leqslant C_{k,\eta}\,N^{\eta-1} \tag{2.8}$$

for some constant $C_{\eta,k}$ *independent of* N. *Also, with* $\hat\lambda:=(\lambda_1^1,...,\lambda_N^d)$, T^N *has the form*

$$(T^N)_i^k(\hat\lambda)=T_0^k(\lambda_i^k)+\frac{1}{N}(T_1^N)_i^k(\hat\lambda)\quad\textit{for all } i=1,...,N \textit{ and } k=1,...,d,$$

where $T_0^k:\mathbb{R}\to\mathbb{R}$ *and* $T_1^N:\mathbb{R}^{dN}\to\mathbb{R}^{dN}$ *are of class* $C^{\sigma-3}$ *and satisfy uniform (in* N*) regularity estimates. More precisely, we have the decomposition*

$$T_1^N=X_{1,1}^N+\frac{1}{N}X_{2,1}^N,$$

where

$$\max_{\substack{1\leqslant k\leqslant d\\1\leqslant i\leqslant N}}\|(X_{1,1}^N)_i^k\|_{L^4(P_\beta^{N,0})}\leqslant C\log N\quad\textit{and}\quad\max_{\substack{1\leqslant k\leqslant d\\1\leqslant i\leqslant N}}\|(X_{2,1}^N)_i^k\|_{L^2(P_\beta^{N,0})}\leqslant C\,(\log N)^2,$$

for some constant $C>0$ *independent of* N. *In addition, with* $P_\beta^{N,0}$*-probability greater than* $1-e^{-c(\log N)^2}$,

$$\begin{aligned}
\max_{i,k}|(X_{1,1}^N)_i^k|&\leqslant C(\log N)N^{1/(\sigma-14)},\\
\max_{i,k}|(X_{2,1}^N)_i^k|&\leqslant C(\log N)^2N^{2/(\sigma-15)},\\
\max_{1\leqslant i,i'\leqslant N}|(X_{1,1}^N)_i^k(\hat\lambda)-(X_{1,1}^N)_{i'}^k(\hat\lambda)|&\leqslant C(\log N)N^{1/(\sigma-15)}\,|\lambda_i^k-\lambda_{i'}^k|\quad\textit{for all } k=1\,...,d,\\
\max_{1\leqslant i,i'\leqslant N}|(X_{2,1}^N)_i^k(\hat\lambda)-(X_{2,1}^N)_{i'}^k(\hat\lambda)|&\leqslant C(\log N)^2N^{2/(\sigma-17)}|\lambda_i^k-\lambda_{i'}^k|\quad\textit{for all } k=1,...,d,\\
\max_{1\leqslant i,j\leqslant N}|\partial_{\lambda_j^\ell}(X_{1,1}^N)_i^k|(\hat\lambda)&\leqslant C(\log N)N^{1/(\sigma-15)}\qquad\textit{for all } k,\ell=1,...,d.
\end{aligned}$$

As explained in §5, the existence of an approximate transport map satisfying regularity properties as above allows us to show universality properties for the local fluctuations of the spectrum. For instance, we can prove the following result.

Corollary 2.6. *Under the hypotheses of Theorem 2.5 the following holds: Let* T_0^k *be as in Theorem 2.5 and denote by* $\widetilde{P}_\beta^{N,aV}$ *the distribution of the increasingly ordered eigenvalues* $(\{\lambda_i^k\}_{i=1}^N)_{k=1}^d$ *under the law* $P_\beta^{N,aV}$. *Also, let* μ_k^0 *and* μ_k^{aV} *be as in Corollary 2.4, and* α *as in Theorem 2.5. Then, for any* $\theta\in\left(0,\frac{1}{6}\right)$ *there exists a constant* $\widehat{C}>0$, *independent of* N, *such that the following two facts hold true provided* $|a|\leqslant\alpha$:

(1) *Let* $\{i_k\}_{k=1}^d\subset[\varepsilon N,(1-\varepsilon)N]$ *for some* $\varepsilon>0$. *Then, choosing* $\gamma_{i_k/N}^k\in\mathbb{R}$ *such that*

$$\mu_k^0((-\infty,\gamma_{i_k/N}^k))=\frac{i_k}{N},$$

if $m \leqslant N^{2/3-\theta}$ *then, for any bounded Lipschitz function* $f\colon \mathbb{R}^{dm} \to \mathbb{R}$,

$$\begin{aligned}\bigg| \int f((N(\lambda^k_{i_k+1}-\lambda^k_{i_k}), ..., N(\lambda^k_{i_k+m}-\lambda^k_{i_k}))_{k=1}^d)\, d\widetilde{P}^{N,aV}_\beta \\ - \int f(((T^k_0)'(\gamma^k_{i_k/N})N(\lambda^k_{i_k+1}-\lambda^k_{i_k}), ..., (T^k_0)'(\gamma^k_{i_k/N})N(\lambda^k_{i_k+m}-\lambda^k_{i_k}))_{k=1}^d)\, d\widetilde{P}^{N,0}_\beta \bigg| \\ \leqslant \widehat{C} N^{\theta-1}\|f\|_\infty + \widehat{C} m^{3/2} N^{\theta-1} \|\nabla f\|_\infty.\end{aligned}$$

(2) *Let* a^0_k *(resp.* a^{aV}_k*) denote the smallest point in the support of* μ^0_k *(resp.* μ^{aV}_k*), so that* $\operatorname{supp}(\mu^0_k) \subset [a^0_k, \infty)$ *(resp.* $\operatorname{supp}(\mu^{aV}_k) \subset [a^{aV}_k, \infty)$*). If* $m \leqslant N^{4/7}$ *then, for any bounded Lipschitz function* $f\colon \mathbb{R}^{dm} \to \mathbb{R}$,

$$\begin{aligned}\bigg| \int f((N^{2/3}(\lambda^k_1 - a^{aV}_k), ..., N^{2/3}(\lambda^k_m - a^{aV}_k))_{k=1}^d)\, d\widetilde{P}^{N,aV}_\beta \\ - \int f(((T^k_0)'(a^0_k)N^{2/3}(\lambda^k_1 - a^0_k), ..., (T^k_0)'(a^0_k)N^{2/3}(\lambda^k_m - a^0_k))_{k=1}^d)\, d\widetilde{P}^{N,0}_\beta \bigg| \\ \leqslant \widehat{C} N^{\theta-1}\|f\|_\infty + \widehat{C}(m^{1/2}N^{\theta-1/3} + m^{7/6} N^{-2/3}) \|\nabla f\|_\infty.\end{aligned}$$

The same bound holds around the largest point in the support of μ^{aV}_k.

Similar results could be derived with functions of both statistics in the bulk and at the edge. Let us remark that for $a=0$ the eigenvalues of the different matrices are uncorrelated and $P^{N,0}_\beta$ becomes a product:

$$dP^{N,0}_\beta = \prod_{k=1}^d dR^{N,W_k}_{\beta,M}.$$

Universality under the latter β-models was already proved in [13], [14], [56], [5]. Also, by the results in [5] we can find approximate transport maps $S^N_k\colon \mathbb{R}^N \to \mathbb{R}^N$ from the law $P^N_{\mathrm{GVE},\beta}$ (this is the law of GUE matrices when $\beta=2$ and GOE matrices when $\beta=1$) to $R^{N,W_k}_{\beta,M}$ for any $k=1,...,d$. Hence $(S^N_1, ..., S^N_d)\colon \mathbb{R}^{dN} \to \mathbb{R}^{dN}$ is an approximate transport from $(P^N_{\mathrm{GVE},\beta})^{\otimes d}$ (i.e., the law of d independent GUE matrices when $\beta=2$ and GOE matrices when $\beta=1$) to $P^{N,0}_\beta$, and this allows us to deduce that the local statistics are in the same universality class as GUE (resp. GOE) matrices.

More precisely, as already observed in [5], the leading orders in the transport can be restated in terms of the equilibrium densities: denoting by

$$\varrho_{\mathrm{sc}}(x) := \frac{1}{2\pi}\sqrt{(4-x^2)_+} \tag{2.9}$$

the density of the semi-circle distribution and by ϱ^0_k the density of μ^0_k, then the leading-order term of S^N_k is given by $(S^k_0)^{\otimes N}$, where $S^k_0\colon \mathbb{R} \to \mathbb{R}$ is the monotone transport from $\varrho_{\mathrm{sc}}\, dx$ to $\varrho^0_k\, dx$ that can be found solving the ordinary differential equation (ODE)

$$(S^k_0)'(x) = \frac{\varrho_{\mathrm{sc}}}{\varrho^0_k(S^k_0)}(x), \quad S^k_0(-2) = a^0_k. \tag{2.10}$$

Also, the transport $T_0^k\colon\mathbb{R}\to\mathbb{R}$ appearing in Corollary 2.6 solves

$$(T_0^k)'(x)=\frac{\varrho_k^0}{\varrho_k^{aV}(T_0^k)}(x),\quad T_0^k(a_k^0)=a_k^{aV}. \tag{2.11}$$

Set

$$c_k^{aV}:=\lim_{x\to-2^+}\frac{\varrho_{\mathrm{sc}}}{\varrho_k^{aV}(T_0^k\circ S_0^k)}(x). \tag{2.12}$$

Due to these observations, we can easily prove the following result.

COROLLARY 2.7. *Let $m\in\mathbb{N}$. Under the hypotheses of Theorem 2.5, the following holds: Denote by $\widetilde{P}_\beta^{N,aV}$ (resp. $(\widetilde{P}^N_{\mathrm{GVE},\beta})^{\otimes d}$) the distribution of the increasingly ordered eigenvalues $(\{\lambda_i^k\}_{i=1}^N)_{k=1}^d$ under the law $P_\beta^{N,aV}$ (resp. $(P^N_{\mathrm{GVE},\beta})^{\otimes d}$). Also, let α be as in Theorem 2.5. Then, for any $\theta\in\left(0,\frac{1}{6}\right)$ and $C_0>0$ there exists a constant $\widehat{C}>0$, independent of N, such that the following two facts hold true provided $|a|\leqslant\alpha$:*

(1) *Given $\{\sigma_k\}_{k=1}^d\subset(0,1)$, let $\gamma_{\sigma_k}\in\mathbb{R}$ be such that $\mu_{\mathrm{sc}}((-\infty,\gamma_{\sigma_k}))=\sigma_k$, and $\gamma_{\sigma_k,k}$ such that $\mu_k^{aV}((-\infty,\gamma_{\sigma_k,k}))=\sigma_k$. Then, if $|i_k/N-\sigma_k|\leqslant C_0/N$ and $m\leqslant N^{2/3-\theta}$, for any bounded Lipschitz function $f\colon\mathbb{R}^{dm}\to\mathbb{R}$ we have*

$$\begin{aligned}&\bigg|\int f((N(\lambda_{i_k+1}^k-\lambda_{i_k}^k),...,N(\lambda_{i_k+m}^k-\lambda_{i_k}^k))_{k=1}^d)\,d\widetilde{P}_\beta^{N,aV}\\&\quad-\int f\bigg(\bigg(\frac{\varrho_{\mathrm{sc}}(\gamma_{\sigma_k})}{\varrho_k^{aV}(\gamma_{\sigma_k,k})}N(\lambda_{i_k+1}^k-\lambda_{i_k}^k),...,\frac{\varrho_{\mathrm{sc}}(\gamma_{\sigma_k})}{\varrho_k^{aV}(\gamma_{\sigma_k,k})}N(\lambda_{i_k+m}^k-\lambda_{i_k}^k)\bigg)_{k=1}^d\bigg)\,d(\widetilde{P}^N_{\mathrm{GVE},\beta})^{\otimes d}\bigg|\\&\qquad\leqslant\widehat{C}N^{\theta-1}\|f\|_\infty+\widehat{C}m^{3/2}N^{\theta-1}\|\nabla f\|_\infty.\end{aligned}$$

(2) *Let c_k^{aV} be as in (2.12). If $m\leqslant N^{4/7}$ then, for any bounded Lipschitz function $f\colon\mathbb{R}^m\to\mathbb{R}$, we have*

$$\begin{aligned}&\bigg|\int f((N^{2/3}(\lambda_1^k-a_k^{aV}),...,N^{2/3}(\lambda_m^k-a_k^{aV}))_{k=1}^d)\,d\widetilde{P}_\beta^{N,aV}\\&\quad-\int f(c_k^{aV}N^{2/3}(\lambda_1^k+2),...,c_k^{aV}\,N^{2/3}(\lambda_m^k+2))_{k=1}^d)\,d(\widetilde{P}^N_{\mathrm{GVE},\beta})^{\otimes d}\bigg|\\&\qquad\leqslant\widehat{C}N^{\theta-1}\|f\|_\infty+\widehat{C}(m^{1/2}\,N^{\theta-1/3}+m^{7/6}\,N^{-2/3})\|\nabla f\|_\infty.\end{aligned}$$

The same bound holds around the largest point in the support of μ_k^{aV}.

While the previous results deal only with bounded test function, in the next theorem we take full advantage of the estimate (2.8) to show averaged energy universality in our multi-matrix setting. Note that, to show this result, we need to consider as test functions averages (with respect to E) of m-points correlation functions of the form $\sum_{i_1\neq\ldots\neq i_m}f(N(\lambda_{i_1}^k-E),...,N(\lambda_{i_m}^k-E))$, where E belongs to the bulk of the spectrum.

In particular, these test functions have L^∞ norm of size N^m. Actually, as in Corollaries 2.6 and 2.7, we can deal with test functions depending at the same time on the eigenvalues of the different matrices.

Here and in the following, we use $⨍_I$ to denote the averaged integral over an interval $I\subset\mathbb{R}$, namely $⨍_I=(1/|I|)\int_I$.

Corollary 2.8. *Fix $m\in\mathbb{N}$ and $\zeta\in(0,1)$, and let α be as in Theorem 2.5. Also, let T_0^k and S_0^k be as in (2.11) and (2.10), and define $R_k:=T_0^k\circ S_0^k$. Then, given $\{E_k\}_{1\leqslant k\leqslant d}\subset(-2,2)$, $\theta\in(0,\min\{\zeta,1-\zeta\})$, and a non-negative Lipschitz function $f:\mathbb{R}^{dm}\to\mathbb{R}^+$ with compact support, there exists a constant $\widehat{C}>0$, independent of N, such that the following holds true provided $|a|\leqslant\alpha$:*

$$\begin{aligned}\bigg|\int\bigg[&⨍_{R_1(E_1)-N^{-\zeta}R_1'(E_1)}^{R_1(E_1)+N^{-\zeta}R_1'(E_1)} d\widetilde{E}_1 \dots ⨍_{R_d(E_d)-N^{-\zeta}R_d'(E_d)}^{R_d(E_d)+N^{-\zeta}R_d'(E_d)} d\widetilde{E}_d \\ &\times\sum_{i_{k,1}\neq\ldots\neq i_{k,m}} f((N(\lambda^k_{i_{k,1}}-\widetilde{E}_k),\dots,N(\lambda^k_{i_{k,m}}-\widetilde{E}_k))_{k=1}^d)\bigg]\,dP_\beta^{N,aV} \\ &-\int\bigg[⨍_{E_1-N^{-\zeta}}^{E_1+N^{-\zeta}} d\widetilde{E}_1\dots⨍_{E_d-N^{-\zeta}}^{E_d+N^{-\zeta}} d\widetilde{E}_d \\ &\times\sum_{i_{k,1}\neq\ldots\neq i_{k,m}} f((R_k'(E_k)N(\lambda^k_{i_{k,1}}-\widetilde{E}_k),\dots,R_k'(E_k)N(\lambda^k_{i_{k,m}}-\widetilde{E}_k))_{k=1}^d)\bigg]\,dP^N_{\mathrm{GVE}}\bigg| \\ &\leqslant\widehat{C}(N^{\theta+\zeta-1}+N^{\theta-\zeta}).\end{aligned}$$

It is worth mentioning that, in the single-matrix case, Bourgade, Erdős, Yau, and Yin [15] have recently been able to remove the average with respect to E and prove the Wigner–Dyson–Mehta conjecture at fixed energy in the bulk of the spectrum for generalized symmetric and Hermitian Wigner matrices. We believe that combining their techniques with ours one should be able to remove the average with respect to E in the previous theorem. However, this would go beyond the scope of this paper and we shall not investigate this here.

Another consequence of our transportation approach is the universality of other observables, such as the minimum spacing in the bulk. The next result is restricted to the case $\beta=2$ since we rely on [6, Theorem 1.4] which is proved in the case $\beta=2$ and is currently unknown for $\beta=1$.

Corollary 2.9. *Let $\beta=2$, fix $k\in\{1,\dots,d\}$, let I_k be a compact subset of $(-a_k^{aV},b_k^{aV})$ with non-empty interior, and denote the renormalized gaps by*

$$\Delta_i^k:=\frac{\lambda^k_{i+1}-\lambda^k_i}{(T_0^k\circ S_0^k)'(\gamma_{i/N})},\quad \lambda_i^k\in I_k,$$

where $\gamma_{i/N}\in\mathbb{R}$ is such that $\mu_{\rm sc}((-\infty,\gamma_{i/N}))=i/N$. Also, denote by $\widetilde{P}^{N,aV}_{\beta,k}$ the distribution of the increasingly ordered eigenvalues $\{\lambda_i^k\}_{i=1}^N$ under $P^{N,aV}_{\beta,k}$, the law of the eigenvalues of the k-th matrix under $P^{N,aV}_\beta$. Then, under the hypotheses of Theorem 2.5, the following statements hold:

- (Smallest gaps) *Let $\tilde t^1_{N,k}<\tilde t^2_{N,k}\ldots<\tilde t^p_{N,k}$ denote the p smallest renormalized spacings Δ_i^k of the eigenvalues of the k-th matrix lying in I, and set*

$$\tilde\tau^p_{N,k}:=\biggl(\frac{1}{144\pi^2}\int_{(T_0^k\circ S_0^k)^{-1}(I)}(4-x^2)^2\,dx\biggr)^{1/3}\tilde t^p_{N,k}.$$

Then, as $N\to\infty$, $N^{4/3}\tilde\tau^p_{N,k}$ converges in law towards τ^p whose density is given by

$$\frac{3}{(p-1)!}x^{3p-1}e^{-x^3}\,dx.$$

- (Largest gaps) *Let $\ell^1_{N,k}(I)>\ell^2_{N,k}(I)>\ldots$ be the largest gaps of the form Δ_i^k with $\lambda_i^k\in I_k$. Let $\{r_N\}_{N\in\mathbb{N}}$ be a family of positive integers such that*

$$\frac{\log r_N}{\log N}\to 0\quad \textit{as } N\to\infty.$$

Then, as $N\to\infty$,

$$\frac{N}{\sqrt{32\log N}}\ell^{r_N}_{N,k}\to 1\quad \textit{in } L^q(\widetilde{P}^{N,aV}_{\beta,k})$$

for any $q<\infty$.

All the above corollaries are proved in §5.

As an important application of our results, we consider the law of the eigenvalues of a self-adjoint polynomials in several GUE or GOE matrices. Indeed, if ε is sufficiently small and $X_1,\ldots,X_d$ are independent GUE or GOE matrices, a change of variable formula shows that the law of the eigenvalues of the d random matrices given by

$$Y_i=X_i+\varepsilon\,P_i(X_1,\ldots,X_d),\quad 1\leqslant i\leqslant d,$$

follows a distribution of the form $P^{N,aV}_\beta$ with $r=2$ and V a convergent series, see §7. Hence we have the following result.

Corollary 2.10. *Let $P_1,\ldots,P_d\in\mathbb{C}\langle x_1,\ldots,x_d,b_1,\ldots,b_m\rangle$ be self-adjoint polynomials. There exists $\varepsilon_0>0$ such that the following holds*: *Let X_i be independent* GUE *or* GOE *matrices and set*

$$Y_i:=X_i+\varepsilon\,P_i(X_1,\ldots,X_d).$$

Then, for $\varepsilon \in [-\varepsilon_0, \varepsilon_0]$, the eigenvalues of the matrices $\{Y_i\}_{i=1}^d$ fluctuate in the bulk or at the edge as when $\varepsilon=0$, up to rescaling. The same result holds for

$$Y_i = X_i + \varepsilon\, P_i(X_1, ..., X_d, B_1, ..., B_m)$$

provided τ_B^N satisfies (2.7). Namely, in both models, the law $\widetilde{P}_\beta^{N,\varepsilon P}$ of the ordered eigenvalues of the matrices Y_k satisfies the same conclusions as $\widetilde{P}_\beta^{N,aV}$ in Corollaries 2.7 and 2.9.

Remark 2.11. Recall that, as already stated at the beginning of §2, when $\beta=1$ the matrices B_i, as well as the coefficients of P, are assumed to be real. In particular, in the statement above, if X_i are GOE then the matrices Y_i must be orthogonal. The reason for this is that we need the map $(X_1, ..., X_d) \mapsto (Y_1, ..., Y_d)$ to be an isomorphism close to identity at least for uniformly bounded matrices. Our result should generalize to mixed polynomials in GOE and GUE which satisfy this property, but it does not include the case of the perturbation of a GOE matrix by a small GUE matrix which is Hermitian but not orthogonal.

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3. Study of the equilibrium measure

In this section we study the macroscopic behavior of the eigenvalues, that is the convergence of the empirical measures and the properties of their limits. Note here that we are restricting ourselves to measures supported on $[-M, M]$ so that the weak topology is equivalent to the topology of moments induced by the norm

$$\|\nu\|_{\zeta M} := \max_{k \geqslant 1} (\zeta M)^{-k} |\nu(x^k)|.$$

As a consequence, a large deviation principle for the law $\Pi_\beta^{N,aV}$ of $(L_1^N, ..., L_d^N)$ under $P_\beta^{N,aV}$ can be proved:

Lemma 3.1. *Assume that $M>1$ is sufficiently large and that τ_B^N converges towards τ_B (see (2.5) and (2.6)). Then the measures $(\Pi_\beta^{N,aV})_{N\geqslant 0}$ on $\mathcal{P}([-M,M])^d$ equipped with the weak topology satisfy a large deviation principle in the scale N^2 with good rate function*

$$I^a(\mu_1, ..., \mu_d) := J^a(\mu_1, ..., \mu_d) - \inf_{\nu_k \in \mathcal{P}([-M,M])} J^a(\nu_1, ..., \nu_d),$$

where

$$J^a(\mu_1, ..., \mu_d) := \frac{1}{2}\sum_{k=1}^{d} \iint [W_k(x)+W_k(y)-\beta\log|x-y|\,]\,d\mu_k(x)\,d\mu_k(y) \\ -F_0^a(\mu_1, ..., \mu_d, \tau_B).$$

Proof. The proof is given in [7], [2] in the case $F_0^a=0$, while the general case follows from the Laplace method (known also as Varadhan lemma) since F_0^a is continuous for the $\|\cdot\|_{\zeta M}$ topology (and therefore for the usual weak topology, which is stronger). $\square$

It follows by the result above that $\{L_k^N\}_{k=1}^d$ converge to the minimizers of I^a. We next prove that, for a small enough, I^a admits a unique minimizer, and show some of its properties. This is an extended and refined version of (1.2) which shall be useful later on.

LEMMA 3.2. *Assume that Hypothesis 2.1 holds. There exists $a_0>0$ such that, for $a\in[-a_0, a_0]$, I^a admits a unique minimizer $(\mu_1^{aV}, ..., \mu_d^{aV})$. Moreover the support of each μ_k^{aV} is connected and strictly contained inside $[-M, M]$, and each μ_k^{aV} has a density which is smooth and strictly positive inside its support except at the two boundary points, where it goes to zero as a square root.*

Proof. We first notice that if $I^a(\mu_1, ..., \mu_k)$ is finite, so is

$$-\int \log|x-y|\,d\mu_k(x)\,d\mu_k(y).$$

In particular the minimizers $\{\mu_i^{aV}\}_{i=1}^d$ of I^a have no atoms. We then consider the small perturbation $I^a(\mu_1^{aV}+\varepsilon\nu_1, ..., \mu_d^{aV}+\varepsilon\nu_d)$ for centered measures $(\nu_1, ..., \nu_d)$ (that is, $\int d\nu_k=0$) such that $\nu_k\geqslant 0$ outside the support of μ_k^{aV} and $\mu_k^{aV}+\varepsilon\nu_k\geqslant 0$ for $|\varepsilon|\ll 1$. Hence, by differentiating $I^a(\mu_1^{aV}+\varepsilon\nu_1, ..., \mu_d^{aV}+\varepsilon\nu_d)$ with respect to ε and setting $\varepsilon=0$, we deduce that

$$0=\int F_k(x)\,d\nu_k(x), \tag{3.1}$$

where

$$F_k(x) := W_k(x)-D_kF_0^a(\mu_1^{aV}, ..., \mu_d^{aV}, \tau_B)[\delta_x]-\beta\int \log|x-y|\,d\mu_k^{aV}(y)$$

and $x\mapsto D_kF_0^a(\mu_1, ..., \mu_d, \tau_B)[\delta_x]$ denotes the function such that, for any measure ν,

$$\frac{d}{d\varepsilon}\bigg|_{\varepsilon=0} F_0^a(\mu_1^{aV}, ..., \mu_{k-1}^{aV}, \mu_k^{aV}+\varepsilon\nu, \mu_{k+1}^{aV}, ..., \mu_d^{aV}, \tau_B) \\ =\int D_kF_0^a(\mu_1^{aV}, ..., \mu_d^{aV}, \tau_B)[\delta_x]\,d\nu(x). \tag{3.2}$$

It is shown in Lemma 6.16 that this function, as well as its derivatives, is smooth and of size a. Since ν_k is centered and $\nu_k \geqslant 0$ outside the support of μ_k, it follows from (3.1) that there exists a constant $C_k \in \mathbb{R}$ such that

$$F_k = C_k \text{ on } \operatorname{supp}(\mu_k^{aV}) \quad \text{and} \quad F_k \geqslant C_k \text{ on } \mathbb{R} \setminus \operatorname{supp}(\mu_k^{aV}).$$

Since $\partial_x^2(D_k F_0^a(\mu_1^{aV}, ..., \mu_k^{aV})[\delta_x])$ is uniformly bounded by $C(M)a$ for some finite constant $C(M)$ which only depends on M, the effective potential

$$W_k^{\text{eff}}(x) := W_k(x) - D_k F_0^a(\mu_1^{aV}, ..., \mu_k^{aV}, \tau_B)[\delta_x] \tag{3.3}$$

is uniformly convex for $a < c_0/C(M)$ due to Hypothesis 2.1. In addition,

$$x \longmapsto -\int \log|x-y| \, d\mu_k^{aV}(y)$$

is convex for $x \in \mathbb{R} \setminus \operatorname{supp}(\mu_k^{aV})$. This implies that the non-negative function $F_k - C_k$ is uniformly convex on $\mathbb{R} \setminus \operatorname{supp}(\mu_k^{aV})$ and vanishes at the boundary of the support of μ_k, and hence μ_k^{aV} necessarily has connected support, which we denote by $[a_k^{aV}, b_k^{aV}]$.

We now consider the measures $\mu_k^\varepsilon := (\mathrm{Id} + \varepsilon f_k)_\# \mu_k^{aV}$, where $f_k: \mathbb{R} \to \mathbb{R}$ is a smooth function. Then, since $I^a(\mu_1^\varepsilon, ..., \mu_d^\varepsilon) \geqslant I^a(\mu_1^{aV}, ..., \mu_d^{aV})$, we deduce by comparing the terms linear in ε that

$$\int (W_k^{\text{eff}})'(x) f(x) \, d\mu_k^{aV}(x) = \iint \frac{f(x) - f(y)}{x - y} \, d\mu_k^{aV}(x) \, d\mu_k^{aV}(y) \tag{3.4}$$

for all $k = 1, ..., d$ and all f. In particular, choosing $f(x) := (z-x)^{-1}$ with $z \in \mathbb{R} \setminus [a_k^{aV}, b_k^{aV}]$, we obtain that

$$G_k(z) := \int (z-x)^{-1} \, d\mu_k^{aV}(x)$$

satisfies the equation

$$G_k(z)^2 = (W_k^{\text{eff}})'(z) G_k(z) + H_k(z), \quad H_k(z) := \int \frac{(W_k^{\text{eff}})'(x) - (W_k^{\text{eff}})'(z)}{z - x} \, d\mu_k^{aV}(x).$$

Solving this quadratic equation so that $G(z) \to 0$ as $|z| \to \infty$ yields

$$G_k(z) = \tfrac{1}{2}\Big((W_k^{\text{eff}})'(z) - \sqrt{(W_k^{\text{eff}})'(z)^2 + 4H_k(z)}\Big)$$

from which it follows (by smoothness of H_k, see also [5, Proof of Lemma 3.2]) that

$$\frac{d\mu_k^{aV}(x)}{dx} = d_k(x) \sqrt{(x - a_k^{aV})(b_k^{aV} - x)},$$

where

$$d_k(x)^2(x-a_k^{aV})(b_k^{aV}-x)=-(W_k^{\mathrm{eff}})'(x)^2-4H_k(x)=:g_k(x) \quad \text{for } x\in[a_k^{aV},b_k^{aV}].$$

Note that g_k is a smooth function. In the case where $a=0$, it is well known that the strict convexity of W_k implies that g_k has simple zeroes in a_k^{aV} and b_k^{aV}, and that d_k does not vanish in an open neighborhood of $[a_k^{aV},b_k^{aV}]$. On the other hand we also know (see e.g. Lemma 6.15) that the measures μ_k^{aV} depend continuously on the parameter a (the set of probability measures being equipped with the weak topology) as they are compactly supported measures with moments depending analytically on a. As a consequence, g_k and g_k' are smooth functions of a, uniformly in the variable x. This implies that, for a small enough, g_k can only vanish in a small neighborhood of a_k^{aV} and b_k^{aV}, where its derivative does not vanish. Hence g_k can only have one simple zero in a small neighborhood of a_k^{aV} (resp. b_k^{aV}), and d_k cannot vanish in an open neighborhood of $[a_k^{aV},b_k^{aV}]$. Also, notice that, since W_k^{eff} and H_k are smooth, so is d_k. In addition, if one chooses $M>\max\{|a_k^0|,|b_k^0|\}$ for all $k=1,...,d$, then by continuity we deduce that $[a_k^{aV},b_k^{aV}]\subset(-M,M)$ for any $a\in[-a_0,a_0]$.

We finally deduce uniqueness: Assume there are two minimizers $(\mu_1,...,\mu_d)$ and $(\mu_1',...,\mu_d')$. By the previous considerations, both μ_i and μ_i' have smooth densities with respect to the Lebesgue measure on $\mathbb{R}$, and we can therefore consider the unique monotone non-decreasing maps $T_i:\mathbb{R}\to\mathbb{R}$ such that that $\mu_i'=(T_i)_{\#}\mu_i$. We then consider

$$j^a(\tau):=J^a((\tau\mathrm{Id}+(1-\tau)T_1)_{\#}\mu_1,...,(\tau\mathrm{Id}+(1-\tau)T_d)_{\#}\mu_d).$$

By concavity of the logarithm and uniform convexity of $W_k-D_kF_0^a(\nu_1,...,\nu_d,\tau_B)[\delta_x]$ (uniform with respect to $\nu_\ell\in\mathcal{P}([-M,M])$), we conclude that j^a is uniformly convex on $[0,1]$, which contradicts the minimality of μ_i and μ_i'. $\square$

We next show that, since the support of each μ_k^{aV} is strictly contained inside $[-M,M]$, the eigenvalues will not touch $\mathbb{R}\setminus[-M,M]$ with large probability.

LEMMA 3.3. *Let Hypothesis 2.1 hold. There exists $a_0>0$ such that the following holds for $a\in[-a_0,a_0]$: if $[a_k^{aV},b_k^{aV}]$ denotes the support of μ_k^{aV} (see Lemma 3.2), then for any $\varepsilon>0$ there exists $c(\varepsilon)>0$ such that, for N large enough,*

$$P_\beta^{N,aV}(\exists i\in\{1,...,N\},\exists k\in\{1,...,d\}:\lambda_i^k\in[a_k^{aV}-\varepsilon,b_k^{aV}+\varepsilon]^c)\leqslant e^{-c(\varepsilon)N}.$$

Proof. By [11, Lemma 3.1] (see also [9] and [10]) we can prove that, for any closed sets F_k,

$$\limsup_{N\to\infty}\frac{1}{N}\log P_\beta^{N,aV}(\exists i,k:\lambda_i^k\in F_k)\leqslant-\inf_{F_1\times...\times F_d}\mathcal{I},$$

where $\mathcal{I}$ is the good rate function

$$\mathcal{I}(x_1, ..., x_i) := \mathcal{J}(x_1, ..., x_k) - \inf_{y_1, ..., y_k \in [-M,M]^d} \mathcal{J}(y_1, ..., y_k)$$

with

$$\mathcal{J}(x_1, ..., x_d) := \sum_{k=1}^{d} \left(W_k^{\mathrm{eff}}(x_k) - \beta \int \log |x_k - y| \, d\mu_k^{aV}(y) \right),$$

where W^{eff} is defined in (3.3). As in the proof of Lemma 3.2 one sees that, for $|a|$ sufficiently small, $\mathcal{J}$ is uniformly convex outside the support of the measure, whereas it is constant on each support. Hence it is strictly greater than its minimal value at positive distance of this support, from which the conclusion follows. $\square$

4. Construction of approximate transport maps: proof of Theorem 2.5

As explained in the introduction, one of the drawbacks of the results in [5] is that it only allows one to deal with bounded test functions. To avoid this, we shall prove a multiplicative closeness result (see (2.8)).

4.1. Simplification of the measures and strategy of the proof

We begin from the measure $P_\beta^{N,V}$ as in (2.2). Because of Theorem 2.3, it makes sense to introduce the probability measures

$$dP_{t,\beta}^{N,aV}(\lambda^1, ..., \lambda^d) := \frac{1}{\widetilde{Z}_{t,\beta}^{N,aV}} e^{N^2 t F_0^a(L_1^N, ..., L_d^N, \tau_B^N) + N t F_1^a(L_1^N, ..., L_d^N, \tau_B^N) + t F_2^a(L_1^N, ..., L_d^N, \tau_B^N)} \times \prod_{k=1}^{d} dR_{\beta,M}^{N,W_k}(\lambda^k)$$

for $t \in [0,1]$, where $R_\beta^{N,W}$ is as in (2.3). Then, it follows by (2.2) and (2.4) that, for any non-negative function $\chi: \mathbb{R}^N \to \mathbb{R}^+$,

$$\frac{1 + \int \chi \, dP_\beta^{N,aV}}{1 + \int \chi \, dP_{1,\beta}^{N,aV}} = \frac{\int (1+\chi) \, dP_\beta^{N,aV}}{\int (1+\chi) \, dP_{1,\beta}^{N,aV}} = 1 + O\left(\frac{1}{N}\right),$$

and therefore

$$\left| \log\left(1 + \int \chi \, dP_\beta^{N,aV}\right) - \log\left(1 + \int \chi \, dP_{1,\beta}^{N,aV}\right) \right| \leqslant \frac{C}{N}. \tag{4.1}$$

Hereafter we do not stress the dependency in β, so $P_{t,\beta}^{N,aV} = P_t^{N,aV}$.

To remove the cutoff in M, let

$$dQ_t^{N,aV}(\lambda^1,...,\lambda^d) := \frac{1}{Z_t^{N,aV}} e^{\sum_{l=0}^2 N^{2-l} t F_l^a(\phi_\#^M L_1^N,...,\phi_\#^M L_d^N,\tau_B^N)} \prod_{k=1}^d dR_{\beta,\infty}^{N,W_k}(\lambda^k),$$

where

$$Z_t^{N,aV} := \int e^{\sum_{l=0}^2 N^{2-l} t F_l^a(\phi_\#^M L_1^N,...,\phi_\#^M L_d^N,\tau_B^N)} \prod_{k=1}^d dR_{\beta,\infty}^{N,W_k}(\lambda^k) \tag{4.2}$$

and $\phi^M:\mathbb{R}\to\mathbb{R}$ is a smooth function equal to x on a neighborhood of the supports $[a_k^{aV}, b_k^{aV}]$, vanishing outside of $[-2M, 2M]$, and bounded by $2M$ everywhere. Then Lemma 3.3 (as well as similar considerations for $Q_t^{N,aV}$) implies that, for some $\delta>0$,

$$\|Q_1^{N,aV} - P_1^{N,aV}\|_{TV} \leqslant e^{-\delta N}. \tag{4.3}$$

Notice that $Q_0^{N,aV} = Q_1^{N,0} = P_\beta^{N,0}$ so, if we can construct an approximate transport map from $Q_0^{N,aV}$ to $Q_1^{N,aV}$ as in the statement of Theorem 2.5, by (4.1) and (4.3) the same map will be an approximate transport from $P_\beta^{N,0}$ to $P_\beta^{N,aV}$. Thus it suffices to prove Theorem 2.5 with $Q_0^{N,aV}$ and $Q_1^{N,aV}$ in place of $P_\beta^{N,0}$ and $P_\beta^{N,aV}$.

For this, we improve the strategy developed in [5]: we construct a 1-parameter family of maps $T_t^N:\mathbb{R}^{dN}\to\mathbb{R}^{dN}$ that approximately sends $Q_0^{N,aV}$ onto $Q_t^{N,aV}$ by solving

$$\partial_t T_t^N = \mathbf{Y}_t^N(T_t^N), \quad T_0^N = \mathrm{Id},$$

where $\mathbf{Y}_t^N = ((\mathbf{Y}_t^N)_1^1, ..., (\mathbf{Y}_t^N)_N^d):\mathbb{R}^{dN}\to\mathbb{R}^{dN}$ is constructed so that the following quantity is small in $L^q(Q_t^{N,aV})$, for any $q<\infty$:

$$\begin{aligned}\mathcal{R}_t^N(\mathbf{Y}^N) :=\ & c_t^N - \beta \sum_k \sum_{i<j} \frac{(\mathbf{Y}_t^N)_i^k - (\mathbf{Y}_t^N)_j^k}{\lambda_i^k - \lambda_j^k} - \sum_{i,k} \partial_{\lambda_i^k} (\mathbf{Y}_t^N)_i^k \\ & - N^2 F_0^a(\phi_\#^M L_1^N, ..., \phi_\#^M L_d^N, \tau_B^N) - N F_1^a(\phi_\#^M L_1^N, ..., \phi_\#^M L_d^N, \tau_B^N) \\ & - F_2^a(\phi_\#^M L_1^N, ..., \phi_\#^M L_d^N, \tau_B^N) + \sum_{i,k} \partial_{\lambda_i^k} H_t(\hat{\lambda}) (\mathbf{Y}_t^N)_i^k,\end{aligned} \tag{4.4}$$

where $\hat{\lambda} := (\lambda^1, ..., \lambda^d) = (\lambda_1^1, ..., \lambda_N^1, ..., \lambda_1^d, ... \lambda_N^d)$, $c_t^N := \partial_t \log Z_t^{N,aV}$,

$$L_k^N := \frac{1}{N} \sum_{i=1}^N \delta_{\lambda_i^k},$$

and

$$\begin{aligned}H_t(\hat{\lambda}) :=\ & N \sum_{i,k} W_k(\lambda_i^k) - t N^2 F_0^a(\phi_\#^M L_1^N, ..., \phi_\#^M L_d^N, \tau_B^N) \\ & - t N F_1^a(\phi_\#^M L_1^N, ..., \phi_\#^M L_d^N, \tau_B^N) - t F_2^a(\phi_\#^M L_1^N, ..., \phi_\#^M L_d^N, \tau_B^N).\end{aligned} \tag{4.5}$$

In [5] it is proved that the flow of $\mathbf{Y}_t^N$ is an approximate transport map provided $\mathcal{R}_t^N(\mathbf{Y}^N)$ is small: more precisely, if X_t^N solves the ODE

$$\dot{X}_t^N = \mathbf{Y}_t^N(X_t^N), \quad X_0^N = \mathrm{Id}, \tag{4.6}$$

and we set $T^N := X_1^N$, then [5, Lemma 2.2] shows that

$$\left| \int \chi \circ T^N \, dQ_0^{N,aV} - \int \chi \, dQ_1^{N,aV} \right| \leqslant \|\chi\|_\infty \int_0^1 \|\mathcal{R}_t^N(\mathbf{Y}^N)\|_{L^1(Q_t^{N,aV})} \, dt \tag{4.7}$$

for any bounded measurable function $\chi: \mathbb{R}^{dN} \to \mathbb{R}$.

Although this result is powerful enough if χ is a bounded test function, it becomes immediately useless if we would like to integrate a function that grows polynomially in N. For this reason we prove here a new estimate that considerably improves [5, Lemma 2.2].

Lemma 4.1. *Assume that, for any $q < \infty$, there exists a constant C_q such that*

$$\|\mathcal{R}_t^N(\mathbf{Y}^N)\|_{L^q(Q_t^{N,aV})} \leqslant C_q \frac{(\log N)^3}{N} \quad \text{for all } t \in [0,1], \tag{4.8}$$

define X_t^N as in (4.6), and set $T^N := X_1^N$. Let $\chi: \mathbb{R}^N \to \mathbb{R}^+$ be a non-negative measurable function satisfying $\|\chi\|_\infty \leqslant N^k$ for some $k \geqslant 0$. Then, for any $\eta > 0$, there exists a constant $C_{k,\eta}$, independent of χ, such that

$$\left| \log\left(1 + \int \chi \, dQ_1^{N,aV}\right) - \log\left(1 + \int \chi \circ T^N \, dQ_0^{N,aV}\right) \right| \leqslant C_{k,\eta} \, N^{\eta - 1}.$$

Notice that this lemma proves the validity of (2.8) with $Q_0^{N,aV}$ and $Q_1^{N,aV}$ in place of $P_\beta^{N,0}$ and $P_\beta^{N,aV}$, respectively, provided we can show that (4.8) holds.

Here, we shall first prove Lemma 4.1 and then we show the validity of (4.8). More precisely, in §4.2 we prove Lemma 4.1, then in §§4.3–4.5 we show that

$$|\mathcal{R}_t^N(\mathbf{Y}^N)| \leqslant C \, \frac{(\log N)^3}{N} \quad \text{on a set } G_t \subset \mathbb{R}^N \text{ satisfying } Q_t^{N,aV}(G_t) \geqslant 1 - N^{-cN}. \tag{4.9}$$

As $\mathcal{R}_t^N(\mathbf{Y}^N)$ is trivially bounded by CN^2 everywhere (being the sum of $O(N^2)$ bounded terms, see (4.4)), (4.9) implies that

$$\|\mathcal{R}_t^N(\mathbf{Y}^N)\|_{L^q(Q_t^{N,aV})} \leqslant C \, \frac{(\log N)^3}{N} + CN^2 (Q_t^{N,aV}(\mathbb{R}^N \setminus G_t))^{1/q} \leqslant C \, \frac{(\log N)^3}{N},$$

proving (4.8).

Finally, in §4.6 we show that $T^N = X_1^N$ satisfies all the properties stated in Theorem 2.5.

4.2. Proof of Lemma 4.1

Let ϱ_t denote the density of $Q_t^{N,aV}$ with respect to the Lebesgue measure $\mathcal{L}$. Then, by a direct computation one can check that ϱ_t, $\mathbf{Y}^N$, and $\mathcal{R}_t^N = \mathcal{R}_t^N(\mathbf{Y}^N)$ are related by the following formula:

$$\partial_t \varrho_t + \operatorname{div}(\mathbf{Y}_t^N \varrho_t) = \mathcal{R}_t^N \varrho_t. \tag{4.10}$$

Now, given a smooth function $\chi: \mathbb{R}^N \to \mathbb{R}^+$ satisfying $\|\chi\|_\infty \leqslant N^k$, we define

$$\chi_t := \chi \circ X_1^N \circ (X_t^N)^{-1} \quad \text{for all } t \in [0,1]. \tag{4.11}$$

Note that with this definition $\chi_1 = \chi$. Also, since $\chi_t \circ X_t^N$ is constant in time, differentiating with respect to t we deduce that

$$0 = \frac{d}{dt}(\chi_t \circ X_t^N) = (\partial_t \chi_t + \mathbf{Y}_t^N \cdot \nabla \chi_t) \circ X_t^N,$$

and hence χ_t solves the transport equation

$$\partial_t \chi_t + \mathbf{Y}_t^N \cdot \nabla \chi_t = 0, \quad \chi_1 = \chi. \tag{4.12}$$

Combining (4.10) and (4.12), we compute

$$\begin{aligned} \frac{d}{dt} \int \chi_t \varrho_t \, d\mathcal{L} &= \int \partial_t \chi_t \varrho_t \, d\mathcal{L} + \int \chi_t \partial_t \varrho_t \, d\mathcal{L} \\ &= -\int \mathbf{Y}_t^N \cdot \nabla \chi_t \varrho_t \, d\mathcal{L} - \int \chi_t \operatorname{div}(\mathbf{Y}_t^N \varrho_t) \, d\mathcal{L} + \int \chi_t \mathcal{R}_t^N \varrho_t \, d\mathcal{L} \\ &= \int \chi_t \mathcal{R}_t^N \varrho_t \, d\mathcal{L}. \end{aligned}$$

We want to control the last term. To this aim we notice that, since $\|\chi\|_\infty \leqslant N^k$, it follows immediately from (4.11) that $\|\chi_t\|_\infty \leqslant N^k$ for any $t \in [0,1]$. Hence, using Hölder's inequality and (4.8), for any $p > 1$ we can bound

$$\begin{aligned} \left| \int \chi_t \, \mathcal{R}_t^N \, \varrho_t \, d\mathcal{L} \right| &\leqslant \|\chi_t\|_{L^p(Q_t^{N,aV})} \|\mathcal{R}_t^N\|_{L^q(Q_t^{N,aV})} \\ &\leqslant \|\chi_t\|_\infty^{(p-1)/p} \|\chi_t\|_{L^1(Q_t^{N,aV})}^{1/p} \|\mathcal{R}_t^N\|_{L^q(Q_t^{N,aV})} \\ &\leqslant N^{k(p-1)/p} \|\chi_t\|_{L^1(Q_t^{N,aV})}^{1/p} \|\mathcal{R}_t^N\|_{L^q(Q_t^{N,aV})} \\ &\leqslant C_q \frac{N^{k(p-1)/p} (\log N)^3}{N} \|\chi_t\|_{L^1(Q_t^{N,aV})}^{1/p}, \end{aligned}$$

where $q := p/(p-1)$. Hence, given $\eta > 0$, we can choose $p := 1 + \eta/2k$ to obtain

$$\left| \int \chi_t \, \mathcal{R}_t^N \varrho_t \, d\mathcal{L} \right| \leqslant C_q N^{\eta - 1} \|\chi_t\|_{L^1(Q_t^{N,aV})}^{1/p} \leqslant C N^{\eta - 1} (1 + \|\chi_t\|_{L^1(Q_t^{N,aV})}),$$

where C depends only on C_q, k, and η. Therefore, setting

$$Z(t) := \int \chi_t \, \varrho_t \, d\mathcal{L} = \|\chi_t\|_{L^1(Q_t^{N,aV})}$$

(recall that $\chi_t \geqslant 0$), we proved that

$$|\dot{Z}(t)| \leqslant CN^{\eta-1}(1+Z(t)),$$

which implies that

$$|\log(1+Z(1)) - \log(1+Z(0))| \leqslant CN^{\eta-1}.$$

Recalling that $T^N = X_1^N$, this proves the desired result when χ is smooth. By approximation the result extends to all measurable functions $\chi: \mathbb{R}^N \to \mathbb{R}^+$ satisfying $\|\chi\|_\infty \leqslant N^k$, concluding the proof.

4.3. Construction of approximate transport maps

Define

$$M_k^N := \sum_{i=1}^N \delta_{\lambda_i^k} - N\mu_{k,t}^*,$$

where $\mu_{k,t}^* := \mu_{k,t}^{aV}$ are the limiting measures for L_k^N under $Q_t^{N,aV}$; their existence and properties are derived exactly as in the case $t=1$, see §3. In analogy with [5, §2.3] we make the following ansatz: we look for a vector field $\mathbf{Y}_t^N$ of the form

$$(\mathbf{Y}_t^N)_i^k(\hat{\lambda}) = \mathbf{y}_{k,t}^0(\lambda_i^k) + \frac{1}{N}\mathbf{y}_{k,t}^1(\lambda_i^k) + \frac{1}{N}\sum_{\ell=1}^d \boldsymbol{\zeta}_{k\ell,t}(\lambda_i^k, M_\ell^N), \tag{4.13}$$

where $\mathbf{y}_{k,t}^0: \mathbb{R} \to \mathbb{R}$, $\mathbf{y}_{k,t}^1: \mathbb{R} \to \mathbb{R}$, $\mathbf{z}_{k\ell,t} = \mathbf{z}_{\ell k,t}: \mathbb{R}^2 \to \mathbb{R}$, and

$$\boldsymbol{\zeta}_{k\ell,t}(x, M_\ell^N) := \int \mathbf{z}_{k\ell,t}(x,y)\, dM_\ell^N(y).$$

With this particular choice of $\mathbf{Y}_t^N$, we see that

$$\begin{aligned}\sum_{i=1}^N \partial_{\lambda_i^k}(\mathbf{Y}_t^N)_i^k(\hat{\lambda}) &= N\int (\mathbf{y}_{k,t}^0)'(x)\, dL_k^N(x) + \int (\mathbf{y}_{k,t}^1)'(x)\, dL_k^N(x) \\ &\quad + \sum_{\ell=1}^d \int \partial_1 \boldsymbol{\zeta}_{k\ell,t}(x, M_\ell^N)\, dL_k^N(x) + \int \partial_2 \mathbf{z}_{kk,t}(x,x)\, dL_k^N(x).\end{aligned}$$

We now expand $\{F^a_l\}_{l=0}^2$ around the stationary measures $\mu^*_{k,t}$ (recall that F^a_l are smooth by Lemma 6.16, and that M_N has mass bounded by $2N$) and use that $\phi^M_{\#}\mu^*_{k,t}=\mu^*_{k,t}$ to get

$$\begin{aligned}
&F^a_l(\phi^M_{\#}L^N_1,\dots,\phi^M_{\#}L^N_d,\tau^N_B)\\
&\quad=F^a_l(\mu^*_{1,t},\dots,\mu^*_{d,t},\tau^N_B)+\frac{1}{N}\sum_k D_kF^a_l(\mu^*_{1,t},\dots,\mu^*_{d,t},\tau^N_B)[\phi^M_{\#}M^N_k]\\
&\qquad+\frac{1}{N^2}\sum_{k,\ell}D^2_{k\ell}F^a_l(\mu^*_{1,t},\dots,\mu^*_{d,t},\tau^N_B)[\phi^M_{\#}M^N_k,\phi^M_{\#}M^N_\ell]\\
&\qquad+\frac{1}{N^3}\sum_{k,\ell,m}D^3_{k\ell m}F^a_l(\mu^*_{1,t},\dots,\mu^*_{d,t},\tau^N_B)[\phi^M_{\#}M^N_k,\phi^M_{\#}M^N_\ell,\phi^M_{\#}M^N_m]+O\biggl(\frac{|\phi^M_{\#}M^N|^4}{N^4}\biggr),
\end{aligned}$$

where

$$O\biggl(\frac{|\phi^M_{\#}M^N|^p}{N^k}\biggr):=O(N^{-k}\|\phi^M_{\#}M^N\|^p_{M\zeta});$$

see Lemma 6.16.

We now use assumption (2.7) and the smoothness of the functions F^a_l (see again Lemma 6.16) to expand $D_kF^a_l$, $D^2_{k\ell}F^a_l$, and $D^3_{k\ell m}F^a_l$ with respect to τ. To simplify notation, we define the following functions:

$$\begin{aligned}
f_{k,l}(x)&:=D_kF^a_l(\mu^*_{1,t},\dots,\mu^*_{d,t},\tau^0_B)[\delta_{\phi^M(x)}],\\
f_{k\tau^1,l}(x)&:=D^2_{k,\tau}F^a_l(\mu^*_{1,t},\dots,\mu^*_{d,t},\tau^0_B)[\delta_{\phi^M(x)},\tau^1_B],\\
f_{k\tau^2,l}(x)&:=D^2_{k,\tau}F^a_l(\mu^*_{1,t},\dots,\mu^*_{d,t},\tau^0_B)[\delta_{\phi^M(x)},\tau^2_B],\\
&\qquad+\tfrac{1}{2}D^3_{k,\tau\tau}F^a_l(\mu^*_{1,t},\dots,\mu^*_{d,t},\tau^0_B)[\delta_{\phi^M(x)},\tau^1_B,\tau^1_B],\\
f_{k\ell,l}(x,y)&:=D^2_{k\ell}F^a_l(\mu^*_{1,t},\dots,\mu^*_{d,t},\tau^0_B)[\delta_{\phi^M(x)},\delta_{\phi^M(y)}]\\
f_{k\ell\tau^1,l}(x,y)&:=D^3_{k\ell,\tau}F^a_l(\mu^*_{1,t},\dots,\mu^*_{d,t},\tau^0_B)[\delta_{\phi^M(x)},\delta_{\phi^M(y)},\tau^1_B],\\
f_{k\ell m,l}(x,y)&:=D^3_{k\ell m}F^a_l(\mu^*_{1,t},\dots,\mu^*_{d,t},\tau^0_B)[\delta_{\phi^M(x)},\delta_{\phi^M(y)},\delta_{\phi^M(z)}].
\end{aligned}$$

We may assume without loss of generality that these functions are symmetric with respect to their arguments. Then we get the following formulas:

$$\begin{aligned}
&F^a_l(\phi^M_{\#}L^N_1,\dots,\phi^M_{\#}L^N_d,\tau^N_B)\\
&\quad=F^a_l(\mu^*_{1,t},\dots,\mu^*_{d,t},\tau^0_B)+\frac{1}{N}\sum_k\int f_{k,l}(x)\,dM^N_k(x)\\
&\qquad+\frac{1}{N^2}\sum_k\int f_{k\tau^1,l}(x)\,dM^N_k(x)+\frac{1}{N^2}\sum_{k,\ell}\iint f_{k\ell,l}(x,y)\,dM^N_k(x)\,dM^N_\ell(y)
\end{aligned}$$

$$+\frac{1}{N^3}\sum_k\int f_{k\tau^2,l}(x)\,dM_k^N(x)+\frac{1}{N^3}\sum_{k,\ell}\iint f_{k\ell\tau^1,l}(x,y)\,dM_k^N(x)\,dM_\ell^N(y)$$
$$+\frac{1}{N^3}\sum_{k,\ell,m}\iiint f_{k\ell m,l}(x,y,z)\,dM_k^N(x)\,dM_\ell^N(y)\,dM_m^N(z)+O\biggl(\frac{|\phi_\#^M M^N|^4}{N^4}\biggr),$$

and

$$\begin{aligned}
&\partial_{\lambda_i^k}F_l^a(\phi_\#^M L_1^N,...,\phi_\#^M L_d^N,\tau_B^N)\\
&\quad=\frac{1}{N}f'_{k,l}(\lambda_i^k)+\frac{1}{N^2}f'_{k\tau^1,\alpha}(\lambda_i^k)+\frac{2}{N^2}\sum_\ell\int\partial_1 f_{k\ell,l}(\lambda_i^k,y)\,dM_\ell^N(y)\\
&\quad+\frac{1}{N^3}f'_{k\tau^2,\alpha}(\lambda_i^k)+\frac{2}{N^3}\sum_\ell\int\partial_1 f_{k\ell\tau^1,\alpha}(\lambda_i^k,y)\,dM_\ell^N(y)\\
&\quad+\frac{3}{N^3}\sum_{\ell,m}\iint\partial_1 f_{k\ell m,l}(\lambda_i^k,y,z)\,dM_\ell^N(y)\,dM_m^N(z)+O\biggl(\frac{|\phi_\#^M M^N|^3}{N^4}\biggr).
\end{aligned}$$

This gives, for H defined in (4.5),

$$\begin{aligned}
\partial_{\lambda_i^k}H_t(\hat\lambda)&=NW_k'(\lambda_i^k)-tNf'_{k,0}(\lambda_i^k)-t[f'_{k\tau^1,0}(\lambda_i^k)-f'_{k,1}(\lambda_i^k)]\\
&\quad-2t\sum_\ell\int\partial_1 f_{k\ell,0}(\lambda_i^k,y)\,dM_\ell^N(y)\\
&\quad-\frac{t}{N}[f'_{k\tau^2,0}(\lambda_i^k)+f'_{k\tau^1,1}(\lambda_i^k)+f'_{k,2}(\lambda_i^k)]\\
&\quad-\frac{2t}{N}\sum_\ell\int[\partial_1 f_{k\ell\tau^1,0}(\lambda_i^k,y)+\partial_1 f_{k\ell,1}(\lambda_i^k,y)]\,dM_\ell^N(y)\\
&\quad-\frac{3t}{N}\sum_{\ell,m}\iint\partial_1 f_{k\ell m,0}(\lambda_i^k,y,z)\,dM_\ell^N(y)\,dM_m^N(z)+O\biggl(\frac{|\phi_\#^M M^N|^2}{N^2}\biggr).
\end{aligned}$$

Also, with this notation, the analogue of (3.3) for $t\in[0,1]$ becomes

$$W_{k,t}^{\mathrm{eff}}(x):=W_k(x)-tf_{k,0}(x). \tag{4.14}$$

Hence, with all this at hand, we can estimate the term $\mathcal{R}_t^N(\mathbf{Y}^N)$ defined in (4.4): using the convention that when we integrate a function of the form

$$\frac{\psi(x)-\psi(y)}{x-y}$$

with respect to $L_k^N \otimes L_k^N$ the diagonal terms give $\psi'(x)$, we get

$$
\begin{aligned}
&\mathcal{R}_t^N(\mathbf{Y}^N) \\
&= c_t^N - \frac{\beta N^2}{2} \sum_k \iint \frac{\mathbf{y}_{k,t}^0(x) - \mathbf{y}_{k,t}^0(y)}{x-y} \, dL_k^N(x) \, dL_k^N(y) \\
&\quad - N\left(1 - \frac{\beta}{2}\right) \sum_k \int (\mathbf{y}_{k,t}^0)' \, dL_k^N \\
&\quad - \frac{\beta N}{2} \sum_k \iint \frac{\mathbf{y}_{k,t}^1(x) - \mathbf{y}_{k,t}^1(y)}{x-y} \, dL_k^N(x) \, dL_k^N(y) - \left(1 - \frac{\beta}{2}\right) \sum_k \int (\mathbf{y}_{k,t}^1)' \, dL_k^N \\
&\quad - \frac{\beta N}{2} \sum_{k,\ell} \iint \frac{\boldsymbol{\zeta}_{k\ell,t}(x, M_\ell^N) - \boldsymbol{\zeta}_{k\ell,t}(y, M_\ell^N)}{x-y} \, dL_k^N(x) \, dL_k^N(y) \\
&\quad - \left(1 - \frac{\beta}{2}\right) \sum_{k,\ell} \int \partial_1 \boldsymbol{\zeta}_{k\ell,t}(x, M_\ell^N) \, dL_k^N - \sum_k \int \partial_2 \mathbf{z}_{kk,t}(x,x) \, dL_k^N(x) \\
&\quad - N^2 F_0^a(\mu_{1,t}^*, \ldots, \mu_{d,t}^*, \tau_B^0) - N \sum_k \int f_{k,0}(x) \, dM_k^N(x) \\
&\quad - \sum_{k,\ell} \iint f_{k\ell,0}(x,y) \, dM_k^N(x) \, dM_\ell^N(y) - N F_1^a(\mu_{1,t}^*, \ldots, \mu_{d,t}^*, \tau_B^0) \\
&\quad - \sum_k \int f_{k,1}(x) \, dM_k^N(x) - F_2^a(\mu_{1,t}^*, \ldots, \mu_{d,t}^*, \tau_B^0) \\
&\quad + N^2 \sum_k \int (W_{k,t}^{\mathrm{eff}})'(x) \mathbf{y}_{k,t}^0(x) \, dL_k^N(x) \\
&\quad + N \sum_k \int (W_{k,t}^{\mathrm{eff}})'(x) \mathbf{y}_{k,t}^1 \, dL_k^N(x) \\
&\quad + N \sum_{k,\ell} \int (W_{k,t}^{\mathrm{eff}})'(x) \, \boldsymbol{\zeta}_{k\ell,t}(x, M_\ell^N) \, dL_k^N(x) \\
&\quad - tN \sum_k \int [f'_{k\tau^1,0} - f'_{k,1}](x) \mathbf{y}_{k,t}^0(x) \, dL_k^N(x) \\
&\quad - t \sum_k \int [f'_{k\tau^1,0} - f'_{k,1}](x) \mathbf{y}_{k,t}^1(x) \, dL_k^N(x) \\
&\quad - t \sum_{k,\ell} \int [f'_{k\tau^1,0} - f'_{k,1}](x) \boldsymbol{\zeta}_{k\ell,t}(x, M_\ell^N) \, dL_k^N(x) \\
&\quad - 2tN \sum_{k,\ell} \iint \partial_1 f_{k\ell,0}(x,y) \mathbf{y}_{k,t}^0(x) \, dM_\ell^N(y) \, dL_k^N(x) \\
&\quad - 2t \sum_{k,\ell} \iint \partial_1 f_{k\ell,0}(x,y) \mathbf{y}_{k,t}^1(x) \, dM_\ell^N(y) \, dL_k^N(x)
\end{aligned}
$$

$$
\begin{aligned}
&-2t\sum_{k,\ell,m}\iint \partial_1 f_{k\ell,0}(x,y)\,\boldsymbol{\zeta}_{km,t}(x,M_m^N)\,dM_\ell^N(y)\,dL_k^N(x)\\
&-3t\sum_{k,\ell,m}\iiint \partial_1 f_{k\ell m,0}(x,y,z)\mathbf{y}^0_{k,t}(x)\,dM_\ell^N(y)\,dM_m^N(z)\,dL_k^N(x)\\
&-2t\sum_{k,\ell}\iint [\partial_1 f_{k\ell\tau^1,0}+\partial_1 f_{k\ell,1}](x,y)\,\mathbf{y}^0_{k,t}(x)\,dM_\ell^N(y)\,dL_k^N(x)\\
&-t\sum_{k}\int [f'_{k\tau^2,0}+f'_{k\tau^1,1}+f'_{k,2}](x)\mathbf{y}^0_{k,t}(x)\,dL_k^N(x)+O\biggl(\frac{|\phi^M_\# M^N|^3}{N}\biggr).
\end{aligned}
$$

Recalling (3.4) we observe that, for any function f,

$$
\begin{aligned}
N^2\int (W^{\mathrm{eff}}_{k,t})' f\,dL_k^N-\frac{\beta N^2}{2}\iint \frac{f(x)-f(y)}{x-y}\,dL_k^N(x)\,dL_k^N(y)&\\
=N\int \Xi_k f\,dM_k^N-\frac{\beta}{2}\iint \frac{f(x)-f(y)}{x-y}\,dM_k^N(x)\,dM_k^N(y),&
\end{aligned}
\tag{4.15}
$$

where

$$
\Xi_k f(x):=-\beta\int \frac{f(x)-f(y)}{x-y}\,d\mu^*_{k,t}(y)+(W^{\mathrm{eff}}_{k,t})'(x)f(x). \tag{4.16}
$$

Also, observe that up to now the term $O(|\phi^M_\# M^N|^3/N)$ does not depend on the smoothness of the functions $\mathbf{y}^0_{k,t},\mathbf{y}^1_{k,t},\mathbf{z}_{k\ell,t}$. However, in order to be able later to quantify the degree of smoothness required on the potentials W_k, we introduce the following notation: we will denote by $O(|\phi^M_\# M^N|^3/N;g_1,g_2,\dots,g_p)$ a quantity bounded by

$$
\sum_{m=1}^p R[g_m]+\frac{C}{N}\|\phi^M_\# M^N\|^3_{M\zeta}, \tag{4.17}
$$

where the functions g_m map $\mathbb{R}^{\ell_m}$ into $\mathbb{R}$ for $\ell_m\in\{1,2\}$, and

$$
\begin{aligned}
R[g_m]:=&\sum_{r_1,r_2,r_3=1}^d\frac{1}{N}\int_0^1 d\alpha\biggl|\iiint g_m(\alpha z_1+(1-\alpha)z_2,z_3)\,dM^N_{r_1}(z_1)\,dM^N_{r_2}(z_2)\,dM^N_{r_3}(z_3)\biggr|\\
&+\sum_{r_1,r_2=1}^d\frac{1}{N}\biggl|\iint g_m(z_1,z_2)\,dM^N_{r_1}(z_1)\,dM^N_{r_2}(z_2)\biggr|\\
&+\sum_{r_1=1}^d\frac{1}{N}\biggl|\int g_m(z_1,z_1)\,dM^N_{r_1}(z_1)\biggr|
\end{aligned}
$$

if $\ell_m=2$, while

$$
\begin{aligned}
R[g_m]:=&\sum_{r_1,r_2=1}^d\frac{1}{N}\int_0^1 d\alpha\biggl|\iint g_m(\alpha z_1+(1-\alpha)z_2)\,dM^N_{r_1}(z_1)\,dM^N_{r_2}(z_2)\biggr|\\
&+\sum_{r_1=1}^d\frac{1}{N}\biggl|\int g_m(z_1)\,dM^N_{r_1}(z_1)\biggr|
\end{aligned}
$$

if $\ell_m=1$. For instance, writing

$$\frac{\mathbf{z}_{k\ell,t}(x,z)-\mathbf{z}_{k\ell,t}(y,z)}{x-y}=\int_0^1 \partial_1\mathbf{z}_{k\ell,t}(\alpha x+(1-\alpha)y,z)\,d\alpha$$

and recalling the definition of $\boldsymbol{\zeta}_{k\ell,t}$, we see that

$$\frac{1}{N}\iint\frac{\boldsymbol{\zeta}_{k\ell,t}(x,M_\ell^N)-\boldsymbol{\zeta}_{k\ell,t}(y,M_\ell^N)}{x-y}\,dM_k^N(x)\,dM_k^N(y)=O\bigg(\frac{|\phi_\#^M M^N|^3}{N};\partial_1\mathbf{z}_{k\ell,t}\bigg).$$

Thus, applying (4.15) to $f=\mathbf{y}^0_{k,t},\mathbf{y}^1_{k,t},\boldsymbol{\zeta}_{k\ell,t}(\cdot,M_\ell^N)$, and using that $L_k^N=\mu^*_{k,t}+M_k^N/N$ (recall that $\mathbf{z}_{k\ell,t}=\mathbf{z}_{\ell k,t}$ for all k and ℓ), we get

$$\begin{aligned}
\mathcal{R}_t^N(\mathbf{Y}^N)=N\sum_k\int\bigg[&\Xi_k\mathbf{y}^0_{k,t}-2t\bigg(\sum_\ell\int\mathbf{y}^0_{\ell,t}(y)\partial_1 f_{k\ell,0}(y,\cdot)\,d\mu^*_{\ell,t}(y)\bigg)-f_{k,0}\bigg]\,dM_k^N\\
+\sum_k\int\bigg[&\Xi_k\mathbf{y}^1_{k,t}-2t\bigg(\sum_\ell\int\mathbf{y}^1_{\ell,t}(y)\partial_1 f_{k\ell,0}(y,\cdot)\,d\mu^*_{\ell,t}(y)\bigg)\\
&-f_{k,1}-t[f'_{k\tau^1,0}-f'_{k,1}]\mathbf{y}^0_{k,t}-\bigg(\frac{\beta}{2}-1\bigg)(\mathbf{y}^0_{k,t})'\\
&-\bigg(1-\frac{\beta}{2}\bigg)\sum_\ell\int\partial_1\mathbf{z}_{k\ell,t}(y,\cdot)\,d\mu^*_{\ell,t}(y)\\
&-t\sum_\ell\int\big[f'_{\ell\tau^1,0}-f'_{\ell,1}\big](y)\,\mathbf{z}_{k\ell,t}(y,\cdot)\,d\mu^*_{\ell,t}(y)\\
&-2t\sum_\ell\int\mathbf{y}^0_{\ell,t}(y)[\partial_1 f_{k\ell\tau^1,0}+\partial_1 f_{k\ell,1}](y,\cdot)\,d\mu^*_{\ell,t}(y)\bigg]\,dM_k^N\\
+\sum_{k,\ell}\iint\bigg[&\Xi_k[\mathbf{z}_{k\ell,t}(\cdot,y)](x)-2t\sum_m\int\mathbf{z}_{km,t}(z,y)\,\partial_1 f_{m\ell,0}(z,x)\,d\mu^*_{m,t}(z)\\
&-f_{k\ell,0}(x,y)-2t\,\partial_1 f_{k\ell,0}(x,y)\mathbf{y}^0_{k,t}(x)-\frac{\beta}{2}1_{k=\ell}\frac{\mathbf{y}^0_{k,t}(x)-\mathbf{y}^0_{k,t}(y)}{x-y}\\
&-3t\sum_m\int\mathbf{y}^0_{m,t}(z)\,\partial_1 f_{k\ell m,0}(x,y,z)\,d\mu^*_{m,t}(z)\bigg]\,dM_k^N(x)\,dM_\ell^N(y)\\
+C_t^N+O\bigg(&\frac{|\phi_\#^M M^N|^3}{N};(\mathbf{y}^1_{k,t})',\partial_1\mathbf{z}_{k\ell,t},\partial_2\mathbf{z}_{kk,t}\bigg)
\end{aligned}$$

where C_t^N is a constant. Let us consider the operator $\boldsymbol{\Xi}_t$ defined on d-tuples of functions by

$$\boldsymbol{\Xi}_t(\Psi_1,...,\Psi_d):=(\boldsymbol{\Xi}_t(\Psi_1,...,\Psi_d)_1,...,\boldsymbol{\Xi}_t(\Psi_1,...,\Psi_d)_d),$$

where

$$\boldsymbol{\Xi}_t(\Psi_1,...,\Psi_d)_k:=\Xi_k\Psi_k-2t\sum_{\ell=1}^d\int\Psi_\ell(y)\,\partial_1 f_{k\ell,0}(y,\cdot)\,d\mu^*_{\ell,t}(y)\quad\text{for all } k=1,...,d. \tag{4.18}$$

Then, for $\mathcal{R}_t^N(\mathbf{Y}^N)$ to be small we want to impose

$$
\begin{aligned}
\Xi_t(\mathbf{y}^0_{1,t}, ..., \mathbf{y}^0_{d,t})_k &= (g^0_1, ..., g^0_d),\\
\Xi_t(\mathbf{z}_{1\ell,t}(\,\cdot\,,y), ..., \mathbf{z}_{d\ell,t}(\,\cdot\,,y))_k &= (g^2_{1\ell}(\,\cdot\,,y), ..., g^2_{d\ell}(\,\cdot\,,y)) \quad \text{for all } \ell=1,...,d \text{ and all } y,\\
\Xi_t\big(\mathbf{y}^1_{1,t}, ..., \mathbf{y}^1_{d,t}\big)_k &= \big(g^1_1, ..., g^1_d\big),
\end{aligned}
\tag{4.19}
$$

where

$$
g^0_k(x) := f_{k,0}(x) + c_k,
$$

$$
\begin{aligned}
g^2_{k\ell}(x,y) := {}& f_{k\ell,0}(x,y) + 2t\,\partial_1 f_{k\ell,0}(x,y)\mathbf{y}^0_{k,t}(x)\\
&+ 3t\sum_m \int \mathbf{y}^0_{m,t}(z)\,\partial_1 f_{k\ell m,0}(x,y,z)\,d\mu^*_{m,t}(z) + c_{k\ell}(y), \quad \text{if } k\neq\ell,\\
g^2_{kk}(x,y) := {}& f_{kk,0}(x,y) + 2t\,\partial_1 f_{kk,0}(x,y)\mathbf{y}^0_{k,t}(x) - \frac{\beta}{2}\,\frac{\mathbf{y}^0_{k,t}(x)-\mathbf{y}^0_{k,t}(y)}{x-y}\\
&+ 3t\sum_m \int \mathbf{y}^0_{m,t}(z)\,\partial_1 f_{kkm,0}(x,y,z)\,d\mu^*_{m,t}(z) + c_{kk}(y),\\
g^1_k(x) := {}& f_{k,1}(x) + t[f'_{k\tau^1,0}(x) - f'_{k,1}(x)]\mathbf{y}^0_{k,t}(x) + \left(\frac{\beta}{2}-1\right)(\mathbf{y}^0_{k,t})'(x)\\
&+ \left(1-\frac{\beta}{2}\right)\sum_\ell \int \partial_1 \mathbf{z}_{k\ell,t}(y,x)\,d\mu^*_{\ell,t}(y) + \sum_\ell \int f'_{\ell,1}(y)\,\mathbf{z}_{k\ell,t}(y,x)\,d\mu^*_{\ell,t}(y)\\
&+ 2t\sum_\ell \int \mathbf{y}^0_{\ell,t}(y)[\partial_1 f_{k\ell\tau^1,0} + \partial_1 f_{k\ell,1}](y,\cdot\,)\,d\mu^*_{\ell,t}(y) + c'_k,
\end{aligned}
$$

where c_k and c'_k are constants to be fixed later, and $c_{k\ell}(y)$ is a family of functions depending only on y also to be fixed.

Indeed, noticing that $\int dM^N_k = 0$ for all k, we see that all constants integrate to zero against M^N, and we conclude that the following holds.

Lemma 4.2. *Let Ξ_t be defined as in (4.18), with $\{\Xi_k\}_{k=1}^d$ as in (4.16). Also, recall the notation (4.17). Assume that we can find functions $\mathbf{y}^0_{k,t}, \mathbf{y}^1_{k,t}, \mathbf{z}_{k\ell,t}$ solving (4.19). Then*

$$
\mathcal{R}_t^N(\mathbf{Y}^N) = C_t^N + O\left(\frac{|\phi^M_\# M^N|^3}{N}; (\mathbf{y}^1_{k,t})', \partial_1 \mathbf{z}_{k\ell,t}, \partial_2 \mathbf{z}_{kk,t}\right),
$$

where C_t^N is a constant.

4.4. Invertibility properties of $\boldsymbol{\Xi}_t$

Lemma 4.2 suggests that, to construct an approximate map, we need to solve an equation of the form

$$\boldsymbol{\Xi}_t(\Psi_1, ..., \Psi_d) = (g_1, ..., g_d).$$

We remind that, in our setting, the functions $\partial_1 f_{k\ell,0}(\cdot, y)$ are smooth and their C^s norms are of size $O(|a|)$ for any $s>0$, where a is a small number. Also, note that the operators Ξ_k defined in (4.16) are continuous with respect to the C^1 topology. This will allow us to show invertibility of $\boldsymbol{\Xi}_t$ using Lemma 4.3 below and a fixed-point argument.

Before stating the result in our setting we recall that, given a function $f:\mathbb{R}\to\mathbb{R}$, the norm C^s is defined as

$$\|f\|_{C^s(\mathbb{R})} := \sum_{j=0}^{s} \|f^{(j)}\|_{L^\infty(\mathbb{R})},$$

where $f^{(j)}$ denotes the jth derivative of f. The next result is contained in [5, Lemma 3.2].

Lemma 4.3. *Let $V:\mathbb{R}\to\mathbb{R}$ be a function of class C^σ with $\sigma\geqslant 4$, assume that μ_V has support given by $[a,b]$ and that*

$$\frac{d\mu_V}{dx}(x) = S(x)\sqrt{(a-x)(x-b)} \quad \text{with } S(x)\geqslant \bar{c}>0 \text{ a.e. on } [a,b]. \tag{4.20}$$

Define the operator

$$\Xi\Psi(x) := -\beta\int \frac{\Psi(x)-\Psi(y)}{x-y}\, d\mu_V(x) + V'(x)\Psi(x),$$

and fix an integer $3\leqslant s\leqslant \sigma-1$. Then, for any function $g:\mathbb{R}\to\mathbb{R}$ of class C^s, there exists a unique constant c_g such that the equation

$$\Xi\Psi(x) = g(x) + c_g$$

has a unique solution $\Psi:\mathbb{R}\to\mathbb{R}$ of class C^{s-2}, also denoted by $\Xi^{-1}g$, which satisfies the estimate

$$\|\Psi\|_{C^{s-2}(\mathbb{R})} \leqslant \widehat{C}_s \|g\|_{C^s(\mathbb{R})}. \tag{4.21}$$

Moreover Ψ (and its derivatives) behaves like

$$\frac{g(x)+c_g}{V'(x)}$$

(and its corresponding derivatives) when $|x|\to\infty$.

We now want to apply this lemma with $V = W_{k,t}^{\mathrm{eff}}$ and $\mu_V = \mu_{k,t}^*$ (so that $\Xi = \Xi_k$, see (4.16)), and prove the invertibility of $\mathbf{\Xi}_t$ by a fixed point argument. We notice that the constants appearing in the above result depend only on the smoothness of V and on the assumption (4.20), which is satisfied by $\mu_{k,t}^*$ due to Lemma 3.2. In particular, when applied with $V = W_{k,t}^{\mathrm{eff}}$ and $\mu_V = \mu_{k,t}^*$, all the constants are uniform for $t \in [0,1]$. Also, as F_0^a is of class C^∞, the smoothness of $W_{k,t}^{\mathrm{eff}}$ is the same as that of W_k (see (4.14)).

Proposition 4.4. *There exists $\alpha > 0$ such that the following holds. Assume that the functions $W_1, ..., W_d: \mathbb{R} \to \mathbb{R}$ are of class C^σ for some $\sigma \geqslant 4$. Suppose that $|a| \leqslant \alpha$, and let $t \in [0,1]$. Then, for any family of functions $g_1, ..., g_d: \mathbb{R} \to \mathbb{R}$ of class C^s with $s \in [3, \sigma - 1]$, there exist a unique family of constants $(c_{g_1}, ..., c_{g_d})$ such that the equation*

$$\mathbf{\Xi}_t(\Psi_1, ..., \Psi_d) = (g_1, ..., g_d) + (c_{g_1}, ..., c_{g_d}) \tag{4.22}$$

has a solution $\Psi_1, ..., \Psi_d: \mathbb{R} \to \mathbb{R}$ of class C^{s-2}. In addition, there exists a finite constant $\overline{C}_0$ such that

$$\max_{k=1}^{d} \|\Psi_k\|_{C^1(\mathbb{R})} \leqslant \overline{C}_0 \max_{k=1}^{d} \|g_k\|_{C^3(\mathbb{R})}. \tag{4.23}$$

Furthermore, there exists $\gamma_s > 0$ such that Ψ_k and its derivatives up to order $s-1$ decay like $O(1/[(W_{k,t}^{\mathrm{eff}})'(x)]^{\gamma_s})$ as $|x| \to \infty$.

Proof. Define the operator

$$\Upsilon_k^{aV}(\Psi_1, ..., \Psi_d) := \sum_{\ell=1}^{d} \int \Psi_\ell(y)\, \partial_1 f_{k\ell,0}(y, \cdot\,)\, d\mu_{\ell,t}^*(y),$$

so that (4.22) can be rewritten as

$$\Xi_k \Psi_k - 2t\, \Upsilon_k^{aV}(\Psi_1, ..., \Psi_d) = g_k + c_{g_k} \quad \text{for all } k = 1, ..., d.$$

Recalling that $\partial_1 f_{k\ell,0}(\,\cdot\,, y)$ is a smooth function with all derivatives of size $O(|a|)$, for any family of bounded functions $\Psi_k: \mathbb{R} \to \mathbb{R}$, one has

$$\big\|\Upsilon_k^{aV}(\Psi_1, ..., \Psi_d)\big\|_{C^3(\mathbb{R})} \leqslant \overline{C} |a| \max_{k=1}^{d} \|\Psi_k\|_{C^1(\mathbb{R})} \tag{4.24}$$

for some universal constant $\overline{C}$. To prove the result we simply apply a fixed point argument: more precisely, we set $(\Psi_{1,(0)}, ..., \Psi_{d,(0)}) = (0, ..., 0)$ and we recursively define, for $j \geqslant 1$,

$$\Psi_{k,(j+1)} := (\Xi_k)^{-1}(2t \Upsilon_k^{aV}(\Psi_{1,(j)}, ..., \Psi_{d,(j)}) + g_k), \quad k = 1, ..., d.$$

Applying Lemma 4.3 with $V = W_{k,t}^{\mathrm{eff}}$ and $\mu_V = \mu_{k,t}^*$ (so that $\Xi = \Xi_k$), we deduce that

$$\Psi_{k,(j)} \in C^1(\mathbb{R}) \quad \text{for all } j \geqslant 1 \text{ and } k = 1, ..., d.$$

Also, by the linearity of Ξ_k and Υ_k^{aV}, we have

$$\Psi_{k,(j+1)}-\Psi_{k,(j)}=(\Xi_k)^{-1}(2t\,\Upsilon_k^{aV}(\Psi_{1,(j)}-\Psi_{1,(j-1)},...,\Psi_{d,(j)}-\Psi_{d,(j-1)})),$$

so it follows from (4.21) and (4.24) that

$$\max_{k=1}^{d}\|\Psi_{k,(j+1)}-\Psi_{k,(j)}\|_{C^1(\mathbb{R})}\leqslant 2t\widehat{C}_3\overline{C}|a|\max_{k=1}^{d}\|\Psi_{k,(j)}-\Psi_{k,(j-1)}\|_{C^1(\mathbb{R})}.$$

Hence, if we choose α small enough so that $\widehat{C}_3\overline{C}\alpha\leqslant\frac{1}{4}$, we deduce that $\{\Psi_{k,(j)}\}_{j\geqslant 1}$ is a Cauchy sequence in C^1 for all $k=1,...,d$. Recalling that the operator Ξ_k are continuous with respect to the C^1 topology, we deduce that the sequence $(\Psi_{1,(j)},...,\Psi_{d,(j)})$ converges to a solution of our problem $(\Psi_1,...,\Psi_d)$.

Applying (4.21) and (4.24) again, we deduce that

$$\begin{aligned}\max_{k=1}^{d}\|\Psi_{k,(j+1)}\|_{C^1(\mathbb{R})}&\leqslant 2t\widehat{C}_3\overline{C}|a|\max_{k=1}^{d}\|\Psi_{k,(j)}\|_{C^1(\mathbb{R})}+\widehat{C}_3\max_{k=1}^{d}\|g_k\|_{C^3(\mathbb{R})}\\&\leqslant\tfrac{1}{2}\max_{k=1}^{d}\|\Psi_{k,(j)}\|_{C^1(\mathbb{R})}+\widehat{C}_3\max_{k=1}^{d}\|g_k\|_{C^3(\mathbb{R})},\end{aligned}$$

so (4.23) follows by letting $j\to\infty$. In addition, Lemma 4.3 implies that Ψ_k decays like $O(1/(W_{k,t}^{\mathrm{eff}})'(x))$ as $|x|\to\infty$. Furthermore, since $\Upsilon_k^{aV}(\Psi_1,...,\Psi_d)\in C^\infty$, it follows by (4.21) that

$$\max_{k=1}^{d}\|\Psi_k\|_{C^s(\mathbb{R})}\leqslant\overline{C}_s,$$

showing that $\Psi_k\in C^s$.

To prove the final statement we note that, since $\|\Psi_k\|_{C^s(\mathbb{R})}\leqslant\overline{C}_s$ and Ψ_k decays like $O(1/(W_{k,t}^{\mathrm{eff}})')$, by interpolation inequalities the derivatives of Ψ_k up to order $s-1$ decay as an inverse power of $(W_{k,t}^{\mathrm{eff}})'$. $\square$

We can now apply the above proposition to invert the first equation in (4.19) and find a solution $\mathbf{y}_{k,t}^0$ of class $C^{\sigma-3}$. Then (since now $\mathbf{y}_{k,t}^0$ is given) we solve the second equation in (4.19) using again the proposition above, and finally we invert the third equation. In this way, in analogy with [5, Lemma 3.3] we obtain the following result (we recall that a function of two variables belongs to $C^{\tau,\tau'}$ if it is τ times continuously differentiable with respect to the first variable and τ' times with respect to the second).

COROLLARY 4.5. *Let α be as in Proposition 4.4. Assume that $W_k:\mathbb{R}\to\mathbb{R}$ are of class C^σ for all $k=1,...,d$ with $\sigma\geqslant 10$, and that $|a|\leqslant\alpha$. Then there exist functions $\mathbf{y}_{k,t}^0,\mathbf{y}_{k,t}^1,\mathbf{z}_{k\ell,t}$ solving (4.19), and a finite universal constant C_σ, such that*

$$\|\mathbf{y}_{k,t}^0\|_{C^{\sigma-3}(\mathbb{R})}+\|\mathbf{y}_{k,t}^1\|_{C^{\sigma-9}(\mathbb{R})}+\sum_{\tau+\tau'\leqslant\sigma-6}\|\mathbf{z}_{k\ell,t}\|_{C^{\tau,\tau'}(\mathbb{R}\times\mathbb{R})}\leqslant C_\sigma\quad\textit{for all } k,\ell=1,...,d.$$

Moreover these functions and their derivatives (except the last ones) decay as an inverse power of $(W_{k,t}^{\mathrm{eff}})'(x)$ as $|x|\to\infty$.

Recalling (4.4), it follows by Lemma 4.2 and Corollary 4.5 that

$$\mathcal{R}_t^N(\mathbf{Y}^N) = C_t^N + O\biggl(\frac{|\phi_\#^M M^N|^3}{N}; (\mathbf{y}_{k,t}^1)', \partial_1 \mathbf{z}_{k\ell,t}, \partial_2 \mathbf{z}_{kk,t}\biggr).$$

But in fact, since $\mathcal{R}_t^N(\mathbf{Y}^N)$ is centered (compare with [5, §3.5]), we deduce that

$$\mathcal{R}_t^N(\mathbf{Y}^N) = O\biggl(\frac{|\phi_\#^M M^N|^3}{N}; (\mathbf{y}_{k,t}^1)', \partial_1 \mathbf{z}_{k\ell,t}, \partial_2 \mathbf{z}_{kk,t}\biggr).$$

The goal of the next section is to control the right-hand side.

4.5. Getting rid of the remainder

We start by using concentration inequalities to control $M_k^N - \mathbb{E}[M_k^N]$.

LEMMA 4.6. *Let Hypothesis 2.1 hold, and let a_0 be as in §3. For $a \in [-a_0, a_0]$, there exists $c' > 0$ such that, for any Lipschitz function $f: \mathbb{R} \to \mathbb{R}$, for all $\delta > 0$, all $t \in [0,1]$ and $k \in \{1, ..., d\}$,*

$$Q_t^{N,aV}\Biggl(\biggl|\sum_{i=1}^N f(\lambda_i^k) - \mathbb{E}\biggl[\sum_{i=1}^N f(\lambda_i^k)\biggr]\biggr| \geqslant \|f\|_L \delta\Biggr) \leqslant 2e^{-c'\delta^2},$$

where $\|f\|_L$ denotes the Lipschitz constant of f.

Proof. $Q_t^{N,aV}$ being a probability measure with uniformly log-concave density (see §3), Bakry–Emery's and Herbst's argument applies (see e.g. [2, §4.4]). □

We now need to control the difference between $\mathbb{E}[L_k^N]$ and its limit $\mu_{k,t}^*$. We shall do this in two steps: we first derive a rough estimate which only provides a bound of order $N^{-1/2}$ following ideas initiated in [48], and in a second step we use loop equations to get a bound of order $(\log N)/N$, see e.g. [55]. This two steps approach was already developed in [9], [10], [11]. To get the rough estimate, we shall use the distance $d(\mu, \mu') = d(\mu - \mu')$ on the space of probability measures on $\mathbb{R}$ defined on centered measures ν by

$$d(\nu) := \biggl(2 \iint \log|x-y|^{-1} d\nu(x)\, d\nu(y)\biggr)^{1/2} = \sqrt{\int_{\mathbb{R}} \frac{1}{|\tau|}\, |\hat{\nu}(\tau)|^2\, d\tau},$$

where $\hat{\nu}$ denotes the Fourier transform of the measure ν. Because this distance blows up on measures with atoms, we shall consider the following regularization of the empirical measure: For a given vector $\lambda := (\lambda_1 < ... < \lambda_N)$, we denote by $\tilde{\lambda} := (\tilde{\lambda}_1 < ... < \tilde{\lambda}_N)$ its transformation given by

$$\tilde{\lambda}_1 := \lambda_1, \quad \tilde{\lambda}_{i+1} := \tilde{\lambda}_i + \max\{\lambda_{i+1} - \lambda_i, N^{-3}\}.$$

We denote by $\tilde{L}_k^N$ the empirical measure of the $\tilde{\lambda}_i^k$, and by $\bar{L}_k^N$ its convolution with the uniform measure on $[0, N^{-4}]$. We then claim the following result.

LEMMA 4.7. *Let Hypothesis 2.1 hold. Then there is an $a_0>0$ so that, for $a\in[-a_0,a_0]$, there exist positive constants c and C such that, for all $\delta>0$ and $t\in[0,1]$, the following bounds hold*:

- *we have*

$$Q_t^{N,aV}\Big(\max_{k=1}^{d} d(\bar{L}_k^N,\mu_{k,t}^*)\geqslant\delta\Big)\leqslant e^{CN\log N-\beta\delta^2N^2/6}+Ce^{-cN^2};$$

- *if $f:\mathbb{R}\to\mathbb{R}$ is Lipschitz and belongs to $L^2(\mathbb{R})$, then*

$$Q_t^{N,aV}\bigg(\bigg|\int f(x)\,d(L_k^N-\mu_{k,t}^*)(x)\bigg|\geqslant\delta\|f\|_{1/2}+\frac{\|f\|_L}{N^2}\bigg)\leqslant e^{CN\log N-\beta\delta^2N^2/8}+Ce^{-cN^2},$$

where $\|f\|_{1/2}:=(\int_{\mathbb{R}}|\tau|\,|\hat{f}(\tau)|^2d\tau)^{1/2}$.

Remark 4.8. Note for later use that if f is supported in $[-M,M]$, then there exists a finite constant $C(M)$ such that

$$\|f\|_{1/2}\leqslant C(M)\|f'\|_\infty.$$

Indeed,

$$\begin{aligned}\|f\|_{1/2}^2&=\int|s|\,|\hat{f}(s)|^2\,ds=\int\frac{1}{|s|}|\widehat{f'}(s)|^2\,ds\\&=-2\iint\log|x-y|f'(x)\,f'(y)\,dx\,dy\leqslant C(M)\|f'\|_\infty^2.\end{aligned}$$

Proof of Lemma 4.7. We just recall the main point of the proof, which is almost identical to that of [11, Corollary 3.5]. In the latter article, the potential is only depending polynomially on the measures rather than being an infinite series. It turns out that the main point is to show that

$$S(\nu):=\frac{\beta}{2}\sum_k d(\nu_k)^2-\sum_{k,\ell}D_{k\ell}^2F_0^a(\mu_{1,t}^*,...,\mu_{d,t}^*,\tau_B^N)[\nu_k,\nu_\ell]$$

is uniformly convex on the set $P([-M,M])^d$ of probability measures on $[-M,M]$, so that its square root defines a Lipschitz distance. Here, we more simply notice that for a small enough

$$S(\nu)\geqslant\frac{\beta}{4}\sum_{k=1}^d d(\nu_k)^2. \tag{4.25}$$

Indeed, the latter amounts to bound from above the second term in the definition of S. But since $D_{k\ell}^2F_0^a(\mu_{1,t}^*,...,\mu_{d,t}^*)[\delta_x,\delta_y]$ is smooth and compactly supported, we can always write

$$D_{k\ell}^2F_0^a(\mu_{1,t}^*,...,\mu_{d,t}^*,\tau_B^N)[\delta_x,\delta_y]=\iint e^{i\xi x+i\zeta y}\,\widehat{D_{k\ell}^2F_0^a}(\mu_{1,t}^*,...,\mu_{d,t}^*,\tau_B^N)(\xi,\zeta)\,d\xi\,d\zeta$$

and for any centered measures ν_k and ν_ℓ we get, by the Cauchy–Schwarz inequality,

$$\begin{aligned}&|D^2_{k\ell}F^a_0(\mu^*_{1,t},...,\mu^*_{d,t},\tau^N_B)[\nu_k,\nu_\ell]|\\&\qquad\leqslant d(\nu_k)\,d(\nu_\ell)\biggl(\iint|\widehat{D^2_{k\ell}F^a_0}(\mu^*_{1,t},...,\mu^*_{d,t},\tau^N_B)(\xi,\zeta)|^2|\xi|\,|\zeta|\,d\xi\,d\zeta\biggr)^{1/2}.\end{aligned}$$

Hence we can always choose a small enough so that the last term is as small as wished, proving (4.25).

Let us sketch the rest of the proof. By localizing the eigenvalues in a very tiny neighborhood around the quantiles of $\mu^*_{k,t}$ it is possible to show (see e.g. [11, Lemma 3.11]) that there exists a finite constant C such that

$$Z^{N,aV}_t\geqslant e^{-N^2J^a_t(\mu^*_{1,t},...,\mu^*_{d,t})-CN\log N}$$

where $Z^{N,aV}_t$ is as in (4.2) and

$$\begin{aligned}&J^a_t(\mu_1,...,\mu_k)\\&\quad:=\frac{1}{2}\sum_{k=1}^d\biggl(\iint[W_k(x)+W_k(y)-\beta\log|x-y|]\,d\mu_k(x)\,d\mu_k(y)\biggr)-tF^a_0(\mu_1,...,\mu_k,\tau^N_B).\end{aligned}$$

Then, writing $L_N:=(L^N_1,...,L^N_d)$, $\bar L_N:=(\bar L^N_1,...,\bar L^N_d)$, and $\mu^*:=(\mu^*_{1,t},...,\mu^*_{d,t})$, one has

$$\begin{aligned}&\frac{\beta}{2}\int_{x\neq y}\log|x-y|\,dL_N(x)\,dL_N(y)-tF^a_0(L_N,\tau^N_B)-\sum_k\int W_k\,dL^N_k+J^a_t(\mu^*_{1,t},...,\mu^*_{d,t})\\&\qquad=\frac{\beta}{2}\int_{x\neq y}\log|x-y|\,d[L_N-\mu^*](x)\,d[L_N-\mu^*](y)+R(L_N-\mu^*)\\&\qquad=\frac{\beta}{2}\int\log|x-y|\,d[\bar L_N-\mu^*](x)\,d[\bar L_N-\mu^*](y)+R(L_N-\mu^*)+O\biggl(\frac{\log N}{N}\biggr),\end{aligned}$$

where we used the regularization $\bar L_N$ of L_N to add the diagonal term $x=y$ in the logarithmic term up to an error of order $N\log N$, we bounded uniformly F^a_1 and F^a_2 up to an error of order N, and we set

$$R(\nu):=\sum_k\int f_k(x)\,d\nu_k(x)-D^3F^a_0(\mu^*+\theta\nu,\tau^N_B)[\nu^{\otimes 3}]$$

for some $\theta\in[0,1]$ and some functions f_k vanishing on the support of the equilibrium measure $\mu^*_{k,t}$, positive outside, and going to infinity like W_k (see [11, Lemma 3.11] for more details). In this way one deduces that

$$\begin{aligned}&Q^{N,aV}_t\Bigl(\max_{k=1}^d d(\bar L^N_k,\mu^*_{k,t})\geqslant\delta\Bigr)\\&\qquad\leqslant e^{CN\log N}\int_{\max_{k=1}^d d(\bar L^N_k,\mu^*_{k,t})\geqslant\delta}e^{-N^2d(\bar L_N,\mu^*)^2-N^2R(\bar L_N-\mu^*)}\prod_{i=1}^N d\lambda^k_i.\end{aligned}$$

By the large deviation principle in Theorem 3.1, we see that the cubic term in R is negligible compared to the quadratic term on a set with probability greater than $1-e^{-cN^2}$. Thus, setting $\overline{M}_k^N := N(\bar{L}_k^N - \mu_{k,t}^*)$, we get

$$\begin{aligned} &Q_t^{N,aV}\Big(\max_{k=1}^{d} d(\overline{M}_k^N) \geqslant N\delta\Big) \\ &\quad\leqslant e^{CN\log N}\bigg[\int_{\max_{k=1}^d d(\overline{M}_k^N)\geqslant N\delta} e^{-(\beta/5)\sum_{k=1}^d d(\overline{M}_k^N)^2 - N^2\sum_k \int f_k(x)\,dL_k^N(x)} \prod_{i=1}^N d\lambda_i^k + e^{-cN^2}\bigg] \\ &\quad\leqslant e^{CN\log N}(e^{-\beta N^2\delta^2/6} + e^{-cN^2}) \end{aligned}$$

This gives the first bound of the lemma, from which the second is easily deduced since

$$\left|\int f(x)\,d\nu(x)\right| = \left|\int \hat{f}(\tau)\hat{\nu}(\tau)\,d\tau\right| \leqslant \|f\|_{1/2}\,d(\nu)$$

and

$$\left|\int f(x)\,d(L_k^N - \bar{L}_k^N)(x)\right| \leqslant \frac{\|f\|_L}{N^2}. \qquad \square$$

We finally improve the previous bounds to get an error of order $(\log N)/N$ instead of $(\log N)/\sqrt{N}$.

Lemma 4.9. *Let Hypothesis 2.1 hold, and given a function $f:\mathbb{R}\to\mathbb{R}$ define the norm*

$$|||f||| := \int (1+|\tau|^7)|\hat{f}(\tau)|\,d\tau. \tag{4.26}$$

There exists $a_0>0$ so that, for all $a\in[-a_0,a_0]$ and all functions $f:\mathbb{R}\to\mathbb{R}$, with $|||f|||<\infty$,

$$\left|\int\left[\int f(x)\,d(L_k^N - \mu_{k,t}^*)(x)\right] dQ_t^{N,aV}\right| \leqslant C|||f|||\,\frac{\log N}{N}$$

for some constant C independent of a and f.

Proof. Before starting the proof, we recall the notation

$$L_N := (L_1^N, ..., L_d^N) \quad \text{and} \quad \mu^* := (\mu_{1,t}^*, ..., \mu_{d,t}^*).$$

To improve the bound we just obtained, we use the loop equation. Such an equation is simply obtained by integration by parts and, for any smooth test function, reads as follows:

$$\begin{aligned} &-\frac{1}{N}\iint f'(x)\,dL_k^N(x)\,dQ_t^{N,aV} \\ &\quad= \frac{1}{N^2}\sum_{i=1}^N \int f(\lambda_i^k)\partial_{\lambda_i^k}\bigg(\frac{dQ_t^{N,aV}}{\prod_{j=1}^N\prod_{\ell=1}^d d\lambda_j^\ell}\bigg)\prod_{j=1}^N\prod_{\ell=1}^d d\lambda_j^\ell \\ &\quad= \int\bigg(\int f(x)(t[\partial_x D_k F^a](L_N,\tau_B^N)[\delta_x] - W_k'(x))\,dL_k^N(x) \\ &\qquad\quad + \frac{\beta}{2}\iint \frac{f(x)-f(y)}{x-y}\,dL_k^N(x)\,dL_k^N(y) - \frac{\beta}{2N}\int f'(x)\,dL_k^N(x)\bigg)\,dQ_t^{N,aV}, \end{aligned}$$

where $F^a := \sum_{l=0}^{2} F_l^a N^{-l}$. Recalling that $M_k^N = N(L_k^N - \mu_{k,t}^*)$ and (4.16), we rewrite the above equation as

$$\int \left[\int \Xi_k f \, dM_k^N - t \sum_{\ell \neq k} \iint \partial_x D_{k\ell} F_0^a(\mu^*, \tau_B^N)[\delta_y, \delta_x] f(x) \, d\mu_{k,t}^*(x) \, dM_\ell^N(y) \right] dQ_t^{N,aV} = \sum_{\gamma=1}^{4} R_\gamma^N(f), \tag{4.27}$$

where

$$\begin{aligned}
R_1^N(f) &:= \left(1 - \frac{\beta}{2}\right) \iint f'(x) \, dL_k^N(x) \, dQ_t^{N,aV}, \\
R_2^N(f) &:= \frac{\beta}{2N} \iiint \frac{f(x) - f(y)}{x - y} \, dM_k^N(x) \, dM_k^N(y) \, dQ_t^{N,aV}, \\
R_3^N(f) &:= Nt \int \left[\int f(x) \, \partial_x D_k (F^a - F_0^a)(L_N, \tau_B^N)[\delta_x] \, dL_k^N(x) \right] dQ_t^{N,aV}, \\
R_4^N(f) &:= \frac{t}{N} \sum_\ell \int \left(\int f(x) \, \partial_x D_{k\ell} F_0^a(\mu^* + \theta(L_N - \mu^*), \tau_B^N)[M_k^N, M_\ell^N] \right) dQ_t^{N,aV},
\end{aligned}$$

and the last term was computed using a Taylor expansion. Writing $f(x) = \int e^{ixs} \hat{f}(s) \, ds$ and noticing that $\|e^{i\lambda \cdot}\|_{1/2} + \|e^{i\lambda \cdot}\|_L \leqslant 2(1+|\lambda|)$ so that Lemma 4.7 entails

$$\int \left| \widehat{(M_k^N)}(\lambda) \right|^2 dQ_t^{N,aV} \leqslant CN \log N (1+|\lambda|)^2,$$

we get

$$\begin{aligned}
|R_1^N(f)| &\leqslant \|f\|_L, \\
|R_2^N(f)| &\leqslant N^{-1} \int d\tau \, |\hat{f}(\tau)| \int_0^1 d\alpha \, |\tau| \int \left| \widehat{(M_k^N)}(\alpha\tau) \right| \left| \widehat{(M_k^N)}\big((1-\alpha)\tau\big) \right| dQ_t^{N,aV} \\
&\leqslant N^{-1} \int d\tau \, |\tau| \, |\hat{f}(\tau)| \int_0^1 d\alpha \int \left| \widehat{(M_k^N)}(\alpha\tau) \right|^2 dQ_t^{N,aV} \\
&\leqslant (\log N) \int (1+|\tau|^3) |\hat{f}(\tau)| \, d\tau, \\
|R_3^N(f)| &\leqslant C \|f\|_\infty, \\
|R_4^N(f)| &\leqslant C (\log N) \|f\|_\infty,
\end{aligned}$$

where we used Lemma 4.7 for the second and fourth terms, and to bound the last term we noticed that, since F_0^a is smooth and it is of size $O(|a|)$ together with its derivatives, we have

$$\max_{k,\ell} \left| \left[\widehat{\partial_x D_{k\ell} F_0^a} \right](\lambda, \zeta) \right| \leqslant \frac{\widehat{C} |a|}{(1+\lambda^2)(1+|\zeta|^{10})}. \tag{4.28}$$

Hence, since

$$\begin{aligned}&\left|\int\left[\int \partial_x D_{k\ell}F_0^a(\mu^*,\tau_B^N)[\delta_y,\delta_x]\, f(x)\, d\mu_{k,t}^*(x)\, dM_\ell^N(y)\right] dQ_t^{N,aV}\right|\\&\qquad\leqslant \iint \left|\widehat{f\cdot d\mu_{k,t}^*}(\zeta)\right| \left|\partial_x\widehat{D_{k\ell}F_0^a}(\xi,\zeta)\right| \left|\int \widehat{(M_\ell^N)}(\xi)\, dQ_t^{N,aV}\right| d\xi\, d\zeta,\end{aligned}$$

we deduce from (4.27) that

$$\begin{aligned}&\left|\iint f(x)\, dM_k^N(x)\, dQ_t^{N,aV}\right|\\&\qquad\leqslant \|\Xi_k^{-1}f\|_\infty \sum_{\ell\neq k}\int \left|\partial_x\widehat{D_{k\ell}F_0^a}(\xi,\zeta)\right| \left|\int \widehat{(M_\ell^N)}(\zeta)\, dQ_t^{N,aV}\right| d\xi\, d\zeta\\&\qquad\qquad + C\|\Xi_k^{-1}f\|_{C^1(\mathbb{R})} + \log N \int (1+|\tau|^3)|\widehat{\Xi_k^{-1}f}(\tau)|\, d\tau.\end{aligned}$$

Applying the above bound with $f(x)=e^{i\lambda x}$ and using (4.21) with $\Xi=\Xi_k$, we get

$$\begin{aligned}\delta_N(\lambda) &:= \max_{k=1}^{d}\left|\int \widehat{M_k^N}(\lambda)\, dQ_t^{N,aV}\right|\\&\leqslant \lambda^2 \int \max_{k,\ell} |\partial_x\widehat{D_{k\ell}F_0^a}(\lambda,\zeta)|\delta_N(\zeta)\, d\zeta + C(1+|\lambda|^7)\log N.\end{aligned}\tag{4.29}$$

By (4.28), we deduce from the above equation that

$$\begin{aligned}&\int \frac{1}{1+|\lambda|^{10}}\, \delta_N(\lambda)\, d\lambda\\&\qquad\leqslant \widehat{C}\,|a| \left(\int \frac{1}{1+|\lambda|^{10}}\, d\lambda\right)\int \frac{1}{1+|\zeta|^{10}}\, \delta_N(\zeta)\, d\zeta + C\left(\int \frac{1+|\lambda|^7}{1+|\lambda|^{10}}\, d\lambda\right)\log N\\&\qquad\leqslant C\widehat{C}|a| \int \frac{1}{1+|\zeta|^{10}}\, \delta_N(\zeta)\, d\zeta + C\log N.\end{aligned}$$

In particular, if a is sufficiently small so that $C\widehat{C}|a|\leqslant\frac{1}{2}$, we can reabsorb the first term in the right-hand side and obtain

$$\int \frac{1}{1+|\lambda|^{10}}\, \delta_N(\lambda)\, d\lambda \leqslant 2C\log N.$$

Plugging back this control in (4.29) and using again (4.28), we finally get the bound

$$\delta_N(\lambda)\leqslant C(1+|\lambda|^7)\log N.$$

Therefore, using the identity $f(x)=\int \hat f(\tau)e^{i\tau x}d\tau$ we conclude that

$$\max_{k=1}^{d}\left|\int\left[\int f(x)\, dM_k^N(x)\right] dQ_t^{N,aV}\right| \leqslant \int |\hat f(\tau)|\delta_N(\tau)\, d\tau \leqslant C\log N\int (1+|\tau|^7)|\hat f(\tau)|\, d\tau,$$

as desired. $\square$

A straightforward corollary of Lemmas 4.6 and 4.9 is the following.

COROLLARY 4.10. *There exists $a_0>0$ such that, for all $a\in[-a_0,a_0]$, there are finite positive constants C and c' such that, for all $f:\mathbb{R}\to\mathbb{R}$ with $|||f|||<\infty$ and all $\delta\geqslant 0$, we have*

$$Q_t^{N,aV}\left(\left|\int f(x)\,dM_k^N(x)\right|\geqslant \delta\|f\|_L+C|||f|||\log N\right)\leqslant 2e^{-c'\delta^2}. \tag{4.30}$$

In particular, for all $p\geqslant 1$ there exists a finite constant C_p such that

$$\big\|M_k^N[f]\big\|_{L^p(Q_t^{N,aV})}=\left\|\int f(x)\,dM_k^N(x)\right\|_{L^p(Q_t^{N,aV})}\leqslant C_p\big(\|f\|_L+|||f|||\log N\big).$$

Thanks to this corollary we get the following result.

COROLLARY 4.11. *Assume that $\phi^M\in C^9(\mathbb{R})$ vanishes outside $[-M,M]$ and that it is bounded by M. There exists $a_0>0$ so that, for all $a\in[-a_0,a_0]$ and for all $\zeta>M$, there are finite constants $c_\zeta,C_\zeta,c>0$ such that, for all $\delta\geqslant 0$, we have*

$$Q_t^{N,aV}(\|\phi_\#^M M_k^N\|_\zeta\geqslant \delta c_\zeta+C_\zeta\log N)\leqslant 2e^{-c\delta^2}. \tag{4.31}$$

Proof. Using Corollary 4.10 with $f(x)=(\phi^M(x))^p$, together with Remark 4.8, we deduce that there exist constants $c_0,C_0>0$, only depending on ϕ^M, such that

$$Q_t^{N,aV}(|M_k^N((\phi^M)^p)|\geqslant c_0pM^{p-1}\delta+C_0M^pp^7\log N)\leqslant 2e^{-c'\delta^2}.$$

Therefore, for $\zeta>M$ we find $c_1,C_1>0$ such that

$$Q_t^{N,aV}(|M_k^N((\phi^M)^p)|\geqslant c_1\zeta^p\delta+C_1\zeta^p\log N)\leqslant 2e^{-c'\delta^2\zeta^{2p}/M^{2p}p^2}.$$

Applying this bound for $p\in[1,e^{cN^2/2}]$, by a union bound we deduce that there exists $c''>0$ such that

$$Q_t^{N,aV}\Big(\max_{1\leqslant p\leqslant e^{cN^2/2}}\zeta^{-p}\big|M_k^N((\phi^M)^p)|\geqslant c_1\delta+C_1\log N\Big)\leqslant 2e^{-c''\delta^2}.$$

On the other hand, for $p\geqslant e^{cN^2/2}$ the bound is trivial as

$$\zeta^{-e^{cN^2/2}}|M_k^N((\phi^M)^{e^{cN^2/2}})|\leqslant N\left(\frac{M}{\zeta}\right)^{e^{cN^2/2}}\leqslant c_1\delta+C_1\log N$$

as soon as N is large enough. This concludes the proof. □

Due to this corollary, we can finally estimate the remainder

$$\mathcal{R}_t^N(\mathbf{Y}^N)=O\bigg(\frac{|\phi_\#^M M^N|^3}{N};(\mathbf{y}_{k,t}^1)',\partial_1\mathbf{z}_{k\ell,t},\partial_2\mathbf{z}_{kk,t}\bigg)$$

with $C(\log N)^3/N$. Indeed, recalling (4.17), using Fourier transform we have

$$\begin{aligned}\iiint \psi(x,y,z)\,dM_k^N(x)\,dM_\ell^N(y)\,dM_m^N(z)\\ =\iiint \hat{\psi}(\xi,\zeta,\theta)M_k^N[e^{i\xi\cdot}]M_\ell^N[e^{i\zeta\cdot}]M_m^N[e^{i\theta\cdot}]\,d\xi\,d\zeta\,d\theta,\end{aligned}$$

so applying Corollaries 4.10 and 4.11, and recalling (4.26), we can bound our remainder by

$$C\frac{(\log N)^3}{N}+C\frac{(\log N)^3}{N}\iiint|\hat{\psi}(\xi,\zeta,\theta)|(1+|\xi|)^7(1+|\zeta|)^7(1+|\theta|)^7\,d\xi\,d\zeta\,d\theta$$

with probability greater than $1-N^{-cN}$. Since all the functions involved decay at infinity, for the above integral to converge it is enough to assume that $\psi\in C^{26}$, as this ensures that

$$|\hat{\psi}(\xi,\zeta,\theta)|\,(1+|\xi|)^7\,(1+|\zeta|)^7\,(1+|\theta|)^7\leqslant\frac{C}{1+|\xi|^5+|\zeta|^5+|\theta|^5}\in L^1(\mathbb{R}^3).$$

Recalling that by assumption ψ is as smooth as $(\mathbf{y}_{k,t}^1)',\partial_1\mathbf{z}_{k\ell,t}$, or $\partial_2\mathbf{z}_{kk,t}$, the assumption is, by Corollary 4.5, satisfied provided $W_k\in C^\sigma$ with $\sigma\geqslant 36$. Due to our Hypothesis 2.1, this concludes the proof of (4.9). As explained at the end of §4.1 this implies (4.8), which combined with (4.1), (4.3), and (4.7) proves (2.8).

Before concluding this section, we prove an additional estimate on the size of the integral of smooth functions against the measure M_N. Corollary 4.10 provides a very strong bound on the probability that $\int f\,dM_N$ is large when f is a fixed function. We now show how to obtain an estimate that holds true when we replace $\int f\,dM_N$ by its supremum over smooth functions.

LEMMA 4.12. *There exists $a_0>0$ so that, for all $a\in[-a_0,a_0]$, the following holds: for any $\ell\geqslant 0$ there are finite positive constants C_ℓ and c_ℓ such that*

$$Q_t^{N,aV}\bigg(\sup_{\|f\|_{C^{\ell+9}(\mathbb{R})}\leqslant 1}\bigg|\int f(x)\,dM_k^N(x)\bigg|\geqslant(\log N)N^{1/(\ell+1)}\bigg)\leqslant C_\ell e^{-c_\ell(\log N)^{2+2/\ell}}. \quad (4.32)$$

Proof. Since the measure $Q_t^{N,aV}$ is supported inside the cube $[-M,M]^N$ (see §4.1), we may assume that all functions f are supported on $[-2M,2M]$. Fix $L\in\mathbb{N}$ and define the points

$$x_{m,L}:=-2M+m\frac{4M}{L},\quad m=0,...,L.$$

Given $f\in C_0^{\ell+9}([-2M,2M])$ with $\|f\|_{C^{\ell+9}}\leqslant 1$, we set $g:=f^{(9)}\in C_0^{\ell}([-2M,2M])$ and define the function

$$g_L(x):=\sum_{j=0}^{\ell-1}\frac{g^{(j)}(x_{m,L})}{j!}(x-x_{m,L})^j \quad \text{for all } x\in[x_{m,L},x_{m+1,L}].$$

Note that, since $\|g\|_{C^\ell}\leqslant 1$,

$$|g(x)-g_L(x)|\leqslant\|g^{(\ell)}\|_\infty(x-x_{m,L})^\ell\leqslant\left(\frac{4M}{L}\right)^\ell$$

for all $x\in[x_{m,L},x_{m+1,L}]$ and all $m=0,...,L-1$. So, by the arbitrariness of x,

$$\|g-g_L\|_{L^\infty([-2M,2M])}\leqslant(4M)^\ell L^{-\ell}.$$

Hence, if we set

$$f_L(x):=\int_{-2M}^x\frac{(x-y)^8}{8!}g_L(y)\,dy,$$

since $f_L^{(9)}=g_L$ and $f^{(j)}(-2M)=0$ for all $j=0,...,8$, we get

$$\|f-f_L\|_{L^\infty([-2M,2M])}\leqslant C_{M,\ell}L^{-\ell}.$$

Recalling that M_N has mass bounded by $2N$, this implies that

$$\left|\int f\,dM_N-\int f_L\,dM_N\right|\leqslant 2\,C_{M,\ell}NL^{-\ell}. \tag{4.33}$$

Fix now a smooth cut-off function $\psi_M:\mathbb{R}\to[0,1]$ satisfying $\psi_M=1$ inside $[-M,M]$ and $\psi_M=0$ outside $[-2M,2M]$, and define

$$f_{L,M}(x)=\sum_{m=0}^{L-1}\sum_{j=0}^{\ell-1}g^{(j)}(x_{m,L})\hat f_{m,j}(x),$$

where

$$\hat f_{m,j}(x):=\psi_M(x)\int_{-2M}^x\frac{(x-y)^8}{8!}(y-x_{m,L})^j\chi_{[x_{m,L},x_{m+1,L}]}(y)\,dy.$$

It is immediate to check that $\hat f_{m,j}\in C_0^{8,1}([-2M,2M])$ (i.e., $\hat f_{m,j}$ has eight derivatives, and its 8th derivative is Lipschitz), and that $f_{L,M}=f_L$ on $[-M,M]$. Also, since $\|f\|_{C^{\ell+9}}\leqslant 1$ we see that $|g^{(j)}(x_{m,L})|\leqslant 1$ for all m,j. Hence, recalling (4.33) and the fact that M_N is supported on $[-M,M]$, this proves that for any function $f\in C_0^{\ell+9}([-2M,2M])$ with $\|f\|_{C^{\ell+9}}\leqslant 1$ there exist some coefficients $\alpha_{m,j}\in[-1,1]$ such that

$$\left|\int f\,dM_N-\sum_{m,j}\alpha_{m,j}\int\hat f_{m,j}\,dM_N\right|\leqslant 2C_{M,\ell}NL^{-\ell}.$$

Since $\#\{\hat{f}_{m,j}\}=\ell L$, this implies that

$$\begin{aligned} &Q_t^{N,aV}\Big(\sup_{\|f\|_{C^{\ell+9}}\leqslant 1}\Big|\int f\,dM_N\Big|>(\log N)N^{1/(\ell+1)}\Big)\\ &\qquad\leqslant \sum_{m,j} Q_t^{N,aV}\Big(\Big|\int \hat{f}_{m,j}\,dM_N\Big|>\frac{(\log N)N^{1/(\ell+1)}-2\,C_{M,\ell}N\,L^{-\ell}}{\ell L}\Big). \end{aligned} \tag{4.34}$$

We now observe that $\|\hat{f}_{m,j}\|_{C^{8,1}}\leqslant A_{M,\ell}$, where $A_{M,\ell}$ is a constant depending only on M and ℓ. Thus, recalling that the functions $\hat{f}_{m,j}$ are supported on $[-2M,2M]$, this yields

$$|||\hat{f}_{m,j}|||\leqslant A'_{M,\ell},$$

where the norm $|||\cdot|||$ is defined in (4.26). Hence, choosing

$$L:=\lfloor \widehat{C}_{M,\ell}N^{1/(\ell+1)}(\log N)^{-1/\ell}\rfloor \tag{4.35}$$

with $\widehat{C}_{M,\ell}$ large enough so that

$$(\log N)N^{1/(\ell+1)}-2\,C_{M,\ell}N\,L^{-\ell}\geqslant \tfrac{1}{2}(\log N)N^{1/(\ell+1)},$$

we can apply Corollary 4.10 to the functions $\hat{f}_{m,j}$, and it follows from (4.34) and (4.35) that

$$\begin{aligned} Q_t^{N,aV}\Big(\sup_{\|f\|_{C^{\ell+9}}\leqslant 1}\Big|\int f\,dM_N\Big|>(\log N)N^{1/(\ell+1)}\Big) &\leqslant C'_{M,\ell}L\,e^{-c'_{M,\ell}((\log N)N^{1/(\ell+1)}/L)^2}\\ &\leqslant C''_{M,\ell}e^{-c''_{M,\ell}(\log N)^{2+2/\ell}}. \end{aligned}$$

□

4.6. Reconstructing the transport map via the flow

In this section we study the properties of the flow $X_t^N:\mathbb{R}^{dN}\to\mathbb{R}^{dN}$ generated by a vector field $\mathbf{Y}_t^N$ as in (4.13), i.e., X_t^N solves the ODE

$$\dot{X}_t^N=\mathbf{Y}_t^N(X_t^N),\quad X_0^N=\mathrm{Id},$$

and we prove that $T^N:=X_1^N$ satisfies all the properties stated in Theorem 2.5.

Recalling the form of $\mathbf{Y}_t^N$ (see (4.13)), it is natural to expect that for all $t\in[0,1]$ we can give an expansion for X_t^N as

$$X_t^N=X_{0,t}+\frac{1}{N}X_{1,t}+\frac{1}{N^2}X_{2,t},$$

where each component $(X_{0,t})_i^k$ of $X_{0,t}$ should flow accordingly to $\mathbf{y}^0_{k,t}$: more precisely, we define $(X_{0,t})_i^k := X^k_{0,t}(\lambda_i^k)$ with $X^k_{0,t}\colon \mathbb{R}\to\mathbb{R}$ the solution of

$$\dot{X}^k_{0,t} = \mathbf{y}^0_{k,t}(X^k_{0,t}), \quad X^k_{0,t}(\lambda) = \lambda. \tag{4.36}$$

Recalling the notation $\hat{\lambda}=(\lambda^1, ..., \lambda^d)$ where $\lambda^k := (\lambda_1^k, ..., \lambda_N^k)$, we define

$$X_{1,t} = ((X_{1,t})_1^1, ..., (X_{1,t})_N^1, ..., (X_{1,t})_1^d, ..., (X_{1,t})_N^d)\colon \mathbb{R}^{dN} \longrightarrow \mathbb{R}^{dN}$$

to be the solution of the linear ODE

$$\begin{aligned}(\dot{X}_{1,t})_i^k(\hat{\lambda}) = (\mathbf{y}^0_{k,t})'(X^k_{0,t}(\lambda_i^k))\cdot(X_{1,t})_i^k(\hat{\lambda}) + \mathbf{y}^1_{k,t}(X^k_{0,t}(\lambda_i^k))\\ +\sum_{\ell=1}^d \int \mathbf{z}_{k\ell,t}(X^k_{0,t}(\lambda_i^k), y)\, dM^N_{X^\ell_{0,t}}(y)\\ +\frac{1}{N}\sum_{\ell=1}^d\sum_{j=1}^N \partial_2 \mathbf{z}_{k\ell,t}(X^k_{0,t}(\lambda_i^k), X^\ell_{0,t}(\lambda_j^\ell))\cdot(X_{1,t})_j^\ell(\hat{\lambda}),\end{aligned} \tag{4.37}$$

with the initial condition $(X_{1,0})_i^k = 0$, where $M^N_{X^\ell_{0,t}}$ is defined as

$$\int f(y)\, dM^N_{X^\ell_{0,t}}(y) = \sum_{i=1}^N \left(f\big(X^\ell_{0,t}(\lambda_i^\ell)\big) - \int f\, d\mu^*_{\ell,t}\right) \quad \text{for all } f\in C_c(\mathbb{R}).$$

Proposition 4.13. *Let α be as in Proposition 4.4. Assume that $W_k\colon\mathbb{R}\to\mathbb{R}$ are of class C^σ for all $k=1, ..., d$, with $\sigma\geqslant 16$, and that $|a|\leqslant\alpha$. Then the flow*

$$X_t^N = ((X_t^N)_1^1, ..., (X_t^N)_N^1, ..., (X_t^N)_1^d, ..., (X_t^N)_N^d)\colon \mathbb{R}^{dN}\longrightarrow\mathbb{R}^{dN}$$

is of class $C^{\sigma-9}$ and the following properties hold: Let $(X_{0,t})_i^k$ and $(X_{1,t})_i^k$ be as in (4.36) and (4.37) above, and define $X_{2,t}\colon\mathbb{R}^{dN}\to\mathbb{R}^{dN}$ via the identity

$$X_t^N = X_{0,t} + \frac{1}{N}X_{1,t} + \frac{1}{N^2}X_{2,t}.$$

Then, for any $t\in[0,1]$,

$$\max_{k,i} \|(X_{1,t})_i^k\|_{L^4(Q_0^{N,aV})} \leqslant C\log N, \quad \max_{k,i}\|(X_{2,t})_i^k\|_{L^2(Q_0^{N,aV})} \leqslant C(\log N)^2. \tag{4.38}$$

Also, there exist constants $C, c>0$ such that, with probability greater than $1-e^{-c(\log N)^2}$, the following bounds hold:

$$\begin{aligned}\sup_{t\in[0,1]}\max_{i,k} |(X_{1,t})_i^k| \leqslant C(\log N)N^{1/(\sigma-14)},\\ \sup_{t\in[0,1]}\max_{i,k}|(X_{2,t})_i^k| \leqslant C\,(\log N)^2 N^{2/(\sigma-15)},\end{aligned} \tag{4.39}$$

and, for all $k,\ell=1,...,d$,

$$\sup_{t\in[0,1]}\max_{i,i'}\big|(X_{1,t})_i^k(\hat\lambda)-(X_{1,t})_{i'}^k(\hat\lambda)\big|\leqslant C\,(\log N)N^{1/(\sigma-15)}|\lambda_i^k-\lambda_{i'}^k|, \tag{4.40}$$

$$\sup_{t\in[0,1]}\max_{i,i'}\big|(X_{2,t})_i^k(\hat\lambda)-(X_{2,t})_{i'}^k(\hat\lambda)\big|\leqslant C\,(\log N)^2\,N^{2/(\sigma-17)}|\lambda_i^k-\lambda_{i'}^k|, \tag{4.41}$$

$$\sup_{t\in[0,1]}\max_{i,j}|\partial_{\lambda_j^\ell}(X_{1,t})_i^k|(\hat\lambda)\leqslant C\,(\log N)N^{1/(\sigma-15)}. \tag{4.42}$$

Proof. Since $\mathbf{Y}_t^N\in C^{\sigma-9}$ (see Corollary 4.5) it follows by Cauchy–Lipschitz theory that X_t^N is of class $C^{\sigma-9}$. Define

$$(X_t^{N,\tau})_i^k(\hat\lambda):=X_{0,t}^k(\lambda_i^k)+\tau\frac{(X_{1,t})_i^k}{N}(\hat\lambda)+\tau\frac{(X_{2,t})_i^k}{N^2}(\hat\lambda)=(1-\tau)X_{0,t}^k(\lambda_i^k)+\tau(X_t^N)_i^k(\hat\lambda).$$

Also, we define the measure $M^N_{(X_t^{N,\tau})^k}$ as

$$\int f(y)\,dM^N_{(X_t^{N,\tau})^k}(y)=\sum_{i=1}^N\bigg[f((1-\tau)X_{0,t}^k(\lambda_i^k)+\tau(X_t^N)_i^k(\hat\lambda))-\int f\,d\mu_{k,t}^*\bigg] \tag{4.43}$$

for all $f\in C_c(\mathbb{R})$. In order to get an ODE for $X_{2,t}$, the strategy is to use the Taylor formula with integral remainder to expand the ODE $\dot X_t^N=\mathbf{Y}_t^N(X_t^N)$, and then use (4.36) and (4.37) to simplify the terms involving $\dot X_{0,t}$ and $\dot X_{1,t}$. In this way we get

$$\begin{aligned}
(\dot X_{2,t})_i^k(\hat\lambda)={}&\int_0^1(\mathbf{y}_{k,t}^0)'((X_t^{N,\tau})_i^k(\hat\lambda))\,d\tau\cdot(X_{2,t})_i^k(\hat\lambda)\\
&+N\int_0^1[(\mathbf{y}_{k,t}^0)'((X_t^{N,\tau})_i^k(\hat\lambda))-(\mathbf{y}_{k,t}^0)'(X_{0,t}^k(\lambda_i^k))]\,d\tau\cdot(X_{1,t})_i^k(\hat\lambda)\\
&+\int_0^1(\mathbf{y}_{k,t}^1)'((X_t^{N,\tau})_i^k(\hat\lambda))\,d\tau\cdot\bigg((X_{1,t})_i^k(\hat\lambda)+\frac{(X_{2,t})_i^k(\hat\lambda)}{N}\bigg)\\
&+\int_0^1\sum_\ell\bigg[\int\partial_1\mathbf{z}_{k\ell,t}((X_t^{N,\tau})_i^k(\hat\lambda),y)\,dM^N_{(X_t^{N,\tau})^\ell}(y)\\
&\qquad-\int\partial_1\mathbf{z}_{k\ell,t}(X_{0,t}^k(\lambda_i^k),y)\,dM^N_{X_{0,t}^\ell}(y)\bigg]\,d\tau\cdot\bigg((X_{1,t})_i^k(\hat\lambda)+\frac{(X_{2,t})_i^k(\hat\lambda)}{N}\bigg)\\
&+\sum_\ell\int\partial_1\mathbf{z}_{k\ell,t}(X_{0,t}^k(\lambda_i^k),y)\,dM^N_{X_{0,t}^\ell}(y)\cdot\bigg((X_{1,t})_i^k(\hat\lambda)+\frac{(X_{2,t})_i^k(\hat\lambda)}{N}\bigg)\\
&+\sum_\ell\sum_{j=1}^N\int_0^1[\partial_2\mathbf{z}_{k\ell,t}((X_t^{N,\tau})_i^k(\hat\lambda),(X_t^{N,\tau})_j^\ell(\hat\lambda))\\
&\qquad-\partial_2\mathbf{z}_{k\ell,t}(X_{0,t}^k(\lambda_i^k),X_{0,t}^\ell(\lambda_j^\ell))]\,d\tau\cdot(X_{1,t})_j^\ell(\hat\lambda)\\
&+\sum_\ell\sum_{j=1}^N\int_0^1[\partial_2\mathbf{z}_{k\ell,t}((X_t^{N,\tau})_i^k(\hat\lambda),(X_t^{N,\tau})_j^\ell(\hat\lambda))]\,d\tau\cdot\frac{(X_{2,t})_j^\ell(\hat\lambda)}{N},
\end{aligned} \tag{4.44}$$

with the initial condition $(X_{2,t})_i^k=0$. Using that

$$\|\mathbf{y}^0_{k,t}\|_{C^{\sigma-3}(\mathbb{R})}\leqslant C$$

(see Corollary 4.5) we obtain

$$\|X_{0,t}\|_{C^{\sigma-4}(\mathbb{R})}\leqslant C. \tag{4.45}$$

We now start by controling $(X_{1,t})_i^k$. First, simply by using that M_N has mass bounded by $2N$ we obtain the rough bound $|(X_{1,t})_i^k|\leqslant CN$. Inserting this bound into (4.44) one easily obtains $|(X_{2,t})_i^k|\leqslant CN^2$.

We now prove finer estimates. First, by Lemma 4.12 together with the fact that $(X_{0,t})_i^k$ and $y\mapsto\mathbf{z}_{k\ell,t}(x,y)$ are of class $C^{\sigma-6}$ uniformly in x and t (see Corollary 4.5), it follows that there exists a finite constant C such that, with probability greater than $1-e^{-c(\log N)^2}$,

$$\sup_{\substack{x\in\mathbb{R}\\ t\in[0,1]}}\left|\int \mathbf{z}_{k\ell,t}(x,\lambda)\,dM^N_{X^\ell_{0,t}}(\lambda)\right|\leqslant C\,(\log N)N^{1/(\sigma-14)}.$$

Hence, using (4.37) we easily deduce the first bound in (4.39).

In order to control $X_{2,t}$ we first estimate $(X_{1,t})_i^k$ in $L^4(Q_0^{N,aV})$: using (4.37) again, we get

$$\begin{aligned}\frac{d}{dt}\max_{i,k}&\|(X_{1,t})_i^k\|_{L^4(Q_0^{N,aV})}\\ &\leqslant C\biggl(\max_{i,k}\|(X_{1,t})_i^k\|_{L^4(Q_0^{N,aV})}+1\\ &\qquad+\max_{i,k,\ell}\biggl\|\int \mathbf{z}_{k\ell,t}(X^k_{0,t}(\lambda_i^k),y)\,dM^N_{X^\ell_{0,t}}(y)\biggr\|_{L^4(Q_0^{N,aV})}\biggr).\end{aligned} \tag{4.46}$$

To bound $(X_{1,t})_i^k$ in $L^4(Q_0^{N,aV})$, and then to be able to estimate $X_{2,t}$, we will use the following lemma.

LEMMA 4.14. *Assume that $s\geqslant 16$. Then, for any $i=1,...,N$ and $k,\ell=1,...,d$,*

$$\left\|\int \mathbf{z}_{k\ell,t}(X^k_{0,t}(\lambda_i^k),y)\,dM^N_{X^\ell_{0,t}}(y)\right\|_{L^4(Q_0^{N,aV})}\leqslant C\log N, \tag{4.47}$$

$$\left\|\int \partial_1\mathbf{z}_{k\ell,t}(X^k_{0,t}(\lambda_i^k),y)\,dM^N_{X^\ell_{0,t}}(y)\right\|_{L^4(Q_0^{N,aV})}\leqslant C\log N. \tag{4.48}$$

Proof. Fix indices i, k, and ℓ and write the Fourier decomposition of

$$\eta_{2,t}(x,y):=\mathbf{z}_{k\ell,t}(X^k_{0,t}(x),X^\ell_{0,t}(y))$$

to get

$$\int \eta_{2,t}(x,y)\,dM_\ell^N(y) = \int \hat{\eta}_{2,t}(x,\xi) \int e^{i\xi y}\,dM_\ell^N(y)\,d\xi.$$

Since $\mathbf{z}_{k\ell,t} \in C^{u,v}$ for $u,v \leqslant \sigma-6$ and $X_{0,t}^k \in C^{\sigma-4}$ (see (4.45)) with derivatives decaying fast at infinity, we deduce that

$$|\hat{\eta}_{2,t}(x,\xi)| \leqslant \frac{C}{1+|\xi|^{\sigma-6}}.$$

Thus, using Corollary 4.10, we get

$$\begin{aligned}
\left\| \sup_x \left| \int \eta_{2,t}(x,y)\,dM_k^N(y) \right| \right\|_{L^4(Q_0^{N,aV})} &\leqslant \int \|\hat{\eta}_{2,t}(\,\cdot\,,\xi)\|_\infty \left\| \int e^{i\xi y} dM_k^N(y) \right\|_{L^4(Q_0^{N,aV})} d\xi \\
&\leqslant C(\log N) \int \|\hat{\eta}_{2,t}(\,\cdot\,,\xi)\|_\infty (1+|\xi|^7)\,d\xi \\
&\leqslant C \log N
\end{aligned}$$

provided $\sigma > 13$. The same argument works for $\partial_1 \mathbf{z}_{k\ell,t}$ provided $\sigma > 15$, which concludes the proof. □

Inserting (4.47) into (4.46), we obtain the validity of the first bound in (4.38).

We now bound the time derivative of $X_{2,t}$: using that M_N has mass bounded by $2N$, in (4.44) we can easily estimate

$$\begin{aligned}
&\left| N \int_0^1 [(\mathbf{y}_{k,t}^0)'((X_t^{N,\tau})_i^k(\hat{\lambda})) - (\mathbf{y}_{k,t}^0)'(X_{0,t}^k(\lambda_i^k))]\,d\tau \cdot (X_{1,t})_i^k(\hat{\lambda}) \right| \\
&\qquad \leqslant C|(X_{1,t})_i^k|^2 + \frac{C}{N}|(X_{1,t})_i^k|\,|(X_{2,t})_i^k|, \\
&\int_0^1 \left| \int \partial_1 \mathbf{z}_{k\ell,t}((X_t^{N,\tau})_i^k(\hat{\lambda}), y)\,dM^N_{(X_t^{N,\tau})^\ell}(y) - \int \partial_1 \mathbf{z}_{k\ell,t}(X_{0,t}^k(\lambda_i^k), y)\,dM^N_{X_{0,t}^\ell}(y) \right| d\tau \\
&\qquad \leqslant C|(X_{1,t})_i^k| + \frac{C}{N}|(X_{2,t})_i^k| + \frac{C}{N}\sum_j \left(|(X_{1,t})_j^\ell| + \frac{1}{N}|(X_{2,t})_j^\ell| \right),
\end{aligned}$$

and

$$\begin{aligned}
&\sum_{j=1}^N \int_0^1 |\partial_2 \mathbf{z}_{k\ell,t}((X_t^{N,\tau})_i^k(\hat{\lambda}), (X_t^{N,\tau})_j^\ell(\hat{\lambda})) - \partial_2 \mathbf{z}_{k\ell,t}(X_{0,t}^k(\lambda_i^k), X_{0,t}^\ell(\lambda_j^\ell))|\,d\tau\,|(X_{1,t})_j^\ell| \\
&\qquad \leqslant \frac{C}{N}\left(|(X_{1,t})_i^k| + \frac{1}{N}|(X_{2,t})_i^k| \right) \sum_j |(X_{1,t})_j^\ell| \\
&\qquad\quad + \frac{C}{N}\sum_j \left(|(X_{1,t})_j^\ell|^2 + \frac{1}{N}|(X_{2,t})_j^\ell|\,|(X_{1,t})_j^\ell| \right),
\end{aligned}$$

and hence, noticing that
$$\frac{d|(X_{2,t})_i^k|}{dt} \leqslant |(\dot X_{2,t})_i^k|,$$
we get
$$\begin{aligned}
&\frac{d}{dt}|(X_{2,t})_i^k| \\
&\leqslant C|(X_{2,t})_i^k|+C|(X_{1,t})_i^k|^2+\frac{C}{N}\,|(X_{1,t})_i^k|\,|(X_{2,t})_i^k|+C|(X_{1,t})_i^k|+\frac{C}{N^2}\,|(X_{2,t})_i^k|^2 \\
&\quad +\frac{C}{N}\sum_{\ell,j}|(X_{1,t})_j^\ell|\,|(X_{1,t})_i^k|+\frac{C}{N^3}\sum_{\ell,j}|(X_{1,t})_i^k|\,|(X_{2,t})_j^\ell|+\frac{C}{N^3}\sum_{\ell,j}|(X_{2,t})_i^k|\,|(X_{2,t})_j^\ell| \\
&\quad +\left|\int \partial_1 \mathbf{z}_{k\ell,t}(X_{0,t}^k(\lambda_i^k),y)\,dM^N_{X_{0,t}^\ell}(y)\right|\,|(X_{1,t})_i^k|+\frac{C}{N}\sum_{\ell,j}|(X_{1,t})_j^\ell|^2 \\
&\quad +\frac{C}{N^2}\sum_{\ell,j}|(X_{2,t})_j^\ell|\,|(X_{1,t})_j^\ell|+\frac{C}{N^2}\sum_{\ell,j}|(X_{1,t})_j^\ell|\,|(X_{2,t})_i^k|+\frac{C}{N}\sum_{\ell,j}|(X_{2,t})_j^\ell|.
\end{aligned}$$

Using the trivial bounds $|(X_{1,t})_i^k|\leqslant CN$ and $|(X_{2,t})_i^k|\leqslant CN^2$, and the elementary inequality $ab\leqslant a^2+b^2$, we obtain
$$\begin{aligned}
\frac{d}{dt}|(X_{2,t})_i^k| \leqslant C\Bigg(&|(X_{2,t})_i^k|+|(X_{1,t})_i^k|^2+\frac{1}{N}\sum_{\ell,j}|(X_{1,t})_j^\ell|^2+\frac{1}{N}\sum_{\ell,j}|(X_{2,t})_j^\ell| \\
&+\left|\int \partial_1 \mathbf{z}_{k\ell,t}(X_{0,t}^k(\lambda_i^k),y)\,dM^N_{X_{0,t}^\ell}(y)\right|^2\Bigg).
\end{aligned} \tag{4.49}$$

In particular, if we set
$$A_{1,t}:=\max_{i,k}|(X_{1,t})_i^k| \quad \text{and} \quad A_{2,t}:=\max_{i,k}|(X_{2,t})_i^k|,$$
we obtain
$$\frac{d}{dt}A_{2,t}\leqslant C\left(A_{2,t}+A_{1,t}^2+\max_{i,k}\left|\int \partial_1 \mathbf{z}_{k\ell,t}(X_{0,t}^k(\lambda_i^k),y)\,dM^N_{X_{0,t}^\ell}(y)\right|^2\right). \tag{4.50}$$

Hence, noticing that
$$\sup_{\substack{x\in\mathbb{R}\\ t\in[0,1]}}\left|\int \partial_1 \mathbf{z}_{k\ell,t}(x,\lambda)\,dM^N_{X_{0,t}^\ell}(\lambda)\right|\leqslant C(\log N)N^{1/(\sigma-15)} \tag{4.51}$$
with probability greater than $1-e^{-c(\log N)^2}$ (see Corollary 4.5 and Lemma 4.12) and recalling the first bound in (4.39), using (4.50) and a Grönwall argument we deduce the validity also of the second bound in (4.39).

Going back to (4.49) and again using the inequality $ab\leqslant a^2+b^2$, we also see that

$$\begin{aligned}\frac{d}{dt}\|(X_{2,t})^k_i\|^2_{L^2(Q_0^{N,aV})}\leqslant C\bigg(&\|(X_{2,t})^k_i\|^2_{L^2(Q_0^{N,aV})}+\|(X_{1,t})^k_i\|^4_{L^4(Q_0^{N,aV})}\\&+\frac{1}{N}\sum_{\ell,j}\|(X_{1,t})^\ell_j\|^4_{L^4(Q_0^{N,aV})}+\frac{1}{N}\sum_{\ell,j}\|(X_{2,t})^\ell_j\|^2_{L^2(Q_0^{N,aV})}\\&+\bigg\|\int\partial_1\mathbf{z}_{k\ell,t}(X^k_{0,t}(\lambda^k_i),y)\,dM^N_{X^\ell_{0,t}}(y)\bigg\|^4_{L^4(Q_0^{N,aV})}\bigg).\end{aligned}$$

Hence, recalling the first bound in (4.38) and (4.48), we get

$$\frac{d}{dt}\|(X_{2,t})^k_i\|^2_{L^2(Q_0^{N,aV})}\leqslant C(\|(X_{2,t})^k_i\|^2_{L^2(Q_0^{N,aV})}+(\log N)^4),$$

so a Grönwall argument concludes the proof of (4.38).

We now prove (4.40). Recalling (4.37), we have

$$\begin{aligned}&|(\dot X_{1,t})^k_i(\hat\lambda)-(\dot X_{1,t})^k_{i'}(\hat\lambda)|\\&\quad\leqslant|(\mathbf{y}^0_{k,t})'(X^k_{0,t}(\lambda^k_i))-(\mathbf{y}^0_{k,t})'(X^k_{0,t}(\lambda^k_{i'}))|\,|(X_{1,t})^k_i(\hat\lambda)|\\&\qquad+|(\mathbf{y}^0_{k,t})'(X^k_{0,t}(\lambda^k_{i'}))|\,|(X_{1,t})^k_i(\hat\lambda)-X^{N,k'}_{1,t}(\lambda^k_{i'})|+|\mathbf{y}^1_{k,t}(X^k_{0,t}(\lambda^k_i))-\mathbf{y}^1_{k,t}(X^k_{0,t}(\lambda^k_{i'}))|\\&\qquad+\sum_\ell\bigg|\int(\mathbf{z}_{k\ell,t}(X^k_{0,t}(\lambda^k_i),y)-\mathbf{z}_{k\ell,t}(X^k_{0,t}(\lambda^k_{i'}),y))\,dM^N_{X^\ell_{0,t}}(y)\bigg|\\&\qquad+\frac{1}{N}\sum_{\ell,j}|\partial_2\mathbf{z}_{k\ell,t}(X^k_{0,t}(\lambda^k_i),X^\ell_{0,t}(\lambda^\ell_j))-\partial_2\mathbf{z}_{k\ell,t}(X^k_{0,t}(\lambda^k_{i'}),X^\ell_{0,t}(\lambda^\ell_j))|\,|(X_{1,t})^\ell_j(\lambda^\ell_j)|.\end{aligned}$$

Hence, using that $|X^k_{0,t}(\lambda^k_i)-X^k_{0,t}(\lambda^k_{i'})|\leqslant C|\lambda^k_i-\lambda^k_{i'}|$, the bounds (4.39) and (4.51), and the Lipschitz regularity of $(\mathbf{y}^0_{k,t})'$, $\mathbf{y}^1_{k,t}$, $\mathbf{z}_{k\ell,t}$, and $\partial_2\mathbf{z}_{k\ell,t}$, we get

$$|(\dot X_{1,t})^k_i(\hat\lambda)-(\dot X_{1,t})^k_{i'}(\hat\lambda)|\leqslant C|(X_{1,t})^k_i(\hat\lambda)-(X_{1,t})^k_{i'}(\hat\lambda)|+C(\log N)N^{1/(\sigma-15)}|\lambda^k_i-\lambda^k_{i'}|$$

outside a set of probability less than $e^{-c(\log N)^2}$, so (4.40) follows from Grönwall's inequality.

By a completely analogous argument, it follows from (4.44), (4.40), (4.39), and estimates analogue to (4.51) for the higher derivatives of $\mathbf{z}_{k\ell,t}$, that

$$|(\dot X_{2,t})^k_i(\hat\lambda)-(\dot X_{2,t})^k_{i'}(\hat\lambda)|\leqslant C|(X_{2,t})^k_i(\hat\lambda)-(X_{2,t})^k_{i'}(\hat\lambda)|+C\,(\log N)^2\,N^{2/(\sigma-17)}|\lambda^k_i-\lambda^k_{i'}|$$

holds outside a set of probability less than $e^{-c(\log N)^2}$. Thus (4.41) follows.

Finally, denoting by δ^ℓ_j the vector with zero entries except at position j,ℓ where there is a one (so that $\hat\lambda+\varepsilon\delta^\ell_j=(\lambda^1_1,...,\lambda^\ell_j+\varepsilon,...\lambda^d_N)$), one can differentiate in time

$$|(X_{1,t})^k_i(\hat\lambda+\varepsilon\delta^\ell_j)-(X_{1,t})^k_{i'}(\hat\lambda)|$$

and argue as above to deduce that

$$|(X_{1,t})^k_i(\hat\lambda+\varepsilon\delta^\ell_j)-(X_{1,t})^k_{i'}(\hat\lambda)|\leqslant C(\log N)N^{1/(\sigma-15)}\varepsilon$$

outside a set of probability less than $e^{-c(\log N)^2}$. Dividing by ε and letting $\varepsilon\to 0$, this proves (4.42). □

5. Universality results

In this section we explain how Corollaries 2.6, 2.7 and 2.9 follow from Theorem 2.5.

Proof of Corollary 2.6. Given $\vartheta>0$, we define the set

$$G_\vartheta:=\{\hat\lambda\in\mathbb{R}^{dN}:|\lambda^\ell_i-\gamma^\ell_{i/N}|\leqslant N^{\vartheta-2/3}\min\{i,N+1-i\}^{1/3}\text{ for all } i \text{ and } \ell\}. \tag{5.1}$$

As proved in [29] in the special case of the Gaussian ensembles and then generalized in [13, Theorem 2.4] to potentials W_k satisfying much weaker conditions than the ones assumed here, the following rigidity estimate holds: for all $\vartheta>0$ there exist $\bar c>0$ and $\overline{C}<\infty$ such that, for all $N\geqslant 0$,

$$\widetilde{P}^{N,0}_\beta(\mathbb{R}^N\setminus G_\vartheta)\leqslant \overline{C}e^{-N^{\bar c}}. \tag{5.2}$$

Also, due to the fact that μ^0_k has a density which is strictly positive inside its support $[a^0_k,b^0_k]$ except at the two boundary points where it goes to zero as a square root (see Lemma 3.2), we deduce that

$$\frac{m}{N}\geqslant\frac{1}{C}\int_{\gamma^k_{i/N}}^{\gamma^k_{(i+m)/N}}\min\bigl\{\sqrt{s-a^0_k},\sqrt{b^0_k-s}\,\bigr\}\,ds,$$

from which it follows easily that

$$|\gamma^k_{(i+m)/N}-\gamma^k_{i/N}|\leqslant\frac{C}{N^{2/3}}\min\biggl\{m^{2/3},\frac{m}{\min\{i,N+1-i\}^{1/3}}\biggr\}. \tag{5.3}$$

Since

$$|\lambda^k_{i+m}-\lambda^k_i|\leqslant|\lambda^k_i-\gamma^k_{i/N}|+|\lambda^k_{i+m}-\gamma^k_{(i+m)/N}|+|\gamma^k_{(i+m)/N}-\gamma^k_{i/N}|, \tag{5.4}$$

using (5.2) and (5.3) and recalling that by assumption $m\ll N$, we deduce that

$$|N(\lambda^k_{i_k+j}-\lambda^k_{i_k})|\leqslant C(N^\vartheta+m)\quad\text{for all }\hat\lambda\in G_\vartheta,\ i_k\in[N\varepsilon,N(1-\varepsilon)],\ j=1,...,m, \tag{5.5}$$

and

$$|N^{2/3}(\lambda^k_j-a^0_k)|\leqslant C(N^\vartheta+m^{2/3})\quad\text{for all }\hat\lambda\in G_\vartheta,\ j=1,...,m. \tag{5.6}$$

Now, given a bounded function $\chi:\mathbb{R}^{dN}\to\mathbb{R}$, applying (2.8) to

$$\frac{1}{2}\left(1+\frac{\chi}{\|\chi\|_\infty}\right)$$

with $k=0$ and $\eta=\vartheta$, we deduce that

$$\left|\int \chi\circ T^N\,dP_\beta^{N,0}-\int \chi\,dP_\beta^{N,aV}\right|\leqslant CN^{\vartheta-1}\|\chi\|_\infty. \tag{5.7}$$

Recall that the map T^N is given by X_1^N, where X_t^N is the flow of the vector field $\mathbf{Y}_t^N$ that has the very special form (4.13) (see Proposition 4.13). In particular, since the functions $\mathbf{y}_{k,t}^0$, $\mathbf{y}_{k,t}^1$, $\zeta_{k\ell,t}(\,\cdot\,,y)$ are uniformly Lipschitz, we see that

$$|(\dot X_t^N)_i^k-(\dot X_t^N)_j^k|\leqslant L\,|(X_t^N)_i^k-(X_t^N)_j^k|\quad \text{for all } i,j=1,...,N \text{ and } k=1,...,d.$$

Hence, since $X_1^N=T^N$ and $X_0^N=\mathrm{Id}$, Grönwall's inequality yields

$$e^{-L}(\lambda_i^k-\lambda_j^k)\leqslant (T^N)_i^k(\hat\lambda)-(T^N)_j^k(\hat\lambda)\leqslant e^{L}(\lambda_i^k-\lambda_j^k)\quad \text{for all } \lambda_i^k\geqslant\lambda_j^k. \tag{5.8}$$

We now remark that the law $\widetilde P_\beta^{N,aV}$ is obtained as the image of the law of $\lambda^k=(\lambda_1^k,...,\lambda_N^k)$, $1\leqslant k\leqslant d$ under $P_\beta^{N,aV}$ under the map

$$\widehat{\mathcal{R}}:\mathbb{R}^{dN}\to\mathbb{R}^{dN},\quad \widehat{\mathcal{R}}(\lambda^1,...,\lambda^k,...,\lambda^d):=(\mathcal{R}(\lambda^1),...,\mathcal{R}(\lambda^k),...,\mathcal{R}(\lambda^d)), \tag{5.9}$$

where $\mathcal{R}:\mathbb{R}^N\to\mathbb{R}^N$ is defined as

$$[\mathcal{R}(x_1,...,x_N)]_i:=\min_{\#J=i}\max_{j\in J}x_j\quad \text{for all } i=1,...,N. \tag{5.10}$$

Hence, due to (5.8), it follows that T^N and $\widehat{\mathcal{R}}$ commute, namely

$$\widehat{\mathcal{R}}\circ T^N=T^N\circ\widehat{\mathcal{R}}. \tag{5.11}$$

We now consider a test function χ of the form

$$\chi(\hat\lambda)=f((N(\lambda_{i_k+1}^k-\lambda_{i_k}^k),...,N(\lambda_{i_k+m}^k-\lambda_{i_k}^k))_{k=1}^d). \tag{5.12}$$

Then

$$\int f((N(\lambda_{i_k+1}^k-\lambda_{i_k}^k),...,N(\lambda_{i_k+m}^k-\lambda_{i_k}^k))_{k=1}^d)\,d\widetilde P_\beta^{N,aV}=\int \chi\circ\widehat{\mathcal{R}}\,dP_\beta^{N,aV},$$

and it follows by (5.7) and (5.11) that

$$\left|\int \chi\,d\widetilde P_\beta^{N,aV}-\int \chi\circ T^N\circ\widehat{\mathcal{R}}\,dP_\beta^{N,0}\right|\leqslant CN^{\vartheta-1}\|f\|_\infty.$$

Let $X_{0,t}$, $X_{1,t}$, and $X_{2,t}$ be as in Proposition 4.13, and note the following fact: whenever $\hat\lambda\in G_\vartheta$ we know that, for any $\ell=1,...,d$, the numbers $\{\lambda_i^\ell\}_{i=1}^N$ are close, up to an error N^ϑ, to the quantiles of the stationary measure $\mu_\ell^0=\mu_{\ell,0}^*$. Hence, given any 1-Lipschitz function ψ,

$$\left|\int \psi\, dM_\ell^N\right| \leqslant CN^\vartheta \quad \text{for all } \ell=1,...,d.$$

Since $X_{0,t}^\ell$ is a smooth diffeomorphism which sends the quantiles of $\mu_{\ell,0}^*$ onto the quantiles of $\mu_{\ell,t}^*$, we deduce that

$$\left|\int \psi\, dM_{X_{0,t}^\ell}^N\right| \leqslant CN^\vartheta \quad \text{for all } \ell=1,...,d \text{ and } t\in[0,1].$$

This implies that

$$\sup_{x,t}\int \mathbf{z}_{k\ell,t}(x,y)\, dM_{X_{0,t}^\ell}^N(y)=O(N^\vartheta) \quad \text{and} \quad \sup_{x,t}\int \partial_1\mathbf{z}_{k\ell,t}(x,\lambda)\, dM_{X_{0,t}^\ell}^N(\lambda)=O\big(N^\vartheta\big),$$

and by the same argument as the one used in the proof of Proposition 4.13 to show (4.39) and (4.40) we get

$$\max_{i,k}|(X_{1,1})_i^k(\hat\lambda)|\leqslant CN^\vartheta \quad \text{and} \quad |(X_{1,1})_i^k(\hat\lambda)-(X_{1,1})_{i'}^k(\hat\lambda)|\leqslant CN^\vartheta\,|\lambda_i^k-\lambda_{i'}^k|. \tag{5.13}$$

for all $\hat\lambda\in G_\vartheta$. Then, noticing that $\|\nabla\chi\|_\infty\leqslant N\|\nabla f\|_\infty$, due to (5.13), (4.41), and (5.5), we get

$$\begin{aligned}
&\left|\int_{G_\vartheta}\chi\circ T^N\circ\widehat{\mathcal{R}}\, dP_\beta^{N,0}-\int_{G_\vartheta}\chi\circ X_{0,1}\circ\widehat{\mathcal{R}}\, dP_\beta^{N,0}\right|\\
&\quad\leqslant \|\nabla\chi\|_\infty\int_{G_\vartheta}\left[\sum_{k=1}^d\sum_{j=1}^m\left(\frac{|(X_{1,1})_{i_k+j}^k-(X_{1,1})_{i_k}^k|}{N}+\frac{|(X_{2,1})_{i_k+j}^k-(X_{2,1})_{i_k}^k|}{N^2}\right)^2\right]^{1/2} d\widetilde{P}_\beta^{N,0}\\
&\quad\leqslant C\|\nabla f\|_\infty\, N^\vartheta\int_{G_\vartheta}\left(\sum_{k=1}^d\sum_{j=1}^m|\lambda_{i_k+j}^k-\lambda_{i_k}^k|^2\right)^{1/2} d\widetilde{P}_\beta^{N,0}\\
&\quad\leqslant C\|\nabla f\|_\infty\,\frac{m^{1/2}N^\vartheta\,(N^\vartheta+m)}{N}.
\end{aligned}$$

Note now that $(X_{0,1})_i^k=T_0^k$ for all $i=1,...,N$, and that

$$e^{-L}\leqslant (T_0^k)'\leqslant e^L \tag{5.14}$$

(this follows by the same proof as the one of (5.8), compare also with [5, Equation (5.2)]). In addition,

$$(T_{0,1})_{i_k+j}^k(\hat\lambda)-(T_{0,1})_{i_k}^k(\hat\lambda)=(T_0^k)'(\lambda_{i_k}^k)[\lambda_{i_k+j}^k-\lambda_{i_k}^k]+O(|\lambda_{i_k+j}^k-\lambda_{i_k}^k|^2),$$

and hence, by the definition of G_ϑ,

$$\begin{aligned}\int_{G_\vartheta} \chi\circ X_{0,1}\circ\widehat{\mathcal{R}}\,dP_\beta^{N,0} &= \int_{G_\vartheta} f(((T_0^k)'(\lambda_{i_k}^k)\,N(\lambda_{i_k+1}^k-\lambda_{i_k}^k),...,(T_0^k)'(\lambda_{i_k}^k)N(\lambda_{i_k+m}^k-\lambda_{i_k}^k))_{k=1}^d)\,d\widetilde{P}_\beta^{N,0}\\ &\quad+O(\|\nabla f\|_\infty\,m^{1/2}\,(N^\vartheta+m)^2\,N^{-1}).\end{aligned}$$

Also, in the integral above we can replace $(T_0^k)'(\lambda_{i_k}^k)$ with $(T_0^k)'(\gamma_{i_k/N}^k)$, up to an error bounded by

$$\begin{aligned}C\|\nabla f\|_\infty \int_{G_\vartheta}\biggl(\sum_{k=1}^d\sum_{j=1}^m|\lambda_{i_k}^k-\gamma_{i/N}^k|^2(N|\lambda_{i_k+j}^k-\lambda_{i_k}^k|)^2\biggr)^{1/2}\,d\widetilde{P}_\beta^{N,0}&\\ =O(\|\nabla f\|_\infty m^{1/2}(N^\vartheta+m)N^{\vartheta-1}).&\end{aligned}$$

Finally, it follows by (5.2) that all integrals on $\mathbb{R}^N\setminus G_\vartheta$ are bounded by $C\|f\|_\infty\,e^{-N^{\bar{c}}}$. Hence, we have proved that([1])

$$\begin{aligned}\biggl|\int f((N(\lambda_{i_k+1}^k-\lambda_{i_k}^k),...,N(\lambda_{i_k+m}^k-\lambda_{i_k}^k))_{k=1}^d)\,d\widetilde{P}_\beta^{N,aV}&\\ -\int f(((T_0^k)'(\gamma_{i_k/N}^k)N(\lambda_{i_k+1}^k-\lambda_{i_k}^k),...,(T_0^k)'(\gamma_{i_k/N}^k)N(\lambda_{i_k+m}^k-\lambda_{i_k}^k))_{k=1}^d)\,d\widetilde{P}_\beta^{N,0}\biggr|&\\ \leqslant\widehat{C}(N^{\vartheta-1}+e^{-N^{\bar{c}}})\|f\|_\infty+\widehat{C}\,\frac{m^{1/2}N^{2\vartheta}+m^{3/2}\,N^\vartheta}{N}\|\nabla f\|_\infty.&\end{aligned}$$

Since $e^{-N^{\bar{c}}}\leqslant CN^{\theta-1}$, choosing $\vartheta\leqslant\frac{1}{2}\theta$ we conclude the validity of the first statement.

For the second statement we choose

$$\chi(\hat{\lambda})=f((N^{2/3}(\lambda_1^k-a_k^{aV}),...,N^{2/3}(\lambda_m^k-a_k^{aV}))_{k=1}^d),$$

([1]) This estimate, as well as the one at the edge that we shall prove below, should be compared with the one obtained in [5, Theorem 1.5]. While the estimates here are considerably stronger than the ones in [5, Theorem 1.5] (this follows from the fact that we have better bounds on our approximate transport maps), as a small "loss" we now have $N^{\vartheta-1}$ instead of a term $(\log N)^3/N$. The reason for this small difference comes from the fact that we decided to apply (2.8) to deduce (5.7). It is worth noticing that the argument in §4 combined with [5, Lemma 2.2] proves that also the stronger bound

$$\biggl|\int\chi\circ T^N\,dP_\beta^{N,0}-\int\chi\,dP_\beta^{N,aV}\biggr|\leqslant C\frac{(\log N)^3}{N}\|\chi\|_\infty$$

holds. However, since in general (2.8) is much more powerful than the estimate above (as it allows to deal with functions that grow polynomially with respect to the dimension) and the improvement between $(\log N)^3/N$ and $N^{\vartheta-1}$ is minimal, we have decided not to state also this second estimate.

and we note that $T_0^k(a_k^0)=a_k^{aV}$. Then, due to (4.39) and (5.13), we get

$$\begin{aligned}\left|\int_{G_\vartheta}\chi\circ T^N\circ\widehat{\mathcal{R}}\,dP_\beta^{N,0}-\int_{G_\vartheta}\chi\circ X_{0,1}\circ\widehat{\mathcal{R}}\,dP_\beta^{N,0}\right|&\\ \leqslant\frac{\|\nabla f\|_\infty}{N^{1/3}}\int_{G_\vartheta}\biggl[\sum_{k=1}^d\sum_{j=1}^m\biggl(|(X_{1,1})_j^k|+\frac{|(X_{2,1})_j^k|}{N}\biggr)^2\circ\widehat{\mathcal{R}}\biggr]^{1/2}dP_\beta^{N,0}&\\ \leqslant\frac{\|\nabla f\|_\infty}{N^{1/3}}\,(dm)^{1/2}\int_{G_\vartheta}\biggl(\max_{i,k}|(X_{1,1})_i^k|+\frac{\max_{i,k}|(X_{2,1})_i^k|}{N}\biggr)\,dP_\beta^{N,0}&\\ \leqslant C\|\nabla f\|_\infty\frac{m^{1/2}N^\vartheta}{N^{1/3}}.&\end{aligned}$$

Also, since

$$T_0^k(\lambda_1^k)-T_0^k(a_k^0)=(T_0^k)'(a_k^0)(\lambda_1^k-a_k^0)+O(|\lambda_1^k-a_k^0|^2),$$

using the rigidity estimate (5.6), we may replace

$$N^{2/3}(T_0^k(\lambda_1^k)-T_0^k(a_k^0))\quad\text{by }(T_0^k)'(a_k^0)\,N^{2/3}(\lambda_1^k-a_k^0)$$

up to an error of size $m^{1/2}\,(N^\vartheta+m^{2/3})\,N^{-2/3}$. Hence, arguing as above we conclude that

$$\begin{aligned}\biggl|\int f((N^{2/3}(\lambda_1^k-a_k^{aV}),\dots,N^{2/3}(\lambda_m^k-a_k^{aV}))_{k=1}^d)\,d\widetilde{P}_\beta^{N,aV}&\\ -\int f(((T_0^k)'(a_k^0)\,N^{2/3}(\lambda_1^k-a_k^0),\dots,(T_0^k)'(a_k^0)\,N^{2/3}(\lambda_m^k-a_k^0))_{k=1}^d)\,d\widetilde{P}_\beta^{N,0}\biggr|&\\ \leqslant\widehat{C}\,N^{\vartheta-1}\|f\|_\infty+\widehat{C}\left(\frac{m^{1/2}\,N^\vartheta}{N^{1/3}}+\frac{m^{1/2}(N^\vartheta+m^{2/3})}{N^{2/3}}\right)\|\nabla f\|_\infty.&\end{aligned}$$

which proves the second statement by choosing $\vartheta\leqslant\theta$. □

Proof of Corollary 2.7. We first note that the proof of Corollary 2.6 could be repeated verbatim in the context of [5] to show that [5, Theorem 1.5] holds with the same estimates as we obtained here. Hence, by combining this result with Corollary 2.6, we have

$$\begin{aligned}\biggl|\int f((N(\lambda_{i_k+1}^k-\lambda_{i_k}^k),\dots,N(\lambda_{i_k+m}^k-\lambda_{i_k}^k))_{k=1}^d)\,d\widetilde{P}_\beta^{N,aV}&\\ -\int f(((T_0^k\circ S_0^k)'(\gamma_{i_k/N})N(\lambda_{i_k+1}^k-\lambda_{i_k}^k),\dots,(T_0^k\circ S_0^k)'(\gamma_{i_k/N})N(\lambda_{i_k+m}^k-\lambda_{i_k}^k))_{k=1}^d)&\\ \times d(\widetilde{P}_{\mathrm{GVE},\beta}^N)^{\otimes d}\biggr|&\\ \leqslant\widehat{C}N^{\theta-1}\|f\|_\infty+\widehat{C}m^{3/2}\,N^{\theta-1}\|\nabla f\|_\infty,&\end{aligned}$$

where $\gamma_{i_k/N}$ satisfies $\mu_{\rm sc}((-\infty,\gamma_{i_k/N}))=i_k/N$. Note that the transport relations (2.10) and (2.11) imply that $T_0^k\circ S_0^k(\gamma_{i_k/N})=\gamma^k_{i_k/N,a}$, where $\gamma^k_{i_k/N,a}$ satisfies

$$\mu_k^{aV}((-\infty,\gamma^k_{i_k/N,a}))=\frac{i_k}{N},$$

and hence (again by (2.10) and (2.11))

$$(T_0^k\circ S_0^k)'(\gamma_{i_k/N})=\frac{\varrho_{sc}(\gamma_{i_k/N})}{\varrho_k^{aV}(\gamma^k_{i_k/N,a})}.$$

Finally, since $|\sigma_k-i_k/N|\leqslant C/N$ and $\sigma_k\in(0,1)$, arguing as we did for proving (5.3), we deduce that $|\gamma_{i_k/N}-\gamma_{\sigma_k}|\leqslant\widetilde{C}/N$, so up to another small error we may replace

$$\frac{\varrho_{sc}(\gamma_{i_k/N})}{\varrho_k^{aV}(\gamma^k_{i_k/N,a})}\quad\text{by}\quad\frac{\varrho_{sc}(\gamma_{\sigma_k})}{\varrho_k^{aV}(\gamma_{\sigma_k,k})}.$$

This concludes the proof of of the first statement, while the second one is just a consequence of Corollary 2.6 (2) and [5, Theorem 1.5 (2)]. □

Proof of Corollary 2.8. As is clear by looking at the proof of Corollaries 2.6 and 2.7, the fact of dealing at the same time with the eigenvalues of different matrices does not complicate the proof. For this reason, since the proof of Corollary 2.8 is already very involved, to make the argument more transparent we shall prove the result when the test function is of the form

$$\fint_{R_k(E)-N^{-\varsigma}R_k'(E)}^{R_k(E)+N^{-\varsigma}R_k'(E)}\sum_{i_j\ \text{distinct}}f\big(N(\lambda^k_{i_1}-\widetilde{E}),...,N(\lambda^k_{i_m}-\widetilde{E})\big)\,d\widetilde{E}$$

for some $E\in(-2,2)$, the proof in the general case being completely analogous and just notationally heavier.

To simplify the notation, we set

$$\begin{aligned}g_{\widetilde{E}}(\hat{\lambda}) &:= \sum_{i_1\neq...\neq i_m}f(N(\lambda^k_{i_1}-\widetilde{E}),...,N(\lambda^k_{i_m}-\widetilde{E})),\\ A_k &:= \int\bigg[\fint_{R_k(E)-N^{-\varsigma}R_k'(E)}^{R_k(E)+N^{-\varsigma}R_k'(E)}g_{\widetilde{E}}\,d\widetilde{E}\bigg]\,dP_\beta^{N,aV}.\end{aligned}$$

It follows by (2.8) with $\eta=\theta$ that

$$|\log(1+A_k)-\log(1+A_{1,k})|\leqslant CN^{\theta-1},\tag{5.15}$$

where

$$
\begin{aligned}
A_{1,k} &:= \int \left[\fint_{R_k(E)-N^{-\zeta}R_k'(E)}^{R_k(E)+N^{-\zeta}R_k'(E)} g_{\widetilde{E}} \circ (T^N)^k \, d\widetilde{E} \right] dP_\beta^{N,0} \\
&= \int \left[\fint_{R_k(E)-N^{-\zeta}R_k'(E)}^{R_k(E)+N^{-\zeta}R_k'(E)} \right. \\
&\qquad \left. \sum_{i_1\neq \ldots \neq i_m} f(N((T^N)_{i_1}^k(\hat{\lambda})-\widetilde{E}), \ldots, N((T^N)_{i_m}^k(\hat{\lambda})-\widetilde{E}))\, d\widetilde{E} \right] dP_\beta^{N,0}.
\end{aligned}
$$

Define the quantiles $\gamma^k_{i/N}\in(S_k^0(-2), S_k^0(2))$ as in Corollary 2.6, and given $\vartheta>0$ small (to be fixed later) we consider the set G_ϑ defined in (5.1).

Since the integrand $g_{\widetilde{E}}\circ(T^N)^k$ is pointwise bounded by $\|f\|_\infty N^m$, it follows by (5.2) that

$$
A_{1,k} = A_{2,k} + O(e^{-N^c}), \tag{5.16}
$$

where

$$
A_{2,k} := \int_{G_\vartheta} \left[\fint_{R_k(E)-N^{-\zeta}R_k'(E)}^{R_k(E)+N^{-\zeta}R_k'(E)} g_{\widetilde{E}} \circ (T^N)^k \, d\widetilde{E} \right] dP_\beta^{N,0}.
$$

Observe that if $\hat{\lambda}\in G_\vartheta$ then, by definition,

$$
|\lambda_i^k-\lambda_j^k| \geqslant |\gamma^k_{i/N}-\gamma^k_{j/N}| - N^{-2/3+\vartheta}\min\{i, N+1-i\}^{-1/3} - N^{-2/3+\vartheta}\min\{j, N+1-j\}^{-1/3}.
$$

Hence, since $\gamma^k_{(i+1)/N}-\gamma^k_{i/N}\geqslant c_0 N^{-2/3}\min\{i, N+1-i\}^{-1/3}$ for all i, we deduce that

$$
|\lambda_i^k-\lambda_j^k| \geqslant N^{\vartheta-1} \quad \text{provided } |i-j|\geqslant C_0 N^\vartheta,
$$

which, combined with (5.8) yields, for $\hat{\lambda}\in G_\vartheta$,

$$
|(T^N)_i^k(\hat{\lambda})-(T^N)_j^k(\hat{\lambda})| \geqslant e^{-L}N^{\vartheta-1} \quad \text{provided } |i-j|\geqslant C_0 N^\vartheta. \tag{5.17}
$$

We now notice that, since f is compactly supported, the quantity

$$
f(N((T^N)_{i_1}^k(\hat{\lambda})-\widetilde{E}), \ldots, N((T^N)_{i_m}^k(\hat{\lambda})-\widetilde{E}))
$$

can be non-zero only if

$$
|(T^N)_{i_j}^k(\hat{\lambda})-\widetilde{E}| \leqslant \frac{C_1}{N} \quad \text{for all } j=1, \ldots, m.
$$

Therefore, if $\bar{i}\in\{1, \ldots, N\}$ is an index (depending on $\hat{\lambda}$ and $\widetilde{E}$) such that

$$
|(T^N)_{\bar{i}}^k(\hat{\lambda})-\widetilde{E}| \leqslant \frac{C_1}{N},
$$

then (5.17) yields

$$|(T^N)^k_i(\hat\lambda)-\widetilde{E}|\leqslant \frac{C_1}{N} \quad\Longrightarrow\quad |i-\bar{i}|\leqslant C_0N^{\vartheta}.$$

This proves that, for any $\hat\lambda\in G_\vartheta$, there exists a set of indices

$$J_{\hat\lambda,\widetilde{E}}\subset\{(i_1,...,i_m)\in\{1,...,N\}^m : i_1\neq...\neq i_m\}$$

such that $\# J_{\hat\lambda,\widetilde{E}}\leqslant CN^{m\vartheta}$ and

$$A_{2,k}=\int_{G_\vartheta}\left[⨍_{R_k(E)-N^{-\zeta}R'_k(E)}^{R_k(E)+N^{-\zeta}R'_k(E)}\hat{g}_{\widetilde{E}}\circ(T^N)^k\,d\widetilde{E}\right]dP^{N,0}_\beta,$$

where

$$\hat{g}_{\widetilde{E}}(\hat\lambda):=\sum_{(i_1,...,i_m)\in J_{\hat\lambda,\widetilde{E}}} f(N(\lambda^k_{i_1}-\widetilde{E}),...,N(\lambda^k_{i_m}-\widetilde{E}))$$

satisfies $|\hat{g}_{T^k_0(\widetilde{E})}|\leqslant C\|f\|_\infty N^{m\vartheta}$.

We now perform the change of variable $\widetilde{E}\mapsto T^k_0(\widetilde{E})$, which gives

$$\int_{R_k(E)-N^{-\zeta}R'_k(E)}^{R_k(E)+N^{-\zeta}R'_k(E)}\hat{g}_{\widetilde{E}}\circ(T^N)^k\,d\widetilde{E}=\int_{(T^k_0)^{-1}[R_k(E)-N^{-\zeta}R'_k(E)]}^{(T^k_0)^{-1}[R_k(E)+N^{-\zeta}R'_k(E)]}\hat{g}_{T^k_0(\widetilde{E})}\circ(T^N)^k\,(T^k_0)'(\widetilde{E})\,d\widetilde{E}.$$

Recalling that $R_k=T^k_0\circ S^k_0$ and that these maps are all smooth diffeomorphisms of $\mathbb{R}$, we see that for

$$\widetilde{E}\in[(T^k_0)^{-1}[R_k(E)-N^{-\zeta}R'_k(E)],(T^k_0)^{-1}[R_k(E)+N^{-\zeta}R'_k(E)]]$$

it holds

$$|(T^k_0)'(\widetilde{E})-(T^k_0)'\circ S^k_0(E)|\leqslant CN^{-\zeta},\quad R'_k(E)=[(T^k_0)'\circ S^k_0(E)]\,(S^k_0)'(E),$$

and

$$(T^k_0)^{-1}[R_k(E)\pm N^{-\zeta}R'_k(E)]=S^k_0(E)\pm N^{-\zeta}(S^k_0)'(E)+O(N^{-2\zeta}).$$

Hence, since $|\hat{g}_{T^k_0(\widetilde{E})}|\leqslant CN^{m\vartheta}$,

$$\begin{aligned}&⨍_{(T^k_0)^{-1}[R_k(E)-N^{-\zeta}R'_k(E)]}^{(T^k_0)^{-1}[R_k(E)+N^{-\zeta}R'_k(E)]}\hat{g}_{T^k_0(\widetilde{E})}\circ(T^N)^k\,(T^k_0)'(\widetilde{E})\,d\widetilde{E}\\&\qquad=⨍_{S^k_0(E)-N^{-\zeta}(S^k_0)'(E)}^{S^k_0(E)-N^{-\zeta}(S^k_0)'(E)}\hat{g}_{T^k_0(\widetilde{E})}\circ(T^N)^k\,d\widetilde{E}+O(N^{m\vartheta-\zeta}),\end{aligned}$$

which proves that

$$A_{2,k}=A_{3,k}+O(N^{m\vartheta-\zeta}), \tag{5.18}$$

where

$$A_{3,k}:=\int_{G_\vartheta}\left[\fint_{S_0^k(E)-N^{-\zeta}(S_0^k)'(E)}^{S_0^k(E)-N^{-\zeta}(S_0^k)'(E)}\hat g_{T_0^k(\widetilde E)}\circ(T^N)^k\,d\widetilde E\right]dP_\beta^{N,0}.$$

We now estimate $A_{3,k}$.

Due to Theorem 2.5, we can write

$$\begin{aligned}\hat g_{T_0^k(\widetilde E)}\circ(T^N)^k(\hat\lambda)=\sum_{(i_1,\dots,i_m)\in J_{\hat\lambda,\widetilde E}} & f(N(T_0^k(\lambda_{i_1}^k)-T_0^k(\widetilde E))+(X_{1,1}^N)_{i_1}^k(\hat\lambda),\\ &\dots,N(T_0^k(\lambda_{i_m}^k)-T_0^k(\widetilde E))+(X_{1,1}^N)_{i_m}^k(\hat\lambda))\\ &+O\biggl(\frac{\|\nabla f\|_\infty}{N}\sum_{(i_1,\dots,i_m)\in J_{\hat\lambda,\widetilde E}}|(X_{2,1}^N)_{i_j}^k|\biggr).\end{aligned}$$

and thus

$$A_{3,k}=A_{4,k}+O\biggl(\frac1N\int_{G_\vartheta}\sum_{(i_1,\dots,i_m)\in J_{\hat\lambda,\widetilde E}}|(X_{2,1}^N)_{i_j}^k|\,dP_\beta^{N,0}\biggr) \tag{5.19}$$

where

$$A_{4,k}:=\int_{G_\vartheta}\left[\fint_{S_0^k(E)-N^{-\zeta}(S_0^k)'(E)}^{S_0^k(E)-N^{-\zeta}(S_0^k)'(E)}h_{\widetilde E}\,d\widetilde E\right]d\widetilde P_{\beta,k}^{N,0},$$

and with

$$\begin{aligned}h_{\widetilde E}(\hat\lambda):=\sum_{(i_1,\dots,i_m)\in J_{\hat\lambda,\widetilde E}} & f(N(T_0^k(\lambda_{i_1}^k)-T_0^k(\widetilde E))+(X_{1,1}^N)_{i_1}^k(\hat\lambda),\\ &\dots,N(T_0^k(\lambda_{i_m}^k)-T_0^k(\widetilde E))+(X_{1,1}^N)_{i_m}^k(\hat\lambda)).\end{aligned}$$

We now want to get rid of the terms $(X_{1,1}^N)_{i_j}^k$ and $|(X_{2,1}^N)_{i_j}^k|$.

Motivated by (4.37), for any $\widetilde E\in\mathbb R$ we define $X_{1,\hat\lambda}^k(\widetilde E)$ as the solution of the ODE

$$\begin{aligned}\dot X_{t,\hat\lambda}^k(\widetilde E)&=(\mathbf y_{k,t}^0)'(X_{0,t}^k(\widetilde E))\cdot X_{t,\hat\lambda}^k(\widetilde E)+\mathbf y_{k,t}^1(X_{0,t}^k(\widetilde E))\\ &\quad+\sum_{\ell=1}^d\int\mathbf z_{k\ell,t}(X_{0,t}^k(\widetilde E),y)\,dM_{X_{0,t}^\ell}^N(y)\\ &\quad+\frac1N\sum_{\ell=1}^d\sum_{j=1}^N\partial_2\mathbf z_{k\ell,t}(X_{0,t}^k(\widetilde E),X_{0,t}^\ell(\lambda_j^\ell))\cdot(X_{1,t})_j^\ell(\hat\lambda),\end{aligned}$$

with $X_{0,\hat\lambda}^k(\widetilde E)=\widetilde E$, and we note the following fact: whenever $\hat\lambda\in G_\vartheta$ we know that $\{\lambda_i^\ell\}_{i=1}^N$ are close, up to an error N^ϑ, to the quantiles of the stationary measure $\mu_\ell^0=\mu_{\ell,0}^*$. Hence, arguing as we did for (5.13), we get

$$|\partial_{\widetilde E}X_{1,\hat\lambda}^k(\widetilde E)|\leqslant CN^\vartheta,\quad|(X_{1,1}^N)_i^k(\hat\lambda)-X_{1,\hat\lambda}^k(\widetilde E)|\leqslant CN^\vartheta\,|\lambda_i^k-\widetilde E|\quad\text{for all }\hat\lambda\in G_\vartheta. \tag{5.20}$$

In addition, by the same reasoning,

$$\max_{i,k}\int \partial_1 \mathbf{z}_{k\ell,t}(X^k_{0,t}(\lambda^k_i),y)\,dM^N_{X^\ell_{0,t}}(y)=O(N^\vartheta)\quad\text{for all }\hat\lambda\in G_\vartheta,$$

and the argument used to prove (4.39) (see in particular (4.50)) yields

$$\max_{i,k}|(X^N_{2,1})^k_i|\leqslant CN^{2\vartheta}\quad\text{for all }\hat\lambda\in G_\vartheta.$$

Hence, since $\#J_{\hat\lambda,\widetilde E}\leqslant CN^{m\vartheta}$ we immediately deduce that

$$O\biggl(\frac{1}{N}\int_{G_\vartheta}\sum_{(i_1,\dots,i_m)\in J_{\hat\lambda,\widetilde E}}|(X^N_{2,1})^k_{i_j}|\,dP^{N,0}_\beta\biggr)=O\bigl(N^{(m+2)\vartheta-1}\bigr).\tag{5.21}$$

Now, to get rid of the term $X^k_{1,\hat\lambda}(\widetilde E)$ inside $h_{\widetilde E}$ we take advantage of (5.20) and the average with respect to $\widetilde E$: more precisely, we consider the change of variable

$$\widetilde E\longmapsto\Phi_{\hat\lambda}(\widetilde E):=(T^k_0)^{-1}\biggl[T^k_0(\widetilde E)+\frac{1}{N}X^k_{1,\hat\lambda}(\widetilde E)\biggr]$$

so that

$$\begin{aligned}&\fint_{S^k_0(E)-N^{-\zeta}(S^k_0)'(E)}^{S^k_0(E)-N^{-\zeta}(S^k_0)'(E)} h_{\widetilde E}\,d\widetilde E\\ &\quad=\fint_{S^k_0(E)-N^{-\zeta}(S^k_0)'(E)}^{S^k_0(E)-N^{-\zeta}(S^k_0)'(E)}\sum_{(i_1,\dots,i_m)\in J_{\hat\lambda,\widetilde E}} f(N(T^k_0(\lambda^k_{i_1})-T^k_0(\widetilde E))+[(X^N_{1,1})^k_{i_1}(\hat\lambda)-X^k_{1,\hat\lambda}(\widetilde E)],\\ &\qquad\qquad\dots,N(T^k_0(\lambda^k_{i_m})-T^k_0(\widetilde E))+[(X^N_{1,1})^k_{i_m}(\hat\lambda)-X^k_{1,\hat\lambda}(\widetilde E)])\partial_{\widetilde E}\Phi_{\hat\lambda}(\widetilde E)\,d\widetilde E.\end{aligned}$$

Therefore, since $\partial_{\widetilde E}\Phi_{\hat\lambda}(\widetilde E)=1+O\bigl(N^{\vartheta-1}\bigr)$ (due to (5.20)), $|h_{\widetilde E}|\leqslant CN^{m\vartheta}$, and the interval $[S^k_0(E)-N^{-\zeta}(S^k_0)'(E),S^k_0(E)-N^{-\zeta}(S^k_0)'(E)]$ has length of order $N^{-\zeta}$, we deduce that

$$\begin{aligned}&\fint_{S^k_0(E)-N^{-\zeta}(S^k_0)'(E)}^{S^k_0(E)-N^{-\zeta}(S^k_0)'(E)} h_{\widetilde E}\,d\widetilde E\\ &\quad=\fint_{S^k_0(E)-N^{-\zeta}(S^k_0)'(E)}^{S^k_0(E)-N^{-\zeta}(S^k_0)'(E)}\sum_{(i_1,\dots,i_m)\in J_{\hat\lambda,\widetilde E}} f(N(T^k_0(\lambda^k_{i_1})-T^k_0(\widetilde E))+[(X^N_{1,1})^k_{i_1}(\hat\lambda)-X^k_{1,\hat\lambda}(\widetilde E)],\\ &\qquad\dots,N(T^k_0(\lambda^k_{i_m})-T^k_0(\widetilde E))+[(X^N_{1,1})^k_{i_m}(\hat\lambda)-X^k_{1,\hat\lambda}(\widetilde E)])\,d\widetilde E+O(N^\zeta N^{m\vartheta}N^{\vartheta-1}).\end{aligned}\tag{5.22}$$

We now observe that, since $T^k_0:\mathbb{R}\to\mathbb{R}$ is a diffeomorphism with $(T^k_0)'\geqslant e^{-L}>0$ (see (5.14)), it follows by (5.20) that

$$|(X^N_{1,1})^k_{i_1}(\hat\lambda)-X^k_{1,\hat\lambda}(\widetilde E)|\leqslant CN^\vartheta\,|T^k_0(\lambda^k_i)-T^k_0(\widetilde E)|.$$

Therefore, since f is compactly supported, we see that the expression

$$f(N(T_0^k(\lambda_{i_1}^k)-T_0^k(\widetilde{E}))+[(X_{1,1}^N)_{i_1}^k(\hat{\lambda})-X_{1,\hat{\lambda}}^k(\widetilde{E})], \\ ..., N(T_0^k(\lambda_{i_m}^k)-T_0^k(\widetilde{E}))+[(X_{1,1}^N)_{i_m}^k(\hat{\lambda})-X_{1,\hat{\lambda}}^k(\widetilde{E})])$$

is non-zero only if

$$|T_0^k(\lambda_{i_j}^k)-T_0^k(\widetilde{E})|\leqslant\frac{C_1}{N}\quad\text{for all } j=1,...,m.$$

In particular, using again that $(T_0^k)'\geqslant e^{-L}>0$, this implies that $|\lambda_{i_j}^k-\widetilde{E}|\leqslant C/N$. Thus

$$|T_0^k(\lambda_{i_j}^k)-T_0^k(\widetilde{E})-(T_0^k)'(E)[\lambda_{i_j}^k-\widetilde{E}]|=O\left(\frac{1}{N^2}\right)$$

and

$$N^\vartheta|T_0^k(\lambda_{i_j}^k)-T_0^k(\widetilde{E})|=O\big(N^{\vartheta-1}\big),$$

and we get

$$\begin{aligned} &f(N(T_0^k(\lambda_{i_1}^k)-T_0^k(\widetilde{E})\big)+[(X_{1,1}^N)_{i_1}^k(\hat{\lambda})-X_{1,\hat{\lambda}}^k(\widetilde{E})], \\ &\qquad ..., N(T_0^k(\lambda_{i_m}^k)-T_0^k(\widetilde{E}))+[(X_{1,1}^N)_{i_m}^k(\hat{\lambda})-X_{1,\hat{\lambda}}^k(\widetilde{E})]) \\ &\quad = f((T_0^k)'(E)\,N(\lambda_{i_j}^k-\widetilde{E}),...,(T_0^k)'(E)\,N(\lambda_{i_j}^k-\widetilde{E}))+O(\|\nabla f\|_\infty N^{\vartheta-1}). \end{aligned}$$

Combining this estimate with (5.22) and the fact that $\#J_{\hat{\lambda},\widetilde{E}}\leqslant CN^{m\vartheta}$ we conclude that

$$\fint_{S_0^k(E)-N^{-\zeta}(S_0^k)'(E)}^{S_0^k(E)-N^{-\zeta}(S_0^k)'(E)} h_{\widetilde{E}}\,d\widetilde{E}=\bar{g}_E+O\big(N^{(m+1)\vartheta+\zeta-1}\big),$$

where

$$\begin{aligned} &\bar{g}_E(\hat{\lambda}) \\ &\ :=\fint_{S_0^k(E)-N^{-\zeta}(S_0^k)'(E)}^{S_0^k(E)-N^{-\zeta}(S_0^k)'(E)} \sum_{(i_1,...,i_m)\in J_{\hat{\lambda},\widetilde{E}}} f((T_0^k)'(E)N(\lambda_{i_j}^k-\widetilde{E}),...,(T_0^k)'(E)N(\lambda_{i_j}^k-\widetilde{E}))\,d\widetilde{E}. \end{aligned}$$

Also, by the argument above it follows that we can add back into the sum all the indices outside $J_{\hat{\lambda},\widetilde{E}}$ (since, up to infinitesimal errors, the function above vanishes on such indices), and therefore

$$\fint_{S_0^k(E)-N^{-\zeta}(S_0^k)'(E)}^{S_0^k(E)-N^{-\zeta}(S_0^k)'(E)} h_{\widetilde{E}}\,d\widetilde{E}=\bar{\bar{g}}_E+O(N^{(m+1)\vartheta+\zeta-1}),$$

with

$$\bar{\bar{g}}_E(\hat{\lambda}) := \fint_{S_0^k(E)-N^{-\zeta}(S_0^k)'(E)}^{S_0^k(E)-N^{-\zeta}(S_0^k)'(E)} \sum_{i_1\neq\ldots\neq i_m} f((T_0^k)'(E)N(\lambda_{i_j}^k-\widetilde{E}),\ldots,(T_0^k)'(E)N(\lambda_{i_j}^k-\widetilde{E}))\,d\widetilde{E}.$$

Combining this bound with (5.15), (5.16), (5.18), (5.19), and (5.21), we conclude that

$$|\log(1+A_k)-\log(1+\bar{\bar{A}}_k)| \leqslant C(N^{m\vartheta-\zeta}+N^{(m+2)\vartheta-1}+N^{(m+1)\vartheta+\zeta-1}), \tag{5.23}$$

where $\bar{\bar{A}}_k := \int \bar{\bar{g}}_E\,dP_\beta^{N,0}$.

We now repeat this very same argument replacing $P_\beta^{N,aV}$, $P_\beta^{N,0}$, and T^N, with $P_\beta^{N,0}$, $(P_{\mathrm{GVE},\beta}^N)^{\otimes d}$, and $S^N=(S_1^N,\ldots,S_d^N)$, respectively (see the discussion before Corollary 2.7), and we deduce that

$$|\log(1+\bar{\bar{A}}_k)-\log(1+\hat{A}_k)| \leqslant C(N^{m\vartheta-\zeta}+N^{(m+2)\vartheta-1}+N^{(m+1)\vartheta+\zeta-1}),$$

where

$$\hat{A}_k := \int\left[\fint_{E-N^{-\zeta}}^{E+N^{-\zeta}} \sum_{i_1\neq\ldots\neq i_m} f(R_k'(E)N(\lambda_{i_1}-\widetilde{E}),\ldots,R_k'(E)N(\lambda_{i_m}-\widetilde{E}))\,d\widetilde{E}\right] dP_{\mathrm{GVE},\beta}^N.$$

Combining this estimate with (5.23), we get

$$|\log(1+A_k)-\log(1+\hat{A}_k)| \leqslant C(N^{m\vartheta-\zeta}+N^{(m+2)\vartheta-1}+N^{(m+1)\vartheta+\zeta-1}).$$

Choosing ϑ small enough so that $(m+2)\vartheta<\theta$, this gives

$$|\log(1+A_k)-\log(1+\hat{A}_k)| \leqslant C(N^{\theta+\zeta-1}+N^{\theta-1/2}+N^{\theta-\zeta}) \leqslant C(N^{\theta+\zeta-1}+N^{\theta-\zeta}),$$

and since $\hat{A}_k$ is uniformly bounded in N (see for instance [65]) and the right-hand side is infinitesimal (recall that $\theta<\min\{\zeta,1-\zeta\}$), we conclude that

$$|A_k-\hat{A}_k| \leqslant C(N^{\theta+\zeta-1}+N^{\theta-\zeta}).$$

Recalling the definition of A_k and $\hat{A}_k$, this proves that

$$\begin{aligned}\bigg|\int\left[\fint_{R_k(E)-N^{-\zeta}R_k'(E)}^{R_k(E)+N^{-\zeta}R_k'(E)} \sum_{i_1\neq\ldots\neq i_m} f(N(\lambda_{i_1}^k-\widetilde{E}),\ldots,N(\lambda_{i_m}^k-\widetilde{E}))\,d\widetilde{E}\right] dP_\beta^{N,aV} \\ -\int\left[\fint_{E-N^{-\zeta}}^{E+N^{-\zeta}} \sum_{i_1\neq\ldots\neq i_m} f\big(R_k'(E)\,N(\lambda_{i_1}-\widetilde{E}),\ldots,R_k'(E)\,N(\lambda_{i_m}-\widetilde{E})\big)\,d\widetilde{E}\right] dP_{\mathrm{GVE}}^N\bigg| \\ \leqslant \widehat{C}(N^{\theta+\zeta-1}+N^{\theta-\zeta}),\end{aligned}$$

which corresponds to our statement when f depends only on the eigenvalues of one matrix. As explained at the beginning of the proof, the very same argument presented above extends also to the general case. □

Proof of Corollary 2.9. We begin by noticing that the proof of Theorem 2.5 could be repeated verbatim in the context of [5] to show that [5, Theorem 1.4] holds with the same estimates as we obtained here.

To prove the gap estimates, it is enough to show that the approximate transport maps do not change gaps in the bulk uniformly (away from the edges). Due to Theorem 2.5 and [5, Theorem 1.4], we have the expansions

$$\begin{aligned}(T^N)_i^k(\hat\lambda) &= T_0^k(\lambda_i^k)+\frac{1}{N}(X_{1,1}^N)_i^k(\hat\lambda)+\frac{1}{N^2}(X_{2,1}^N)_i^k(\hat\lambda),\\ (S_k^N)_i(\lambda^k) &= S_0^k(\lambda_i^k)+\frac{1}{N}(S_{k,1})_i(\lambda^k)+\frac{1}{N^2}(S_{k,2})_i(\lambda^k),\end{aligned}$$

where $(S_{k,1})_i$ and $(S_{k,2})_i$ satisfy the same estimates as $(X_1^N)_i^k$ and $(X_2^N)_i^k$. Hence, by the formulas above we deduce that

$$\begin{aligned}&(T^N)_i^k\big(S_1^N(\lambda^1),...,S_d^N(\lambda^d)\big)\\ &\quad = T_0^k\circ S_0^k(\lambda_i^k)+\frac{1}{N}[(T_0^k)'\circ S_0^k(\lambda_i^k)](S_{k,1})_i(\lambda^k)\\ &\qquad +\frac{1}{N}(X_{1,1}^N)_i^k\bigg(S_0^1(\lambda_1^1)+\frac{1}{N}(S_{1,1})_1(\lambda^1),...,S_0^d(\lambda_N^d)+\frac{1}{N}(S_{N,d})_N(\lambda^d)\bigg)+\mathcal{E}_i,\end{aligned}\tag{5.24}$$

where the error $\mathcal{E}_i$ satisfies (due to the bounds in Theorem 2.5 and [5, Theorem 1.4])

$$\sqrt{\sum_i \|\mathcal{E}_i\|^2_{L^2(P^N_{\mathrm{GVE},\beta})}} = O\bigg(\frac{(\log N)^2}{N^{3/2}}\bigg)\tag{5.25}$$

Also, by using again Theorem 2.5 and [5, Theorem 1.4], with probability greater than $1-e^{-c(\log N)^2}$ and uniformly with respect to $i\in\{1,...,N\}$, we have

$$\begin{aligned}&|[(T_0^k)'\circ S_0^k(\lambda_{i+1}^k)](S_{k,1})_{i+1}(\lambda^k)-[(T_0^k)'\circ S_0^k(\lambda_i^k)](S_{k,1})_i(\lambda^k)|\\ &\qquad\qquad \leqslant C(\log N)N^{1/(\sigma-15)}|\lambda_{i+1}^k-\lambda_i^k|,\end{aligned}$$

and

$$\begin{aligned}&|(X_{1,1}^N)_{i+1}^k-(X_{1,1}^N)_i^k|\circ\bigg((S_0^1)^{\otimes N}+\frac{1}{N}S_{1,1},...,(S_0^d)^{\otimes N}+\frac{1}{N}S_{d,1}\bigg)(\hat\lambda)\\ &\quad\leqslant C(\log N)N^{1/(\sigma-15)}\bigg(|S_0^k(\lambda_{i+1}^k)-S_0^k(\lambda_i^k)|+\frac{1}{N}|(S_{k,1})_{i+1}(\lambda^k)-(S_{k,1})_i(\lambda^k)|\bigg)\\ &\quad\leqslant C\log N\, N^{1/(\sigma-15)}|\lambda_{i+1}^k-\lambda_i^k|,\end{aligned}$$

while

$$T_0^k\circ S_0^k(\lambda_{i+1}^k)-T_0^k\circ S_0^k(\lambda_i^k)=(T_0^k\circ S_0^k)'(\lambda_i^k)[\lambda_{i+1}^k-\lambda_i^k]+O(|\lambda_{i+1}^k-\lambda_i^k|^2).$$

Recalling that, with probability greater than $1-e^{-N^{\bar{c}}}$, $|\lambda_{i+1}^k-\lambda_i^k|\leqslant CN^{\theta-1}$ when the $\{\lambda_i^k\}_{i=1}^N$ are ordered and $i\in[\varepsilon N,(1-\varepsilon)N]$ (see (5.2) and (5.4)), we conclude that, with probability greater than $1-e^{-c(\log N)^2}$, and uniformly with respect to $i\in[\varepsilon N,(1-\varepsilon)N]$, we have

$$\begin{aligned}[(T^N)_{i+1}^k-(T^N)_i^k](S_1^N(\lambda^1),...,S_d^N(\lambda^d))\\ =(T_0^k\circ S_0^k)'(\lambda_i^k)[\lambda_{i+1}^k-\lambda_i^k]+O\biggl(\frac{(\log N)N^{1/(\sigma-15)}}{N^{2-\theta}}\biggr).\end{aligned}$$

Combining this estimate with (5.25) and noticing that

$$N^{4/3}\biggl(\frac{(\log N)N^{2/(\sigma-15)}}{N^{2-\theta}}+\frac{(\log N)^2}{N^{3/2}}\biggr)\to 0\quad\text{as } N\to\infty,$$

provided $\theta<\frac{1}{6}$ (recall that by assumption $\sigma\geqslant 36$; see Hypothesis 2.1), the two statements follow from the fact that $T^N\circ(S_1^N,...S_d^N):\mathbb{R}^{dN}\to\mathbb{R}^{dN}$ is an approximate transport map from $(P_{\mathrm{GVE},\beta}^N)^{\otimes d}$ to $P_\beta^{N,aV}$ and that the results are true under $P_{\mathrm{GVE},\beta}^N$ due to [6, Theorem 1.3 and Corollary 1.5]. $\square$

6. Matrix integrals

In this section, we consider the integral

$$I_\beta^{N,V}(A_1,...,A_d,B_1,...,B_m):=\int e^{N^{2-r}\mathrm{Tr}^{\otimes r}V(U_1A_1U_1^*,...,U_dA_dU_d^*,B_1,...,B_m)}\,dU_1\,...\,dU_d,$$

where $\beta=2$ (resp. $\beta=1$) corresponds to integration over the unitary (resp. the orthogonal) group $U(N)$ (resp. $O(N)$). Here $A_1,...,A_d$ and $B_1,...,B_m$ are $m+d$ Hermitian (resp. symmetric) matrices such that

$$\max_{i=1}^d\|A_i\|_\infty\leqslant 1\quad\text{and}\quad\max_{i=1}^m\|B_i\|_\infty\leqslant 1,\tag{6.1}$$

and V belongs to the tensor product $\mathbb{C}\langle x_1,...,x_d;b_1,...,b_m\rangle^{\otimes r}$ (or more generally to its closure for the norm defined below), where $\mathbb{C}\langle x_1,...,x_d;b_1,...,b_m\rangle$ denotes the set of polynomial in $d+m$ self-adjoint variables.

We see V as a Laurent polynomial in $\{u_i,u_i^*,a_i\}_{i=1}^d$ and $\{b_i\}_{i=1}^m$, where $x_i=u_ia_iu_i^{-1}$. The set $\mathscr{L}$ of Laurent polynomials is equipped with the involution $*$ given by $u_i^*=u_i^{-1}$, $a_i^*=a_i$, $b_i^*=b_i$, and for any Laurent polynomials p and q one has $(zpq)^*=\bar{z}q^*p^*$. We let $\sum\langle p,q_1\otimes...\otimes q_r\rangle q_1\otimes...\otimes q_r$ be the decomposition of a polynomial p in

$$\mathscr{L}^{\otimes r}:=\mathbb{C}\langle u_1,u_1^*,...,u_d,u_d^*;a_1,...,a_d;b_1,...,b_m\rangle^{\otimes r}$$

in the basis of tensor products of monomials, and for $\xi,\zeta\geqslant 1$ we set

$$\|p\|_{\xi,\zeta}:=\sum|\langle p,q_1\otimes\ldots\otimes q_r\rangle|\xi^{\sum_{i=1}^r \deg_U(q_i)}\zeta^{\sum_{i=1}^r \deg_{A,B}(q_i)},$$

where $\deg_U(q)$ (resp. $\deg_{A,B}(q)$) is the number of letters in $\{u_i,u_i^*\}_{i=1}^d$ (resp. $\{a_i\}_{i=1}^d$ and $\{b_i\}_{i=1}^m$) in the word q. We let $\mathscr{L}^r_{\xi,\zeta}:=\overline{\mathscr{L}^{\otimes r}}^{\|\cdot\|_{\xi,\zeta}}$ be the closure of $\mathscr{L}^{\otimes r}$ for the norm $\|\cdot\|_{\xi,\zeta}$. We endow the space of linear forms $\mathcal{L}^r_{\xi,\zeta}$ on $\mathscr{L}^r_{\xi,\zeta}$ with the weak topology, that can be recast in terms of the norm

$$\|\tau\|_{\xi,\zeta}:=\sup_{\|p\|_{\xi,\zeta}\leqslant 1}|\tau(p)|.$$

Notice that, by abuse of notation, we use $\|\cdot\|_{\xi,\zeta}$ to denote both the norm and the dual norm. It will always be clear from the context which one we are referring to. For later purpose, observe that $\xi,\zeta\mapsto\|p\|_{\xi,\zeta}$ is increasing for any $p\in\mathscr{L}^r_{\xi,\zeta}$, whereas $\xi,\zeta\mapsto\|\tau\|_{\xi,\zeta}$ is decreasing for any $\tau\in\mathcal{L}^r_{\xi,\zeta}$. In the case where $r=1$, we denote in short $\mathscr{L}_{\xi,\zeta}$, $\mathcal{L}_{\xi,\zeta}$, etc.

We denote by $\mathcal{L}(\mathscr{S})$ the set of linear forms on a vector subspace $\mathscr{S}$ of $\mathscr{L}$, and endow it with the weak norm $\|\cdot\|_{\xi,\zeta}$. In particular if $\mathscr{A}\mathscr{B}$ is the algebra generated by $\{a_1,\ldots,a_d,b_1,\ldots,b_m\}$, the parameter ξ does not appear and we write in short $\|\cdot\|_\zeta$. In case of a linear form on the algebra generated by a single self-adjoint variable, that corresponds simply to measure on the real line, this is

$$\|\nu\|_\zeta:=\sup_k \zeta^{-k}|\nu(x^k)|.$$

We denote by $\mathcal{M}(K)$ (resp. $\mathcal{P}(K)$) the set of Borel measures (resp. probability measure) on the set $K\subset\mathbb{R}$ and by $\mathscr{B}$ the algebra generated by $\{b_1,\ldots,b_m\}$, and we write

$$\|\nu\|_\zeta:=\sum_{i=1}^d\|\nu_i\|_\zeta+\|\tau\|_\zeta$$

for $(d+1)$-tuples consisting of d probability measures on $[-1,1]$ and one linear form in $\mathcal{L}(\mathscr{B})$. Notice that, for $\tau\in\mathcal{L}(\mathscr{B})$,

$$\|\tau\|_\zeta:=\sup_{\substack{k\\ i_j\in\{1,\ldots,m\}}}\zeta^{-k}|\tau(b_{i_1}\ldots b_{i_k})|$$

as in this case the degree $\deg_{A,B}$ is simply the degree in $\{b_i\}_{i=1}^m$. We assume, without loss of generality, that V is symmetric, in the sense that for any permutation σ of $\{1,\ldots,r\}$

$$\sum\langle V,q_1\otimes\ldots\otimes q_r\rangle q_1\otimes\ldots\otimes q_r=\sum\langle V,q_{\sigma(1)}\otimes\ldots\otimes q_{\sigma(r)}\rangle q_{\sigma(1)}\otimes\ldots\otimes q_{\sigma(r)}.$$

Compared to the notation used in (2.1), we have rescaled V so that the A_i are bounded by 1 instead of M, but otherwise we can compare the norms as the diverse degrees are related by $\deg_U(q)\leqslant\frac{1}{2}\deg_X(q)$ and $\deg_{A,B}(q)=\deg_X(q)+\deg_B(q)$. In particular, the norm $\|V\|_{\xi,\zeta}$ used in this section can be compared to the norm $\|V\|_{M\xi^{1/2}\zeta,\zeta}$ used in (2.1). Once this is said, the two notions are sufficiently close that we keep the same notation.

The following is the main result of this section.

THEOREM 6.1. *Let $\beta=2$ (resp. $\beta=1$). Let $\{\alpha^i_j\}_{1\leqslant i\leqslant d,1\leqslant j\leqslant N}\subset[-1,+1]^{dN}$ and set*

$$L_i^N := \frac{1}{N}\sum_{j=1}^N \delta_{\alpha_j^i}.$$

Let $A_1, ..., A_d$ be Hermitian (resp. symmetric) matrices with eigenvalues $(\alpha^i_1, ..., \alpha^i_N)$, let $B_1, ..., B_m$ be Hermitian (resp. symmetric) matrices, and let

$$p\longmapsto \tau_B^N(p) := \frac{1}{N}\operatorname{Tr}\bigl(p(B_1 ..., B_k)\bigr)$$

be the non-commutative distribution of $B_1, ..., B_m$. Let $V\in\mathscr{L}^r_{\|\cdot\|_{\xi,\zeta}}$ be self-adjoint. Then, if $\|V\|_{\xi,\zeta}$ is finite for some ξ large enough and $\zeta\geqslant1$, there exists $a_0>0$ such that, for all $a\in[-a_0,a_0]$,

$$I_\beta^{N,aV}(A_1,...,A_d,B_1,...,B_m) = e^{\sum_{l=0}^2 N^{2-l}F_{l,\beta}^{aV}(L_1^N,...,L_d^N,\tau_B^N)}\left(1+O\left(\frac{1}{N}\right)\right),$$

where the error is uniform on the set of matrices satisfying (6.1) and F^{aV}_l are smooth functions on $\mathcal{P}([-1,1])^d\times\mathcal{L}(\mathscr{B})$. More precisely, for any $\ell\geqslant0$, the ℓ-th derivative of $F^{aV}_{l,\beta}$ at $\mu\in\mathcal{P}([-1,1])^d\times\mathcal{L}(\mathscr{B})$ in the direction ν is such that

$$|D^\ell F^{aV}_{l,\beta}[\mu](\nu)^{\otimes\ell}|\leqslant C_\ell|a|\,\|\nu\|^\ell_\zeta,$$

where C_ℓ is a finite constant, uniform with respect to μ.

The proof of this theorem is split over the next sections. For notational convenience, instead of adding a small parameter a in front of V we rather write down our hypotheses in terms of the smallness of the norms of V.

6.1. Integrals over the unitary or orthogonal group

The goal of this section is to prove Theorem 6.1. Recall that $\mathscr{L}_{\xi,\zeta}$ and $\mathscr{L}^r_{\xi,\zeta}$ denote the completion of $\mathscr{L}$ and $\mathscr{L}^{\otimes r}$, respectively, with respect to the norm $\|\cdot\|_{\xi,\zeta}$.

We shall prove Theorem 6.1 in two steps. First we extend the results of [37] to the case $\beta=1$ and $r\geqslant1$.

Proposition 6.2. *Let $\beta\in\{1,2\}$. Let τ^N_{AB} be the non-commutative distribution of $(A_1,...,A_d,B_1,...,B_m)$, that is, the linear form on $\mathscr{A}\mathscr{B}$ given by*

$$\tau^N_{AB}(p):=\frac{1}{N}\mathrm{Tr}(p(A_1,...,A_d,B_1,...,B_m))\quad \text{for all } p\in\mathscr{L}.$$

There exist $\xi_0>1$, $\zeta\geqslant 1$, and $\varepsilon_0>0$ such that if $\|V\|_{\xi_0,\zeta}\leqslant\varepsilon_0$ then, uniformly on the set of matrices $A_1,...,A_d$ and $B_1,...,B_m$ satisfying (6.1) and with respect to the dimension N, we have

$$I^{N,V}_\beta(A_1,...,A_d,B_1,...,B_m)=e^{N^2G^V_{0,\beta}(\tau^N_{AB})+NG^V_{1,\beta}(\tau^N_{AB})+G^V_{2,\beta}(\tau^N_{AB})}\bigg(1+O\bigg(\frac{1}{N}\bigg)\bigg),$$

where the $G^V_{l,\beta}$ are real-valued functions on $\mathscr{L}(\mathscr{A}\mathscr{B})$ and the error is uniform in the norm $\|\cdot\|_\zeta$.

Next, we show that the functions $\{G^V_{l,\beta}\}_{l=0}^2$ depend only on the spectral measures of the matrices A_i and on τ^N_B. More precisely, let $\mathcal{T}$ be the set of tracial states on $\mathscr{L}$, that is, the set of linear forms τ on $\mathscr{L}$ satisfying

$$\tau(pp^*)\geqslant 0,\quad \tau(pq)=\tau(qp),\quad \text{and}\quad \tau(1)=1. \tag{6.2}$$

Also, denote by $\mathcal{T}(\mathscr{B})\subset\mathscr{L}(\mathscr{B})$ the set of tracial states on $\mathscr{B}$.

Recall that, given $\nu=(\nu^1,...,\nu^{d+1})\in\mathcal{M}([-1,1])^d\times\mathcal{L}(\mathscr{B})$, we have

$$\|\nu\|_\zeta=\sum_{i=1}^d\|\nu^i\|_\zeta+\|\nu^{d+1}\|_\zeta,$$

where

$$\|\mu\|_\zeta=\begin{cases}\max_{k\geqslant 1}\zeta^{-k}|\nu(x^k)|, & \text{if } \mu\in\mathcal{P}([-1,1]),\\ \max_{i_1,...,i_k}\zeta^{-k}|\mu(B_{i_1}...B_{i_k})|, & \text{if } \mu\in\mathcal{T}(\mathscr{B}).\end{cases} \tag{6.3}$$

Lemma 6.3. *The functions $\{G^V_l\}_{l=0}^2$ are absolutely summable series whose coefficients depend only on τ^N_B and the moments*

$$L^N_i(x^k)=\frac{1}{N}\mathrm{Tr}[(A_i)^k],\quad 1\leqslant i\leqslant m,\ k\in\mathbb{N}.$$

In other words, there exists a function $F^V_{l,\beta}\colon\mathcal{P}([-1,1])^d\times\mathcal{T}(\mathscr{B})\to\mathbb{R}$ such that

$$G^V_{l,\beta}(\tau^N_{AB})=F^V_{l,\beta}(L^N_1,...,L^N_d,\tau^N_B).$$

Moreover, $F^V_{l,\beta}$ is Fréchet differentiable and its derivatives are bounded by

$$|D^\ell F^V_{l,\beta}[\mu](\nu_1,...,\nu_\ell)|\leqslant C_\ell\|\nu_1\|_\zeta...\|\nu_\ell\|_\zeta.$$

As in [35], [36], [18], [9], [37], the derivation of the expansion for large N of the free energy

$$F_\beta^{N,V}(A_1,...,A_d,B_1,...,B_m) := \frac{1}{N^2}\log I_\beta^{N,V}(A_1,...,A_d,B_1,...,B_m)$$

is based on the expansion of the function given, for any polynomial $p\in\mathscr{L}$, by

$$\mathcal{W}_{1N}^{V,\beta}(p) := \int \operatorname{Tr}(p(U_1,...,U_d,U_1^*,...,U_d^*,A_1,...,A_d,B_1,...,B_m))\, d\mathbb{Q}_\beta^{N,V}(U_1,...,U_d), \tag{6.4}$$

where $d\mathbb{Q}_\beta^{N,V}$ is the measure on $U(N)^d$ defined by

$$d\mathbb{Q}_\beta^{N,V}(U_1,...,U_d) := \frac{1}{I_\beta^{N,V}} e^{N^{2-r}\operatorname{Tr}^{\otimes r} V(U_1A_1U_1^*,...,U_dA_dU_d^*,B_1,...,B_m)}\, dU_1 \ldots dU_d. \tag{6.5}$$

The main step to prove Proposition 6.2 is the following large dimension expansion.

PROPOSITION 6.4. *Let $\beta=1$ (resp. $\beta=2$). Let $A_1,...,A_d$ be symmetric (resp. Hermitian) matrices with real eigenvalues $(\alpha_i^1,...,\alpha_i^N)_{i=1}^d$ and satisfying (6.1). Let V be a self-adjoint polynomial in $\mathscr{L}_{\xi,\zeta}^r$ for some $\xi>1$ and $\zeta\geqslant1$. There exist $\xi_0>1$, and $\varepsilon_0>0$ so that, if $\xi\geqslant\xi_0$ and $\|V\|_{\xi,\zeta}\leqslant\varepsilon_0$, then*

$$\mathcal{W}_{1N}^{V,\beta}(p) = N\tau_{10}^\beta(p)+\tau_{11}^\beta(p)+\frac{1}{N}\tau_{12}^\beta(p)+O\left(\frac{1}{N^2}\right) \quad \textit{for all } p\in\mathscr{L},$$

for some $\tau_{10}^\beta,\tau_{11}^\beta,\tau_{12}^\beta\in\mathscr{L}_{\xi,\zeta}$. Moreover, the error is uniform in $\|\cdot\|_{\xi,\zeta}$.

Notice that this result implies Proposition 6.2 provided we prove also the convergence of the second correlator $\mathcal{W}_{2N}^{V,\beta}$, see (6.8) and §6.2.1.

Hereafter we will drop the index β, but all our results will remain true both for $\beta=1$ and $\beta=2$.

The proof of Proposition 6.4 is based on Schwinger–Dyson's equation and a-priori concentration of measure properties, which depend on differentials acting on the space $\mathscr{L}$ of Laurent polynomial in letters $\{u_1,...,u_d,u_1^{-1},...,u_d^{-1},a_1,...,a_d,b_1,...,b_m\}$. Recall that $\mathscr{A}\mathscr{B}$ denotes the Laurent polynomial with degree zero, that is the linear span of words in $\{a_1,...,a_d,b_1...,b_m\}$. We now introduce some notation.

• The non-commutative derivative with respect to the ith variable u_i is defined by its action on monomials of $\mathscr{L}$:

$$\partial_i p := \sum_{p=p_1u_ip_2} p_1u_i\otimes p_2 - \sum_{p=p_1u_i^{-1}p_2} p_1\otimes u_i^{-1}p_2. \tag{6.6}$$

• The cyclic derivative with respect to u_i is defined as the endomorphism of $\mathscr{L}$ which acts on monomials according to

$$\mathcal{D}_i p := \sum_{p=p_1 u_i p_2} p_2 p_1 u_i - \sum_{p=p_1 u_i^{-1} p_2} u_i^{-1} p_2 p_1.$$

We can think of $\mathcal{D}_i$ as $\mathcal{D}_i = m \circ \partial_i$ with $m(p \otimes q) := qp$ for all $p, q \in \mathscr{L}$. We will set $\widetilde{m}(p \otimes q) := q^* p$.

Note that $\mathcal{D}_i$ appears naturally when differentiating the trace of a polynomial. More precisely, if we let $u_j(t) = u_j$ for $j \neq i$ and $u_i(t) = u_i e^{tB}$ then, for any Laurent polynomial p and any tracial state τ, we have

$$\frac{d}{dt}\bigg|_{t=0} \tau(p(u(t))) = \tau(\mathcal{D}_i p(u(0)) B).$$

As we shall apply it to differentiate quantities of the form $\mathrm{Tr}^{\otimes r} V(U(t))$, let us introduce the following notation: for $p \in \mathscr{L}^{\otimes r}$ with $p = p_1 \otimes p_2 \otimes \ldots \otimes p_r$ and a tracial state τ, we set

$$\mathcal{D}_{i,\tau} p := \sum_{k=1}^{r} \biggl(\prod_{j=1}^{k-1} \tau(p_j) \biggr) \mathcal{D}_i p_k \biggl(\prod_{j=k+1}^{r} \tau(p_j) \biggr).$$

Hence, if B is a anti-symmetric matrix (that is $B = -B^*$) and $U_j(t) = U_j e^{t \mathbf{1}_{j=i} B}$,

$$\frac{d}{dt}\Big|_{t=0} \frac{1}{N^r} \mathrm{Tr}^{\otimes r} V(U(t)) = \frac{1}{N} \mathrm{Tr}(B \mathcal{D}_{i,(1/N)\mathrm{Tr}} V).$$

• We will consider linear transformations

$$\mathsf{T}\colon (\mathscr{L}^{\otimes k_1}, \|\cdot\|_{\xi_1,\zeta}) \longrightarrow (\mathscr{L}^{\otimes k_2}, \|\cdot\|_{\xi_2,\zeta})$$

mapping between the various tensor powers of $\mathscr{L}$. A linear transformation

$$\mathsf{T}\colon \mathscr{L}^{\otimes k_1} \longrightarrow \mathscr{L}^{\otimes k_2}$$

is $(\xi_1, \xi_2; \zeta)$-continuous if and only if there exists a constant C such that

$$\|\mathsf{T}(p_1 \otimes \ldots \otimes p_{k_1})\|_{\xi_2,\zeta} \leqslant C \|p_1 \otimes \ldots \otimes p_{k_1}\|_{\xi_1,\zeta}$$

for all monomials $p_1 \otimes \ldots \otimes p_{k_1} \in \mathscr{L}^{\otimes k_1}$. The operator norm of T, denoted $\|\mathsf{T}\|_{\xi_1,\xi_2,\zeta}$, can be calculated by considering the smallest constant C for which the above inequality holds.

Allowing different instances of the ξ-norm on the source and target of our linear maps is useful for the following reason: certain linear transformations that we will need to deal with are not $(\xi, \xi; \zeta)$-continuous for any $\xi \geqslant 1$, but are $(\xi_1, \xi_2; \zeta)$-continuous, and even contractive, if the ratio ξ_1/ξ_2 is large enough. When $\xi_1=\xi_2$ we simplify the notation by putting only one index ξ.

- Recall that for ν a multilinear form on $\mathscr{L}^{\otimes k}$, we set

$$\|\nu\|_{\xi,\zeta}=\max_{\|p\|_{\xi,\zeta}\leqslant 1}|\nu(p)|,$$

and denote by $\mathcal{L}^{k,k'}_{\xi,\zeta}$ the set of linear maps from $(\mathscr{L}^{\otimes k},\|\cdot\|_{\xi,\zeta})$ into $(\mathscr{L}^{\otimes k'},\|\cdot\|_{\xi,\zeta})$, and $\mathcal{L}^{k}_{\xi,\zeta}$ denotes the set of linear maps from $(\mathscr{L}^{\otimes k},\|\cdot\|_{\xi,\zeta})$ into $\mathbb{C}$. Also, if $\mathscr{S}$ is a vector subspace of $(\mathscr{L}^{\otimes k},\|\cdot\|_{\xi,\zeta})$, then $\mathcal{L}(\mathscr{S})$ is the set of linear forms on $\mathscr{S}$ (if $\mathscr{S}=\mathscr{L}$, we simply denote it by $\mathcal{L}$). One can check that $\mathcal{L}^{k,k'}_{\xi,\zeta}$, $\mathcal{L}^{k}_{\xi,\zeta}$, and $\mathcal{L}(\mathscr{S})$ are Banach spaces (see for instance [37, Proposition 7] to see that $\|\cdot\|_{\xi,\zeta}$ is a vector space norm on $\mathscr{L}^{\otimes k}$, and in fact an algebra norm). We denote by $\mathcal{T}^{k}_{\xi,\zeta}$ the subset of tracial states on $(\mathscr{L}^{\otimes k},\|\cdot\|_{\xi,\zeta})$.

The basis of the Schwinger–Dyson equation is the following equation.

LEMMA 6.5. *Let V be a self-adjoint polynomial, $p\in\mathscr{L}$, and $i\in\{1,...,d\}$. Then*

$$\mathbb{E}\left[\frac{1}{N}\mathrm{Tr}\otimes\frac{1}{N}\mathrm{Tr}(\partial_i p)+\frac{1+\mathbf{1}_{\beta=1}}{N}\mathrm{Tr}(\mathcal{D}_{i,(1/N)\mathrm{Tr}}Vp)\right]=\mathbf{1}_{\beta=1}\frac{1}{N}\mathbb{E}\left[\frac{1}{N}\mathrm{Tr}(\widetilde{m}\circ\partial_i p)\right], \tag{6.7}$$

where $\mathbb{E}$ denotes the expectation under $\mathbb{Q}^V_{\beta,N}$ (see (6.5)).

Proof. We focus on the case $\beta=1$, the proof for $\beta=2$ is similar and detailed in [37] for the case $r=1$. This equation is derived by performing an infinitesimal change of variable $U_i\mapsto U_i(t):=U_ie^{tD_i}$, where D_i is a $N\times N$ matrix with real entries such that $D_i^*=-D_i$, and writing that for any polynomial function $p\in\mathscr{L}$, and any $k,\ell\in\{1,...,N\}$,

$$\begin{aligned}\frac{d}{dt}\bigg|_{t=0}\int p(U_1(t),...,U_d(t),U_1^*(t),...,U_d^*(t),A_1,...,A_d,B_1,...,B_m)_{k\ell}&\\ \times d\mathbb{Q}^V_{1,N}(U_1(t),...,U_d(t))=0.&\end{aligned}$$

Taking $D_j:=\mathbf{1}_{j=i}(\Delta(k,\ell)-\Delta(\ell,k))$, with $\Delta(k,\ell)$ the matrix with zero entries except at (k,ℓ) where the entry equals 1, and summing over $k,\ell\in\{1,...,N\}$, yields

$$\mathbb{E}\left[\frac{1}{N}\mathrm{Tr}\otimes\frac{1}{N}\mathrm{Tr}(\partial_i p)+\frac{1}{N}\mathrm{Tr}((\mathcal{D}_{i,(1/N)\mathrm{Tr}}V-(\mathcal{D}_{i,(1/N)\mathrm{Tr}}V)^*)p)\right]=\frac{1}{N}\mathbb{E}\left[\frac{1}{N}\mathrm{Tr}(\widetilde{m}\circ\partial_i p)\right].$$

The last thing to check is that $(\mathcal{D}_{i,(1/N)\mathrm{Tr}}V)^*=-\mathcal{D}_{i,(1/N)\mathrm{Tr}}V$. Indeed, it is enough to check it for $r=1$. Then, for all i and $p\in\mathscr{L}$ we have

$$\begin{aligned}\mathcal{D}_i p&=\sum\langle p,q\rangle\,\mathcal{D}_i q=\sum\langle p,q\rangle\left[\sum_{q=q_1u_iq_2}q_2q_1u_i-\sum_{q=q_1u_i^*q_2}u_i^*q_2q_1\right],\\ \mathcal{D}_i(p^*)&=\sum\langle p,q\rangle\left[-\sum_{q=q_1u_iq_2}u_i^*q_1^*q_2^*+\sum_{q=q_1u_i^*q_2}q_1^*q_2^*u_i\right]=-(\mathcal{D}_ip)^*.\end{aligned}$$

Since V is self-adjoint, the proof is complete. □

Equation (6.7) can be reinterpreted as a relation between the "correlators" $\mathcal{W}_{kN}^V$ defined as (see also (6.4))

$$\begin{aligned}\mathcal{W}_{kN}^V(p_1, ..., p_k) &:= \frac{d}{dt_1} \cdots \frac{d}{dt_k}\bigg|_{t_1=0,\dots,t_k=0} \log I_{\beta,N}^{V+(t_1/N)p_1+\dots+(t_k/N)p_k} \\ &= \frac{d}{dt_2} \cdots \frac{d}{dt_k}\bigg|_{t_2=0,\dots,t_k=0} \mathcal{W}_{1,N}^{V+(t_2/N)p_2+\dots+(t_k/N)p_k}(p_1).\end{aligned} \tag{6.8}$$

Notice that here the p_i's belong to $\mathscr{L}$, but we can identify them with $p_i \otimes \mathbf{1}^{\otimes(r-1)} \in \mathscr{L}^{\otimes r}$. Observe that we can always write the following expansion

$$\mathbb{E}\bigg[\prod_{j=1}^r \operatorname{Tr}(q_j)\bigg] = \prod_{j=1}^r \mathcal{W}_{1N}^V(q_j) + \sum_{j\neq k} \mathcal{W}_{2N}^V(q_j, q_k) \prod_{\ell\neq j,k} \mathcal{W}_{1N}^V(q_\ell) + R_N(q_1, ..., q_r)$$

where $R_N(q_1, ..., q_r)$ is a sum of products of correlators, each of which contains either a correlator of order at least 3, or two correlators of order 2. We define

$$\begin{aligned}\mathsf{S}_{V,\tau}^i p := \sum_{j=1}^r \sum \langle V, q_1 \otimes \dots \otimes q_r\rangle \sum_{k\neq j} \bigg[\bigg(\prod_{\ell\neq k,j} \tau(q_\ell)\bigg) \mathcal{D}_i q_j p \otimes q_k \\ + \sum_{k,\,j,\,m \text{ distinct}} \bigg(\prod_{\ell\neq k,j,m} \tau(q_\ell)\bigg) \tau(\mathcal{D}_i q_j p) q_m \otimes q_k\bigg].\end{aligned} \tag{6.9}$$

Using this expansion, we can rewrite (6.7) as follows.

COROLLARY 6.6. *Let V be a self-adjoint polynomial, $p\in\mathscr{L}$, and $i\in\{1, ..., d\}$. Then the first Schwinger–Dyson equation reads*

$$\begin{aligned}\frac{1}{N}\mathcal{W}_{1N}^V \otimes \frac{1}{N}\mathcal{W}_{1N}^V(\partial_i p) &+ \frac{1+\mathbf{1}_{\beta=1}}{N}\mathcal{W}_{1N}^V(\mathcal{D}_{i,(1/N)\mathcal{W}_{1N}} V p) \\ &= \frac{\mathbf{1}_{\beta=1}}{N^2}\mathcal{W}_{1N}^V(\widetilde{m}\circ\partial_i p) - \frac{1}{N^2}\mathcal{W}_{2N}^V(\partial_i p) \\ &\quad - \frac{\mathbf{1}_{r\geqslant 2}}{N^2}\mathcal{W}_{2N}^V(\mathsf{S}_{V,\frac{1}{N}\mathcal{W}_{1N}^V}^i p) + \frac{1}{N^r} R(\mathcal{W}_{1N}^V, ..., \mathcal{W}_{rN}^V : p),\end{aligned}$$

where R is a sum (independent of N) of products of correlators of polynomials extracted from p and V, each of which contains either a correlator of order at least 3, or two correlators of order 2.

To derive asymptotics from the Schwinger–Dyson equations we shall use a-priori upper bounds on the correlators $\mathcal{W}_{kN}^V$. The next result (proved in Appendix 8) is a direct consequence of concentration of measures and states as follows.

LEMMA 6.7. *Let $p_1, ..., p_k$ be monomials in $\mathscr{L}$. Then there exists a finite constant C_k, independent of N and the p_i's, such that, for $k \geqslant 2$,*

$$|\mathcal{W}_{kN}^V(p_1, ..., p_k)| \leqslant C_k \prod_{i=1}^{k} \deg_U(p_i) \quad \textit{and} \quad |\mathcal{W}_{1N}^V(p)| \leqslant N.$$

In particular, $\|\mathcal{W}_{kN}^V\|_{\xi,\zeta} \leqslant C_k(\max_{\ell\geqslant 1} \xi^{-\ell}\ell)^k$ is finite for all $\xi>1$, $\zeta \geqslant 1$, and $k \geqslant 2$, whereas $\|\mathcal{W}_{1N}^V(p)\|_{\xi,\zeta} \leqslant N$ for any $\xi, \zeta \geqslant 1$.

We now deduce the expansion of $\mathcal{W}_{1N}^V$ up to order $O(N^{-2})$, and of $\mathcal{W}_{2N}^V$ up to $O(N^{-1})$.

As $N^{-1}\mathcal{W}_{1N}^V(p)$ is bounded by 1 for all $p \in \mathscr{L}$, we deduce that $N^{-1}\mathcal{W}_{1N}^V$ has limit points. Let τ be such a limit point. As $N^{-1}\mathcal{W}_{2N}^V(\partial_i p)$ goes to zero for any polynomial $p \in \mathscr{L}$ (see Lemma 6.7), we deduce from the Schwinger–Dyson equation (see Corollary 6.6) that the limit point τ satisfies the limiting Schwinger–Dyson equation

$$\tau \otimes \tau(\partial_i p) + (1+\mathbf{1}_{\beta=1})\tau(\mathcal{D}_{i,\tau} V\, p) = 0 \quad \text{for all } p \in \mathscr{L}. \tag{6.10}$$

Hereafter we let

$$V_\beta := (1+\mathbf{1}_{\beta=1})V,$$

and we show uniqueness of the solutions to such an equation whenever τ restricted to $\mathscr{A}\mathscr{B}$ is prescribed, $\|\tau\|_{1,1} \leqslant 1$, and $\|V\|_{\xi,\zeta}$ is small enough. In our application $\tau_1 := \tau|_{\mathscr{A}\mathscr{B}}$ will simply be given by τ_{AB}^N, the non-commutative distribution of $(A_1, ..., A_d, B_1, ..., B_m)$. It could also be given by its limit, if any, but we prefer to take it dependent on the dimension N.

To show uniqueness, we apply the above equation to $p_i = \mathcal{D}_i q$ and sum over $i \in \{1, ..., d\}$. We will use that (see [37, Proposition 10])

$$\tau \otimes \tau\left(\sum_{i=1}^{d} \partial_i \mathcal{D}_i q\right) = \tau(\mathsf{D}q) + \tau \otimes \tau\left(\sum_{i=1}^{d} \Delta_i q\right), \tag{6.11}$$

where

- D is the degree operator: $\mathsf{D}p := \deg_U(p)p$;
- Δ_i acts on monomials according to

$$\Delta_i p := \partial_i \mathcal{D}_i p - \sum_{p=p_1 u_i p_2} p_2 p_1 u_i \otimes \mathbf{1} - \sum_{p=p_1 u_i^{-1} p_2} \mathbf{1} \otimes u_i^{-1} p_2 p_1,$$

that is,

$$\begin{aligned} \Delta_i p = & \sum_{p=p_1 u_i p_2} \left(\sum_{p_2 p_1 u_i = q_1 u_i q_2 u_i} q_1\, u_i \otimes q_2\, u_i - \sum_{p_2 p_1 u_i = q_1 u_i^{-1} q_2 u_i} q_1 \otimes q_2 \right) \\ & - \sum_{p=p_1 u_i^{-1} p_2} \left(\sum_{u_i^{-1} p_2 p_1 = u_i^{-1} q_1 u_i q_2} q_1 \otimes q_2 - \sum_{u_i^{-1} p_2 p_1 = u_i^{-1} q_1 u_i^{-1} q_2} u_i^{-1}\, q_1 \otimes u_i^{-1}\, q_2 \right), \end{aligned} \tag{6.12}$$

where the sum is over all possible decompositions as specified.

We write in short $\Delta:=\sum_{i=1}^d \Delta_i$, and we rewrite equation (6.10) as

$$\tau\big(\big(\mathsf{D}+\tfrac{1}{2}\mathsf{T}_\tau+\mathsf{P}_\tau^{V_\beta}\big)q\big)=0 \tag{6.13}$$

where T_τ and $\mathsf{P}_\tau^{V_\beta}$ are the following operators:

- T_τ arises as the analogue of the Laplacian:

$$\mathsf{T}_\tau:=(\mathsf{Id}\otimes\tau+\tau\otimes\mathsf{Id})\Delta.$$

- The operator $\mathsf{P}_\tau^{V_\beta}$ is the dot product of the cyclic gradient of V_β with the cyclic gradient of p:

$$\mathsf{P}_\tau^{V_\beta}p:=\mathcal{D}_\tau V_\beta\cdot\mathcal{D}p=\sum_{i=1}^d \mathcal{D}_{i,\tau}V_\beta\cdot\mathcal{D}_i p.$$

More generally, for linear forms $\tau_1,...,\tau_{r-1}$ on $\mathscr{L}$, we define

$$\mathsf{P}_{\tau_1,...,\tau_{r-1}}^{V_\beta}p:=\sum_{i=1}^d\sum_{j=1}^r\sum\langle V_\beta,q_1\otimes...\otimes q_r\rangle\Bigg(\prod_{k=1}^{j-1}\tau_k(q_k)\Bigg)\mathcal{D}_i q_j\cdot\mathcal{D}_i p\Bigg(\prod_{k=j+1}^{r}\tau_{k-1}(q_k)\Bigg).$$

When $r\geqslant 2$, we also define a companion operator $\mathsf{Q}_{\tau_1,...,\tau_{r-1}}^{V_\beta}$ to $\mathsf{P}_{\tau_1,...,\tau_{r-1}}^{V_\beta}$:

$$\begin{aligned}&\mathsf{Q}_{\tau_1,...,\tau_{r-1}}^{V_\beta}p\\ &\quad:=\sum_{i=1}^d\sum_{1\leqslant j<\ell\leqslant r}\sum\langle V_\beta,q_1\otimes...\otimes q_r\rangle\Bigg(\prod_{k\in\{j,\ell\}^c}\tau_{k-\mathbf{1}_{k>\ell}}(q_k)\Bigg)\tau_{j-\mathbf{1}_{j=r}}(\mathcal{D}_i q_j\cdot\mathcal{D}_i p)\,q_\ell.\end{aligned}$$

We set Π' (resp. Π) to be the orthogonal projection onto (resp. onto the complement of) the algebra $\mathscr{A}\mathscr{B}$ generated by $\{a_1,...,a_d,b_1,...,b_m\}$. For any linear transformation T with domain $\mathscr{L}$, we define its *degree regularization* by

$$\bar{\mathsf{T}}:=\mathsf{T}\mathsf{D}^{-1},$$

where D is the degree operator defined above. It is understood that the domain of the regularized operator $\bar{\mathsf{T}}$ is restricted to $(\mathscr{A}\mathscr{B})^\perp$. We recall that, for our applications, we assume that the restriction of τ to $\mathscr{A}\mathscr{B}$ is given and equal to τ_1, and therefore

$$\tau=\tau\Pi+\tau_1\Pi'.$$

Hence, we can see (6.13) as a fixed point equation for $\tau\in\mathcal{L}_{\xi,\zeta}$ given by

$$F[\tau;\tau_1,V_\beta]=0,\quad \tau|_{\mathscr{A}\mathscr{B}}=\tau_1, \tag{6.14}$$

where

$$F: \mathcal{L}_{\xi,\zeta} \times (\mathcal{T}(\mathscr{A}\mathscr{B}), \|\cdot\|_{\zeta}) \times (\mathscr{L}^{\otimes r}, \|\cdot\|_{\xi,\zeta}) \longrightarrow \mathcal{L}_{\xi,\zeta}$$

is given by $F[\tau;\tau_1,V_\beta] := G[\tau\Pi + \tau_1\Pi'; V_\beta]$ with

$$G[\tau;V_\beta](q) := \tau\big(\big(\mathsf{Id} + \tfrac{1}{2}\bar{\mathsf{T}}_\tau + \bar{\mathsf{P}}_\tau^{V_\beta}\big)\Pi q\big) \quad \text{for all } q \in \mathscr{L}_{\xi,\zeta} \text{ and } \tau \in \mathcal{L}_{\xi,\zeta}. \tag{6.15}$$

When $V=0$ and $\tau_1 \in \mathcal{T}(\mathscr{A}\mathscr{B})$, the equation $F[\tau;\tau_1,0]=0$ has a unique solution τ_{10}^{0,τ_1} since the moments of τ are defined recursively from those of τ_1. In this case, τ is the non-commutative distribution of $(\{a_i, u_i, u_i^*\}_{i=1}^d, \{b_j\}_{j=1}^m)$ so that $(a_1, ..., a_d, b_1, ..., b_m)$ has law τ_1, and is free from the d free unitary variables $(\{u_i, u_i^*\}_{i=1}^d)$, see [67] and [2, Theorem 5.4.10].

Observe that we know that solutions exist in $\mathcal{T}(\mathscr{A}\mathscr{B})$ as limit points of $N^{-1}\mathcal{W}_{N1}^V$ (which is tight in any $\mathcal{L}_{\xi,\zeta}$ by Lemma 6.7); we shall prove uniqueness of such solutions for V small by applying ideas similar to those of the implicit function theorem.

To state our result precisely, for $\xi>1$ and $\zeta \geqslant 1$ we define

$$\delta_{\xi,\zeta}(V) := \frac{8}{\xi-1} + \sum |\langle V_\beta, q_1 \otimes ... \otimes q_r\rangle| \Bigg(\sum_{j=1}^r \deg_U(q_j)\Bigg)\Bigg[\sum_{\ell=1}^r \xi^{\deg_U(q_\ell)}\zeta^{\deg_{A,B}(q_\ell)}\Bigg]. \tag{6.16}$$

Observe that for $\xi \geqslant \xi_0$, with ξ_0 sufficiently large so that

$$\frac{8}{\xi_0 - 1} \leqslant \frac{1}{2(1+\max\{2,r\})},$$

if $\|V\|_{\xi,\zeta}$ is finite one can choose a_0 small enough so that $\delta_{\xi,\zeta}(aV) < 1/(1+\max\{2,r\})$ for all $a \in [-a_0, a_0]$.

Lemma 6.8. *Assume that there exist $\zeta \geqslant 1$ and $\xi > 1$ such that*

$$\delta_{\xi,\zeta}(V) < \frac{1}{1+\max\{2,r\}}. \tag{6.17}$$

Then, for any law $\tau_1 \in \mathcal{T}(\mathscr{A}\mathscr{B})$, there exists a unique solution $\tau_{10}^{V,\tau_1} \in \mathcal{T} \cap \mathcal{L}_{\xi,\zeta}$ to

$$F[\,\cdot\,;\tau_1,V_\beta] = 0$$

such that $\tau|_{\mathscr{A}\mathscr{B}} = \tau_1$ and $\|\tau\|_{1,1} \leqslant 1$. Also, the map $\mathcal{T}(\mathscr{A}\mathscr{B}) \ni \tau_1 \mapsto \tau_{10}^{V,\tau_1} \in \mathcal{T}_{\xi,\zeta}$ is Fréchet differentiable at all orders, and its derivatives $D^\ell \tau_{10}^{V,\tau_1}$ satisfy, for any $\nu_1, ..., \nu_\ell \in \mathcal{L}_\zeta(\mathscr{A}\mathscr{B})$,

$$\|D^\ell \tau_{10}^{V,\tau_1}[\nu_1, ..., \nu_\ell]\|_{\xi,\zeta} \leqslant C_{\xi,\zeta,\ell} \|\nu_1\|_\zeta ... \|\nu_\ell\|_\zeta$$

for some finite constant $C_{\xi,\zeta,\ell}$. Finally,

$$\lim_{N\to\infty} \|N^{-1}\mathcal{W}_{1N} - \tau_{10}^{V,\tau_{AB}^N}\|_{\xi,\zeta} = 0.$$

Before proving Lemma 6.8, we need the following technical result.

Lemma 6.9. *Let $\xi>1$, $\tilde\xi\geqslant 1$ and $\zeta,\tilde\zeta\geqslant 1$. Then the following statements hold*:

- *Let* $\mathsf{f}\in\mathcal{L}_{\tilde\xi,\tilde\zeta}$ *and* $\xi>\tilde\xi$ *and* $\zeta\geqslant\tilde\zeta$. *Then*

$$\|\bar{\mathsf{T}}_{\mathsf{f}}\|_{\xi,\zeta}<8\|\mathsf{f}\|_{\tilde\xi,\tilde\zeta}\frac{\tilde\xi}{\xi-\tilde\xi}. \tag{6.18}$$

- *Let* $\mathsf{f}_1,...,\mathsf{f}_{r-1}\in\mathcal{L}$. *Then, for any* $V\in\mathscr{L}^r_{\xi,\zeta}$ *self-adjoint and any* $\tilde\xi,\tilde\zeta\geqslant 1$, *we have*

$$\big\|\bar{\mathsf{P}}^{V_\beta}_{\mathsf{f}_1,...,\mathsf{f}_{r-1}}\big\|_{\xi,\zeta}\leqslant\prod_{j=1}^{r-1}\|\mathsf{f}_j\|_{\tilde\xi,\tilde\zeta}\,\big\||\Pi V_\beta|\big\|_{\xi,\zeta,\tilde\xi,\tilde\zeta}, \tag{6.19}$$

with

$$\begin{aligned}\big\||\Pi V_\beta|\big\|_{\xi,\zeta,\tilde\xi,\tilde\zeta}:=&\sum|\langle V_\beta,q_1\otimes...\otimes q_r\rangle|\\&\times\sum_{j=1}^{r}\deg_U(q_j)\xi^{\deg_U(q_j)}\zeta^{\deg_{A,B}(q_j)}\tilde\xi^{\sum_{i\neq j}\deg_U(q_i)}\tilde\zeta^{\sum_{i\neq j}\deg_B(q_i)}.\end{aligned}$$

- *Let* $\mathsf{f}_1,...,\mathsf{f}_{r-1}\in\mathcal{L}$. *Then, for any* $V\in\mathscr{L}^r_{\xi,\zeta}$ *self-adjoint and any* $\tilde\xi,\tilde\zeta\geqslant 1$ *with* $\tilde\xi\leqslant\xi$ *and* $\tilde\zeta\leqslant\zeta$, *we have*

$$\|\bar{\mathsf{Q}}^{V_\beta}_{\mathsf{f}_1,...,\mathsf{f}_{r-1}}\|_{\xi,\zeta}\leqslant\prod_{j=1}^{r-1}\|\mathsf{f}_j\|_{\tilde\xi,\tilde\zeta}\big\||\Pi V_\beta|\big\|_{\xi,\zeta,\tilde\xi,\tilde\zeta;2}, \tag{6.20}$$

with

$$\begin{aligned}\big\||\Pi V_\beta|\big\|_{\xi,\zeta,\tilde\xi,\tilde\zeta;2}:=&\sum|\langle V_\beta,q_1\otimes...\otimes q_r\rangle|\\&\times\sum_{j\neq\ell}\tilde\xi^{\sum_{i\neq\ell}\deg_U(q_i)}\tilde\zeta^{\sum_{i\neq\ell}\deg_{A,B}(q_i)}\deg_U(q_j)\xi^{\deg_U(q_\ell)}\zeta^{\deg_{A,B}(q_\ell)}.\end{aligned}$$

- *Let* $\mathsf{f}_1,...,\mathsf{f}_r\in\mathcal{L}$, *and for* $V\in\mathscr{L}^r_{\xi,\zeta}$ *self-adjoint set*

$$\begin{aligned}&\mathsf{S}^V_{\mathsf{f}_1,...,\mathsf{f}_{r-2}}p\\&\quad:=\sum\langle V,q_1\otimes...\otimes q_r\rangle\\&\qquad\times\sum_{i=1}^{d}\sum_{j,k}\bigg[\bigg(\prod_{\ell\neq k,j}\mathsf{f}_{\ell-\mathbf{1}_{k\leqslant\ell}-\mathbf{1}_{j\leqslant\ell}}(q_\ell)\bigg)(1_{j<k}\mathcal{D}_iq_j\cdot\mathcal{D}_ip\otimes q_k+1_{k<j}q_k\otimes\mathcal{D}_iq_j\cdot\mathcal{D}_ip)\\&\qquad\quad+\sum_{s\neq j,k}\bigg(\prod_{\ell\neq k,j,s}\mathsf{f}_{\ell-\mathbf{1}_{k\leqslant\ell}-\mathbf{1}_{s\leqslant\ell}-\mathbf{1}_{m\leqslant\ell}}(q_\ell)\bigg)f_{r-2}(\mathcal{D}_iq_j\cdot\mathcal{D}_ip)\,q_s\otimes q_k\bigg].\end{aligned} \tag{6.21}$$

Then, we have

$$|\mathsf{f}_{r-1}\otimes\mathsf{f}_r(\bar{\mathsf{S}}^V_{\mathsf{f}_1,\dots,\mathsf{f}_{r-2}}(p))|\leqslant\prod_{j=1}^r\|\mathsf{f}_j\|_{\tilde\xi,\tilde\zeta}\sum_{k=r-2}^r\frac{\|\mathsf{f}_k\|_{\xi,\zeta}}{\|\mathsf{f}_k\|_{\tilde\xi,\tilde\zeta}}\big\||\Pi V_\beta|\big\|_{\xi,\zeta,\tilde\xi,\tilde\zeta;3}\|p\|_{\xi,\zeta},\tag{6.22}$$

where

$$\big\||\Pi V_\beta|\big\|_{\xi,\zeta,\tilde\xi,\tilde\zeta;3}=r\big\||\Pi V_\beta|\big\|_{\xi,\zeta,\tilde\xi,\tilde\zeta}+r\big\||\Pi V_\beta|\big\|_{\xi,\zeta,\tilde\xi,\tilde\zeta;2}.$$

Proof. The proof of (6.18) is done by considering term by term the norm of $1\otimes\mathsf{f}\Delta_i p$. For instance, if p has degree d_i in u_i and u_i^*, and $d=\deg_U(p)$, then we have that

$$\begin{aligned}
&\Big\|\sum_{p=p_1u_ip_2}\ \sum_{p_2p_1u_i=q_1u_iq_2u_1}q_1u_i\mathsf{f}(q_2u_i)\Big\|_{\xi,\zeta}\\
&\qquad\leqslant\|f\|_{\tilde\xi,\tilde\zeta}\sum_{p=p_1u_ip_2}\ \sum_{p_2p_1u_i=q_1u_iq_2u_1}\|q_1u_i\|_{\xi,\zeta}\|q_2u_i\|_{\tilde\xi,\tilde\zeta}\\
&\qquad\leqslant d_i\|f\|_{\tilde\xi,\tilde\zeta}\sum_{p=0}^{d-1}\xi^p\,\tilde\xi^{d-p}\zeta^{\deg_{AB}(p)}\\
&\qquad\leqslant d_i\|f\|_{\tilde\xi,\tilde\zeta}\|p\|_{\xi,\zeta}\frac{\tilde\xi}{\xi-\tilde\xi},
\end{aligned}$$

where we used $\zeta\geqslant\tilde\zeta$ and the fact that q_1 and q_2 have degree smaller than $d-1$. Proceeding for each term similarly (and noting a degree reduction of each term) yields the claim, after summing over i and dividing by d. More details are given in [37, Proposition 17] in the case $\zeta=1$.

We next prove (6.19). Take a monomial p in $(\mathscr{A}\mathscr{B})^\perp$. Then, with $\varepsilon,\varepsilon_j\in\{-1,1\}$,

$$\begin{aligned}
&\|\bar{\mathsf{P}}^{V_\beta}_{\mathsf{f}_1,\dots,\mathsf{f}_{r-1}}p\|_{\xi,\zeta}\\
&\quad=\Big\|\frac{1}{\deg_U(p)}\sum_{i=1}^d\sum\langle V_\beta,q_1\otimes\dots\otimes q_r\rangle\sum_{j=1}^r\Big(\prod_{k=1}^{j-1}\mathsf{f}_k(q_k)\Big)\,\mathcal{D}_iq_j\cdot\mathcal{D}_ip\Big(\prod_{k=j+1}^r\mathsf{f}_{k-1}(q_k)\Big)\Big\|_{\xi,\zeta}\\
&\quad\leqslant\frac{1}{\deg_U(p)}\sum_{i=1}^d\sum|\langle V_\beta,q_1\otimes\dots\otimes q_r\rangle|\sum_{j=1}^r\Big(\prod_{k=1}^{j-1}|\mathsf{f}_k(q_k)|\Big)\,\Big(\prod_{k=j+1}^r|\mathsf{f}_{k+1}(q_k)|\Big)\\
&\qquad\times\sum_{q_j=q_j^1u_i^{\varepsilon j}\,q_j^2}\ \sum_{p=p^1u_i^\varepsilon p^2}\|u_i^{-\mathbf{1}_{\varepsilon j=-1}}q_j^2q_j^1u_i^{\mathbf{1}_{\varepsilon j=1}}u_i^{-\mathbf{1}_{\varepsilon=-1}}p^2p^1u_i^{\mathbf{1}_{\varepsilon=1}}\|_{\xi,\zeta}\\
&\quad\leqslant\sum|\langle V_\beta,q_1\otimes\dots\otimes q_r\rangle|\sum_{j=1}^r\Big(\prod_{k=1}^{r-1}\|\mathsf{f}_k\|_{\tilde\xi,\tilde\zeta}\Big)\tilde\xi^{\sum_{i\neq j}\deg_U(q_i)}\tilde\zeta^{\sum_{i\neq j}\deg_B(q_i)}\\
&\qquad\times\deg_U(q_j)\,\xi^{\deg_U(p)+\deg_U(q_j)}\zeta^{\deg_{A,B}(p)+\deg_{A,B}(q_j)},
\end{aligned}$$

where we have used the facts that $\xi,\zeta\geqslant 1$, that the degree of

$$u_i^{-\mathbf{1}_{\varepsilon^j=-1}} q_j^2 q_j^1 u_i^{\mathbf{1}_{\varepsilon^j=1}} u_i^{-\mathbf{1}_{\varepsilon=-1}} p^2 p^1 u_i^{\mathbf{1}_{\varepsilon=1}}$$

is at most $\deg_U(p)+\deg_U(q_j)$ in the u_i's (and similarly in the a_i's and b_i's), and that the sum contained at most $\deg_U(p)\times\deg(q_j)$ terms. We thus obtain (6.19).

To prove (6.20) we note that $\|\bar{\mathsf{Q}}^{V_\beta}_{\mathsf{f}_1,\ldots,\mathsf{f}_{r-1}} p\|_{\xi,\zeta}$ is equal to

$$\begin{aligned}
&\biggl\| \frac{1}{\deg_U(p)} \sum_{i=1}^d \sum \langle V_\beta, q_1\otimes\ldots\otimes q_r\rangle \sum_{\ell\neq j} \biggl(\prod_{\substack{k=1\\k\neq\ell}}^{j-1} \mathsf{f}_k(q_k)\biggr) \mathsf{f}_j(\mathcal{D}_i q_j\cdot\mathcal{D}_i p) q_\ell \biggl(\prod_{\substack{k=j+1\\k\neq\ell}}^{r} \mathsf{f}_{k-1}(q_k)\biggr)\biggr\|_{\xi,\zeta}\\
&\qquad\leqslant \sum |\langle V_\beta, q_1\otimes\ldots\otimes q_r\rangle| \sum_{j\neq\ell} \biggl(\prod_{k=1}^{r-1} \|\mathsf{f}_k\|_{\tilde\xi,\tilde\zeta}\biggr) \tilde\xi^{\sum_{i\neq\ell}\deg_U(q_i)+\deg_U(p)}\\
&\qquad\qquad\times \tilde\zeta^{\sum_{i\neq\ell}\deg_B(q_i)+\deg_{A,B}(p)} \deg_U(q_j) \xi^{\deg_U(q_\ell)} \zeta^{\deg_B(q_\ell)}\\
&\qquad\leqslant \bigl\||\Pi V_\beta|\bigr\|_{\xi,\zeta,\tilde\xi,\tilde\zeta;2} \|p\|_{\xi,\zeta},
\end{aligned}$$

where we used in the last line that $\tilde\xi\leqslant\xi$ and $\tilde\zeta\leqslant\zeta$. The bound (6.22) is analogous and left to the reader. $\square$

Proof of Lemma 6.8. Following the implicit function theorem, let us consider F as a function from $X\times Y$ to Y, with $X:=\mathcal{L}(\mathscr{A}\mathscr{B})_\zeta\times\mathscr{L}^r_{\xi,\zeta}$ and $Y:=\mathcal{L}(\mathscr{A}\mathscr{B}^\perp)_{\xi,\zeta}$. (Here $\mathcal{L}(\mathscr{A}\mathscr{B}^\perp)$ is the set of linear functionals over $\mathscr{A}\mathscr{B}^\perp$. Even though $\mathscr{A}\mathscr{B}^\perp$ is not an algebra, this is a well-defined Banach space once equipped with $\|\cdot\|_{\xi,\zeta}$.)

Recall that F has a unique solution τ_{10}^{0,τ_1} on the subset $\mathcal{T}(\mathscr{A}\mathscr{B})\times\{0\}$ of X, given by the law of free variables, as discussed above. To show that this unique solution extends to a neighborhood of $\mathcal{T}(\mathscr{A}\mathscr{B})\times\{0\}$, it is enough to check that F is differentiable along the variable $\tau\in Y$, and its derivative is a Banach space isomorphism from $\mathcal{L}(\mathscr{A}\mathscr{B}^\perp)_{\xi,\zeta}$ into $\mathcal{L}(\mathscr{A}\mathscr{B}^\perp)_{\xi,\zeta}$ at $(\tau_1,0)$. But this is clear as for any $q\in\mathscr{A}\mathscr{B}^\perp$,

$$DF[\tau;\tau_1,V_\beta](\mu;0)(q):=\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}(F[\tau+\varepsilon\mu;\tau_1,V_\beta]-F[\tau;\tau_1,V_\beta])(q)=\mu((\mathsf{Id}+\Pi[\bar{\mathsf{T}}_{\tau_{10}^{0,\tau_1}}])q),$$

where $\mathsf{Id}+\Pi\bar{\mathsf{T}}_{\tau_{10}^{0,\tau_1}}$ is invertible, as a triangular operator. Hence, by the implicit function theorem there exists a unique solution of $F(\tau;\tau_1,V_\beta)$ for $\|V_\beta\|_{\xi,\zeta}$ small enough and $\tau_1\in\mathcal{T}(\mathscr{A}\mathscr{B})$. However, for further use we shall reprove this result "by hand".

If τ and τ' are two solutions of (6.14) we see that $\delta:=\tau-\tau'$ satisfies

$$\delta((\mathsf{Id}+\Xi^V_{\tau,\tau_1})p)=\delta\otimes\delta(\bar\Delta p+\mathsf{R}^V_{\tau,\delta}p), \tag{6.23}$$

where

$$\Xi^V_{\tau,\tau_1}:=\Pi[\bar{\mathsf{T}}_{\tau\Pi+\tau_1\Pi'}+\bar{\mathsf{P}}^{V_\beta}_{\tau\Pi+\tau_1\Pi'}+\bar{\mathsf{Q}}^{V_\beta}_{\tau\Pi+\tau_1\Pi'}],$$

$$\mathsf{R}^V_{\tau,\delta} := -\int_0^1 \bar{\mathsf{S}}_{V,\tau'+s\delta}\, s\, ds \quad \text{and} \quad \mathsf{S}_{V,\tau}(p) := \sum_{i=1}^d \mathsf{S}^i_{V,\tau}(\mathcal{D}_i p)$$

where $\mathsf{S}^i_{V,\tau}$ is defined in (6.9). Indeed, this follows by the identity

$$\tau\otimes\tau-\tau'\otimes\tau' = \delta\otimes\tau+\tau\otimes\delta-\delta\otimes\delta$$

and the expansion

$$\begin{aligned}
&\tau(\bar{\mathsf{P}}^{V_\beta}_{\tau\Pi+\tau_1\Pi'}p)-\tau'(\bar{\mathsf{P}}^{V_\beta}_{\tau'\Pi+\tau_1\Pi'}p)\\
&\quad= \int_0^1 \frac{d}{ds}((\tau'+s\delta)(\bar{\mathsf{P}}^{V_\beta}_{(\tau'+s\delta)\Pi+\tau_1\Pi'}p))\, ds\\
&\quad= \delta\biggl(\int_0^1 (\Pi[\bar{\mathsf{P}}^{V_\beta}_{(\tau'+s\delta)\Pi+\tau_1\Pi'}+\bar{\mathsf{Q}}^{V_\beta}_{(\tau'+s\delta)\Pi+\tau_1\Pi'}]p)\, ds\biggr)\\
&\quad= \delta(\Pi[\bar{\mathsf{P}}^{V_\beta}_{\tau\Pi+\tau_1\Pi'}+\bar{\mathsf{Q}}^{V_\beta}_{\tau\Pi+\tau_1\Pi'}]p)+\delta\otimes\delta\biggl(\int_0^1\int_s^1 \Pi(\bar{\mathsf{S}}_{V,\tau'+\sigma\delta}p)\, d\sigma\, ds\biggr)\\
&\quad= \delta(\Pi[\bar{\mathsf{P}}^{V_\beta}_{\tau\Pi+\tau_1\Pi'}+\bar{\mathsf{Q}}^{V_\beta}_{\tau\Pi+\tau_1\Pi'}]p)+\delta\otimes\delta\biggl(\int_0^1 \sigma\Pi(\bar{\mathsf{S}}_{V,\tau'+\sigma\delta}p)\, d\sigma\biggr),
\end{aligned}$$

which proves the desired formula noticing that $\delta=\delta\circ\Pi$.

We next claim that $\mathsf{Id}+\Xi^V_{\tau,\tau_1}$ is invertible and with bounded inverse in

$$((\mathscr{A}\mathscr{B})^\perp, \|\cdot\|_{\xi,\zeta}).$$

We begin by noticing that (6.18), (6.19), and (6.20) imply the following: if $\tau,\tau_1\in\mathcal{T}$, as $\tau\Pi+\tau_1\Pi'$ is a tracial state which has $\|\cdot\|_{1,1}$ norm bounded by 1, we have (by taking $\tilde\xi=\tilde\zeta=1$)

$$\|\Xi^V_{\tau,\tau_1}\|_{\xi,\zeta} \leqslant \frac{8}{\xi-1}+\bigl\||\Pi V_\beta|\bigr\|_{\xi,\zeta,1,1}+\bigl\||\Pi V_\beta|\bigr\|_{\xi,\zeta,1,1;2} = \delta_{\xi,\zeta}(V) \tag{6.24}$$

(see (6.16)). Therefore, since $\delta_{\xi,\zeta}(V)<1$ (by (6.17)), it follows that $\mathsf{Id}+\Xi^V_{\tau,\tau_1}$ is invertible on $(\mathcal{L}((\mathscr{A}\mathscr{B})^\perp), \|\cdot\|_{\xi,\zeta})$, with inverse bounded by $(1-\delta_{\xi,\zeta}(V))^{-1}$.

By (6.18) and because $\|\delta\|_{1,1}\leqslant\|\tau\|_{1,1}+\|\tau'\|_{1,1}\leqslant 2$, as well as $\|\tau'+s\delta\|_{1,1}\leqslant 1$,

$$|\delta\otimes\delta(\bar\Delta p)| = |\delta(\mathsf{T}_\delta p)| \leqslant \frac{16}{\xi-1}\|\delta\|_{\xi,\zeta}\, \|p\|_{\xi,\zeta},$$

and similarly, by (6.22), we find that for $\xi,\zeta\geqslant 1$, since $\|p\|_{1,1}\leqslant\|p\|_{\xi,\zeta}$,

$$|\delta\otimes\delta(\mathsf{R}^V_{\tau,\delta}(p))| \leqslant \bigl\||\Pi V_\beta|\bigr\|_{\xi,\zeta,1,1;3}\|\delta\|_{\xi,\zeta}\, \|p\|_{\xi,\zeta}.$$

It follows from (6.24) and (6.23) that

$$\|\delta\|_{\xi,\zeta} \leqslant \frac{\max\{2,r\}}{1-\delta_{\xi,\zeta}(V)}\delta_{\xi,\zeta}(V)\|\delta\|_{\xi,\zeta},$$

and recalling (6.17) we conclude that $\|\delta\|_{\xi,\zeta}=0$, that is $\tau=\tau'$ as desired.

We let τ_{10}^{V,τ_1} denote our unique solution. Notice that if τ_1 is not necessarily a tracial state, but an element of $\mathscr{L}_{\xi,\zeta}$ which still satisfies $\|\tau_1\|_1\leqslant 1$ and such that $\|\tau_1-\tau_1^0\|_\zeta\leqslant\varepsilon$ for some $\tau_1^0\in\mathcal{T}(\mathscr{A}\mathscr{B})$ with ε small enough, then the very same argument as before shows that there exists a unique τ_{10}^{V,τ_1} in a small neighborhood of τ_{10}^{V,τ_1^0} solving (6.7).

By the implicit function theorem, since the function F is smooth, the solution τ_{10}^{V,τ_1} is smooth both in V and τ_1. For $\nu_1, ..., \nu_\ell\in\mathcal{L}_{\xi,\zeta}$, we denote by $D^\ell\tau_{0,1}^{V,\tau_1}$ the ℓth derivative of $\tau_{0,1}^{V,\tau_1}$ with respect to τ_1, which is given by

$$D^\ell\tau_{0,1}^{V,\tau_1}[\nu_1, ..., \nu_\ell]=\frac{d}{d\varepsilon_1}\ ...\ \frac{d}{d\varepsilon_\ell}\bigg|_{\varepsilon_1=0,...,\varepsilon_\ell=0}\big[\tau_{0,1}^{V,\tau_1+\sum_i\varepsilon_i\nu_i}\big],$$

and is defined inductively by the formula, valid for all $q\in(\mathscr{A}\mathscr{B})^\perp$,

$$D^1\tau_{0,1}^{V,\tau_1}[\nu]\big(\big(\mathsf{Id}+\Xi^V_{\tau_{01}^{V,\tau_1}}\big)q\big)=-\nu\big(\Pi'\big[\overline{\mathsf{T}}_{\tau_{01}^{V,\tau_1}\Pi+\tau_1\Pi'}+\bar{\mathsf{P}}^{V_\beta}_{\tau_{01}^{V,\tau_1}\Pi+\tau_1\Pi'}+\bar{\mathsf{Q}}^{V_\beta}_{\tau_{01}^{V,\tau_1}\Pi+\tau_1\Pi'}\big]q\big), \tag{6.25}$$

where we use the simplified notation $\Xi^V_{\tau_{01}^{V,\tau_1}}=\Xi^V_{\tau_{01}^{V,\tau_1},\tau_1}$. Hence, if we set $K=\{1, ..., \ell\}$ and $D_I\tau:=D^{|I|}\tau_{0,1}^{V,\tau_{AB}^N}[\nu_i, i\in I]$, then

$$\begin{aligned}D_K\tau((\mathsf{Id}+\Xi^V_{\tau_{01}^{V,\tau_1}})q)=-\frac{1}{2}&\sum_{\substack{I\cup J=K\\ I,J\neq\varnothing}}D_I\tau\otimes D_J\tau(\bar\Delta p)\\ &-\sum_{i=1}^{\ell}(\nu_i\otimes D_{K\setminus\{i\}}\tau(\bar\Delta p)-\mathbf{1}_{\ell=2}\nu_i\otimes\nu_{K\setminus\{i\}}(\bar\Delta p))\\ &-\sum_{J_i}\sum_{\theta_i\in D_{J_i}\tau}\theta_1(\bar{\mathsf{P}}^V_{\theta_2,...,\theta_r}q),\end{aligned}\tag{6.26}$$

where in the last term we sum over all choices J_i, with $\bigcup_{i=1}^r J_i=K$ and $J_1\neq\varnothing, K$, and all θ_i in the set

$$D_{J_i}\tau=\begin{cases}\nu_{J_i}, & \text{if } |J_i|=1,\\ \tau, & \text{if } J_i=\varnothing.\end{cases}$$

From this formula and the invertibility of $\mathsf{Id}+\Xi^V_{\tau_{01}^{V,\tau_1}}$, we deduce by induction that for all ξ satisfying (6.17) and for all $\ell\in\mathbb{N}$, there exists a finite constant $C_{\xi,\zeta,\ell}$ such that

$$\|D^\ell\tau_{10}^{V,\tau_1}[\nu_1, ..., \nu_\ell]\|_{\xi,\zeta}\leqslant C_{\xi,\zeta,\ell}\|\nu_1\|_\zeta\ ...\ \|\nu_\ell\|_\zeta.$$

Finally, we apply the above uniqueness result with $\tau_1:=\tau_{AB}^N$, that is, to the non-commutative distribution of $(A_1, ..., A_d, B_1, ..., B_m)$; see Proposition 6.2. Indeed, by the discussion after Lemma 6.7, any limit point of $N^{-1}\mathcal{W}_{1N}^V\in\mathcal{L}_{\xi,\zeta}$ satisfies the limiting Schwinger–Dyson equation, so this lemma ensures that this limit is unique and that $N^{-1}\mathcal{W}_{1N}^V$ converge to $\tau_{10}^{V,\tau_{AB}^N}$, which concludes the proof. □

In order to simplify the notation, we use τ_{10} to denote $\tau_{10}^{V,\tau_{AB}^N}$. We next develop similar arguments to expand $\mathcal{W}_{1N}^V$ as a function of N^{-1}. Let us first consider the first error term and rewrite the first Schwinger–Dyson equation by taking $P=\mathcal{D}_i p$ in Corollary 6.6. Summing over i, we get $\delta_N:=\mathcal{W}_{1N}^V-N\tau_{10}$,

$$\delta_N((\mathsf{Id}+\bar{\mathsf{T}}_{\tau_{10}}+\bar{\mathsf{P}}_{\tau_{10}}^{V_\beta}+\bar{\mathsf{Q}}_{\tau_{10}}^{V_\beta})p)=\frac{\mathbf{1}_{\beta=1}}{N}\mathcal{W}_{1N}^V(\tilde{\Delta}p)-\frac{1}{N}\mathcal{W}_{2N}^V(\bar{\Delta}p)+R_N(p), \tag{6.27}$$

where

$$\tilde{\Delta}:=\sum_{i=1}^d \widetilde{m}\circ\partial_i\mathcal{D}_i\mathsf{D}^{-1}$$

and $R_N(p)$ contains the terms which are at least quadratic in δ_N, or depending on cumulants of order greater than or equal to 2:

$$\begin{aligned}
R_N(p):=&-\delta_N(\bar{\mathsf{T}}_{N^{-1}\delta_N}p)\\
&-\frac{1}{N^{r-1}}\sum_{i=1}^d\sum_{k=1}^r\sum\langle V_\beta,q_1\otimes\ldots\otimes q_r\rangle\\
&\quad\times\sum_{\substack{I\subset\{1,\ldots,r\}\setminus k\\|I|\geqslant 1}}\delta_N(\mathcal{D}_i q_k\cdot\mathcal{D}_i\mathsf{D}^{-1}p)\biggl(\prod_{j\in I}\delta_N(q_j)\biggr)\biggl(\prod_{j\in(I\cup k)^c}\mathcal{W}_{1N}^V(q_j)\biggr)\\
&-\frac{1}{N^{r-1}}\sum_i\sum\langle V_\beta,q_1\otimes\ldots\otimes q_r\rangle\\
&\quad\times\sum_{\substack{I_1\cup I_2\cup\ldots\cup I_k=\{1,\ldots,r\}\\k\leqslant r-1}}\mathcal{W}_{|I_1|N}^V(\mathcal{D}_i q_{i_1}\cdot\mathcal{D}_i\mathsf{D}^{-1}p,\{q_j\}_{j\in I_1\setminus\{i_1\}})\\
&\quad\times\prod_{\ell=2}^k\mathcal{W}_{|I_\ell|N}(\{q_s\}_{s\in I_\ell}),
\end{aligned}$$

where in the above sum at least one set I_j has at least two elements.

In order to control the right-hand side of (6.27) we use the following estimate (compare with [37, Proposition 18]).

LEMMA 6.10. *For any $\zeta\geqslant 1$ and $\xi_1>\xi_2$, the operator $\bar{\Delta}$ is a bounded mapping from $((\mathscr{A}\mathscr{B})^\perp,\|\cdot\|_{\xi_1,\zeta})$ into $(\mathscr{L}^{\otimes 2},\|\cdot\|_{\xi_2,\zeta})$. Moreover $\tilde{\Delta}$ is a bounded mapping from $(\mathscr{L}((\mathscr{A}\mathscr{B})^\perp),\|\cdot\|_{\xi_1,\zeta})$ into $(\mathscr{L},\|\cdot\|_{\xi_2,\zeta})$.*

The proof of this result simply follows by using (6.12) and noticing that there exists a constant $C_{\xi_1,\xi_2}>1$ such that $n\xi_2^n\leqslant C_{\xi_1,\xi_2}\xi_1^n$ for all $n\geqslant 0$: one deduces that, for any monomial p,

$$\|\bar{\Delta}p\|_{\xi_2,\zeta}\leqslant\deg_U(p)\xi_2^{\deg_U(p)}\zeta^{\deg_{A,B}(p)}\leqslant C_{\xi_1,\xi_2}\xi_1^{\deg_U(p)}\zeta^{\deg_{A,B}(p)}=C_{\xi_1,\xi_2}\|p\|_{\xi_1,\zeta}.$$

The proof for $\tilde{\Delta}$ is similar.

Next, we prove the following convergence result for δ_N.

LEMMA 6.11. *Assume that there exist* $\xi_2<\xi_1$ *and* $\zeta\geqslant 1$ *such that, for both* $\xi=\xi_1$ *and* $\xi=\xi_2$,

$$\delta_{\xi,\zeta}(V)<\frac{1}{1+\max\{2,r\}}.$$

Then, for any $p\in\mathscr{L}_{\xi_1,\zeta}$, *we have*

$$\lim_{N\to\infty}\delta_N(p)=\mathbf{1}_{\beta=1}\tau_{10}(\tilde{\Delta}(\mathsf{Id}+\bar{\mathsf{T}}_{\tau_{10}}+\bar{\mathsf{P}}^{V_\beta}_{\tau_{10}}+\bar{\mathsf{Q}}^{V_\beta}_{\tau_{10}})^{-1}p)=:\tau_{11}(p),$$

and $N\|\delta_N-\tau_{11}\|_{\xi_1,\zeta}$ *is uniformly bounded in* N.

Proof. First notice that for $\xi=\xi_1$ or $\xi=\xi_2$, our hypothesis ensures that

$$\Psi^{V_\beta}_\tau:=\mathsf{Id}+\bar{\mathsf{T}}_{\tau_{10}}+\bar{\mathsf{P}}^{V_\beta}_{\tau_{10}}+\bar{\mathsf{Q}}^{V_\beta}_{\tau_{10}}$$

is invertible in $\mathscr{L}_{\xi,\zeta}$ with norm smaller than $(1-\delta_{\xi,\zeta}(V))^{-1}$ (see the proof of Lemma 6.8). Therefore, it follows from (6.27) that, for $p\in(\mathscr{A}\mathscr{B})^\perp$,

$$\delta_N(p)=\frac{1}{N}\mathcal{W}^V_{1N}(\tilde{\Delta}(\Psi^{V_\beta}_{\tau_{10}})^{-1}p)-\frac{1}{N}\mathcal{W}^V_{2N}(\bar{\Delta}(\Psi^{V_\beta}_{\tau_{10}})^{-1}p)+R_N((\Psi^{V_\beta}_{\tau_{10}})^{-1}p). \tag{6.28}$$

We next bound each term separately. For the first one, we get

$$\begin{aligned}\left|\frac{1}{N}\mathcal{W}^V_{1N}(\tilde{\Delta}(\Psi^{V_\beta}_{\tau_{10}})^{-1}p)\right| &\leqslant \left\|\frac{1}{N}\mathcal{W}^V_{1N}\right\|_{\xi_2,\zeta}\|\tilde{\Delta}(\Psi^{V_\beta}_{\tau_{10}})^{-1}p\|_{\xi_2,\zeta}\\ &\leqslant \left\|\frac{1}{N}\mathcal{W}^V_{1N}\right\|_{\xi_2,\zeta}\|\tilde{\Delta}\|_{\xi_2,\xi_1,\zeta}\,\|(\Psi^{V_\beta}_{\tau_{10}})^{-1}p\|_{\xi_1,\zeta}\\ &\leqslant \left\|\frac{1}{N}\mathcal{W}^V_{1N}\right\|_{\xi_2,\zeta}\|\tilde{\Delta}\|_{\xi_2,\xi_1,\zeta}\,\|(\Psi^{V_\beta}_{\tau_{10}})^{-1}\|_{\xi_1,\zeta}\,\|p\|_{\xi_1,\zeta}.\end{aligned}$$

A similar bound holds for the second term. For R_N, note first that (6.18) with $\tilde{\xi}=\xi_2$ yields

$$|\delta_N(\bar{\mathsf{T}}_{N^{-1}\delta_N}p)|\leqslant 8N^{-1}\frac{\xi_2}{\xi_1-\xi_2}\|\delta_N\|_{\xi_2,\zeta}\,\|\delta_N\|_{\xi_1,\zeta}\,\|p\|_{\xi_1,\zeta},$$

and noticing that similar bounds hold for the other terms in R_N, we obtain

$$\begin{aligned}\|\delta_N\|_{\xi_1,\zeta}\leqslant &\left\|\frac{1}{N}\mathcal{W}^V_{1N}\right\|_{\xi_2,\zeta}\|\tilde{\Delta}\|_{\xi_2,\xi_1,\zeta}\,\|(\Psi^{V_\beta}_{\tau_{10}})^{-1}\|_{\xi_1,\zeta}\\ &+\left\|\frac{1}{N}\mathcal{W}^V_{2N}\right\|_{\xi_2,\zeta}\|\bar{\Delta}\|_{\xi_2,\xi_1,\zeta}\|(\Psi^{V_\beta}_{\tau_{10}})^{-1}\|_{\xi_1,\zeta}\\ &+C(1+\|(\Psi^{V_\beta}_{\tau_{10}})^{-1}\|_{\xi_1,\zeta})\frac{1}{N}\|\delta_N\|_{\xi_2,\zeta}\,\|\delta_N\|_{\xi_1,\zeta},\end{aligned}$$

where we bounded the last term using Lemma 6.7. As $N^{-1}\|\delta_N\|_{\xi_2,\zeta}\to 0$ (see Lemma 6.8), for N sufficiently large we can reabsorb the last term and deduce that $\|\delta_N\|_{\xi_1,\zeta}$ is bounded.

Moreover, this implies also that the last term is of order N^{-1}. In addition, the second term is of order N^{-1} by Lemma 6.7. Hence, going back to (6.28) we see that the first term in the right-hand side converges towards the desired limit by Lemma 6.8, provided $\tilde{\Delta}(\Psi^{V_\beta}_{\tau_{10}})^{-1}p\in\mathscr{L}_{\xi_2,\zeta}$, which is true as soon as $p\in\mathscr{L}_{\xi_1,\zeta}$ (see Lemma 6.10).

Finally, to prove the last statement, it is enough to notice that the above reasoning implies that $\|\delta_N\|_{\xi_3,\zeta}$ is bounded for some $\xi_3\in(\xi_2,\xi_1)$ (notice that the assumption on $\delta_{\xi_3,\zeta}$ still holds for ξ_3 close enough to ξ_2 or ξ_1 by continuity of $\delta_{\cdot,\zeta}$) so that the previous arguments (in particular the fact that $\mathcal{W}^V_{2N}$ and R_N are bounded) imply that there exists a finite constant C such that

$$N\|\delta_N-\tau_{11}\|_{\xi_1,\zeta}\leqslant C\|\delta_N\|_{\xi_3,\zeta}\|\tilde{\Delta}\|_{\xi_3,\xi_1,\zeta}\|(\Psi^{V_\beta}_{\tau_{10}})^{-1}\|_{\xi_1,\zeta}+C$$

which concludes the proof. □

The second-order correction to $\mathcal{W}^V_{1N}$ depends on the limit of $\mathcal{W}^V_{2N}$ that we now derive by using the second Schwinger–Dyson equation. The latter is simply derived from the first Schwinger–Dyson equation (see Lemma 6.5), by changing the potential V into $V+tq\otimes\mathbf{1}^{r-1}$ and differentiating with respect to t at $t=0$. This results in the equation, valid for all $p,q\in\mathscr{L}$,

$$\begin{aligned}\mathbb{E}\bigg[(\operatorname{Tr} q-\mathbb{E}[\operatorname{Tr} q])\bigg(\frac{1}{N}\operatorname{Tr}\otimes\frac{1}{N}\operatorname{Tr}(\partial_i p)+\frac{1+\mathbf{1}_{\beta=1}}{N}\operatorname{Tr}((\mathcal{D}_{i,(1/N)\operatorname{Tr}}V)p)\bigg)\bigg]&\\ +\frac{1+\mathbf{1}_{\beta=1}}{N}\mathbb{E}[\operatorname{Tr}((\mathcal{D}_i q)\,p)]&\\ =\frac{1}{N}\mathbb{E}\bigg[(\operatorname{Tr} q-\mathbb{E}[\operatorname{Tr} q])\bigg(\frac{1}{N}\operatorname{Tr}(\widetilde{m}\circ\partial_i p)\bigg)\bigg].&\end{aligned}$$

We next rearrange the above expression in terms of correlators $\mathcal{W}^V_{kN}$, $k=1,2$, replace p by $\mathcal{D}_i p$, and sum over i, to deduce the second Schwinger–Dyson equation:

$$\mathcal{W}^V_{2N}(q,p)=-\frac{1+\mathbf{1}_{\beta=1}}{N}\mathcal{W}^V_{1N}(\bar{\mathsf{P}}^q_{\tau_{10}}(\Psi^{V_\beta}_{\tau_{10}})^{-1}p)+\widehat{R}_N((\Psi^{V_\beta}_{\tau_{10}})^{-1}p),$$

where $\widehat{R}_N$ only depends on correlators of order greater than or equal to 3, or on δ_N to a power greater than or equal to 3. We can therefore see that $\widehat{R}_N$ will be negligible provided $(\Psi^{V_\beta}_{\tau_{10}})^{-1}p$ belongs to a space in which all the previous convergences hold. This allows us to prove the following lemma.

LEMMA 6.12. *Let $\zeta\geqslant 1$. Assume there exist $1<\xi_3<\xi_2<\xi_1$ such that, for $\xi=\xi_1,\xi_2,\xi_3$,*

$$\delta_{\xi,\zeta}(V)<\frac{1}{1+\max\{2,r\}}.$$

Then, for any $p, q \in \mathscr{L}_{\xi_1,\zeta}$, *we have*

$$\lim_{N\to\infty} \mathcal{W}_{2N}^V(p,q) = -(1+\mathbf{1}_{\beta=1})\tau_{10}(\bar{\mathsf{P}}^q_{\tau_{10}}(\Psi^{V_\beta}_{\tau_{10}})^{-1}p) =: \tau_{20}(p,q),$$

and $N\|\mathcal{W}_{2N}^V - \tau_{20}\|_{\xi_1,\zeta}$ *is uniformly bounded in* N.

We can finally derive the correction of order 1 for $\mathcal{W}_{1N}^V$ by going back to the first Schwinger–Dyson equation. Indeed, if we let $\delta_N^2 := N(\mathcal{W}_{1N}^V - N\tau_{10} - \tau_{11})$, the first Schwinger–Dyson equation reads

$$\delta_N^2(\Psi^{V_\beta}_{\tau_{10}}p) = 1_{\beta=1}\delta_N(\tilde{\Delta}p) - [\mathcal{W}_{2N}^{V_\beta} + \delta_N \otimes \delta_N](\bar{\mathsf{S}}^{V_\beta}p + \bar{\Delta}p) + \widetilde{R}_N(p),$$

where $\widetilde{R}_N(p)$ depends on correlators of order three or higher, which are negligible by Lemma 6.7, and S^V is defined in (6.21). Then, arguing as previously, we infer the following result.

Lemma 6.13. *Assume there exist* $1 < \xi_4 < \xi_3 < \xi_2 < \xi_1$ *such that, for* $\xi = \xi_1, \xi_2, \xi_3, \xi_4$,

$$\delta_{\xi,\zeta}(V) < \frac{1}{1+\max\{2,r\}}.$$

Then

$$\lim_{N\to\infty} \delta_N^2(p) = \tau_{11}(\tilde{\Delta}(\Psi^{V_\beta}_{\tau_{10}})^{-1}p) - [\tau_{20} + \tau_{11}\otimes\tau_{11}](\bar{\Delta}(\Psi^{V_\beta}_{\tau_{10}})^{-1}p + \bar{\mathsf{S}}^{V_\beta}(\Psi^{V_\beta}_{\tau_{10}})^{-1}p) =: \tau_{12}(p)$$

and $N\|\delta_N^2 - \tau_{12}\|_{\xi_1,\zeta}$ *is uniformly bounded in* N.

This concludes the proof of Proposition 6.4. We can now prove Proposition 6.2 and Lemma 6.3.

6.2. Proof of Proposition 6.2 and Lemma 6.3.

We first show that the free energy is a function of the correlators, and then that the correlators only depend on $\{L_i^N(x^\ell)\}_{\ell\geqslant 0,\, 1\leqslant i\leqslant d}$ and τ_B^N. Finally, we deduce the large N expansion of the free energy as well as its smoothness.

6.2.1. The free energy in terms of the correlators

Recalling the definition of free energy, (6.5), and (6.4), we have

$$\begin{aligned}
F_\beta^{N,aV}(A_1, ..., A_d, B_1, ..., B_m) &:= \log I_\beta^{N,aV} \\
&= \int_0^a \frac{d}{du} \log I_\beta^{N,uV}\, du \\
&= N^2 \int_0^a \int \frac{1}{N^r} \mathrm{Tr}^{\otimes r} V\, d\mathbb{Q}_\beta^{N,uV}\, du \\
&= N^{2-r} \int_0^a (\mathcal{W}_{1N}^{uV})^{\otimes r}(V)\, du + r(r-1)N^{2-r} \int_0^a \mathcal{W}_{2N}^{uV} \otimes (\mathcal{W}_{1N}^{uV})^{r-2}(V)\, du + \bar{R}_N,
\end{aligned}$$

where $\bar{R}_N$ has terms either with two cumulants of order 2, or a cumulant of order greater or equal to 3. By Lemma 6.7 (note that it applies uniformly in $u\in[-a_0, a_0]$, for some a_0 universally small), this latter term is at most of order $1/N$, and is therefore negligible. Moreover, using Corollary 6.8 and Lemmas 6.12 and 6.13, we find that

$$F_\beta^{N,aV}(A_1, ..., A_d, B_1, ..., B_m) = N^2 \int_0^a f_0^u\, du + N \int_0^a f_1^u\, du + \int_0^a f_2^u\, du + O\left(\frac{1}{N}\right),$$

with

$$\begin{aligned} f_0^u &:= (\tau_{10}^{uV})^{\otimes r}(V), \\ f_1^u &:= r\tau_{11}^{uV} \otimes (\tau_{10}^{uV})^{\otimes (r-1)}(V), \\ f_2^u &:= r(r-1)\big[(\tau_{11}^{uV})^{\otimes 2} + \tau_{20}^{uV}\big] \otimes (\tau_{10}^{uV})^{\otimes (r-2)}(V) + r\tau_{12}^{uV} \otimes (\tau_{10}^{uV})^{\otimes (r-1)}(V), \end{aligned} \tag{6.29}$$

where we have used that V is symmetric and such that $\|V\|_{\xi_1,\zeta}$ is finite for ξ_1 big enough, so that $\delta_{\xi_1,\zeta}(uV) < (1+\max\{2, r\})^{-1}$ provided $u\in[-a_0, a_0]$ with a_0 sufficiently small. In particular this implies that, for a_0 small enough and any $1<\xi_4<\xi_3<\xi_2<\xi_1$,

$$\delta_{\xi_i,\zeta}(uV) < (1+\max\{2, r\})^{-1} \quad \text{for all } u \in [-a_0, a_0],$$

so that the previous lemmas apply. Hence, we deduce the following result.

LEMMA 6.14. *Let $\|V\|_{\xi_1,\zeta_1}$ be finite for some ξ_1 large enough and let $\zeta_1 \geqslant 1$. Then, there exists $a_0>0$ so that, for $a\in[-a_0, a_0]$, and uniformly on Hermitian matrices $\{A_i\}_{i=1}^d$ and $\{B_i\}_{1\leqslant i\leqslant m}$ whose operator norm is bounded by* 1, *we have*

$$F_\beta^{N,aV}(A_1, ..., A_d, B_1, ..., B_m) = \sum_{l=0}^{2} N^{2-l} F_l^a + O\left(\frac{1}{N}\right),$$

with

$$F_l^a = \int_0^a f_l^u\, du$$

and f_l^u given by (6.29).

6.2.2. The correlators as functions of $\{L_i^N\}_{i=1}^d$ and τ_B^N

Let us define the space

$$\mathcal{P} := \{Q(u_1 a_1 u_1^{-1}, ..., u_d a_d u_d^{-1}, b_1, ..., b_k) : Q \in \mathbb{C}\langle x_1, ..., x_d, b_1, ..., b_m\rangle\}.$$

As the functions F_l^a only depend on the restriction to $\mathcal{P}$ of τ_{10}^{uV}, τ_{11}^{uV}, τ_{12}^{uV}, and τ_{20}^{uV} for $u\in[-a, a]$, we shall first prove that the latter only depend on

$$M_{A,B} := \left\{\frac{1}{N}\sum_{j=1}^{N} (a_j^i)^\ell : \ell \geqslant 0 \text{ and } 1 \leqslant i \leqslant d\right\} \cup \{\tau_B^N\}.$$

• *The restriction* $\tau_{01}^{aV}|_{\mathcal{P}}$ *depends only on* $M_{A,B}$. We start by showing that τ_{01}^{aV} can be defined inductively, as is the case when $V=0$, since it depends analytically on the potential V in the following sense.

LEMMA 6.15. *Let* $p\in\mathscr{L}$ *and* V *be a potential such that, for some* $\xi>1$ *and* $\zeta\geqslant 1$,

$$\delta_{\xi,\zeta}(V)<\frac{1}{1+\max\{2,r\}}.$$

Then, for all $a\in[-1,1]$, *the solution* τ_{10}^{aV} *of*

$$\tau\otimes\tau(\partial_i p)+a(1+\mathbf{1}_{\beta=1})\tau(\mathcal{D}_{i,\tau}Vp)=0 \tag{6.30}$$

is uniquely defined. Moreover, we have the decomposition

$$\tau_{10}^{aV}=\sum_{n\geqslant 0}a^n\tau_n^V,$$

with $\tau_n^V\in\mathcal{L}_{\xi,\zeta}$ *satisfying* $\|\tau_n^V\|_{\xi,\zeta}\leqslant C_nD^n$, *where* $\{C_n\}_{n\geqslant 0}$ *denote the Catalan numbers and* D *is a positive constant.*

Proof. This result can be seen to be a consequence of the implicit function theorem. However we will soon need additional information on the τ_n^V, and therefore give a proof "by hand".

By uniqueness of solutions, it is enough to show that there exists a solution of (6.30), or more precisely of (6.13), which is analytic in a. Let us therefore look for such a solution and write $\tau^{aV}(p):=\sum_{n\geqslant 0}a^n\tau_n^V(p)$. We then find that τ^{aV} satisfies (6.13) if and only if

$$\begin{aligned}\tau_n^V(p)+&\sum_{k\in\{0,n\}}\tau_k^V\otimes\tau_{n-k}^V(\Pi\bar{\Delta}p)\\ &=-\sum_{k=1}^{n-1}\tau_k^V\otimes\tau_{n-k}^V(\Pi\bar{\Delta}p)\\ &\quad-\sum\langle V,q_1\otimes\ldots\otimes q_r\rangle\sum_{i=1}^d\sum_{\ell=1}^r\sum_{\sum_i k_i=n-1}\biggl(\prod_{j\neq\ell}\tau_{k_j}^V(q_j)\biggr)\tau_{k_\ell}^V(\mathcal{D}_iq_\ell\cdot\mathcal{D}_i\mathsf{D}^{-1}p)\end{aligned} \tag{6.31}$$

As $\bar{\Delta}$ splits monomials p into simple tensors $q_1\otimes q_2$ each of whose factors has degree strictly smaller than that of p, we see that there exists a unique solution to this equation. Moreover, we prove by induction that there exists finite constant $D>0$ such that, if C_n denote the Catalan numbers, then

$$\|\tau_n^V\|_{\xi,\zeta}\leqslant C_nD^n.$$

Indeed, for $n=0$, we simply have the law of free variables bounded by 1, so that the result is clear. Using the inductive hypothesis until $n-1$ to bound the right-hand side in (6.31), and (6.18) to bound the second term in the left-hand side of (6.31), we deduce that

$$(1-\delta_{\xi,\zeta}(V))\|\tau_n^V\|_{\xi,\zeta} \leqslant \frac{8}{\xi-1}D^n\sum_{k=1}^{n-1}C_kC_{n-k}+D^{n-1}\sum|\langle V,q_1\otimes...\otimes q_r\rangle| \\ \times\bigg(\sum_{i=1}^r \deg q_i\bigg)\zeta^{\sum_{i=1}^r \deg_{A,B}(q_i)}\xi^{\sum_{i=1}^r \deg_U q_i}\sum_{\sum_{i=1}^r k_i\leqslant n-1}\prod_{i=1}^r C_{k_i}.$$

Using the fact that $\sum_{k=0}^n C_kC_{n-k}=C_{n+1}\leqslant 4C_n$, we find recursively that

$$\sum_{\sum_{i=1}^r k_i\leqslant n-1}\prod_{i=1}^r C_{k_i}\leqslant C_{n+r-1}\leqslant 4^{r-1}C_n.$$

Thus we can bound the last term by $4^{r-1}C_nD^{n-1}\big\||V|\big\|_\xi$, which implies that

$$\|\tau_n^V\|_{\xi,\zeta}\leqslant C_nD^n$$

provided D is chosen sufficiently large. As $C_n\leqslant 4^n$, this implies that $\tau^{aV}=\sum_{n\geqslant 0}a^n\tau_n^V$ is absolutely converging provided $|a|<1/4D$ and it satisfies (6.30), so we get $\tau^{aV}=\tau_{01}^{aV}$ as desired. $\square$

We finally show that $\tau_n^V|_{\mathcal{P}}$ only depends on $M_{A,B}$. Again, we can argue by induction. As already mentioned, this is clear when $n=0$ as τ_0^V is the law of free variables. Also, if $p\in\mathcal{P}$ and $\deg(p)=0$ then p depends only on $b_1,...,b_k$, and therefore $\tau_n^V(p)$ only depends on τ_B^N for all $n\geqslant 0$. Thus, by the inductive hypothesis, we can assume that the result is true for $\tau_k^V(p)$ when $k\leqslant n-1$ and $p\in\mathcal{P}$, and for $\tau_n^V(p)$ when $p\in\mathcal{P}$ and $\deg(p)\leqslant\ell$.

To show that this property propagates we shall use the fact that (6.31) can be seen as an induction relation where all monomials belong to $\mathcal{P}$. To this end, first note that $\{\tau_n^V\}_{n\geqslant 0}$ are tracial, that is

$$\tau_n^V(pq)=\tau_n^V(qp)\quad\text{for all } p,q\in\mathcal{P}.$$

Indeed this property is clear as it is satisfied by τ^{aV}, and $\{\tau_n^V\}_{n\geqslant 0}$ are derivatives of τ^{aV} with respect to a.

Next, observe that D^{-1} keeps $\mathcal{P}$ stable. Moreover, if $p=Q(\{u_ia_iu_i^{-1}\}_{i=1}^m)$, where Q is a monomial, then

$$\mathcal{D}_ip=\sum_{Q=q_1x_iq_2}(a_iu_i^{-1}q_2q_1u_i-u_i^{-1}q_2q_1u_ia_i),$$

so that, up to cyclic symmetry, $\mathcal{D}_i p\cdot\mathcal{D}_i q\in\mathcal{P}$ for each i and $q\subset\mathcal{P}$. (Here and in the sequel, cyclic symmetry is just the action of exchanging pq into qp.) We also show that $\bar\Delta$ maps $\mathcal{P}$ into $\mathcal{P}\otimes\mathcal{P}$ up to cyclic symmetry. Indeed, it follows from (6.12) that, for $p\in\mathcal{P}$,

$$\begin{aligned}\bar\Delta_i p = \sum_{p=p_1u_ia_iu_i^{-1}p_2}\Bigg(&\sum_{a_iu_i^{-1}p_2p_1u_i=a_iu_i^{-1}q_1u_ia_iu_i^{-1}q_2u_i} a_iu_i^{-1}q_1u_i\otimes a_iu_i^{-1}q_2u_i\\ &-\sum_{a_iu_i^{-1}p_2p_1u_i=a_iu_i^{-1}q_1u_ia_iu_i^{-1}q_2u_i}(a_iu_i^{-1}q_1u_ia_i\otimes q_2-a_i\otimes p_2p_1-p_2p_1\otimes a_i)\\ &-\sum_{u_i^{-1}p_2p_1u_ia_i=u_i^{-1}q_1u_ia_iu_i^{-1}q_2u_ia_i} q_1\otimes a_iu_i^{-1}q_2u_ia_i\\ &+\sum_{u_i^{-1}p_2p_1u_ia_i=u_i^{-1}q_1u_ia_iu_i^{-1}q_2u_ia_i} u_i^{-1}q_1u_ia_i\otimes u_i^{-1}q_2u_ia_i\Bigg),\end{aligned}$$

so that, up to cyclic symmetry, $\bar\Delta_i p\in\mathcal{P}\otimes\mathcal{P}$ for all $i\in\{1,...,m\}$ and $p\in\mathcal{P}$.

Hence, by induction we see that τ_n^V restricted to $\mathcal{P}$ only depends on the restriction of $\{\tau_k^V\}_{k\leqslant n-1}$ to $\mathcal{P}$, therefore only on the restriction of τ_0^V to $\mathcal{P}$. Since we have already seen that $\tau_0^V|_{\mathcal{P}}$ only depends on $M_{A,B}$, the conclusion follows.

- τ_{11}^{aV} *depends only on* $M_{A,B}$. A direct inspection shows that $\tilde\Delta$ maps $\mathcal{P}$ into $\mathcal{P}$ up to cyclic symmetry. Indeed, $\tilde\Delta=\sum_i\tilde\Delta_i$ with

$$\begin{aligned}\tilde\Delta_i p = \sum_{p=p_1u_ia_iu_i^{-1}p_2}\Bigg(&\sum_{a_iu_i^{-1}p_2p_1u_i=a_iu_i^{-1}q_1u_ia_iu_i^{-1}q_2u_i} u_i^{-1}q_2^*u_ia_i^2u_i^{-1}q_1u_i\\ &-\sum_{a_iu_i^{-1}p_2p_1u_i=a_iu_i^{-1}q_1u_ia_iu_i^{-1}q_2u_i}(u_i^{-1}q_2^*u_ia_iu_i^{-1}q_1u_ia_i-u_i^{-1}p_1^*p_2^*u_ia_i-a_iu_i^{-1}p_2p_1u_i)\\ &-\sum_{u_i^{-1}p_2p_1u_ia_i=u_i^{-1}q_1u_ia_iu_i^{-1}q_2u_ia_i} a_iu_i^{-1}q_2^*u_ia_iu_i^{-1}q_1u_i\\ &+\sum_{u_i^{-1}p_2p_1u_ia_i=u_i^{-1}q_1u_ia_iu_i^{-1}q_2u_ia_i} a_iu_i^{-1}q_2^*q_1u_ia_i\Bigg).\end{aligned}$$

Moreover, the previous considerations showed that $\Psi_{\tau_{10}}^{aV}$ maps $\mathcal{P}$ into $\mathcal{P}$ for a small, and therefore

$$\tau_{11}^{aV}(p)=\mathbf{1}_{\beta=1}\tau_{10}^{aV}(\tilde\Delta(\Psi_{\tau_{10}}^{aV})^{-1}(p))$$

only depends on $\tau_{10}^{aV}|_{\mathcal{P}}$. Since we just checked that the latter only depends on $M_{A,B}$, this proves the result.

- τ_{20}^{aV} *depends only on* $M_{A,B}$. By Lemma 6.12,

$$\tau_{20}^{aV}(\Psi_{\tau_{10}}^{aV}p,q)=-(1+\mathbf{1}_{\beta=1})\tau_{10}^{aV}(\bar{\mathsf{P}}^q_{\tau_{10}^{aV}}p),$$

and recalling that τ_{10}^{aV} expands in a convergent series in a, we see that so does τ_{20}^{aV}. We only need to check that the operators which appear in the equation defining τ_{20}^{aV} keep $\mathcal{P}$ stable. But we have already seen that both the operators $\bar{\Delta}$ and P^V keep $\mathcal{P}$ stable, and hence $\tau_{20}^{aV}(p,q)$ only depends on $M_{A,B}$ and it is in fact a convergent series in such elements.

- τ_{12}^{aV} *depends only on* $M_{A,B}$. By Lemma 6.13,

$$\tau_{12}^{aV}(\Psi_{\tau_{10}}^{aV}p) = \tau_{11}^{aV}(\tilde{\Delta}p) - [\tau_{20}^{aV} + \tau_{11}^{aV} \otimes \tau_{11}^{aV}](\bar{\Delta}p + \bar{\mathsf{S}}^{aV_\beta}p),$$

from which we see that $\tau_{12}^{aV}(p)$ is a convergent series in a (recall that we already proved that $\tau_{10}^{aV}(p), \tau_{11}^{aV}(p)$ and $\tau_{20}^{aV}(p)$ are convergent series in a). So the main point is to prove that, up to cyclic symmetry, $\bar{\Delta}p + \bar{\mathsf{S}}^{aV_\beta}p \in \mathcal{P} \otimes \mathcal{P}$ whenever $p \in \mathcal{P}$.

We already proved that this is the case for $\bar{\Delta}p$, so we focus on $\bar{\mathsf{S}}^{aV_\beta}p$. We notice that it is the sum of two parts. One part is linear over tensors of two monomials appearing in the decomposition of aV, and as $aV \in \mathcal{P}^{\otimes r}$ this part clearly belongs to $\mathcal{P}^{\otimes 2}$. The other part is linear over tensors of one monomial appearing in the decomposition of aV (which therefore belongs to $\mathcal{P}$) and $\mathcal{D}_i p \cdot \mathcal{D}_i q_j$ with q_j appearing in the decomposition of aV (which we have seen belongs to $\mathcal{P}$ up to cyclic symmetry). Hence also this second part satisfies the desired property, which concludes the proof.

6.2.3. Smoothness of the functions F_2, F_1, and F_0

By Lemma 6.14 and the discussion in the previous subsection, we know that

$$F_\beta^{N,aV} = \sum_{l=0}^{2} N^{2-l} F_l^a(L_1^N, ..., L_d^N, \tau_B^N) + O\left(\frac{1}{N}\right),$$

where the functionals F_0^a, F_1^a and F_2^a depend on $\{L_i^N\}_{i=1}^d$ and on τ_B^N through the asymptotic correlators $\{\tau_{1g}^{uV}\}_{g=0}^2$ and τ_{20}^{uV}. We finally prove that they are smooth functions of these measures.

Recall the notation introduced in (6.3). We show the following.

LEMMA 6.16. *There exists $\xi_0 > 1$ large enough such that the following holds: let V have finite $\|\cdot\|_{\xi,\zeta}$ norm for some $\xi > \xi_0$ and $\zeta \geqslant 1$. Then there exists $a_0 > 0$ such that, for all $a \in [-a_0, a_0]$, F_l^a is Fréchet differentiable ℓ times for all $\ell \in \mathbb{N}$, and if $\nu_j = (\nu_1^j, ..., \nu_d^j, \tau_j) \in P([-1,1])^d \times \mathcal{T}(\mathscr{B})$, we have*

$$|D^\ell F_l^a(L_1^N, ..., L_d^N, \tau_B^N)[\nu_1, ..., \nu_\ell]| \leqslant C_\ell |a| \, \|\nu_1\|_\zeta \, ... \, \|\nu_\ell\|_\zeta.$$

Moreover, the derivative $D_kF_0^a(L_1^N,...,L_d^N,\tau_B^N)=DF_0^a(L_1^N,...,L_d^N,\tau_B^N)[0,...,0,\delta_x,0,...,0]$ *of* F *in the direction of the measure* L_k^N *is a function on the real line with finite* $\|\cdot\|_\zeta$ *norm for any* $k\in\{1,...,d\}$*. As a consequence, it is of class* C^∞ *in an open neighborhood of* $[-1,1]$*.*

Proof. First, fix ξ_0 sufficiently large so that all previous results apply. By the previous section it is enough to show that $\{\tau_{1g}^{uV}\}_{g=0}^2$ and τ_{20}^{uV} depend smoothly on $(\{L_i^N\}_{i=1}^d,\tau_B^N)$, uniformly with respect to $u\in[-a,a]$. Indeed, by (6.29), F_0^a is the integral of $(\tau_{10}^{uV})^{\otimes r}(V)$ over $u\in[0,a]$. We have seen in Lemma 6.8 that $\tau_{AB}^N\mapsto\tau_{01}^{uV,\tau_{AB}^N}$ is ℓ times Fréchet differentiable. Moreover, we have also seen that, once restricted to $\mathcal{P}$, it depends only on $\{L_i^N\}_{i=1}^d$ and τ_B^N, and not the full distribution τ_{AB}^N. As a consequence, the smoothness of $\tau_{01}^{uV,\tau_{AB}^N}$ as a function of τ_{AB}^N reduces to the smoothness as a function of the probability measures $\{L_i^N\}_{i=1}^d$ and τ_B^N. The fact that DF_0^a is C^∞ is a direct consequence of formulas (6.25) and (6.26). For instance, if we denote by D_k the derivative along L_k^N, and Π'_k is the projection onto the algebra generated by $\{a_k\}$, for any $p\in\mathscr{L}_{\xi,\zeta}\cap\mathcal{P}$ we have

$$D_k\tau_{0,1}^{V,\tau_1}[p]=-\Pi'_k[\bar{\mathsf{T}}_{\tau\Pi+\tau_1\Pi'}+\bar{\mathsf{P}}^{V_\beta}_{\tau\Pi+\tau_1\Pi'}+\bar{\mathsf{Q}}^{V_\beta}_{\tau\Pi+\tau_1\Pi'}](\mathsf{Id}+\Xi^V_{\tau_{01}^{V,\tau_1}})^{-1}p\in\mathcal{P}, \tag{6.32}$$

where we use the fact (see Lemma 6.2) that

$$[\bar{\mathsf{T}}_{\tau\Pi+\tau_1\Pi'}+\bar{\mathsf{P}}^{V_\beta}_{\tau\Pi+\tau_1\Pi'}+\bar{\mathsf{Q}}^{V_\beta}_{\tau\Pi+\tau_1\Pi'}](\mathsf{Id}+\Xi^V_{\tau_{01}^{V,\tau_1}})^{-1}(\mathcal{P})\subset\mathcal{P}$$

so that once we project it on $\mathscr{A}\mathscr{B}$ we get only polynomials either in the a_i or in the b_i, and hence differentiating in the direction of L_k^N we only keep those in a_k.

The same argument holds for F_1^u and F_0^u, since also τ_{10}^{uV}, τ_{11}^{uV}, and τ_{20}^{uV} are smooth and only depend on $\{L_i^N\}_{i=1}^d$ and τ_B^N. □

7. Law of polynomials of random matrices

Let us consider the equation

$$Y_i=X_i+a\,F_i(X_1,...,X_d,B_1,...,B_m)$$

with $X_1,...,X_d$, (resp. $B_1,...,B_m$) self-adjoint operators with norm bounded by ξ (resp. ζ) and F_i smooth functions (eventually polynomial functions) on such operators. We assume that F_i are self-adjoint and that $F_i=\sum_i\beta_i^q q$, where the sum is over monomials in X_i's and B_i's with total degree $\deg_X(q)$ (resp. $\deg_B(q)$) in $X_1,...,X_d$ (resp. in $B_1,...,B_m$). We also assume that for $\zeta\geqslant1$ and ξ large enough

$$\|F_i\|_{\xi,\zeta}:=\sum_i|\beta_i^q|\,\xi^{\deg_X(q)}\zeta^{\deg_B(q)}<\infty.$$

By the implicit function theorem, see [38, Corollary 2.4], for any fixed ξ and ζ there exist $A<A'<\xi$ such that for a small enough (e.g., so that $A+|a|\,\|F_i\|_{\xi,\zeta}\leqslant A'$) there exist analytic functions G_i, with $\|G_i\|_{A,\zeta}=O(|a|)$, satisfying

$$X_i = Y_i + G_i(Y_1, ..., Y_d, B_1, ..., B_m),$$

for all operators Y_i whose norm is bounded by A.

To be precise, notice that [38] only consider the case where the B_i's are constant, but the proof extends readily to the case where some additional fixed matrices B_i are present, as it is based on a fixed point argument showing that the sequence

$$X_i^0 = Y_i, \quad X_i^{n+1} = Y_i - a\, F_i(X_1^n, ..., X_d^n, B_1, ..., B_m)$$

is Cauchy for $\|\cdot\|_{A,\zeta}$ provided a is small enough. As the closure $\mathbb{C}\langle x_1, ..., x_d; b_1, ..., b_m\rangle_{A,\zeta}$ of the space of polynomials under $\|\cdot\|_{A,\zeta}$ is complete, it follows that the sequence $\{X_i^n\}_{n\in\mathbb{N}}$ converges in this space for all $1\leqslant i\leqslant d$. This construction also shows that there exist functions $G_i\in\overline{\mathbb{C}\langle x_1, ..., x_d; b_1, ..., b_m\rangle}^{\|\cdot\|_{A,\zeta}}$ satisfying the desired properties.

We next consider the law $\mathbb{P}_Y^N$ of the random matrices

$$Y_i^N = X_i^N + a\, F_i(X_1^N, ..., X_d^N, B_1^N, ..., B_m^N)$$

for d independent GUE matrices $X_1^N, ..., X_d^N$ and m deterministic matrices $B_1^N, ..., B_m^N$. Our goal in this section is to show that the law of $Y_1^N, ..., Y_d^N$ satisfies our previous hypotheses.

First, notice that by Lemma 3.3 applied to the current situation where the equilibrium density is the semi-circle law, see (2.9), the matrices X_i^N have norms bounded by 3 with probability greater than $1-e^{-cN}$. Hence, if we fix $\xi=4$ and

$$F_i \in \mathbb{C}\langle x_1, ..., x_d; b_1, ..., b_m\rangle_{\xi,\zeta}$$

we see that, with probability greater than $1-e^{-cN}$, for a small enough we have

$$X_i^N = Y_i^N + G_i(Y_1^N, ..., Y_d^N, B_1^N, ..., B_m^N),$$

for some $G_i\in\overline{\mathbb{C}\langle x_1, ..., x_d; b_1, ..., b_m\rangle}^{\|\cdot\|_{A,\zeta}}$ with $3\leqslant A<A'<\xi$.

Therefore, up to an error of order e^{-cN} in the total variation norm, we have

$$\begin{aligned}\mathbb{P}_Y^N(dY_1^N, ..., dY_d^N) = \frac{1}{Z_N} & e^{-N\sum_{i=1}^d \operatorname{Tr}(Y_i^N + G_i(Y_1^N, ..., Y_d^N, B_1^N, ..., B_m^N))^2} \\ & \times \operatorname{Jac} G(Y_1^N, ..., Y_d^N, B_1^N, ..., B_m^N) \prod_{i=1}^d dY_i^N,\end{aligned}$$

where $\mathrm{Jac}\, G(Y_1^N, ..., Y_d^N, B_1^N, ..., B_m^N)$ denotes the Jacobian of the change of variable

$$X_i = Y_i + G_i(Y_1, ..., Y_d, B_1, ..., B_m).$$

It turns out that, in the case $\beta=2$,

$$\log \mathrm{Jac}\, G(Y_1^N, ..., Y_d^N, B_1^N, ..., B_m^N) = \mathrm{Tr}_d(\mathrm{Tr}\otimes\mathrm{Tr}(\log(\mathsf{Id}+\mathcal{J}G))),$$

where Tr is the trace over $N\times N$ matrices, Tr_d is the trace over $d\times d$ matrices, and

$$(\mathcal{J}G)_{ij,k\ell;t,s} = \partial_{Y_t^N(k\ell)} G_s(ij) = (\hat{\partial}_t G_s)_{ik,\ell j}, \quad i,j,k,\ell \in \{1, ..., N\},\ \ s,t\in\{1, ..., d\},$$

where $\hat{\partial}_t$ denotes the non-commutative derivative over polynomial of self-adjoint variables defined as

$$\hat{\partial}_t p := \sum_{p=q_1 Y_t q_2} q_1 \otimes q_2.$$

Indeed, the above formula follows from the fact that $\hat{\partial}_t p$ lives in the tensor product space (in other words, on the algebra of left multiplication tensored with the right multiplication) and

$$\partial_{Y_t^N(k\ell)} G_s(ij) = (\hat{\partial}_t G_s \sharp \Delta_{k\ell})(ij) = (\hat{\partial}_t G_s)_{ik,\ell j},$$

where $\Delta_{k\ell}$ is the matrix with null entries except at position (ℓ,k) where there is a 1 (here $A\otimes B\sharp C = ACB$).

As G is small for a small enough (at least when restricted to matrices with universally bounded operator norm), the singularity of the logarithm is away from our support of integration and we deduce that the law of $Y_1^N, ..., Y_d^N$ can be approximated in the total variation distance by

$$\frac{1}{Z_N} e^{N\,\mathrm{Tr}\, F_1(Y_1^N, ..., Y_d^N, B_1^N, ..., B_m^N) + \mathrm{Tr}\otimes\mathrm{Tr}\, F_2(Y_1^N, ..., Y_d^N, B_1^N, ..., B_m^N)} \prod_{i=1}^d dY_i^N$$

for two smooth functions F_1 and F_2, belonging respectively to the closure of

$$\mathbb{C}\langle x_1, ..., x_d, b_1, ..., b_m\rangle \quad \text{and} \quad \mathbb{C}\langle x_1, ..., x_d, b_1, ..., b_m\rangle^{\otimes 2}$$

with respect to the norms $\|\cdot\|_{\xi,\zeta}$, where

$$\|F_2\|_{\xi,\zeta} := \sum_{q_1,q_2} |\langle F, q_1\otimes q_2\rangle|\, \|q_1\|_{\xi,\zeta}\, \|q_2\|_{\xi,\zeta}$$

whenever $F = \sum_{q_1,q_2} \langle F, q_1\otimes q_2\rangle q_1\otimes q_2$ and the sum runs over monomials. This proves the result when $\beta=2$.

Next, we consider the random matrices

$$Y_i^N = X_i^N + a\,F_i(X_1^N, ..., X_d^N, B_1^N, ..., B_k^N)$$

for d independent GOE matrices $(X_1^N, ..., X_d^N)$ and m deterministic symmetric matrices $B_1^N, ..., B_m^N$. The Jacobian is slightly changed and reads

$$(\mathcal{J}G)_{ij,k\ell;t,s} = (\hat{\partial}_t G_s)_{ik,\ell j} + (\hat{\partial}_t G_s)_{i\ell,kj}, \quad i,j,k,\ell \in \{1, ..., d\},$$

where the second term comes from the fact that $\partial_{X_{\ell k}} X_{\ell k}$ does not vanish (as in the complex case) but is equal to 1. Notice that we can write the second term as $\Sigma(\hat{\partial}_t G_s)$, where Σ acts on basic tensor products by

$$\Sigma(A \otimes B)_{ik,\ell j} := A_{i\ell} B_{kj}.$$

Considering the logarithm of the determinant of $(I+\mathcal{J}G)$, we see that it expands in moments of $\mathcal{J}G$ as

$$\begin{aligned}\log\det(I+\mathcal{J}G) = \mathrm{Tr}_d \mathrm{Tr}\otimes\mathrm{Tr}\log(I+\mathcal{J}G) &= \sum_{n\geqslant 1} \frac{(-1)^{n+1}}{n} \mathrm{Tr}_d \mathrm{Tr}\otimes\mathrm{Tr}(\mathcal{J}G)^n \\ &= \sum_{n\geqslant 1} \frac{(-1)^{n+1}}{n} \mathrm{Tr}_d \mathrm{Tr}\otimes\mathrm{Tr}(\nabla G + \Sigma(\nabla G))^n\end{aligned}$$

with $\nabla G_{ij,k\ell;t,s} = (\hat{\partial}_t G_s)_{ik,\ell j}$. When expanding the above moments, it turns out that the moments with an odd number of Σ result in the trace of a single polynomial, whereas even numbers result in tensor products of two traces. For instance, when $n=1$,

$$\mathrm{Tr}_d \mathrm{Tr}\otimes\mathrm{Tr}(\Sigma(\nabla G)) = \sum_t \sum_{i,j} (\hat{\partial}_t G_t)_{ij,ji} = \sum_t \mathrm{Tr}(m(\hat{\partial}_t G_t))$$

whereas $\mathrm{Tr}_d \mathrm{Tr}\otimes\mathrm{Tr}((\nabla G)) = \sum_t \sum_{i,j} (\hat{\partial}_t G_t)_{ii,jj}$. Hence, also in this case there exist convergent series F_1, F_0 such that

$$\begin{aligned}\log \mathrm{Jac}\, G(Y_1, ..., Y_d, B_1, ..., B_m) &= \mathrm{Tr}\otimes\mathrm{Tr}\, F_0 + \mathrm{Tr}\, F_1 \\ &= \mathrm{Tr}\otimes\mathrm{Tr}\left(F_0 + \frac{1}{2N}(F_1 \otimes \mathsf{Id} + \mathsf{Id} \otimes F_1)\right),\end{aligned}$$

and we conclude as before.

8. Appendix: Concentration lemma

In this section we prove Lemma 6.7. As already mentioned, it follows from standard results on concentration of measure.

Indeed, thanks to Gromov, it is well known that the groups

$$\mathrm{SU}(N):=\{U\in U(N):\det(U)=1\},\quad \mathrm{SO}(N):=O(N)\cap\mathrm{SU}(N)$$

can be seen as submanifolds of the set of $N\times N$ matrices that have a Ricci curvature bounded below by $\frac{1}{4}\beta(N+2)-1$, see e.g. [2, Theorem 4.4.27] and [2, Corollary 4.4.31]. In particular, this implies concentration of measure under the Haar measures on these groups. To lift this result to $\mathbb{Q}^V_{\beta,N}$, let us first notice that, by definition, the potential V is balanced, in the sense that it is invariant under the maps $U_j\mapsto U_je^{i\theta_j}$ for any $\theta_j\in[0,2\pi)$, being a sum of words each containing the same number of letters U_i and U_i^*. Recalling that $\mathbb{Q}^V_{\beta,N}$ is a measure on $O(N)$ (resp. $U(N)$) when $\beta=1$ (resp. $\beta=2$), it follows that, for any balanced polynomial P,

$$\mathbb{Q}^V_{\beta,N}(|\mathrm{Tr}(P)-\mathbb{Q}^V_{\beta,N}(\mathrm{Tr}(P))\big|\geqslant\delta)=\widetilde{\mathbb{Q}}^V_{\beta,N}(|\mathrm{Tr}(P)-\widetilde{\mathbb{Q}}^V_{\beta,N}(\mathrm{Tr}(P))|\geqslant\delta),$$

where $\widetilde{\mathbb{Q}}^V_{\beta,N}$ is the restriction of $\mathbb{Q}^V_{\beta,N}$ to $\mathrm{SO}(N)$ (resp. $\mathrm{SU}(N)$) when $\beta=1$ (resp. $\beta=2$).

On the other hand, if P is a word which is not balanced and we write U_j as $U_j=e^{i\theta_j}\widetilde{U}_j$ with $\widetilde{U}_j$ in $\mathrm{SU}(N)$, then $\mathrm{Tr}P(U)=e^{i\theta}\mathrm{Tr}P(\widetilde{U})$ for some θ which is a linear combination of the θ_j. As θ_j follows the uniform measure on $[0,2\pi]$, we deduce that $\mathbb{Q}^V_{\beta,N}(\mathrm{Tr}(P))=0$. Hence, if P is not balanced,

$$\mathbb{Q}^V_{\beta,N}(|\mathrm{Tr}(P)-\mathbb{Q}^V_{\beta,N}(\mathrm{Tr}(P))|\geqslant\delta)=\widetilde{\mathbb{Q}}^V_{\beta,N}(\big|\mathrm{Tr}(P)\big|\geqslant\delta),$$

Therefore in both cases we can use concentration inequalities on the special groups.

We then notice that $N^{1-r}\mathrm{Tr}^{\otimes r}V$ has a bounded Hessian, going to zero when $\|V\|_{\xi,\zeta}$ goes to zero. Hence, we can use the Bakry–Emery criterion to conclude that, for any $\xi>1$, if $\|V\|_{\xi,\zeta}$ is small enough then

$$\mathbb{Q}^V_{\beta,N}(|\mathrm{Tr}(P)-\mathbb{Q}^V_{\beta,N}\big(\mathrm{Tr}(P))|\geqslant\delta)\leqslant 2e^{-\beta\delta^2/8\|P\|^2_{\mathcal{L}}},\tag{8.1}$$

where $\|P\|_{\mathcal{L}}$ is the Lipschitz constant of $\mathrm{Tr}P$, which can be bounded as

$$\|P\|^2_{\mathcal{L}}\leqslant\sup_{u_j,u_j^*,a_j}\sum_{i=1}^d\tau\big(|\mathcal{D}_iP|^2(u_j,u_j^*,a_j)\big)$$

where the supremum is taken over all unitary operators u_i, all operators a_i with norm bounded by 1, and all tracial states τ. Note that if P is a word, then we simply have $\|P\|_{\mathcal{L}}\leqslant\deg_U(p)$, and more in general

$$\|P\|_{\mathcal{L}}\leqslant\sum_q|\langle P,q\rangle|\;\deg_U(q)\leqslant C_\xi\|P\|_{\xi,1},$$

where C_ξ is a finite constant so that $s \leqslant C_\xi \, \xi^s$ for all $s \in \mathbb{N}$. Therefore, due to (8.1), we deduce that, for any monomials $q_1, ..., q_k$,

$$\left| \mathbb{Q}^V_{\beta,N} \left(\prod_{\ell=1}^k \left(\operatorname{Tr}(q_\ell) - \mathbb{Q}^V_{\beta,N}(\operatorname{Tr}(q_\ell)) \right) \right) \right| \leqslant C_k \prod_{\ell=1}^k \deg_U(q_\ell). \tag{8.2}$$

As correlators can be decomposed as the sum of products of such moments, it follows that, for any words $q_1, ..., q_k$ and any $\xi > 1$,

$$|\mathcal{W}^V_{kN}(q_1, ..., q_k)| \leqslant C_k \prod_{\ell=1}^k \deg_U(q_\ell) \leqslant C_k (C_\xi)^k \prod_{\ell=1}^k \|q_\ell\|_\xi,$$

which concludes the proof of Lemma 6.7.

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