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Blow up for the critical generalized Korteweg–de Vries equation. I: Dynamics near the soliton

by

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1. Introduction

1.1. Setting of the problem

We consider the L^2-critical generalized Korteweg–de Vries (gKdV) equation

$$\begin{cases} u_t+(u_{xx}+u^5)_x=0, & (t,x)\in[0,T)\times\mathbb{R}, \\ u(0,x)=u_0(x), & x\in\mathbb{R}. \end{cases} \tag{1.1}$$

The Cauchy problem is locally well posed in the energy space H^1 by Kenig, Ponce and Vega [11], and given $u_0\in H^1$, there exists a unique(1) maximal solution $u(t)$ of (1.1) in $C([0,T),H^1)$ with either $T=\infty$, or $T<\infty$ and then $\lim_{t\to T}\|u_x(t)\|_{L^2}=\infty$. The mass and the energy are conserved by the flow, for all $t\in[0,T)$,

$$M(u(t))=\int_{\mathbb{R}}u^2(t)\,dx=M_0 \quad\text{and}\quad E(u(t))=\frac{1}{2}\int_{\mathbb{R}}u_x^2(t)\,dx-\frac{1}{6}\int_{\mathbb{R}}u^6(t)\,dx=E_0,$$

where $M_0=M(u_0)$ and $E_0=E(u_0)$, and the scaling symmetry ($\lambda>0$)

$$u_\lambda(t,x)=\lambda^{1/2}u(\lambda^3t,\lambda x)$$

leaves invariant the L^2 norm so that the problem is *mass critical.*

(1) In a certain sense.

The family of traveling wave solutions (called *solitons*)

$$u(t,x)=\lambda_0^{-1/2}Q(\lambda_0^{-1}(x-\lambda_0^{-2}t-x_0)),\quad (\lambda_0,x_0)\in\mathbb{R}_+^*\times\mathbb{R},$$

with

$$Q(x)=\left(\frac{3}{\cosh^2 2x}\right)^{1/4},\quad Q''+Q^5=Q \quad \text{and} \quad E(Q)=0, \tag{1.2}$$

plays a distinguished role in the analysis. From a variational argument [43], H^1 initial data with subcritical mass $\|u_0\|_{L^2}<\|Q\|_{L^2}$ generate global and H^1 bounded solutions with $T=\infty$.

For $\|u_0\|_{L^2}\geqslant\|Q\|_{L^2}$, the existence of blow-up solutions has been a long standing open problem. In particular, unlike for the analogous Schrödinger problem, there exists no simple obstruction to global existence. The study of singularity formation for small supercritical mass H^1 initial data

$$\|Q\|_{L^2}\leqslant\|u_0\|_{L^2}<\|Q\|_{L^2}+\alpha^*,\quad \alpha^*\ll 1, \tag{1.3}$$

has been developed in a series of works by Martel and Merle [16]–[19], [26], where two new sets of tools are introduced:

– a monotonicity formula and L^2-type localized virial identities to control the flow near the solitary wave;

– rigidity Liouville-type theorems to classify the asymptotic dynamics of the flow.

In particular, the first proof of blow up in finite or infinite time is obtained for initial data

$$u_0\in H^1 \quad \text{with (1.3) and } E(u_0)<0. \tag{1.4}$$

The proof is indirect and based on a classification argument: the solitary wave is characterized as the unique universal attractor of the flow in the singular regime. If $u(t)$ blows up in finite or infinite time T with (1.3), then the flow admits near blow-up time a decomposition

$$u(t,x)=\frac{1}{\lambda^{1/2}(t)}(Q+\varepsilon)\left(t,\frac{x-x(t)}{\lambda(t)}\right),\quad \text{with } \varepsilon(t)\to 0 \text{ in } L^2_{\mathrm{loc}} \text{ as } t\to T. \tag{1.5}$$

Then, in [18], for well-localized initial data

$$u_0 \text{ satisfying (1.4) and } \int_{x'>x} u_0^2(x')\,dx'<\frac{C}{x^6} \text{ for } x>0, \tag{1.6}$$

blow up is proved to occur in finite time T with an upper bound on a sequence $t_n\to T$:

$$\|u_x(t_n)\|_{L^2}\leqslant\frac{C(u_0)}{T-t_n}, \tag{1.7}$$

by a dynamical proof.(2)

For the critical mass problem $\|u_0\|_{L^2}=\|Q\|_{L^2}$, assuming in addition the decay

$$\int_{x'>x} u_0^2(x')\,dx' < \frac{C}{x^3} \quad \text{for } x>0,$$

it was proved in [19] that the solution is global and does not blow up in infinite time.

1.2. Generic blow up for critical problems

In the continuation of these works, the program developed by Merle and Raphaël [7], [27]–[31], [38] for the mass critical non-linear Schrödinger NLS equation(3)

$$\begin{cases} i\partial_t u+\Delta u+|u|^{4/N}u=0, \\ u|_{t=0}=u_0, \end{cases} \qquad (t,x)\in[0,T)\times\mathbb{R}^N, \tag{1.8}$$

in dimensions $1\leqslant N\leqslant 5$, has led to a complete description of the stable blow-up scenario near the solitary wave Q which is the unique H^1 non-negative solution up to translation to $\Delta Q-Q+Q^{1+4/N}=0$. This problem displays a similar structure as the critical gKdV. Initial data in H^1 with $\|u_0\|_{L^2}<\|Q\|_{L^2}$ are global and bounded, [43]. For $u_0\in H^1$ with $\|u_0\|_{L^2}=\|Q\|_{L^2}$, Merle [25] proved that the only blow-up solution (up to the symmetries of the equation) is

$$S(t,x)=\frac{1}{(T-t)^{N/2}}e^{-i|x|^2/4(T-t)}e^{i/(T-t)}Q\Big(\frac{x}{T-t}\Big). \tag{1.9}$$

For small supercritical mass H^1 initial data

$$\|Q\|_{L^2}<\|u_0\|_{L^2}<\|Q\|_{L^2}+\alpha^*, \quad \alpha^*\ll 1, \tag{1.10}$$

an H^1 open set of solutions is exhibited where solutions blow up in finite time at log-log speed:

$$\|\nabla u(t)\|_{L^2}\sim C^*\sqrt{\frac{\log|\log(T-t)|}{T-t}}. \tag{1.11}$$

Moreover, non-positive energy solutions belong to this set of generic blow up. This double log correction to self-similarity for stable blow up was conjectured from numerics by Landman, Papanicolou, Sulem and Sulem [15], and a family of such solutions was rigorously constructed by a different approach by Perelman in dimension $N=1$, [37]. Blow-up solutions of type (1.9) ($\|u(t)\|_{H^1}\sim 1/t$), constructed by Bourgain and Wang [1]

(2) Arguing directly on the solution itself.
(3) Here and in (1.9), but not later, i denotes the imaginary unit.

(see also Krieger and Schlag [13]) correspond to an *unstable threshold dynamics* as proved in Merle, Raphaël and Szeftel [33]. Finally, under (1.10), the quantization of the focused mass at blow up is proved:

$$|u(t)|^2 \rightharpoonup \|Q\|_{L^2}^2 \delta_{x=x(T)} + |u^*|^2, \quad u^* \in L^2. \tag{1.12}$$

More recently, natural connections have been made between mass critical problems and *energy critical* problems. For the energy critical wave map problem, after the work [42], a complete description of a generic finite-time blow-up dynamics (log correction to the self-similar speed) was given by Raphaël and Rodnianski [39], while *unstable regimes* with different speeds were constructed by Krieger, Schlag and Tataru [14]. See also Merle, Raphaël and Rodnianski [32] for the treatment of the Schrödinger map system and Raphaël and Schweyer [40] for the parabolic harmonic heat flow.

The general outcome of these works is twofold.

First the *sharp* derivation of the blow-up speed in the *generic* regime relies on a detailed analysis of the structure of the solution near collapse, and takes in particular into account slowly decaying tails in the computation of the leading order blow-up profile. These tails correspond to the leading order dispersive phenomenon which drives the speed of concentration and the rate of dispersion, both being intimately linked.

Second, a robust analytic approach has been developed in a nowadays more unified framework. In particular, the control of the solution in the singular regime relies on mixed energy/Morawetz or virial type estimates adapted to the flow which have been used in various settings, see in particular [31], [32], [39] and [41].

1.3. Statement of the results

The aim of the paper is to classify the gKdV dynamics for H^1 solutions close to the soliton and with decay on the right. In particular, we aim at recovering the more refined description of the flow obtained for the L^2 critical NLS equation.

More precisely, let us define the L^2 modulated tube around the soliton manifold by

$$\mathcal{T}_{\alpha^*} = \left\{ u \in H^1 : \inf_{\substack{\lambda_0>0\\ x_0\in\mathbb{R}}} \left\| u - \frac{1}{\lambda_0^{1/2}} Q\left(\frac{\cdot - x_0}{\lambda_0}\right)\right\|_{L^2} < \alpha^* \right\} \tag{1.13}$$

and consider the set of initial data

$$\mathcal{A} = \left\{ u_0 = Q + \varepsilon_0 : \|\varepsilon_0\|_{H^1} < \alpha_0 \text{ and } \int_{y>0} y^{10} \varepsilon_0^2\, dy < 1 \right\}.$$

Here α_0 and α^* are universal constants with

$$0<\alpha_0 \ll \alpha^* \ll 1. \tag{1.14}$$

Our aim is to classify the flow for data $u_0\in\mathcal{A}$. First, we fully describe the blow-up solutions in the tube $\mathcal{T}_{\alpha^*}$: there is only one blow-up type, which is stable. We then show that in fact only three scenarios occur:

– stable blow up with $1/(T-t)$ speed;

– convergence to a solitary wave in large time;

– stable defocusing behavior (the solution leaves the tube $\mathcal{T}_{\alpha^*}$ in finite time).

More precisely, we state the following result.

THEOREM 1.1. (Blow up near the soliton in $\mathcal{A}$) *There exist universal constants* $0<\alpha_0\ll\alpha^*\ll1$ *such that the following holds. Let* $u_0\in\mathcal{A}$.

(i) (Non-positive energy blow up) *If* $E(u_0)\leqslant 0$ *and* u_0 *is not a soliton, then* $u(t)$ *blows up in finite time* T *and, for all* $t\in[0,T)$, $u(t)\in\mathcal{T}_{\alpha^*}$.

(ii) (Description of blow up) *Assume that* $u(t)$ *blows up in finite time* T *and that, for all* $t\in[0,T)$, $u(t)\in\mathcal{T}_{\alpha^*}$. *Then there exists* $\ell_0=\ell_0(u_0)>0$ *such that*

$$\|u_x(t)\|_{L^2}\sim\frac{\|Q'\|_{L^2}}{\ell_0(T-t)}\quad \textit{as } t\to T. \tag{1.15}$$

Moreover, there exist $\lambda(t)$, $x(t)$ *and* $u^*\in H^1$, $u^*\neq0$, *such that*

$$u(t,x)-\frac{1}{\lambda^{1/2}(t)}Q\left(\frac{x-x(t)}{\lambda(t)}\right)\to u^*\quad \textit{in } L^2 \textit{ as } t\to T, \tag{1.16}$$

where

$$\lambda(t)\sim\ell_0(T-t)\ \ \textit{and}\ \ x(t)\sim\frac{1}{\ell_0^2(T-t)}\quad \textit{as } t\to T, \tag{1.17}$$

and

$$\int_{x>R}(u^\star)^2(x)\,dx\sim\frac{\|Q\|_{L^1}^2}{8\ell_0R^2}\quad \textit{as } R\to\infty. \tag{1.18}$$

(iii) (Openness of the stable blow up) *Assume that* $u(t)$ *blows up in finite time* T *and that, for all* $t\in[0,T)$, $u(t)\in\mathcal{T}_{\alpha^*}$. *Then there exists* $\varrho_0=\varrho_0(u_0)>0$ *such that, for all* $v_0\in\mathcal{A}$ *with* $\|v_0-u_0\|_{H^1}<\varrho_0$, *the corresponding solution* $v(t)$ *blows up in finite time* $T(v_0)$ *as in* (ii).

Comments on Theorem 1.1.

• *Blow-up speed.* An important feature of Theorem 1.1 is the derivation of *the* stable blow-up speed for $u_0\in\mathcal{A}$:

$$\|u_x(t)\|_{L^2}\sim\frac{C}{T-t} \tag{1.19}$$

which implies that $x(t) \to \infty$ as $t \to T$. Such a blow-up rate confirms the conjecture formulated in [18] for $E_0 < 0$. Recall that, for $u_0 \in \mathcal{A}$ and $E_0 < 0$, assuming some a-priori global information on the $\dot{H}^1$ norm for all time in [18], one could deduce (1.19). The derivation of such a bound is the key to the proof of Theorem 1.1. This blow-up speed is very far above the scaling law $\|u_x\|_{L^2} \sim 1/(T-t)^{1/3}$ (see [32] and [40] for a similar phenomenon for energy critical geometrical problems).

• *Structure of* u^*. The decay of u^* in L^2 described in (1.18) is directly related to the blow-up speed $\|Q'\|_{L^2}/\ell_0(T-t)$, itself related to the speed of ejection of mass in time from the rescaled soliton, similarly as for the critical NLS, see [30]. Note that the Cauchy problem is well-posed in L^2, so that the L^2 convergence (1.16) is relevant. It is an open question but very likely that the convergence in (1.16) holds in H^1 since the left-hand side is bounded in H^1 and u^* is in H^1. Note that $u^* \notin \mathcal{A}$.

The fact that $u^* \in H^1$ is in contrast to the stable regime for critical NLS, where the accumulation of ejected mass from the rescaled soliton implies that $u^* \notin L^p$, $p > 2$. Here we still observe some ejection of mass from the soliton, but since the concentration point $x(t)$ of the soliton is going to infinity, the mass does not accumulate at a fixed point and gives the tail of u^*. More generally, the regularity of u^* is directly connected to the blow-up speed and the strength of deviation from self-similarity, see [32] and [40].

• *On localization on the right.* Let us stress the importance of the decay assumption on the right in space for the initial data which was already essential in [18] and [19]. Indeed, in contrast to the NLS equation, the universal dynamics cannot be seen in H^1 since an additional assumption of decay to the right is required:

– In part II of this work [22], we construct a minimal mass blow-up solution with $1/(T-t)$ blow up. The initial data is in H^1 and decays slowly on the right.(4) Thus, the blow-up set without decay assumption on the right is *not open* in H^1.

– We prove in [23] the existence of H^1 blow-up solutions with different blow-up speeds, in the range $1/(T-t)^\nu$ for any $\nu > \frac{11}{13}$ for initial data with slow decay on the right (so that Theorem 1.1 and [18] do not apply). We also prove the existence of blow up in infinite time for H^1 data close to the soliton.

These examples justify the existence of a theory in the energy space H^1 (see [16], [20] and [26]), where blow up in finite or infinite time is possible, with a large range of possible blow-up rates, together with a theory for initial data with decay on the right ([18] and the present paper), where the universal blow up is described in Theorem 1.1.

However, we do not claim sharpness in the y^{10} weight in Theorem 1.1.

• *Dynamical characterization of* Q. Recall from the variational characterization

(4) This is mandatory from [19]: there is no minimal mass blow up for data with decay on the right.

of Q that $E(u_0)\leqslant 0$ implies $\|u_0\|_{L^2}>\|Q\|_{L^2}$, unless $u_0\equiv Q$ up to scaling and translation symmetries. Theorem 1.1 therefore recovers the dynamical classification of Q as the unique global zero-energy solution in $\mathcal{A}$ like for the mass critical NLS, see [31]. The proof of this type of result is delicate, and one needs to rule out a scenario of vanishing of the energy of the radiation specific to the zero-energy case. Here, we expect this result to hold without decay assumption (no global H^1 zero-energy solution close to Q exists except Q).

We now state the following rigidity result of the flow for data in $\mathcal{A}$.

THEOREM 1.2. (Rigidity of the dynamics in $\mathcal{A}$) *There exist universal constants*

$$0<\alpha_0\ll\alpha^*\ll 1$$

such that the following holds. Let $u_0\in\mathcal{A}$.

Then, one of the following three scenarios occurs:

– (Exit) *There exists $t^*\in(0,T)$ such that $u(t^*)\notin\mathcal{T}_{\alpha^*}$.*

– (Blow up) *For all $t\in[0,T)$ one has $u(t)\in\mathcal{T}_{\alpha^*}$ and the solution blows up in finite time $T<\infty$ in the regime described by Theorem 1.1.*

– (Soliton) *The solution is global, for all $t\geqslant 0$ one has $u(t)\in\mathcal{T}_{\alpha^*}$, and there exist $\lambda_\infty>0$ and $x(t)$ such that*

$$\lambda_\infty^{1/2}u(t,\lambda_\infty\cdot+x(t))\to Q \qquad \textit{in } H^1_{\mathrm{loc}} \textit{ as } t\to\infty, \tag{1.20}$$

$$|\lambda_\infty-1|\leqslant o_{\alpha_0\to 0}(1) \;\; \textit{and} \;\; x(t)\sim\frac{t}{\lambda_\infty^2} \qquad \textit{as } t\to\infty. \tag{1.21}$$

Comments on Theorem 1.2.

• *Stable/unstable manifold.* All three possibilities are known to occur for an infinite set of initial data. Moreover, the sets of initial data leading to (Exit) and (Blow up) are both open in $\mathcal{A}$ by perturbation of the data in H^1. For $\int_{\mathbb{R}}u_0^2\,dx<\int_{\mathbb{R}}Q^2\,dx$ only the (Exit) case can occur, and for $E_0<0$ only (Blow up) can occur. From the proof of Theorem 1.2, the (Soliton) dynamics can be achieved as threshold dynamics between the two stable regimes (Exit) and (Blow up) as in [3], [8] and [32]. More precisely, given $b\in\mathbb{R}$ small, let Q_b be the suitable perturbation of Q built in Lemma 2.4, and ε_0 be a suitable small perturbation satisfying the orthogonality conditions (2.20). Then there exists $b_0=b(\varepsilon_0)$ such that the solution to gKdV with initial data $Q_{b_0}+\varepsilon_0$ satisfies (Soliton). The Lipschitz regularity of the flow $\varepsilon_0\to b(\varepsilon_0)$ needed to build a smooth manifold remains to be proved; see [13] for related constructions. Note also that solutions that scatter to Q in the regime (Soliton) where constructed dynamically by Côte [2].

• *Classification of the flow in $\mathcal{A}$.* Theorem 1.2 is a first step towards a complete classification of the flow for initial data in $\mathcal{A}$. Its structure is reminiscent from classification results obtained by Nakanishi and Schlag [34]–[36], for Klein Gordon and supercritical Schrödinger equations. These results were proved using in particular classification arguments based on the Kenig–Merle concentration compactness approach [10], the classification of critical dynamics by Duyckaerts and Merle [5] (see also [6]), and eventually a no return lemma. In the analogue of the (Exit) regime, this lemma shows that the solution cannot come back close to solitons and in fact scatters. In the critical situations, such an analysis is more delicate and incomplete (see Krieger, Nakanishi and Schlag [12]). Moreover, in [34] and [35], both the blow-up statements and the no return lemma rely on a specific algebraic structure—the virial identity—which does not exist for gKdV.

In the continuation of Theorem 1.2, what remains to be done to describe the flow for data $u_0 \in \mathcal{A}$ is to answer the following question:

What happens after t^* in the (Exit) regime?

In [22], the second part of this work, we propose a new approach to answer this question related to the understanding of the threshold dynamics. We proceed in two steps:

(1) We prove the *existence and uniqueness in* H^1 of a minimal mass blow-up solution $\|u_0\|_{L^2} = \|Q\|_{L^2}$. From [19], this solution has slow decay to the right and is global on the left in time.

(2) We then show that in the (Exit) case of Theorem 1.2, the solution is at time t^* L^2 close to the unique minimal mass blow-up solution.

Having in mind the properties of threshold solutions for H^1 critical NLS and wave equations ([4], [5]), and the case of the L^2 critical NLS equation (the solution $S(t)$ in (1.9) scatters), it is natural to expect that the minimal mass blow-up solution of gKdV also scatters in negative time. *Assuming this* and because scattering is open in the critical L^2 space, we obtain that (Exit) implies scattering. In other words, we prove in [22] that all solutions scatter in the (Exit) regime if and only if the unique H^1 minimal mass blow-up solution scatters to the left. This ends the classification of the flow in $\mathcal{A}$, in particular the only blow-up regime is the $1/(T-t)$ universal blow-up regime of Theorem 1.1 and it is stable.

• *Finite/infinite-dimensional dynamics.* The proof of Theorem 1.2 relies on a detailed description of the flow. We will show that, before the (Exit) time t^*, the solution admits a decomposition

$$u(t,x) = \frac{1}{\lambda^{1/2}(t)} (Q_{b(t)} + \varepsilon)\left(t, \frac{x - x(t)}{\lambda(t)}\right),$$

where Q_b is a suitable $O(b)$ deformation of the solitary wave profile, and the bound

$$\|\varepsilon\|_{H^1_{\mathrm{loc}}} \ll b$$

holds. We then extract the following universal finite-dimensional system which drives the geometrical parameters:

$$\frac{ds}{dt} = \frac{1}{\lambda^3}, \quad -\frac{\lambda_s}{\lambda} = b, \quad b_s + 2b^2 = 0. \tag{1.22}$$

It is easily seen that, starting from $\lambda(0)=1$ and $b(0)=b_0$, the phase portrait of the dynamical system (1.22) is

(1) for $b_0<0$, $\lambda(t)=1+|b_0|t$, $t\geqslant 0$, stable;

(2) for $b_0=0$, $\lambda(t)=1$, $t\geqslant 0$, unstable;

(3) for $b_0>0$, $\lambda(t)=b_0(T-t)$ with $T=1/b_0$, stable.

We may then reword Theorem 1.2 by saying that the infinite-dimensional system gKdV for data $u_0\in\mathcal{A}$ is governed to leading order by the universal finite-dimensional dynamics (1.22). This is a non-trivial claim due to the non-linear structure of the problem, and the proof relies on a *rigidity* formula when measuring the interaction of the radiative term ε with the ordinary differential equations (ODEs) (1.22), see Lemma 4.3. Let us stress that the assumption of decay to the right is fundamental here, and we expect that slow decaying tails may force a different coupling with new leading order ODEs.

Finally, note that like for the finite-dimensional system (1.22), the three scenarios of Theorem 1.2 can be seen on $\lambda(t)$ only and are equivalently characterized by

– (Soliton) for all t, $\lambda(t)\in\left[\frac{1}{2},2\right]$;

– (Exit) there exists $t_0>0$ such that $\lambda(t_0)>2$;

– (Blow up) there exists $t_0>0$ such that $\lambda(t_0)<\frac{1}{2}$.

We expect that results such as Theorem 1.2 (classification of the dynamics close to the solitary waves) can be proved similarly for other problems such as, for example, the mass critical non-linear Schrödinger equation and the energy critical Schrödinger and wave equations.

Notation

Let the linearized operator close to Q be

$$Lf = -f'' + f - 5Q^4 f. \tag{1.23}$$

We introduce the generator of L^2 scaling

$$\Lambda f = \tfrac{1}{2} f + y f'.$$

For a given generic small constant $0<\alpha^*\ll 1$ we let $\delta(\alpha^*)$ denote a generic small constant with

$$\delta(\alpha^*)\to 0 \quad \text{as } \alpha^*\to 0.$$

We denote the L^2 scalar product by

$$(f,g)=\int_{\mathbb{R}} f(x)g(x)\,dx.$$

From now on, to simplify notation, we will write $\int$ to denote $\int_{\mathbb{R}}$, and will often omit dx and dy in integrals.

1.4. Strategy of the proof

We give in this section a brief insight into the proofs of Theorems 1.1 and 1.2. As mentioned before, we are pushing further the dynamical analysis of the problem initiated in [18]. We will not use rigidity arguments as for the theory in H^1 (see [20] and [26]). Nevertheless, we will use tools introduced to prove such rigidity arguments, such as modulation theory, L^2 and energy monotonicity, local virial identities and weighted estimates for $x>0$. However, the proofs here are self-contained, except for the virial estimates, for which we refer to [16] and [20].

(i) *Formal derivation of the law.* We start as in [27], [31] and [39] by refining the blow-up profile and considering an approximation to the renormalized equation. We look for a solution to gKdV of the form

$$u(t,x)=\frac{1}{\lambda^{1/2}(t)}Q_{b(s)}\left(\frac{x-x(t)}{\lambda(t)}\right),\quad \frac{ds}{dt}=\frac{1}{\lambda^3},\quad \frac{x_s}{\lambda}=1,\quad b=-\frac{\lambda_s}{\lambda},\tag{1.24}$$

which leads to the slowly modulated self-similar equation

$$b_s\frac{\partial Q_b}{\partial b}+b\Lambda Q_b+(Q_b''-Q_b+Q_b^5)'=0.\tag{1.25}$$

A formal derivation of the generic blow-up speed can be obtained as follows: look for a modulated ansatz

$$Q_b=Q+bP+b^2P_2+\ldots,\quad b_s=-c_2b^2+c_3b^3+\ldots,$$

where the unknowns are $P,P_2,\ldots$ and $c_2,c_3,\ldots$. Let the linearized operator close to Q be given by (1.23). Then the order b expansion leads to the equation

$$(LP)'=\Lambda Q.$$

Due to the critical orthogonality condition $(Q,\Lambda Q)=0$, it can be solved for a function P that decays exponentially to the right, but displays a non-trivial tail on the left $\lim_{y\to-\infty} P(y)\neq 0$. At the level b^2, a similar *flux type* computation(5) reveals that the P_2 equation can be solved with a similar profile for the value $c_2=2$ only.(6) This corresponds to the formal dynamical system

$$-\frac{\lambda_s}{\lambda}=b, \quad b_s+2b^2=\lambda^2\frac{d}{ds}\left(\frac{b}{\lambda^2}\right)=0, \quad \frac{ds}{dt}=\frac{1}{\lambda^3}, \tag{1.26}$$

which after reintegration yields finite-time blow up for $b(0)>0$ with

$$\lambda(t)=c(u_0)(T-t).$$

(ii) *Decomposition of the flow and modulation equations* (§2). For the analysis, it is enough to work with the localized approximate self-similar profile

$$Q_b=Q+\chi(|b|^\gamma y)P(y)$$

for some well chosen(7) $\gamma>0$. As long as the solution remains in the tube $\mathcal{T}_{\alpha^*}$, we may introduce the non-linear decomposition of the flow

$$u(t,x)=\frac{1}{\lambda^{1/2}(t)}(Q_{b(t)}+\varepsilon)\left(t,\frac{x-x(t)}{\lambda(t)}\right), \tag{1.27}$$

where the three time-dependent parameters are adjusted to ensure suitable orthogonality conditions(8) for ε. A specific feature of the KdV flow is that the generalized null space of the full linearized operator L' close to Q involves non-localized functions, and hence the modulation equations driving the parameters are roughly speaking of the form

$$\frac{\lambda_s}{\lambda}+b=\frac{dJ_1}{ds}+O(\|\varepsilon\|^2_{H^1_{\mathrm{loc}}}) \quad \text{and} \quad b_s+b^2\sim\frac{dJ_2}{ds}+O(\|\varepsilon\|^2_{H^1_{\mathrm{loc}}}), \tag{1.28}$$

with

$$|J_i|\lesssim\|\varepsilon\|_{H^1_{\mathrm{loc}}}+\int_{y>0}|\varepsilon|, \quad i=1,2.$$

This explains the need for a control of radiation on the right as slow tails and large J_i might otherwise perturb the formal system (1.26) (see also [18]).

(iii) *The mixed energy/virial estimate* (§3). The main new input of our analysis is the derivation of a dispersive control on the local norm $\|\varepsilon\|_{H^1_{\mathrm{loc}}}$ which is relevant in all

(5) See (2.43).
(6) Otherwise, P_2 *grows* exponentially on the right or the left.
(7) See Lemma 2.4, we can take $\gamma=\frac{3}{4}$.
(8) See (2.20).

three regimes, and therefore must display some scaling-invariant structure. For this, we adapt and revisit the construction of mixed energy/virial functionals as introduced in [24], [39], [41] and [42]. Indeed, we build a non-linear functional

$$\mathcal{F} \sim \int \left(\psi \varepsilon_y^2 + \varphi \varepsilon^2 - \frac{1}{3} \psi [(\varepsilon + Q_b)^6 - Q_b^6 - 6 Q_b^5 \varepsilon] \right)$$

for well-chosen cut-off functions (ψ, φ) which are exponentially decaying to the left, and polynomially growing to the right. The leading-order quadratic term relates to the linearized Hamiltonian and is coercive from our choice of orthogonality conditions:

$$\mathcal{F} \gtrsim \|\varepsilon\|_{H^1_{\mathrm{loc}}}^2.$$

The essential feature now is the structure of the cut off which is manufactured to also reproduce on the ground state the leading-order virial quadratic form which measures some repulsivity properties of the linearized operator L' as derived in [20], and leads to the *Lyapunov monotonicity*:

$$\frac{d}{ds}\left(\frac{\mathcal{F}}{\lambda^{2j}} \right) + \frac{\|\varepsilon\|_{H^1_{\mathrm{loc}}}^2}{\lambda^{2j}} \lesssim \frac{|b|^4}{\lambda^{2j}}, \quad j = 0, 1. \tag{1.29}$$

The b^4 term relates to the error in the construction of the Q_b profile as an approximate solution to (1.25). The case $j=0$ in (1.29) is a scaling-invariant estimate which will be crucial in all three regimes to control the dynamics, and the case $j=1$ is an H^1 improvement in the blow-up regime $\lambda \to 0$.

(iv) *Rigidity* (§4). Combining the modulation equations (1.28) with the dispersive bound (1.29) leads essentially to(9)

$$\frac{b(t)}{\lambda^2(t)} \sim \ell \tag{1.30}$$

for some constant ℓ. Then the selection of the dynamics depends on

– either $|b(t)| \lesssim \|\varepsilon(t)\|_{H^1_{\mathrm{loc}}}^2$ for all t,

– or there exists a time $t_1^* \geqslant 0$ such that $|b(t_1^*)| \gg \|\varepsilon(t_1^*)\|_{H^1_{\mathrm{loc}}}^2$.

The second condition means that the finite-dimensional dynamics measured by b takes control over the infinite-dimensional dynamics at some time t_1^*. We claim that *this regime is trapped* and that $|b(t)| \gg \|\varepsilon(t)\|_{H^1_{\mathrm{loc}}}^2$ for $t \geqslant t_1^*$ as long as the solution remains in the tube $\mathcal{T}_{\alpha^*}$. Reintegrating the modulation equations driven to leading order by (1.26), we show that this leads to (Blow up) if $b(t_1^*) > 0$, and to (Exit) if $b(t_1^*) < 0$. The first case

(9) See (4.14).

leads to the threshold (Soliton) dynamics. The condition on $b(t_1)$ which determines the (Blow up) and (Exit) regimes is by continuity of the flow an open condition on the data.

(v) *End of the proof of Theorem* 1.1. The case $E_0 \leqslant 0$ is treated in §5. Here the variational characterization of Q and a standard concentration compactness ensures that the solution must remain in $\mathcal{T}_{\alpha^*}$, and then we show (Blow up) by proving that (Soliton) cannot happen. For $E_0 < 0$, this is a classical consequence of the energy conservation law and local dispersive estimates (asymptotic stability) obtained in the previous step. The case $E_0 = 0$ is substantially more subtle, and we show that (Soliton) behavior at zero energy implies L^2 compactness, and hence asymptotic stability implies that the solution has minimal mass, and thus is exactly a solitary wave.

Finally, we complete in §6 the sharp description of the singularity formation and the universality of the focusing bubble stated by Theorem 1.1. This requires propagating the dispersive estimates, which involve local norms around the soliton, further away to the left of the soliton, in particular to compute the trace of the reminder (1.18). This is done using a suitable H^1 monotonicity formula in the spirit of the analysis in [18] and [26].

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2. Non-linear profiles and decomposition close to the soliton

In this section, we introduce refined non-linear profiles following the strategy developed in [27] and [39]. The strategy is to produce approximate solutions to the renormalized flow (1.25) which are as well localized as possible, which turns out to lead to a strong rigidity for the scaling law.

2.1. Structure of the linearized operator

Denote by $\mathcal{Y}$ the set of functions $f \in C^\infty(\mathbb{R}, \mathbb{R})$ such that, for all $k \in \mathbb{N}$, there exist $C_k, r_k > 0$ such that, for all $y \in \mathbb{R}$,

$$|f^{(k)}(y)| \leqslant C_k (1+|y|)^{r_k} e^{-|y|}. \tag{2.1}$$

We recall without proof the following standard result (see, e.g., [17] and [44]).

Lemma 2.1. (Properties of the linearized operator L) *The self-adjoint operator L on L^2 satisfies the following properties*:

(i) (*eigenfunctions*) $LQ^3=-8Q^3$, $LQ'=0$ *and* $\ker L=\{aQ':a\in\mathbb{R}\}$;

(ii) (*scaling*) $L(\Lambda Q)=-2Q$;

(iii) *for any function $h\in L^2(\mathbb{R})$ orthogonal to Q' for the L^2 scalar product, there exists a unique function $f\in H^2(\mathbb{R})$ orthogonal to Q' such that $Lf=h$; moreover, if h is even* (*resp. odd*), *then f is even* (*resp. odd*);

(iv) *if $f\in L^2(\mathbb{R})$ is such that $Lf\in\mathcal{Y}$, then $f\in\mathcal{Y}$*;

(v) (*coercivity of L*) *for all $f\in H^1$*,

$$(f,Q^3)=(f,Q')=0 \quad\Longrightarrow\quad (Lf,f)\geqslant \|f\|_{L^2}^2; \tag{2.2}$$

moreover, there exists $\mu_0>0$ such that for all $f\in H^1$,

$$(Lf,f)\geqslant \mu_0\|f\|_{H^1}^2-\frac{1}{\mu_0}[(f,Q)^2+(f,y\Lambda Q)^2+(f,\Lambda Q)^2]. \tag{2.3}$$

2.2. Definition and estimates of localized profiles

We now look for a slowly modulated approximate solution to the renormalized flow (1.24), (1.25). In fact, in our setting, an order-b expansion is enough.

Proposition 2.2. (Non-localized profiles) *There exists a unique smooth function P such that $P'\in\mathcal{Y}$ and*

$$(LP)'=\Lambda Q,\quad \lim_{y\to-\infty}P(y)=\frac{1}{2}\int Q,\quad \lim_{y\to\infty}P(y)=0, \tag{2.4}$$

$$(P,Q)=\frac{1}{16}\left(\int Q\right)^2>0 \quad \textit{and} \quad (P,Q')=0. \tag{2.5}$$

Moreover,

$$\widetilde{Q}_b=Q+bP$$

is an approximate solution to (1.25) *in the sense that*

$$\|(\widetilde{Q}_b''-\widetilde{Q}_b+\widetilde{Q}_b^5)'+b\Lambda\widetilde{Q}_b\|_{L^\infty}\lesssim b^2. \tag{2.6}$$

Proof. We look for P of the form $P=\widetilde{P}-\int_y^\infty\Lambda Q$. Since $\int\Lambda Q=-\frac{1}{2}\int Q$, the function $y\mapsto\int_y^\infty\Lambda Q$ is bounded and has decay only as $y\to\infty$. Then, P solves (2.4) if

$$(L\widetilde{P})'=\Lambda Q+\left(L\int_y^\infty\Lambda Q\right)'=R',\quad \text{where } R=(\Lambda Q)'-5Q^4\int_y^\infty\Lambda Q.$$

Note that $R\in\mathcal{Y}$. Since $\int(\Lambda Q)Q=0$ and $LQ'=0$, we have $\int RQ'=-\int R'Q=0$ and so, from Lemma 2.1, there exists a unique (smooth) $\widetilde{P}\in\mathcal{Y}$ orthogonal to Q', such that $L\widetilde{P}=R$. Then $P=\widetilde{P}-\int_y^\infty \Lambda Q$ satisfies (2.4) and $\int PQ'=0$. We now compute, from $L(\Lambda Q)=-2Q$,

$$2\int PQ=-\int(LP)\Lambda Q=\int \Lambda Q\int_y^\infty \Lambda Q=\frac{1}{2}\biggl(\int \Lambda Q\biggr)^2=\frac{1}{8}\biggl(\int Q\biggr)^2. \tag{2.7}$$

Finally, for $\widetilde{Q}_b=Q+bP$, we have

$$\begin{aligned}(\widetilde{Q}_b''-\widetilde{Q}_b+\widetilde{Q}_b^5)'+b\Lambda\widetilde{Q}_b=b(-(LP)'+\Lambda Q)+b^2((10Q^3P^2)'+\Lambda P)\\ +b^3(10Q^2P^3)'+b^4(5QP^4)'+b^5(P^5)',\end{aligned}$$

which yields (2.6). □

Remark 2.3. Since $\int\Lambda Q=-\frac{1}{2}\int Q\neq 0$, a solution P of $(LP)'=\Lambda Q$ cannot belong to $L^2(\mathbb{R})$. We have chosen the only solution P which converges to 0 at ∞ and which is orthogonal to Q'. The fact that P displays a non-trivial tail on the left from (2.4) is an essential feature of the critical gKdV problem and will be central in the derivation of the blow-up speed; see the proof of (2.37). Such a non-local profile is a substitute to a dispersive tail (see a similar use in [21]).

We now proceed to a simple localization of the profile to avoid some artificial growth at $-\infty$. Let $\chi\in\mathcal{C}^\infty(\mathbb{R})$ be such that $0\leqslant\chi\leqslant 1$, $\chi'\geqslant 0$ on $\mathbb{R}$ and

$$\chi\equiv\begin{cases}1 & \text{on } [-1,\infty),\\ 0 & \text{on } (-\infty,-2].\end{cases}$$

We fix

$$\gamma=\tfrac{3}{4} \tag{2.8}$$

(note that any $\gamma\in\left(\frac{2}{3},1\right)$ works and $\frac{3}{4}$ has no specific meaning here), and define the localized profile

$$\chi_b(y)=\chi(|b|^\gamma y)\quad\text{and}\quad Q_b(y)=Q(y)+b\chi_b(y)P(y). \tag{2.9}$$

Lemma 2.4. (Definition of localized profiles and properties) *For $|b|<b^*$ small enough, the following properties hold*:

(i) (Estimates on Q_b) *For all $y\in\mathbb{R}$,*

$$|Q_b(y)|\lesssim e^{-|y|}+|b|(\mathbf{1}_{[-2,0]}(|b|^\gamma y)+e^{-|y|/2}), \tag{2.10}$$

$$|Q_b^{(k)}(y)|\lesssim e^{-|y|}+|b|e^{-|y|/2}+|b|^{1+k\gamma}\mathbf{1}_{[-2,-1]}(|b|^\gamma y)\quad\textit{for } k\geqslant 1, \tag{2.11}$$

where $\mathbf{1}_I$ *denotes the characteristic function of the interval* I.

(ii) (Equation of Q_b) *Let*

$$-\Psi_b=(Q_b''-Q_b+Q_b^5)'+b\Lambda Q_b. \tag{2.12}$$

Then, for all $y\in\mathbb{R}$,

$$|\Psi_b(y)|\lesssim|b|^{1+\gamma}\mathbf{1}_{[-2,-1]}(|b|^\gamma y)+b^2(e^{-|y|/2}+\mathbf{1}_{[-2,0]}(|b|^\gamma y)), \tag{2.13}$$

$$|\Psi_b^{(k)}(y)|\lesssim|b|^{1+(k+1)\gamma}\mathbf{1}_{[-2,-1]}(|b|^\gamma y)+b^2e^{-|y|/2} \qquad \textit{for } k\geqslant 1. \tag{2.14}$$

(iii) (Mass and energy properties of Q_b)

$$\left|\int Q_b^2-\left(\int Q^2+2b\int PQ\right)\right|\lesssim|b|^{2-\gamma}, \tag{2.15}$$

$$\left|E(Q_b)+b\int PQ\right|\lesssim b^2. \tag{2.16}$$

Proof. (i) First, from (1.2), for all $k\geqslant 0$ one has $|Q^{(k)}(y)|\lesssim e^{-|y|}$ on $\mathbb{R}$. Since $P'\in\mathcal{Y}$ and $\lim_{y\to\infty}P(y)=0$, we have $|P(y)|\lesssim e^{-|y|/2}$ for $y>0$. The estimates (2.10) and (2.11) then follow from the definition of χ.

(ii) Expanding $Q_b=Q+b\chi_bP$ in the expression of Ψ_b, and using $Q''-Q+Q^5=0$ and $(LP)'=\Lambda Q$, we find that

$$\begin{aligned}-\Psi_b=b(1-\chi_b)\Lambda Q&+b((\chi_b)_{yyy}P+3(\chi_b)_{yy}P'+3(\chi_b)_yP''-(\chi_b)_yP+5(\chi_b)_yQ^4P)\\&+b^2((10Q^3\chi_b^2P^2)_y+P\Lambda\chi_b+\chi_byP')+b^3(10Q^2\chi_b^3P^3)_y\\&+b^4(5Q\chi_b^4P^4)_y+b^5(\chi_b^5P)_y.\end{aligned} \tag{2.17}$$

Therefore, the estimates (2.13) and (2.14) follow from the properties of Q, χ and P. In particular, note that

$$\begin{aligned}|b(1-\chi_b)\Lambda Q|&\lesssim|b|e^{-3|y|/4}\mathbf{1}_{(-\infty,-1]}(|b|^\gamma y)\lesssim|b|e^{-|b|^{-\gamma}/4}e^{-|y|/2}\lesssim|b|^2e^{-|y|/2},\\ b^2|P\Lambda\chi_b|&\lesssim b^2(e^{-|y|/2}+\mathbf{1}_{[-2,-1]}(|b|^\gamma y)).\end{aligned}$$

(iii) We first estimate, from the explicit form of P,

$$\int\chi_b^2P^2\sim_{b\to 0}C_0^2|b|^{-\gamma}$$

for some universal constant $C_0>0$. Estimate (2.15) now follows from

$$\int Q_b^2=\int Q^2+2b\int\chi_bPQ+b^2\int\chi_b^2P^2,$$

and then

$$\int Q_b^2\geqslant\int Q^2+2b\int PQ-C_0^2|b|^{2-\gamma}\quad\text{and}\quad\|Q_b-Q\|_{L^2}\sim_{b\sim 0}C_0|b|^{1-\gamma/2}.$$

Finally, expanding $Q_b=Q+b\chi_bP$ in $E(Q_b)$, we get

$$E(Q_b)=E(Q)-b\int\chi_bP(Q''+Q^5)+O(b^2),$$

and using $E(Q)=0$ and $Q''+Q^5=Q$ yields (2.16). □

2.3. Decomposition of the solution using refined profiles

In this paper, we work with an H^1 solution u to (1.1) which is a priori in the modulated tube $\mathcal{T}_{\alpha^*}$ of functions near the soliton manifold. More explicitely, we assume that there exist $(\lambda_1(t), x_1(t)) \in \mathbb{R}_+^* \times \mathbb{R}$ and $\varepsilon_1(t)$ such that

$$u(t,x) = \frac{1}{\lambda_1^{1/2}(t)}(Q+\varepsilon_1)\left(t, \frac{x-x_1(t)}{\lambda_1(t)}\right) \quad \text{for all } t\in[0,t_0),$$

with, for all $t\in[0,t_0)$,

$$\|\varepsilon_1(t)\|_{L^2} \leqslant \varkappa \leqslant \varkappa_0 \tag{2.18}$$

for a small enough universal constant $\varkappa_0>0$. We then have the following standard refined modulation lemma.

LEMMA 2.5. (Refined modulated flow) *Assuming* (2.18), *there exist continuous functions* $(\lambda, x, b)\colon [0,t_0] \to (0,\infty)\times\mathbb{R}^2$ *such that*

$$\varepsilon(t,y) = \lambda^{1/2}(t) u(t, \lambda(t)y + x(t)) - Q_{b(t)}(y), \quad \textit{for all } t\in[0,t_0], \tag{2.19}$$

satisfies the orthogonality conditions

$$(\varepsilon(t), y\Lambda Q) = (\varepsilon(t), \Lambda Q) = (\varepsilon(t), Q) = 0. \tag{2.20}$$

Moreover,

$$\|\varepsilon(t)\|_{L^2} + |b(t)| + \left|1 - \frac{\lambda(t)}{\lambda_1(t)}\right| \lesssim \delta(\varkappa) \quad \textit{and} \quad \|\varepsilon(t)\|_{H^1} \lesssim \delta(\|\varepsilon(0)\|_{H^1}). \tag{2.21}$$

Remark 2.6. The main novelty here with respect to [18], [20] and [26] is the use of the modulation parameter b which allows for the extra degeneracy $(\varepsilon, Q)=0$. At the formal level, the parameter b now plays the role of (ε, Q) in the previous work [18].

Proof. Lemma 2.5 is a standard consequence of the implicit function theorem applied in L^2. We omit the details and refer for example to [27] for a proof with similar Q_b profiles for the NLS case. The heart of the proof is the non-degeneracy of the Jacobian matrix:

$$\begin{vmatrix} (\Lambda Q, \Lambda Q) & (\Lambda Q, Q) \\ (P, \Lambda Q) & (P, Q) \end{vmatrix} = (\Lambda Q, \Lambda Q)(P,Q) \neq 0,$$

from

$$\frac{\partial}{\partial \lambda}[\lambda^{1/2} Q_b(\lambda y)]\Big|_{\lambda=1, b=0} = \Lambda Q \quad \text{and} \quad \frac{\partial}{\partial b}[\lambda^{1/2} Q_b(\lambda y)]\Big|_{\lambda=1, b=0} = P,$$

and the explicit computations

$$(\Lambda Q, Q) = 0 \quad \text{and} \quad (P,Q) = \frac{1}{16}\int Q^2 \neq 0. \qquad \square$$

2.4. Modulation equations

In the framework of Lemma 2.5, we introduce the new time variable

$$s=\int_0^t \frac{dt'}{\lambda^3(t')}, \quad \text{or equivalently} \quad \frac{ds}{dt}=\frac{1}{\lambda^3}. \tag{2.22}$$

All functions depending on $t\in[0,t_0]$, for some $t_0>0$, can now be seen as depending on $s\in[0,s_0]$, where $s_0=s(t_0)$. We now claim the following properties of the decomposition of $u(t)$, possibly taking a smaller universal $\varkappa_0>0$.

LEMMA 2.7. (Modulation equations) *Assume that, for all* $t\in[0,t_0)$,

$$\|\varepsilon(t)\|_{L^2}\leqslant\varkappa\leqslant\varkappa_0 \quad \textit{and} \quad \int \varepsilon_y^2(t,y)e^{-|y|/2}\,dy\leqslant\varkappa_0 \tag{2.23}$$

for a small enough universal constant $\varkappa_0>0$. *Then the map* $s\in[0,s_0]\mapsto(\lambda(s),x(s),b(s))$ *is* $\mathcal{C}^1$ *and the following holds*:

(i) (Equation of ε) *For all* $s\in[0,s_0]$,

$$\begin{aligned}\varepsilon_s-(L\varepsilon)_y+b\Lambda\varepsilon=\left(\frac{\lambda_s}{\lambda}+b\right)(\Lambda Q_b+\Lambda\varepsilon)+\left(\frac{x_s}{\lambda}-1\right)(\varepsilon+Q_b)_y\\ +\Phi_b+\Psi_b-(R_b(\varepsilon))_y-(R_{\rm NL}(\varepsilon))_y,\end{aligned} \tag{2.24}$$

where Ψ_b *was defined in* (2.12),

$$\Phi_b=-(Q_b)_s=-b_s(\chi_b+\gamma y(\chi_b)_y)P, \tag{2.25}$$

and

$$R_b(\varepsilon)=5(Q_b^4-Q^4)\varepsilon \quad \textit{and} \quad R_{\rm NL}(\varepsilon)=(\varepsilon+Q_b)^5-5Q_b^4\varepsilon-Q_b^5. \tag{2.26}$$

(ii) (Estimates induced by the conservation laws) *On* $[0,s_0]$,

$$\|\varepsilon\|_{L^2}^2\lesssim|b|^{1/2}+\left|\int u_0^2-\int Q^2\right|, \tag{2.27}$$

$$\left|2\lambda^2E_0+\tfrac{1}{8}b\|Q\|_{L^1}^2-\|\varepsilon_y\|_{L^2}^2\right|\lesssim b^2+\|\varepsilon(s)\|_{L^2}^2+\delta(\|\varepsilon\|_{L^2})\|\varepsilon_y\|_{L^2}^2. \tag{2.28}$$

(iii) (H^1 modulation equations) *For all* $s\in[0,s_0]$,

$$\left|\frac{\lambda_s}{\lambda}+b\right|+\left|\frac{x_s}{\lambda}-1\right|\lesssim\left(\int\varepsilon^2e^{-|y|/10}\right)^{1/2}+b^2, \tag{2.29}$$

$$|b_s|\lesssim\int\varepsilon^2e^{-|y|/10}+b^2. \tag{2.30}$$

(iv) (Refined modulation equations in $\mathcal{A}$) *Assuming the following uniform L^1 control on the right*:

$$\int_{y>0} |\varepsilon(t)| \lesssim \delta(\varkappa_0) \quad \textit{for all } t \in [0, t_0), \tag{2.31}$$

the quantities J_1 and J_2 below are well-defined and satisfy the following laws:

- (Law of λ) *Let*

$$\varrho_1(y) = \frac{4}{(\int Q)^2} \int_{-\infty}^{y} \Lambda Q \quad \textit{and} \quad J_1(s) = (\varepsilon(s), \varrho_1); \tag{2.32}$$

then, for some universal constant c_1,

$$\left| \frac{\lambda_s}{\lambda} + b + c_1 b^2 - 2\left((J_1)_s + \frac{1}{2}\frac{\lambda_s}{\lambda} J_1 \right) \right| \lesssim \int \varepsilon^2 e^{-|y|/10} + |b| \left(\int \varepsilon^2 e^{-|y|/10} \right)^{1/2} + |b|^3. \tag{2.33}$$

- (Law of b) *Let*

$$\varrho_2 = \frac{16}{(\int Q)^2} \left(\frac{(\Lambda P, Q)}{\|\Lambda Q\|_{L^2}^2} \Lambda Q + P - \frac{1}{2} \int Q \right) - 8\varrho_1 \quad \textit{and} \quad J_2(s) = (\varepsilon(s), \varrho_2); \tag{2.34}$$

then, for some universal constant c_2,

$$\left| b_s + 2b^2 + c_2 b^3 + b\left((J_2)_s + \frac{1}{2}\frac{\lambda_s}{\lambda} J_2 \right) \right| \lesssim \int \varepsilon^2 e^{-|y|/10} + |b|^4. \tag{2.35}$$

- (Law of b/λ^2) *Let*

$$\varrho = 4\varrho_1 + \varrho_2 \in \mathcal{Y} \quad \textit{and} \quad J(s) = (\varepsilon(s), \varrho); \tag{2.36}$$

then, for $c_0 = c_2 - 2c_1$,

$$\left| \frac{d}{ds}\left(\frac{b}{\lambda^2} \right) + \frac{b}{\lambda^2}\left(J_s + \frac{1}{2}\frac{\lambda_s}{\lambda} J \right) + c_0 \frac{b^3}{\lambda^2} \right| \lesssim \frac{1}{\lambda^2} \left(\int \varepsilon^2 e^{-|y|/10} + |b|^4 \right). \tag{2.37}$$

Remark 2.8. It is a remarkable algebraic fact that the equation of b/λ^2 (2.37) is related to $\varrho \in \mathcal{Y}$, which means that J is an L^2 quantity which is easier to control than J_1 and J_2 separately.

The equations (2.33) and (2.35) correspond to a sharp improvement—after integration in time—of the rough estimates of (iii). However, they hold for initial data in weighted spaces such as $\mathcal{A}$. Here we are facing an intrinsic difficulty of the gKdV equation, which is that the null space of the full linearized operators L' involves badly localized terms, and hence getting geometrical parameters which are quadratic forcing terms of the ε equation (2.24) requires some L^1 control of the solution on the right. Formally, (2.33) and (2.35) are the sharp analogues of the leading-order dynamical system

$$\frac{\lambda_s}{\lambda} = -b, \quad \left(\frac{b}{\lambda^2} \right)_s = \frac{b_s + 2b^2}{\lambda^3} = 0.$$

Proof. (i) The equation of ε, λ, x and b follows by direct computations from the equation of $u(t)$. In particular, we use

$$\frac{\partial}{\partial b}Q_b=\frac{\partial}{\partial b}(b\chi_b(y))P=(\gamma|b|^\gamma y\chi'(|b|^\gamma y)+\chi(|b|^\gamma y))P=(\chi_b+\gamma y(\chi_b)_y)P.$$

The rest of the computation is done in [17, Lemma 1] for example.

(ii) We write down the L^2 conservation law

$$\int Q_b^2-\int Q^2+\int \varepsilon^2+2(\varepsilon,Q_b)=\int u_0^2-\int Q^2,$$

and we deduce from (2.15), using the orthogonality condition (2.20), that

$$\int \varepsilon^2\lesssim |b|+|b|^{1-\gamma}\|\varepsilon\|_{L^2}+\left|\int u_0^2-\int Q^2\right|.$$

Then (2.27) follows since $\gamma=\frac{3}{4}$.

Now, we write down the conservation of energy and use (2.16), the equation of Q and the orthogonality condition $(\varepsilon,Q)=0$ to estimate

$$\begin{aligned}2\lambda^2E(u_0)&=2E(Q_b)-2\int\varepsilon(Q_b)_{yy}+\int\varepsilon_y^2-\frac{1}{3}\int[(Q_b+\varepsilon)^6-Q_b^6]\\&=-2b(P,Q)+O(b^2)+\int\varepsilon_y^2-2\int\varepsilon[(Q_b-Q)_{yy}+(Q_b^5-Q^5)]\\&\quad-\frac{1}{3}\int[(Q_b+\varepsilon)^6-Q_b^6-6Q_b^5\varepsilon].\end{aligned}$$

We estimate all terms in the above identity. By the properties of Q_b,

$$\begin{aligned}\left|\int\varepsilon[(Q_b-Q)_{yy}+(Q_b^5-Q^5)]\right|&\lesssim|b|\left(\int\varepsilon^2e^{-|y|/10}\right)^{1/2}+|b|^{1+2\gamma}\int_{-2|b|^{-\gamma}<y<0}|\varepsilon|\\&\lesssim b^2+\|\varepsilon\|_{L^2}^2.\end{aligned}$$

The non-linear terms are estimated by the homogeneity of the non-linearity which implies that

$$\left|\int[(Q_b+\varepsilon)^6-Q_b^6-6Q_b^5\varepsilon]\right|\lesssim\int|Q_b|^4\varepsilon^2+|\varepsilon|^6\lesssim\|\varepsilon\|_{L^2}^2+\|\varepsilon_y\|_{L^2}^2\|\varepsilon\|_{L^2}^4.$$

The collection of the above estimates yields (2.28).

(iii) We sketch the standard computations([10]) leading to (2.29) and (2.30). Differentiating the orthogonality conditions $(\varepsilon,\Lambda Q)=(\varepsilon,y\Lambda Q)=0$, using the equation of ε and

([10]) See, e.g., [17, Lemma 4] for similar computations.

estimate (2.13), we obtain

$$\begin{aligned}\left|\left(\frac{\lambda_s}{\lambda}+b\right)-\frac{(\varepsilon,L(\Lambda Q)')}{\|\Lambda Q\|_{L^2}^2}\right|+\left|\left(\frac{x_s}{\lambda}-1\right)-\frac{(\varepsilon,L(y\Lambda Q)')}{\|\Lambda Q\|_{L^2}^2}\right|\\ \lesssim\left(\left|\frac{\lambda_s}{\lambda}+b\right|+\left|\frac{x_s}{\lambda}-1\right|+|b|\right)\left(|b|+\left(\int\varepsilon^2e^{-|y|/10}\right)^{1/2}\right)\\ +|b_s|+\int\varepsilon^2e^{-|y|/10}+\int|\varepsilon|^5e^{-9|y|/10}.\end{aligned}$$

We estimate the non-linear term using the Sobolev bound([11]) and the smallness (2.23):

$$\|\varepsilon e^{-|y|/4}\|_{L^\infty}^2\lesssim\int(|\partial_y\varepsilon|^2+|\varepsilon|^2)e^{-|y|/2},$$

so that

$$\int|\varepsilon|^5e^{-9|y|/10}\lesssim\|\varepsilon e^{-|y|/4}\|_{L^\infty}^3\int\varepsilon^2e^{-|y|/10}. \tag{2.38}$$

Thus (2.23) holds and, for $\varkappa_0$ small enough,

$$\left|\left(\frac{\lambda_s}{\lambda}+b\right)-\frac{(\varepsilon,L(\Lambda Q)')}{\|\Lambda Q\|_{L^2}^2}\right|+\left|\left(\frac{x_s}{\lambda}-1\right)-\frac{(\varepsilon,L(y\Lambda Q)')}{\|\Lambda Q\|_{L^2}^2}\right|\lesssim|b|^2+|b_s|+\int\varepsilon^2e^{-|y|/10} \tag{2.39}$$

and

$$\left|\frac{\lambda_s}{\lambda}+b\right|+\left|\frac{x_s}{\lambda}-1\right|\lesssim|b|^2+|b_s|+\left(\int\varepsilon^2e^{-|y|/10}\right)^{1/2}. \tag{2.40}$$

Next, differentiating in time s the relation $(\varepsilon,Q)=0$, using the ε equation, the algebraic facts $LQ'=0$, $(Q,\Lambda Q)=(Q,Q')=0$ and $(\varepsilon,\Lambda Q)=0$, the non-degeneracy $(P,Q)\neq0$ and the bounds (2.13) and (2.14), we find, after integration by parts and using the Sobolev estimate (2.38), that

$$|b_s|\lesssim\left|\frac{\lambda_s}{\lambda}+b\right|^2+\left|\frac{x_s}{\lambda}-1\right|^2+|b|^2+\int\varepsilon^2e^{-|y|/10} \tag{2.41}$$

(see below for a much more detailed computation of b_s).

Combining (2.40) and (2.41) yields (2.29) and (2.30).

(iv) To begin with, we claim the following sharp equation for b:

$$\begin{aligned}b_s+2b^2+cb^3-\frac{16}{(\int Q)^2}b\left[\frac{(\Lambda P,Q)}{\|\Lambda Q\|_{L^2}^2}(\varepsilon,L(\Lambda Q)')+20(\varepsilon,PQ^3Q')\right]\\ =O(|b|^4)+O\left(\int\varepsilon^2e^{-|y|/10}\right),\end{aligned} \tag{2.42}$$

([11]) Which follows by integration by parts.

where c is a universal constant.

To prove (2.42), we take the scalar product of the equation of ε by Q and we keep track of all terms up to order $|b|^3$.

In this proof, c will denote various universal constants. First, we use the explicit formula (2.17) to derive

$$\begin{aligned}(\Psi_b, Q) &= -b^2((10Q^3\chi_b^2P^2)_y + \chi_b\Lambda P, Q) - b^3(10Q^2\chi_b^3P^3, Q') + O(|b|^4)\\ &= -b^2((10P^2Q^3)' + \Lambda P, Q) - b^3(10Q^2P^3, Q') + O(|b|^4)\\ &= -\tfrac{1}{8}b^2\|Q\|_{L^1}^2 + c_0b^3 + O(|b|^4),\end{aligned}$$

where $c_0 = -10\int P^3Q^2Q'$, and where in the last step we have used the following fundamental *flux computation*:

$$\begin{aligned}(\Lambda P, Q) &= -(P, \Lambda Q) = -(P, (LP)') = (P, (P'' - P + 5Q^4P)')\\ &= (P, P''' - P') + 10\int Q^3Q'P^2,\end{aligned}$$

from which we indeed obtain

$$((10P^2Q^3)' + \Lambda P, Q) = \frac{1}{2}\lim_{y\to-\infty} P^2(y) = \frac{1}{8}\left(\int Q\right)^2. \tag{2.43}$$

This computation is the key to the derivation of the blow-up speed.

From (2.5),

$$(\Phi_b, Q) = -(b_s(\chi_b + \gamma y\chi_b')P, Q) = -b_s(P, Q) + O(b^{10}) = -\frac{b_s}{16}\left(\int Q\right)^2 + O(b^{10}).$$

Next, from (2.5),

$$\left|\left(\frac{x_s}{\lambda} - 1\right)(Q_b, Q')\right| + \left|\int(\Lambda Q_b)Q - b(\Lambda P, Q)\right| \lesssim |b|^{10}.$$

We estimate the small linear term as

$$\int R_b(\varepsilon)Q' = 20b\int PQ^3Q'\varepsilon + b^2O\left(\int \varepsilon^2e^{-|y|/10}\right)^{1/2},$$

and non-linear terms in ε are simply treated as before by (2.38).

Therefore, we have obtained

$$b_s + 2b^2 + cb^3 - \frac{16}{(\int Q)^2}b\left[\left(\frac{\lambda_s}{\lambda} + b\right)(\Lambda P, Q) + 20(\varepsilon, PQ^3Q')\right] = O(|b|^4) + O\left(\int \varepsilon^2e^{-|y|/10}\right). \tag{2.44}$$

Moreover, we check that when estimating $\lambda_s/\lambda+b$, using

$$|b_s+2b^2|\leqslant |b|^3+|b|\left(\int \varepsilon^2 e^{-|y|/10}\right)^{1/2}+\int \varepsilon^2 e^{-|y|/10},$$

and, keeping track of all b^2 terms, we can improve (2.39) into

$$\left|\left(\frac{\lambda_s}{\lambda}+b\right)-\frac{(\varepsilon, L(\Lambda Q)')}{\|\Lambda Q\|_{L^2}^2}-cb^2\right|\lesssim \int \varepsilon^2 e^{-|y|/10}+|b|\left(\int \varepsilon^2 e^{-|y|/10}\right)^{1/2}+|b|^3. \tag{2.45}$$

Estimate (2.42) follows from (2.44) and (2.45).

Due to the L^1 bound (2.31), for any $f\in\mathcal{Y}$, $\left(\varepsilon, \int_{-\infty}^y f\right)$ is well defined for all time and by direct computations we have the following general formula:

$$\begin{aligned}\frac{d}{ds}\left(\varepsilon, \int_{-\infty}^y f\right) &= -(\varepsilon, Lf)+\left(\frac{\lambda_s}{\lambda}+b\right)\left(\Lambda Q_b, \int_{-\infty}^y f\right)+\frac{\lambda_s}{\lambda}\left(\Lambda\varepsilon, \int_{-\infty}^y f\right)\\ &\quad -\left(\frac{x_s}{\lambda}-1\right)(Q_b, f)-\left(\frac{x_s}{\lambda}-1\right)(\varepsilon, f)-b_s\left((\chi_b+\gamma y\chi_b')P, \int_{-\infty}^y f\right)\\ &\quad +\left(\Psi_b, \int_{-\infty}^y f\right)+(R_b(\varepsilon)+R_{\mathrm{NL}}(\varepsilon), f).\end{aligned} \tag{2.46}$$

Using (2.29), (2.30), (2.13) and (2.42), we obtain, from (2.46),

$$\begin{aligned}\frac{d}{ds}\left(\varepsilon, \int_{-\infty}^y f\right) &= -(\varepsilon, Lf)+\left(\frac{\lambda_s}{\lambda}+b\right)\left(\Lambda Q, \int_{-\infty}^y f\right)-\left(\frac{x_s}{\lambda}-1\right)(f, Q)\\ &\quad -\frac{1}{2}\frac{\lambda_s}{\lambda}\left(\varepsilon, \int_{-\infty}^y f\right)+cb^2+O\left(\int \varepsilon^2 e^{-|y|/10}\right)\\ &\quad +O\left(|b|\left(\int \varepsilon^2 e^{-|y|/10}\right)^{1/2}\right)+O(|b|^3)\end{aligned} \tag{2.47}$$

for some constant c depending on f.

– Equation of J_1: We apply (2.47) to $f=\Lambda Q$, using the algebraic relations

$$L\Lambda Q=-2Q, \quad \left(\Lambda Q, \int_{-\infty}^y \Lambda Q\right)=\frac{1}{8}\left(\int Q\right)^2 \quad \text{and} \quad \left(Q', \int_{-\infty}^y \Lambda Q\right)=0,$$

to prove that

$$\begin{aligned}2(J_1)_s &= \frac{16(\varepsilon, Q)}{(\int Q)^2}+\left(\frac{\lambda_s}{\lambda}+b\right)-\frac{\lambda_s}{\lambda}J_1+cb^2\\ &\quad +O\left(\int \varepsilon^2 e^{-|y|/10}\right)+O\left(|b|\left(\int \varepsilon^2 e^{-|y|/10}\right)^{1/2}\right)+O(|b|^3).\end{aligned}$$

The orthogonality conditions (2.20) now yield (2.33).

– Equation of J_2. We now apply (2.47) to $\int_{-\infty}^{y} f=\varrho_2$, $f=\varrho_2'$. We need some computation related to ϱ_2. Using $\int \Lambda Q=-\frac{1}{2}\int Q$,

$$\begin{aligned}(\Lambda Q,\varrho_2) &= \frac{16}{(\int Q)^2}\left(\frac{(\Lambda P,Q)}{\|\Lambda Q\|_{L^2}^2}\Lambda Q+P-\frac{1}{2}\int Q,\Lambda Q\right)-\frac{32}{(\int Q)^2}\left(\Lambda Q,\int_{-\infty}^{y}\Lambda Q\right)\\ &= \frac{16}{(\int Q)^2}[(\Lambda P,Q)+(\Lambda Q,P)]+\frac{4}{(\int Q)^2}\left(\int Q\right)^2-\frac{16}{(\int Q)^2}\left(\int \Lambda Q\right)^2=0,\end{aligned}$$

and similarly

$$(\varrho_2',Q)=\frac{16}{(\int Q)^2}\left(\frac{(\Lambda P,Q)}{\|\Lambda Q\|_{L^2}^2}(\Lambda Q)'+P',Q\right)-8(\varrho_1',Q)=0.$$

Next, the algebra

$$L(P')=(LP)'+20Q^3Q'P=\Lambda Q+20Q^3Q'P,$$

and the orthogonality relations $(\varepsilon,\Lambda Q)=0$ and $(P,Q')=0$ yield

$$\begin{aligned}(\varepsilon,L\varrho_2') &= \frac{16}{(\int Q)^2}\left(\varepsilon,L\left[\frac{(\Lambda P,Q)}{\|\Lambda Q\|_{L^2}^2}(\Lambda Q)'+P'\right]\right)-8(\varepsilon,L\varrho_1')\\ &= \frac{16}{(\int Q)^2}\left[\frac{(\Lambda P,Q)}{\|\Lambda Q\|_{L^2}^2}(\varepsilon,L(\Lambda Q)')+20(\varepsilon,PQ^3Q')\right].\end{aligned}$$

Inserting these relations into (2.46) yields

$$\begin{aligned}\frac{d}{ds}J_2 &= -\frac{16}{(\int Q)^2}\left[\frac{(\Lambda P,Q)}{\|\Lambda Q\|_{L^2}^2}(\varepsilon,L(\Lambda Q)')+20\int \varepsilon PQ^3Q'\right]-\frac{1}{2}\frac{\lambda_s}{\lambda}J_2\\ &\quad +cb^2+O\left(\int \varepsilon^2e^{-|y|/10}\right)+O\left(|b|\left(\int \varepsilon^2e^{-|y|/10}\right)^{1/2}\right)+O(|b|^3).\end{aligned}\tag{2.48}$$

Combining (2.42) and (2.48) yields (2.35).

– Equation of J. We now compute, from (2.33) and (2.35),

$$\begin{aligned}\frac{d}{ds}\left(\frac{b}{\lambda^2}\right) &= \frac{b_s}{\lambda^2}-2\frac{\lambda_s}{\lambda}\frac{b}{\lambda^2}=\frac{b_s+2b^2}{\lambda^2}-\frac{2b}{\lambda^2}\left(\frac{\lambda_s}{\lambda}+b\right)\\ &= -\frac{b}{\lambda^2}\left[(J_2)_s+\frac{1}{2}\frac{\lambda_s}{\lambda}J_2\right]-\frac{2b}{\lambda^2}\left[2(J_1)_s+\frac{\lambda_s}{\lambda}J_1\right]\\ &\quad +(2c_1-c_2)\frac{b^3}{\lambda^2}+\frac{1}{\lambda^2}O\left(\int \varepsilon^2e^{-|y|/10}+b^4\right)\\ &= -\frac{b}{\lambda^2}\left[J_s+\frac{1}{2}\frac{\lambda_s}{\lambda}J\right]+(2c_1-c_2)\frac{b^3}{\lambda^2}+\frac{1}{\lambda^2}O\left(\int \varepsilon^2e^{-|y|/10}+b^4\right),\end{aligned}$$

which is (2.37).

Finally, we check that $\varrho = 4\varrho_1 + \varrho_2 \in \mathcal{Y}$. Indeed, ϱ_1 and ϱ_2 are exponentially localized at $-\infty$ from (2.4). We thus only need check that $\lim_{y\to\infty} \varrho(y) = 0$, but it is immediate from their definitions that

$$\lim_{y\to\infty} \varrho_1(y) = -\frac{2}{\int Q} \quad \text{and} \quad \lim_{y\to\infty} \varrho_2(y) = \frac{8}{\int Q}.$$

This concludes the proof of Lemma 2.7. □

2.5. Kato-type identities

We recall the following standard identities which correspond to the localization of conservation laws.

CLAIM 1. (Kato localization identities [9]) *Let g be any C^3 function and $v(t,x)$ be a solution of* (1.1). *Then, the following identities hold*:

(i) (L^2 identity)

$$\frac{d}{dt}\int v^2 g = -3\int v_x^2 g' + \int v^2 g''' + \frac{5}{3}\int v^6 g'. \tag{2.49}$$

(ii) (Energy identity)

$$\frac{d}{dt}\int \left(v_x^2 - \frac{1}{3}v^6\right) g = -\int (v_{xx} + v^5)^2 g' - 2\int v_{xx}^2 g' + 10\int v^4 v_x^2 g' + \int v_x^2 g'''. \tag{2.50}$$

3. Monotonicity formulas

This section is devoted to the derivation of the monotonicity tools for solutions near the soliton manifold which are the key technical arguments of our analysis for initial data in $\mathcal{A}$. We exhibit a Lyapunov functional based on a suitable localization of the linearized Hamiltonian, which will both control pointwise dispersion around the soliton, and display some monotonicity due to the coercivity of the virial quadratic form proved in [20]. A related strategy originated in [24], [32], [33] and [39], but is implemented here in a new optimal way. Such dispersive estimates coupled with the modulation equation for b will lead to the key rigidity property for the proof of the main results of this paper.

3.1. Pointwise monotonicity

Let $\varphi_1, \varphi_2, \psi \in C^\infty(\mathbb{R})$ be such that

$$\varphi_i(y) = \begin{cases} e^y & \text{for } y < -1, \\ 1+y & \text{for } -\frac{1}{2} < y < \frac{1}{2}, \\ y^i & \text{for } y > 2, \end{cases} \quad \varphi_i'(y) > 0 \text{ for all } y \in \mathbb{R}, \quad i = 1, 2, \tag{3.1}$$

$$\psi(y) = \begin{cases} e^{2y} & \text{for } y < -1, \\ 1 & \text{for } y > -\frac{1}{2}, \end{cases} \qquad \psi'(y) \geqslant 0 \text{ for all } y \in \mathbb{R}. \tag{3.2}$$

Let $B > 100$ be a large universal constant to be chosen in Proposition 3.1, let

$$\psi_B(y) = \psi\left(\frac{y}{B}\right) \quad \text{and} \quad \varphi_{i,B} = \varphi_i\left(\frac{y}{B}\right), \ i = 1, 2,$$

and define the following norms on ε:

$$\mathcal{N}_i(s) = \int \varepsilon_y^2(s,y)\psi_B(y)\,dy + \int \varepsilon^2(s,y)\varphi_{i,B}(y)\,dy, \quad i = 1, 2. \tag{3.3}$$

We also define the following L^2 weighted norms for ε:

$$\mathcal{N}_{i,\mathrm{loc}}(s) = \int \varepsilon^2(s,y)\varphi_{i,B}'(y)\,dy, \quad i = 1, 2. \tag{3.4}$$

The heart of our analysis is the following monotonicity property.

PROPOSITION 3.1. (Monotonicity formula) *There exist $\mu > 0$, $B > 100$ and $0 < \varkappa^* < \varkappa_0$ such that the following holds. Assume that $u(t)$ is a solution of* (1.1) *which satisfies* (2.18) *on $[0, t_0]$ and thus on $[0, t_0]$ admits a decomposition* (2.19) *as in Lemma* 2.5. *Let $s_0 = s(t_0)$, and assume the following a-priori bounds, for all $s \in [0, s_0]$*:

(H1) (smallness)

$$\|\varepsilon(s)\|_{L^2} + |b(s)| + \mathcal{N}_2(s) \leqslant \varkappa^*; \tag{3.5}$$

(H2) (bound related to scaling)

$$\frac{|b(s)| + \mathcal{N}_2(s)}{\lambda^2(s)} \leqslant \varkappa^*; \tag{3.6}$$

(H3) (L^2 weighted bound on the right)

$$\int_{y>0} y^{10}\varepsilon^2(s,y)\,dy \leqslant 10\left(1 + \frac{1}{\lambda^{10}(s)}\right). \tag{3.7}$$

Consider, for $(i,j)\in\{1,2\}^2$, *the energy-virial Lyapunov functionals*

$$\mathcal{F}_{i,j}=\int\left[\varepsilon_y^2\psi_B+\varepsilon^2(1+\mathcal{J}_{i,j})\varphi_{i,B}-\frac{1}{3}((\varepsilon+Q_b)^6-Q_b^6-6\varepsilon Q_b^5)\psi_B\right], \tag{3.8}$$

with

$$\mathcal{J}_{i,j}=(1-J_1)^{-(4(j-1)+2i)}-1. \tag{3.9}$$

Then the following estimates hold on $[0,s_0]$:

(i) (Scaling invariant Lyapunov control) *for* $i=1,2$,

$$\frac{d\mathcal{F}_{i,1}}{ds}+\mu\int(\varepsilon_y^2+\varepsilon^2)\varphi'_{i,B}\lesssim|b|^4. \tag{3.10}$$

(ii) (Scaling weighted H^1 Lyapunov control) *for* $i=1,2$,

$$\frac{d}{ds}\left(\frac{\mathcal{F}_{i,2}}{\lambda^2}\right)+\frac{\mu}{\lambda^2}\int(\varepsilon_y^2+\varepsilon^2)\varphi'_{i,B}\lesssim\frac{|b|^4}{\lambda^2}. \tag{3.11}$$

(iii) (Coercivity of $\mathcal{F}_{i,j}$ and pointwise bounds) *for* $(i,j)\in\{1,2\}^2$,

$$\mathcal{N}_i\lesssim\mathcal{F}_{i,j}\lesssim\mathcal{N}_i, \tag{3.12}$$

$$|J_i|+|\mathcal{J}_{i,j}|\lesssim\mathcal{N}_2^{1/2}. \tag{3.13}$$

Remark 3.2. The L^2 weighted bound (3.7) is fundamental for the analysis and will be further dynamically bootstrapped for an initial data in $\mathcal{A}$. Also, one should think of (3.10) as a scaling-invariant L^2 bound, which is sharpened in the singular regime $\lambda\to0$ by the H^1 control (3.11). Finally, an important feature of Proposition 3.1 is that we do not assume any a-priori control on the scaling parameter $\lambda(s)$.

We will use several times in the proof the fact that in the definition of $\mathcal{F}_{i,j}$, the weight on ε_y at $-\infty$ is stronger than the weight on ε. It follows in particular that $\mathcal{F}_{i,j}$ does not control $\int\varepsilon_y^2\varphi'_{i,B}$. See Remark 3.5 below.

Proof. Step 1. Weighted L^2 controls at the right.

We first claim, for all $s\in[0,s_0]$, the controls

$$\int_{y>0}y\varepsilon^2(s,y)\,dy\lesssim\left(1+\frac{1}{\lambda^{10/9}(s)}\right)\mathcal{N}_{1,\mathrm{loc}}^{8/9}(s), \tag{3.14}$$

$$\int_{y>0}y^2\varepsilon^2(s,y)\,dy\lesssim\left(1+\frac{1}{\lambda^{10/9}(s)}\right)\mathcal{N}_{2,\mathrm{loc}}^{8/9}(s), \tag{3.15}$$

$$\int_{y>0}|\varepsilon(s,y)|\,dy\lesssim\mathcal{N}_2^{1/2}(s). \tag{3.16}$$

From (3.7), for all $A>0$,

$$\int_{y>0} y\varepsilon^2 \leqslant A\int_{0\leqslant y\leqslant A}|\varepsilon|^2+\frac{1}{A^9}\int_{y>A}y^{10}|\varepsilon|^2\lesssim A\mathcal{N}_{1,\mathrm{loc}}+\frac{1}{A^9}\left(1+\frac{1}{\lambda^{10}}\right),$$

and so the optimal choice

$$A^{10}\mathcal{N}_{1,\mathrm{loc}}=1+\frac{1}{\lambda^{10}}$$

leads, using the smallness (3.5), to the bound

$$\int_{y>0} y\varepsilon^2 \lesssim \frac{(1+\lambda^{10})^{1/10}}{\lambda}\mathcal{N}_{1,\mathrm{loc}}^{9/10}\lesssim\left(1+\frac{1}{\lambda}\right)\mathcal{N}_{1,\mathrm{loc}}^{9/10}\lesssim\left(1+\frac{1}{\lambda^{10/9}}\right)\mathcal{N}_{1,\mathrm{loc}}^{8/9},$$

and (3.14) is proved. Similarly,

$$\int_{y>0} y^2\varepsilon^2 \leqslant A\int_{0\leqslant y\leqslant A}y|\varepsilon|^2+\frac{1}{A^8}\int_{y>A}y^{10}|\varepsilon|^2\lesssim A\mathcal{N}_{2,\mathrm{loc}}+\frac{1}{A^8}\left(1+\frac{1}{\lambda^{10}}\right),$$

and thus the choice

$$A^9\mathcal{N}_{2,\mathrm{loc}}=1+\frac{1}{\lambda^{10}}$$

leads to the bound

$$\int_{y>0} y^2\varepsilon^2 \lesssim \mathcal{N}_{2,\mathrm{loc}}^{8/9}\frac{(1+\lambda^{10})^{1/9}}{\lambda^{10/9}}\lesssim\left(1+\frac{1}{\lambda^{10/9}}\right)\mathcal{N}_{2,\mathrm{loc}}^{8/9},$$

and (3.15) is proved.

The bound (3.16) follows from

$$\int_{y>0}|\varepsilon|\lesssim\|(1+y)\varepsilon\|_{L^2(y>0)}\lesssim\mathcal{N}_2^{1/2}.$$

Finally, we observe that (3.16) implies (3.13). In particular, the quantities $\mathcal{J}_{i,j}$ are well defined, and so are $\mathcal{F}_{i,j}$.

Step 2. Algebraic computations on $\mathcal{F}_{i,j}$.

We compute

$$\begin{aligned}\lambda^{2(j-1)}\frac{d}{ds}\left(\frac{\mathcal{F}_{i,j}}{\lambda^{2(j-1)}}\right)=2\int\psi_B(\varepsilon_y)_s\varepsilon_y+2\varepsilon_s((1+\mathcal{J}_{i,j})\varepsilon\varphi_{i,B}-\psi_B[(\varepsilon+Q_b)^5-Q_b^5])\\+(\mathcal{J}_{i,j})_s\int\varphi_{i,B}\varepsilon^2-2\int\psi_B(Q_b)_s[(\varepsilon+Q_b)^5-Q_b^5-5\varepsilon Q_b^4]\\-2(j-1)\frac{\lambda_s}{\lambda}\mathcal{F}_{i,j},\end{aligned}$$

which we rewrite as

$$\lambda^{2(j-1)}\frac{d}{ds}\left(\frac{\mathcal{F}_{i,j}}{\lambda^{2(j-1)}}\right)=f_1^{(i)}+f_2^{(i,j)}+f_3^{(i,j)}+f_4^{(i)}, \tag{3.17}$$

where

$$\begin{aligned}
f_1^{(i)}&=2\int\left(\varepsilon_s-\frac{\lambda_s}{\lambda}\Lambda\varepsilon\right)(-(\psi_B\varepsilon_y)_y+\varepsilon\varphi_{i,B}-\psi_B[(\varepsilon+Q_b)^5-Q_b^5]),\\
f_2^{(i,j)}&=2\int\left(\varepsilon_s-\frac{\lambda_s}{\lambda}\Lambda\varepsilon\right)\varepsilon\mathcal{J}_{i,j}\varphi_{i,B},\\
f_3^{(i,j)}&=2\frac{\lambda_s}{\lambda}\int\Lambda\varepsilon(-(\psi_B\varepsilon_y)_y+(1+\mathcal{J}_{i,j})\varepsilon\varphi_{i,B}-\psi_B[(\varepsilon+Q_b)^5-Q_b^5])\\
&\quad+(\mathcal{J}_{i,j})_s\int\varphi_{i,B}\varepsilon^2-2(j-1)\frac{\lambda_s}{\lambda}\mathcal{F}_{i,j},\\
f_4^{(i)}&=-2\int\psi_B(Q_b)_s((\varepsilon+Q_b)^5-Q_b^5-5\varepsilon Q_b^4).
\end{aligned}$$

We claim the following estimates on the above terms: for some $\mu_0>0$,

$$f_1^{(i)}\leqslant-\mu_0\int(\varepsilon_y^2+\varepsilon^2)\varphi'_{i,B}+C|b|^4, \tag{3.18}$$

$$|f_k^{(i)}|\leqslant\frac{\mu_0}{10}\int(\varepsilon_y^2+\varepsilon^2)\varphi'_{i,B}+C|b|^4, \qquad \text{for } k=2,3,4. \tag{3.19}$$

Note that, in (3.18), we obtained a negative term $-\mu\int(\varepsilon_y^2+\varepsilon^2)\varphi'_{i,B}$, related both to the smoothing effect of the gKdV equation and to a virial estimate for the linearization of the gKdV equation close to the soliton. Inserting (3.18) and (3.19) into (3.17) indeed yields (3.10) and (3.11).

In Steps 3–6, we prove (3.18) and (3.19). Observe that the definitions of φ_i and ψ imply the following estimates:

$$|\varphi_i'''(y)|+|\varphi_i''(y)|+|\psi'''(y)|+|y\psi'(y)|+|\psi(y)|\lesssim\varphi_i'(y)\lesssim\varphi_i(y) \quad \text{for all } y\in\mathbb{R}, \tag{3.20}$$

$$e^{|y|}\psi(y)+e^{|y|}\psi'(y)+\varphi_i(y)\lesssim\varphi_i'(y) \quad \text{for all } y\in(-\infty,2], \tag{3.21}$$

$$\varphi_2'(y)\lesssim\varphi_1(y)\lesssim\varphi_2'(y) \quad \text{for all } y\in\mathbb{R}. \tag{3.22}$$

In particular,

$$\mathcal{N}_{1,\mathrm{loc}}(s)\lesssim\mathcal{N}_{2,\mathrm{loc}}(s)\lesssim\mathcal{N}_1(s)\lesssim\mathcal{N}_2(s) \quad \text{and} \quad \int\varepsilon^2(s,y)\varphi_{1,B}(y)\,dy\lesssim\mathcal{N}_{2,\mathrm{loc}}(s). \tag{3.23}$$

Step 3. Control of $f_1^{(i)}$. Proof of (3.18).

We compute $f_1^{(i)}$ using the ε equation (2.24) in the form

$$\begin{aligned}\varepsilon_s-\frac{\lambda_s}{\lambda}\Lambda\varepsilon=(-\varepsilon_{yy}+\varepsilon-(\varepsilon+Q_b)^5+Q_b^5)_y\\+\left(\frac{\lambda_s}{\lambda}+b\right)\Lambda Q_b+\left(\frac{x_s}{\lambda}-1\right)(\varepsilon+Q_b)_y+\Phi_b+\Psi_b,\end{aligned}\tag{3.24}$$

where

$$\Phi_b=-b_s(\chi_b+\gamma y(\chi_b)_y)P \quad \text{and} \quad -\Psi_b=(Q_b''-Q_b+Q_b^5)'+b\Lambda Q_b.$$

This yields

$$\begin{aligned}f_1^{(i)}&=2\int(-\varepsilon_{yy}+\varepsilon-[(\varepsilon+Q_b)^5-Q_b^5])_y(-(\psi_B\varepsilon_y)_y+\varepsilon\varphi_{i,B}-\psi_B[(\varepsilon+Q_b)^5-Q_b^5])\\&\quad+2\left(\frac{\lambda_s}{\lambda}+b\right)\int\Lambda Q_b(-(\psi_B\varepsilon_y)_y+\varepsilon\varphi_{i,B}-\psi_B[(\varepsilon+Q_b)^5-Q_b^5])\\&\quad+2\left(\frac{x_s}{\lambda}-1\right)\int(\varepsilon+Q_b)_y(-(\psi_B\varepsilon_y)_y+\varepsilon\varphi_{i,B}-\psi_B[(\varepsilon+Q_b)^5-Q_b^5])\\&\quad+2\int\Phi_b(-(\psi_B\varepsilon_y)_y+\varepsilon\varphi_{i,B}-\psi_B[(\varepsilon+Q_b)^5-Q_b^5])\\&\quad+2\int\Psi_b(-(\psi_B\varepsilon_y)_y+\varepsilon\varphi_{i,B}-\psi_B[(\varepsilon+Q_b)^5-Q_b^5])\\&=f_{1,1}^{(i)}+f_{1,2}^{(i)}+f_{1,3}^{(i)}+f_{1,4}^{(i)}+f_{1,5}^{(i)}.\end{aligned}$$

Term $f_{1,1}^{(i)}$. This term contains the leading-order negative quadratic terms due to our choice of orthogonality conditions and suitable repulsivity properties of the virial quadratic form([12]) on the soliton core, and intrinsic monotoninicity properties of the renormalized KdV flow in the moving frame at speed 1 which expulses energy to the left and leads to positive terms induced by localization of both mass and energy.

Let us first integrate by parts in order to obtain a more manageable formula:

$$\begin{aligned}f_{1,1}^{(i)}=2\int(-\varepsilon_{yy}+\varepsilon-[(\varepsilon+Q_b)^5-Q_b^5])_y(-\varepsilon_{yy}+\varepsilon-[(\varepsilon+Q_b)^5-Q_b^5])\psi_B\\+2\int(-\varepsilon_{yy}+\varepsilon-[(\varepsilon+Q_b)^5-Q_b^5])_y(-\psi_B'\varepsilon_y+\varepsilon(\varphi_{i,B}-\psi_B)).\end{aligned}$$

([12]) See Lemma 3.4.

We compute the various terms separately:

$$\begin{aligned}
&2\int(-\varepsilon_{yy}+\varepsilon-[(\varepsilon+Q_b)^5-Q_b^5])_y\psi_B(-\varepsilon_{yy}+\varepsilon-[(\varepsilon+Q_b)^5-Q_b^5])\\
&\quad=-\int\psi_B'(-\varepsilon_{yy}+\varepsilon-[(\varepsilon+Q_b)^5-Q_b^5])^2\\
&\quad=-\int\psi_B'(-\varepsilon_{yy}+\varepsilon)^2-\int\psi_B'((-\varepsilon_{yy}+\varepsilon-[(\varepsilon+Q_b)^5-Q_b^5])^2-(-\varepsilon_{yy}+\varepsilon)^2)\\
&\quad=-\left[\int\psi_B'(\varepsilon_{yy}^2+2\varepsilon_y^2)+\int\varepsilon^2(\psi_B'-\psi_B''')\right]\\
&\qquad-\int\psi_B'((-\varepsilon_{yy}+\varepsilon-[(\varepsilon+Q_b)^5-Q_b^5])^2-(-\varepsilon_{yy}+\varepsilon)^2).
\end{aligned}$$

Next, after integration by parts,

$$\begin{aligned}
2\int(-\varepsilon_{yy}+\varepsilon)_y[-\psi_B'\varepsilon_y+\varepsilon(\varphi_{i,B}-\psi_B)]=-2\bigg[&\int\psi_B'\varepsilon_{yy}^2+\int\varepsilon_y^2\left(\frac{3}{2}\varphi_{i,B}'-\frac{1}{2}\psi_B'-\frac{1}{2}\psi_B'''\right)\\
&+\int\varepsilon^2\left(\frac{1}{2}(\varphi_{i,B}-\psi_B)'-\frac{1}{2}(\varphi_{i,B}-\psi_B)'''\right)\bigg].
\end{aligned}$$

Similarly,

$$\begin{aligned}
&-2\int[(\varepsilon+Q_b)^5-Q_b^5]_y(\varphi_{i,B}-\psi_B)\varepsilon\\
&\quad=-\frac{1}{3}\int(\varphi_{i,B}-\psi_B)'([(\varepsilon+Q_b)^6-Q_b^6-6Q_b^5\varepsilon]-6[(\varepsilon+Q_b)^5-Q_b^5]\varepsilon)\\
&\qquad-2\int(\varphi_{i,B}-\psi_B)(Q_b)_y[(\varepsilon+Q_b)^5-Q_b^5-5Q_b^4\varepsilon],
\end{aligned}$$

and, by direct expansion,

$$\int[(\varepsilon+Q_b)^5-Q_b^5]_y\psi_B'\varepsilon_y=5\int\psi_B'\varepsilon_y((Q_b)_y[(\varepsilon+Q_b)^4-Q_b^4]+(\varepsilon+Q_b)^4\varepsilon_y).$$

We collect the above computations and obtain

$$\begin{aligned}
f_{1,1}^{(i)}=&-\int[3\psi_B'\varepsilon_{yy}^2+(3\varphi_{i,B}'+\psi_B'-\psi_B''')\varepsilon_y^2+(\varphi_{i,B}'-\varphi_{i,B}''')\varepsilon^2]\\
&-2\int\left[\frac{(\varepsilon+Q_b)^6-Q_b^6}{6}-Q_b^5\varepsilon-[(\varepsilon+Q_b)^5-Q_b^5]\varepsilon\right](\varphi_{i,B}'-\psi_B')\\
&+2\int[(\varepsilon+Q_b)^5-Q_b^5-5Q_b^4\varepsilon](Q_b)_y(\psi_B-\varphi_{i,B})\\
&+10\int\psi_B'\varepsilon_y((Q_b)_y[(\varepsilon+Q_b)^4-Q_b^4]+(\varepsilon+Q_b)^4\varepsilon_y)\\
&-\int\psi_B'((-\varepsilon_{yy}+\varepsilon-[(\varepsilon+Q_b)^5-Q_b^5])^2-(-\varepsilon_{yy}+\varepsilon)^2)\\
=&(f_{1,1}^{(i)})^{<}+(f_{1,1}^{(i)})^{\sim}+(f_{1,1}^{(i)})^{>},
\end{aligned}$$

where $(f_{1,1}^{(i)})^<$, $(f_{1,1}^{(i)})^\sim$ and $(f_{1,1}^{(i)})^>$ correspond to integration over $y<-\frac{1}{2}B$, $|y|\leqslant\frac{1}{2}B$ and $y>\frac{1}{2}B$, respectively.

For the region $y<-\frac{1}{2}B$, we rely on monotonicity type arguments and estimate, using (3.20),

$$\int_{y<-B/2}\varepsilon^2|\varphi_{i,B}'''|\lesssim\frac{1}{B^2}\int_{y<-B/2}\varepsilon^2\varphi_{i,B}'\leqslant\frac{1}{100}\int_{y<-B/2}\varepsilon^2\varphi_{i,B}',$$
$$\int_{y<-B/2}\varepsilon_y^2|\psi_B'''|\lesssim\frac{1}{B^2}\int_{y<-B/2}\varepsilon_y^2\varphi_{i,B}'\leqslant\frac{1}{100}\int_{y<-B/2}\varepsilon_y^2\varphi_{i,B}',$$

by choosing B large enough. Next, we recall the Sobolev bound,([13]) for all $B\geqslant 1$,

$$\begin{aligned}\Big\|\varepsilon^2\sqrt{\varphi_{i,B}'}\Big\|_{L^\infty(y<-B/2)}^2&\lesssim\|\varepsilon\|_{L^2}^2\bigg(\int_{y<-B/2}\varepsilon_y^2\varphi_{i,B}'+\int_{y<-B/2}\varepsilon^2\frac{(\varphi_{i,B}'')^2}{\varphi_{i,B}'}\bigg)\\&\lesssim\delta(\varkappa^*)\int_{y<-B/2}(\varepsilon_y^2+\varepsilon^2)\varphi_{i,B}'.\end{aligned}\tag{3.25}$$

Remark 3.3. This estimate is linked to the L^2 critical nature of the problem and the smallness relies on the global L^2 smallness (3.5) only, and requires no smallness of derivatives. This is the key to control the pure ε^6 non-linear term in the functionals $\mathcal{F}_{i,j}$.

The homogeneity of the power non-linearity then ensures (for B large and $\varkappa^*$ small) that

$$\begin{aligned}\bigg|\int_{y<-B/2}&\bigg[\frac{(\varepsilon+Q_b)^6-Q_b^6}{6}-Q_b^5\varepsilon-[(\varepsilon+Q_b)^5-Q_b^5]\varepsilon\bigg](\varphi_{i,B}'-\psi_B')\bigg|\\&\lesssim\int_{y<-B/2}(\varepsilon^6+|Q_b|^4\varepsilon^2)\varphi_{i,B}'\\&\lesssim(\delta(\varkappa^*)+e^{-B/10})\int_{y<-B/2}\varphi_{i,B}'(\varepsilon^2+\varepsilon_y^2)\\&\leqslant\frac{1}{100}\int_{y<-B/2}(\varepsilon_y^2+\varepsilon^2)\varphi_{i,B}',\end{aligned}$$

and similarly, for $\varkappa^*$ small depending on B,

$$\begin{aligned}&\bigg|\int_{y<-B/2}[(\varepsilon+Q_b)^5-Q_b^5-5Q_b^4\varepsilon](Q_b)_y(\psi_B-\varphi_{i,B})\bigg|\\&\quad\lesssim B\int_{y<-B/2}(\varepsilon^2|Q_b|^3+|\varepsilon|^5)(|Q_y|+|b|\,|(P\chi_b)'|)\varphi_{i,B}'\leqslant\frac{1}{100}\int_{y<-B/2}(\varepsilon_y^2+\varepsilon^2)\varphi_{i,B}'+b^4.\end{aligned}$$

([13]) See the proof of [26, Lemma 6].

We further estimate, using (3.25) and $\psi'\lesssim(\varphi_i')^2$, for $y<-\frac{1}{2}$,

$$\begin{aligned}\left|\int_{y<-B/2}\psi_B'\varepsilon_y((Q_b)_y[(\varepsilon+Q_b)^4-Q_b^4]+(\varepsilon+Q_b)^4\varepsilon_y)\right|&\\ \lesssim e^{-B/2}\int_{y<-B/2}\varphi_{i,B}'(\varepsilon_y^2+\varepsilon^2)+\int\psi_B'|\varepsilon|^4|\varepsilon_y|^2&\\ \leqslant\frac{1}{100}\int\varepsilon_{yy}^2\psi_B'+\frac{1}{100}\int_{y<-B/2}(\varepsilon_y^2+\varepsilon^2)\varphi_{i,B}'.&\end{aligned}$$

Note that, for the term $\int\psi_B'|\varepsilon|^4|\varepsilon_y|^2$, we have proceeded as follows:

$$\begin{aligned}\int\psi_B'\varepsilon_y^2\varepsilon^4&\lesssim\|\varepsilon^2(\psi_B')^{1/4}\|_{L^\infty}^2\int\varepsilon_y^2(\psi_B')^{1/2}\\ &\lesssim\|\varepsilon\|_{L^2}^2\left(\int(\varepsilon_y^2+\varepsilon^2)(\psi_B')^{1/2}\right)\int\varepsilon_y^2(\psi_B')^{1/2}\\ &\lesssim\delta(\alpha^*)\left(\int\varepsilon_y^2(\psi_B')^{1/2}\right)^2+\delta(\alpha^*)\int\varepsilon_y^2\varphi_{i,B}'\end{aligned}$$

and

$$\left(\int\varepsilon_y^2(\psi_B')^{1/2}\right)^2=\left(-\int\varepsilon\varepsilon_{yy}(\psi_B')^{1/2}+\frac{1}{2}\int\varepsilon^2((\psi_B')^{1/2})''\right)^2\lesssim\left(\int\varepsilon^2\right)\int(\varepsilon_{yy}^2+\varepsilon^2)\psi_B'.$$

Thus,

$$\int\psi_B'\varepsilon_y^2\varepsilon^4\lesssim\delta(\alpha^*)\int(\varepsilon_{yy}^2+\varepsilon^2)\psi_B'+\delta(\alpha^*)\int\varepsilon_y^2\varphi_{i,B}'.$$

The remaining non-linear term is estimated using the local H^2 control provided by localization:

$$\begin{aligned}&\left|\int_{y<-B/2}\psi_B'((-\varepsilon_{yy}+\varepsilon-[(\varepsilon+Q_b)^5-Q_b^5])^2-(-\varepsilon_{yy}+\varepsilon)^2)\right|\\ &\quad=\left|\int_{y<-B/2}\psi_B'(-2\varepsilon_{yy}+2\varepsilon-[(\varepsilon+Q_b)^5-Q_b^5])[(\varepsilon+Q_b)^5-Q_b^5]\right|\\ &\quad\lesssim\frac{1}{100}\int_{y<-B/2}\psi_B'(|\varepsilon_{yy}|^2+|\varepsilon|^2)+100\int_{y<-B/2}(\varphi_{i,B}')^2(|\varepsilon|\,|Q_b|^4+|\varepsilon|^5)^2\\ &\quad\lesssim\frac{1}{100}\int_{y<-B/2}[\varepsilon_{yy}^2\psi_B'+(\varepsilon_y^2+\varepsilon^2)\varphi_{i,B}'].\end{aligned}$$

In the region $y>\frac{1}{2}B$, one has $\psi_B(y)=1$. We rely on (3.20) to estimate

$$\int_{y>B/2}\varepsilon^2|\varphi_{i,B}'''|\lesssim\frac{1}{B^2}\int_{y>B/2}\varepsilon^2\varphi_{i,B}'\leqslant\frac{1}{100}\int_{y>B/2}\varepsilon^2\varphi_{i,B}',$$

and we use the exponential localization of Q_b to the right and the Sobolev bound

$$\|\varepsilon\|_{L^\infty(y>0)} \lesssim \|\varepsilon\|_{H^1(y>0)} \lesssim \mathcal{N}_2^{1/2} \lesssim \delta(\boldsymbol{\varkappa}^*)$$

to control

$$\begin{aligned}
&\left|\int_{y>B/2}\left(\frac{(\varepsilon+Q_b)^6-Q_b^6}{6}-Q_b^5\varepsilon-[(\varepsilon+Q_b)^5-Q_b^5]\varepsilon\right)\varphi'_{i,B}\right| \\
&\qquad \lesssim \int_{y>B/2}(\varepsilon^6+|Q_b|^4\varepsilon^2)\varphi'_{i,B} \\
&\qquad \lesssim (\delta(\boldsymbol{\varkappa}^*)+e^{-B/10})\int_{y>B/2}\varphi'_{i,B}(\varepsilon^2+\varepsilon_y^2) \\
&\qquad \leqslant \frac{1}{100}\int_{y>B/2}(\varepsilon_y^2+\varepsilon^2)\varphi'_{i,B}
\end{aligned}$$

and

$$\begin{aligned}
&\left|\int_{y>B/2}[(\varepsilon+Q_b)^5-Q_b^5-5Q_b^4\varepsilon](Q_b)_y(\psi_B-\varphi_{i,B})\right| \\
&\qquad \lesssim \int_{y>B/2}(\varepsilon^2|Q_b|^3+|\varepsilon|^5)(|Q_y|+|b|e^{-|y|}) \leqslant \frac{1}{100}\int_{y>B/2}(\varepsilon_y^2+\varepsilon^2)\varphi'_{i,B}.
\end{aligned}$$

In the region $|y|<\frac{1}{2}B$, one has $\varphi_{i,B}(s,y)=1+y/B$ and $\psi_B(y)=1$. In particular, $\varphi'''_{i,B}=\psi'_B=0$ in this region, and we obtain

$$\begin{aligned}
(f_{1,1}^{(i)})^\sim &= -\frac{1}{B}\int_{|y|<B/2}\Bigg[3\varepsilon_y^2+\varepsilon^2+2\left(\frac{(\varepsilon+Q_b)^6-Q_b^6}{6}-Q_b^5\varepsilon-[(\varepsilon+Q_b)^5-Q_b^5]\varepsilon\right) \\
&\qquad\qquad +2[(\varepsilon+Q_b)^5-Q_b^5-5Q_b^4\varepsilon]y(Q_b)_y\Bigg] \\
&= -\frac{1}{B}\int_{|y|<B/2}(3\varepsilon_y^2+\varepsilon^2-5Q^4\varepsilon^2+20yQ'Q^3\varepsilon^2)+R_{\mathrm{Vir}}(\varepsilon),
\end{aligned}$$

where

$$\begin{aligned}
R_{\mathrm{Vir}}(\varepsilon) = -\frac{1}{B}\int_{|y|<B/2}\Bigg[&-5(Q_b^4-Q^4)\varepsilon^2+20y((Q_b)_yQ_b^3-Q'Q^3)\varepsilon^2-\frac{40}{3}Q_b^3\varepsilon^3-15Q_b^2\varepsilon^4 \\
&-8Q_b\varepsilon^5-\frac{5}{3}\varepsilon^6+20y(Q_b)_yQ_b^2\varepsilon^3+10y(Q_b)_yQ_b\varepsilon^4+2y(Q_b)_y\varepsilon^5\Bigg].
\end{aligned}$$

We now claim the following coercivity result which is the main tool to measure dispersion (related to the virial estimate, see §A.2).

LEMMA 3.4. (Localized virial estimate) *There exists* $B_0>100$ *and* $\mu_3>0$ *such that, if* $B\geqslant B_0$, *then*

$$\int_{|y|<B/2}(3\varepsilon_y^2+\varepsilon^2-5Q^4\varepsilon^2+20yQ'Q^3\varepsilon^2)\geqslant \mu_3\int_{|y|<B/2}(\varepsilon_y^2+\varepsilon^2)-\frac{1}{B}\int \varepsilon^2 e^{-|y|/2}.$$

We further estimate, by Sobolev's inequality,

$$|R_{\mathrm{Vir}}(\varepsilon)|\lesssim \frac{1}{B}(|b|+\|\varepsilon\|_{L^\infty(|y|<B/2)})\int_{|y|<B/2}(\varepsilon_y^2+\varepsilon^2)\lesssim \frac{1}{B}\delta(\varkappa^*)\int_{|y|<B/2}(\varepsilon_y^2+\varepsilon^2),$$

and thus, for $\varkappa^*$ small enough,

$$(f_{1,1}^{(i)})^{\sim}\leqslant -\frac{\mu_3}{2B}\int_{|y|<B/2}(\varepsilon_y^2+\varepsilon^2)+\frac{1}{B^2}\int \varepsilon^2 e^{-|y|/2}.$$

The collection of the above estimates yields the bound

$$f_{1,1}^{(i)}\leqslant -\frac{\mu_4}{B}\int[\psi_B'\varepsilon_{yy}^2+\varphi_{i,B}'(\varepsilon_y^2+\varepsilon^2)]+Cb^4 \tag{3.26}$$

for some universal $\mu_4>0$ independent of B.

Term $f_{1,2}^{(i)}$. We integrate by parts to express $f_{1,2}$:

$$\begin{aligned} f_{1,2}^{(i)}=\;&2\left(\frac{\lambda_s}{\lambda}+b\right)\int \Lambda Q(L\varepsilon)-2\left(\frac{\lambda_s}{\lambda}+b\right)\int \varepsilon(1-\varphi_{i,B})\Lambda Q\\ &+2b\left(\frac{\lambda_s}{\lambda}+b\right)\int \Lambda(\chi_b P)(-(\psi_B\varepsilon_y)_y+\varepsilon\varphi_{i,B}-\psi_B[(\varepsilon+Q_b)^5-Q_b^5)])\\ &+2\left(\frac{\lambda_s}{\lambda}+b\right)\int \Lambda Q(-(\psi_B)_y\varepsilon_y-(1-\psi_B)\varepsilon_{yy}+(1-\psi_B)[(\varepsilon+Q_b)^5-Q_b^5])\\ &+2\left(\frac{\lambda_s}{\lambda}+b\right)\int \Lambda Q[(\varepsilon+Q_b)^5-Q_b^5-5Q^4\varepsilon]. \end{aligned}$$

Observe, from (2.20), that

$$\int \Lambda Q(L\varepsilon)=(\varepsilon,L\Lambda Q)=-2(\varepsilon,Q)=0.$$

We now use the orthogonality conditions $(\varepsilon,y\Lambda Q)=0$ and the definition of $\varphi_{i,B}$ to estimate

$$\left|\int \Lambda Q\varepsilon(1-\varphi_{i,B})\right|=\left|\int \Lambda Q\varepsilon\Big(1-\varphi_{i,B}+\frac{y}{B}\Big)\right|\lesssim e^{-B/8}\mathcal{N}_{i,\mathrm{loc}}^{1/2},$$

so that, by (2.29) and for B large enough,

$$\left|\left(\frac{\lambda_s}{\lambda}+b\right)\int \Lambda Q\varepsilon(1-\varphi_{i,B})\right|\lesssim (\mathcal{N}_{i,\mathrm{loc}}^{1/2}+b^2)e^{-B/8}\mathcal{N}_{i,\mathrm{loc}}^{1/2}\leqslant \frac{1}{500}\frac{\mu_4}{B}\mathcal{N}_{i,\mathrm{loc}}+Cb^4.$$

For the next term in $f_{1,2}^{(i)}$, we first integrate by parts to remove all derivatives on ε. Then, by (2.29), the weighted Sobolev bound (3.25) and the properties of $\varphi_{i,B}$, ψ_B, P and χ_b (2.9), we obtain, for $\varkappa^*$ small,

$$\begin{aligned}\left|2b\left(\frac{\lambda_s}{\lambda}+b\right)\int\Lambda(\chi_bP)(-(\psi_B\varepsilon_y)_y+\varepsilon\varphi_{i,B}-\psi_B[(\varepsilon+Q_b)^5-Q_b^5)])\right|&\\ \lesssim|b|(\mathcal{N}_{i,\mathrm{loc}}^{1/2}+b^2)\left(\int_{y<0}e^{y/B}+1\right)^{1/2}\mathcal{N}_{i,\mathrm{loc}}^{1/2}&\\ \lesssim|b|(\mathcal{N}_{i,\mathrm{loc}}^{1/2}+b^2)B^{1/2}\mathcal{N}_{i,\mathrm{loc}}^{1/2}&\\ \leqslant\frac{1}{500}\frac{\mu_4}{B}\mathcal{N}_{i,\mathrm{loc}}(s)+Cb^4.&\end{aligned}$$

Next, integrating by parts, using the exponential decay of Q and since $\psi_B(y)\equiv1$ on $\left[-\frac{1}{2}B,\infty\right)$,

$$\begin{aligned}\left|\left(\frac{\lambda_s}{\lambda}+b\right)\int\Lambda Q(-(\psi_B)_y\varepsilon_y-(1-\psi_B)\varepsilon_{yy}+(1-\psi_B)[(\varepsilon+Q_b)^5-Q_b^5])\right|&\\ \lesssim(\mathcal{N}_{i,\mathrm{loc}}^{1/2}+b^2)(e^{-B/10}+\delta(\varkappa^*))\mathcal{N}_{i,\mathrm{loc}}^{1/2}\leqslant\frac{1}{500}\frac{\mu_4}{B}\mathcal{N}_{i,\mathrm{loc}},&\end{aligned}$$

and finally

$$\left|\left(\frac{\lambda_s}{\lambda}+b\right)\int\Lambda Q[(\varepsilon+Q_b)^5-Q_b^5-5Q^4\varepsilon]\right|\lesssim(\mathcal{N}_{i,\mathrm{loc}}^{1/2}+b^2)\delta(\varkappa^*)\mathcal{N}_{i,\mathrm{loc}}^{1/2}\leqslant\frac{1}{500}\frac{\mu_4}{B}\mathcal{N}_{i,\mathrm{loc}}.$$

The collection of the above estimates yields the bound

$$|f_{1,2}^{(i)}|\leqslant\frac{1}{100}\frac{\mu_4}{B}\mathcal{N}_{i,\mathrm{loc}}+Cb^4.$$

Term $f_{1,3}^{(i)}$. We use the identity

$$\begin{aligned}\int\psi_B(Q_b)_y[(\varepsilon+Q_b)^5-Q_b^5-5Q_b^4\varepsilon]+\int\psi_B\varepsilon_y[(\varepsilon+Q_b)^5-Q_b^5]&\\ =\frac{1}{6}\int\psi_B\partial_y[(\varepsilon+Q_b)^6-Q_b^6-6Q_b^5\varepsilon]=-\frac{1}{6}\int\psi_B'[(\varepsilon+Q_b)^6-Q_b^6-6Q_b^5\varepsilon]&\end{aligned}$$

to compute

$$\begin{aligned}f_{1,3}^{(i)}=2\left(\frac{x_s}{\lambda}-1\right)&\int\frac{1}{6}\psi_B'[(\varepsilon+Q_b)^6-Q_b^6-6Q_b^5\varepsilon]\\ &+2\left(\frac{x_s}{\lambda}-1\right)\int(b\chi_bP+\varepsilon)_y[-\psi_B'\varepsilon_y-\psi_B\varepsilon_{yy}+\varepsilon\varphi_{i,B}]\\ &+2\left(\frac{x_s}{\lambda}-1\right)\int Q'[L\varepsilon-\psi_B'\varepsilon_y+(1-\psi_B)\varepsilon_{yy}-\varepsilon(1-\varphi_{i,B})]\\ &+10\left(\frac{x_s}{\lambda}-1\right)\int\varepsilon\psi_B(Q_b^4(Q_b)_y-Q^4Q_y).\end{aligned}$$

Since $|(\varepsilon+Q_b)^6-Q_b^6-6Q_b^5\varepsilon|\lesssim|\varepsilon|^2+|\varepsilon|^6$, by (3.25) and $|x_s/\lambda-1|\leqslant\delta(\varkappa^*)$, we have

$$\left|2\left(\frac{x_s}{\lambda}-1\right)\int\frac{1}{6}\psi_B'[(\varepsilon+Q_b)^6-Q_b^6-6Q_b^5\varepsilon]\right|\lesssim\delta(\varkappa^*)\int\psi_B'(|\varepsilon|^2+|\varepsilon|^6)$$
$$\leqslant\frac{1}{500}\frac{\mu_4}{B}\int(\varepsilon_y^2+\varepsilon^2)\varphi_{i,B}'.$$

Then, as before, integrating by parts, and using the Cauchy–Schwarz inequality,

$$\left|2b\left(\frac{x_s}{\lambda}-1\right)\int(\chi_bP)_y[-\psi_B'\varepsilon_y-\psi_B\varepsilon_{yy}+\varepsilon\varphi_{i,B}]\right|\lesssim|b|(\mathcal{N}_{i,\mathrm{loc}}^{1/2}+b^2)B^{1/2}\mathcal{N}_{i,\mathrm{loc}}$$
$$\leqslant\frac{1}{500}\frac{\mu_4}{B}\mathcal{N}_{i,\mathrm{loc}}+b^4$$

and

$$\left|2\left(\frac{x_s}{\lambda}-1\right)\int\varepsilon_y[-\psi_B'\varepsilon_y-\psi_B\varepsilon_{yy}+\varepsilon\varphi_{i,B}]\right|\lesssim\delta(\varkappa^*)\int(\varepsilon_y^2+\varepsilon^2)\varphi_{i,B}'$$
$$\leqslant\frac{1}{500}\frac{\mu_4}{B}\int(\varepsilon_y^2+\varepsilon^2)\varphi_{i,B}'.$$

The next term is treated using the cancellation $LQ'=0$ and the orthogonality conditions $(\varepsilon,\Lambda Q)=(\varepsilon,Q)=0$, so that $(yQ',\varepsilon)=0$. Thus, by the definitions of $\varphi_{i,B}$ and ψ_B,

$$\begin{aligned}&\left|2\left(\frac{x_s}{\lambda}-1\right)\int Q'[L\varepsilon-\psi_B'\varepsilon_y+(1-\psi_B)\varepsilon_{yy}-\varepsilon(1-\varphi_{i,B})]\right|\\&\quad=\left|2\left(\frac{x_s}{\lambda}-1\right)\int Q'\left[-\psi_B'\varepsilon_y+(1-\psi_B)\varepsilon_{yy}-\varepsilon\left(1+\frac{y}{B}-\varphi_{i,B}\right)\right]\right|\\&\quad\lesssim(\mathcal{N}_{i,\mathrm{loc}}^{1/2}+b^2)e^{-B/10}\mathcal{N}_{i,\mathrm{loc}}^{1/2}\\&\quad\leqslant\frac{1}{500}\frac{\mu_4}{B}\mathcal{N}_{i,\mathrm{loc}}+b^4.\end{aligned}$$

Finally,

$$\left|10\left(\frac{x_s}{\lambda}-1\right)\int\varepsilon\psi_B(Q_b^4(Q_b)_y-Q^4Q_y)\right|\lesssim|b|(\mathcal{N}_{i,\mathrm{loc}}^{1/2}+b^2)B^{1/2}\mathcal{N}_{i,\mathrm{loc}}^{1/2}\leqslant\frac{1}{500}\frac{\mu_4}{B}\mathcal{N}_{i,\mathrm{loc}}+Cb^4.$$

In conclusion, for $f_{1,3}^{(i)}$,

$$|f_{1,3}^{(i)}|\leqslant\frac{1}{100}\frac{\mu_4}{B}\int(\varepsilon_y^2+\varepsilon^2)\varphi_{i,B}'+Cb^4,$$

for B large enough and $\varkappa^*$ small enough.

Term $f_{1,4}^{(i)}$. We compute explicitely

$$f_{1,4}^{(i)}=-2b_s\int(\chi_b+\gamma y(\chi_b)_y)P(-\psi_B\varepsilon_{yy}-\psi_B'\varepsilon_y+\varepsilon\varphi_{i,B}-\psi_B[(\varepsilon+Q_b)^5-Q_b^5]).$$

We estimate, after integrations by parts,

$$\left|\int(\chi_b+\gamma y(\chi_b)_y)P(-\psi_B\varepsilon_y)_y\right|\lesssim\int|\varepsilon|\,|(\psi_B((\chi_b+\gamma y(\chi_b)_y)P)_y)_y|\lesssim B^{1/2}\mathcal{N}_{i,\mathrm{loc}}^{1/2},$$
$$\left|\int(\chi_b+\gamma y(\chi_b)_y)P\varepsilon\varphi_{i,B}\right|\lesssim B^{1/2}\mathcal{N}_{i,\mathrm{loc}}^{1/2}.$$

The estimate of the non-linear term follows from the weighted Sobolev estimate (3.25) with $\psi\leqslant(\varphi_i')^2$, for $y<-\frac{1}{2}$,

$$\begin{aligned}\left|\int(\chi_b+\gamma y(\chi_b)_y)P\psi_B[(\varepsilon+Q_b)^5-Q_b^5]\right|&\lesssim\int\psi_B(|Q_b|^4|\varepsilon|+|\varepsilon|^5)\\&\lesssim B^{1/2}\left(\int(|\varepsilon|^2+|\varepsilon|^6)\psi_B\right)^{1/2}\\&\lesssim B^{1/2}\left(\int(\varepsilon_y^2+\varepsilon^2)\varphi_{i,B}'\right)^{1/2}.\end{aligned}$$

Together with (2.30), these estimates yield the bound

$$|f_{1,4}|\leqslant\frac{1}{500}\frac{\mu_4}{B}\int(\varepsilon_y^2+\varepsilon^2)\varphi_{i,B}'+C|b|^4.$$

Term $f_{1,5}^{(i)}$. This term generates the leading-order term in b through the error term Ψ_b in the construction of the approximate Q_b profile. Recall that

$$f_{1,5}^{(i)}=2\int\Psi_b(-(\psi_B\varepsilon_y)_y+\varepsilon\varphi_{i,B}-\psi_B[(\varepsilon+Q_b)^5-Q_b^5]).$$

We now rely on (2.14) to estimate, by integration by parts and Cauchy–Schwarz's inequality,

$$\left|\int(\Psi_b)_y\psi_B\varepsilon_y\right|\lesssim B^{1/2}b^2\mathcal{N}_{i,\mathrm{loc}}^{1/2}\leqslant\frac{1}{500}\frac{\mu_4}{B}\mathcal{N}_{i,\mathrm{loc}}+C|b|^4.$$

By (2.13), $|\Psi_b|\leqslant b^2+|b|^{1+\gamma}\mathbf{1}_{[-2,-1]}(|b|^\gamma y)$, and so, by the exponential decay of $\varphi_{i,B}$ in the left,

$$\left|\int\Psi_b\varphi_{i,B}\varepsilon\right|\lesssim(b^2B^{1/2}+e^{-1/2|b|^\gamma})|b|^{1+\gamma}\mathcal{N}_{i,\mathrm{loc}}^{1/2}\leqslant\frac{1}{500}\frac{\mu_4}{B}\mathcal{N}_{i,\mathrm{loc}}+C|b|^4.$$

For the non-linear term, similarly and using (3.25),

$$\left|\int\Psi_b\psi_B[(\varepsilon+Q_b)^5-Q_b^5]\right|\leqslant\frac{1}{500}\frac{\mu_4}{B}\int(\varepsilon_y^2+\varepsilon^2)\varphi_{i,B}'+C|b|^4.$$

The collection of the above estimates yields the bound

$$|f_{1,5}^{(i)}|\leqslant\frac{1}{100}\frac{\mu_4}{B}\int(\varepsilon_y^2+\varepsilon^2)\varphi_{i,B}'+|b|^4.$$

Step 4. $f_2^{(i,j)}$ term.

We integrate by parts using (3.24):

$$f_2^{(i,j)} = 2\mathcal{J}_{i,j}\int \varepsilon\varphi_{i,B}\Big[(-\varepsilon_{yy}+\varepsilon-(\varepsilon+Q_b)^5+Q_b^5)_y + \Big(\frac{\lambda_s}{\lambda}+b\Big)\Lambda Q_b+\Big(\frac{x_s}{\lambda}-1\Big)(\varepsilon+Q_b)_y+\Phi_b+\Psi_b\Big].$$

We integrate by parts, estimate all terms like for $f_1^{(i)}$ and use (3.13), which implies

$$|\mathcal{J}_{i,j}| \lesssim \delta(\varkappa^*),$$

to conclude that

$$|f_2^{(i,j)}| \lesssim \delta(\varkappa^*)\Big[\int(\varepsilon_y^2+\varepsilon^2)\varphi'_{i,B}+|b|^4\Big].$$

Step 5. $f_3^{(i,j)}$ term.

Recall that

$$f_3^{(i,j)} = 2\frac{\lambda_s}{\lambda}\int \Lambda\varepsilon(-(\psi_B\varepsilon_y)_y+(1+\mathcal{J}_{i,j})\varepsilon\varphi_{i,B}-\psi_B[(\varepsilon+Q_b)^5-Q_b^5]) + (\mathcal{J}_{i,j})_s\int\varphi_{i,B}\varepsilon^2-2(j-1)\frac{\lambda_s}{\lambda}\mathcal{F}_{i,j}.$$

We integrate by parts to compute

$$\int\Lambda\varepsilon(\psi_B\varepsilon_y)_y = -\int\varepsilon_y^2\psi_B+\frac{1}{2}\int\varepsilon_y^2 y\psi'_B,$$

$$\int(\Lambda\varepsilon)\varepsilon\varphi_{i,B} = -\frac{1}{2}\int\varepsilon^2 y\varphi'_{i,B},$$

$$\int\Lambda\varepsilon\psi_B[(\varepsilon+Q_b)^5-Q_b^5] = \frac{1}{6}\int(2\psi_B-y\psi'_B)[(\varepsilon+Q_b)^6-Q_b^6-6Q_b^5\varepsilon] - \int\psi_B\Lambda Q_b[(\varepsilon+Q_b)^5-Q_b^5-5Q_b^4\varepsilon].$$

Thus,

$$\begin{aligned} f_3^{(i,j)} = {} & \frac{\lambda_s}{\lambda}\int[(2-2(j-1))\psi_B-y\psi'_B]\varepsilon_y^2 \\ & -\frac{1}{3}\frac{\lambda_s}{\lambda}\int[(2-2(j-1))\psi_B-y\psi'_B][(\varepsilon+Q_b)^6-Q_b^6-6Q_b^5\varepsilon] \\ & +2\frac{\lambda_s}{\lambda}\int\psi_B\Lambda Q_b[(\varepsilon+Q_b)^5-Q_b^5-5Q_b^4\varepsilon] \\ & +(\mathcal{J}_{i,j})_s\int\varphi_{i,B}\varepsilon^2-\frac{\lambda_s}{\lambda}(1+\mathcal{J}_{i,j})\int y\varphi'_{i,B}\varepsilon^2-2(j-1)\frac{\lambda_s}{\lambda}(1+\mathcal{J}_{i,j})\int\varphi_{i,B}\varepsilon^2 \end{aligned}$$

$$\begin{aligned}
&= \frac{\lambda_s}{\lambda}\int[2(2-j)\psi_B-y\psi_B']\varepsilon_y^2-\frac{1}{3}\frac{\lambda_s}{\lambda}\int[2(2-j)\psi_B-y\psi_B'][(\varepsilon+Q_b)^6-Q_b^6-6Q_b^5\varepsilon]\\
&\quad+2\frac{\lambda_s}{\lambda}\int\psi_B\Lambda Q_b[(\varepsilon+Q_b)^5-Q_b^5-5Q_b^4\varepsilon]\\
&\quad+\frac{1}{i}\biggl[(\mathcal{J}_{i,j})_s-2(j-1)(1+\mathcal{J}_{i,j})\frac{\lambda_s}{\lambda}\biggr]\int(i\varphi_{i,B}-y\varphi_{i,B}')\varepsilon^2\\
&\quad+\frac{1}{i}\biggl[(\mathcal{J}_{i,j})_s-(2(j-1)+i)(1+\mathcal{J}_{i,j})\frac{\lambda_s}{\lambda}\biggr]\int y\varphi_{i,B}'\varepsilon^2\\
&= f_{3,1}^{(i,j)}+f_{3,2}^{(i,j)},
\end{aligned}$$

where

$$f_{3,2}^{(i,j)}=\frac{1}{i}\biggl[(\mathcal{J}_{i,j})_s-(2(j-1)+i)(1+\mathcal{J}_{i,j})\frac{\lambda_s}{\lambda}\biggr]\int y\varphi_{i,B}'\varepsilon^2.$$

We estimate all terms in the above expression using again the notation $(f_{3,k}^{(i,j)})^<$, $(f_{3,k}^{(i,j)})^\sim$ and $(f_{3,k}^{(i,j)})^>$, corresponding to integration over $y<-\frac{1}{2}B$, $|y|<\frac{1}{2}B$ and $y>\frac{1}{2}B$, respectively. The middle term is easily estimated by brute force using (3.13), (2.33), (2.29) and the a-priori bound (3.5), getting

$$|(f_3^{(i,j)})^\sim|\lesssim\delta(\boldsymbol{\varkappa}^*)\int(\varepsilon_y^2+\varepsilon^2)\varphi_{i,B}'.$$

For $y<-B$, we use the exponential decay of ψ_B and $\varphi_{i,B}$, and (3.20) to estimate

$$\begin{aligned}
&\int_{y<-B/2}(\psi_B+|y|\psi_B'+\varphi_{i,B})(\varepsilon_y^2+\varepsilon^2)+|y|\varphi_{i,B}'\varepsilon^2\\
&\qquad\lesssim\int_{y<-B/2}\varepsilon_y^2\varphi_{i,B}'+\int_{y<-B/2}|y|\varphi_{i,B}'\varepsilon^2\\
&\qquad\lesssim\int\varepsilon_y^2\varphi_{i,B}'+\biggl(\int_{y<-B/2}|y|^{100}e^{y/B}\varepsilon^2\biggr)^{1/100}\biggl(\int_{y<-B/2}e^{y/B}\varepsilon^2\biggr)^{99/100}\\
&\qquad\lesssim\int\varepsilon_y^2\varphi_{i,B}'+\mathcal{N}_{i,\mathrm{loc}}^{9/10},
\end{aligned}$$

where we have used that $\int_{y<-B/2}|y|^{100}e^{y/B}\varepsilon^2\leqslant\|\varepsilon\|_{L^2}^2\leqslant\delta(\boldsymbol{\varkappa}^*)$.

Remark 3.5. We see in the above estimate why we need to impose a stronger exponential weight on ε_y than on ε at $-\infty$ in the definition of $\mathcal{F}_{i,j}$. Indeed, since the global L^2 norm of ε_y is not controlled,(14) we cannot estimate $\int_{y<0}|y|\psi_B'\varepsilon_y^2$ as we did for $\int_{y<0}|y|\varphi_{i,B}'\varepsilon^2$.

(14) Because λ becomes large in the (Exit) regime.

Together with (2.29) and the weighted Sobolev bound (3.25), this yields the bound

$$|(f_3^{(i,j)})^<| \lesssim (b+\mathcal{N}_{i,\mathrm{loc}}^{1/2})\left(\int \varepsilon_y^2\varphi'_{i,B}+\mathcal{N}_{i,\mathrm{loc}}^{9/10}\right) \lesssim \delta(\varkappa^*)\int(\varepsilon_y^2+\varepsilon^2)\varphi'_{i,B}+b^4.$$

For $y>B$, we estimate by brute force, using (3.20),

$$i\varphi_{i,B}-y\varphi'_{i,B}=0 \quad \text{for } y>B,$$

and (3.25),

$$|(f_{3,1}^{(i,j)})^>| \lesssim (b+\mathcal{N}_{i,\mathrm{loc}}^{1/2})\int(\varepsilon_y^2+\varepsilon^2)\varphi'_{i,B} \lesssim \delta(\varkappa^*)\int(\varepsilon_y^2+\varepsilon^2)\varphi'_{i,B}.$$

It only remains to estimate $(f_{3,2}^{(i,j)})^>$. This is a dangerous term which requires

– the weighted bound (3.7) and in particular its consequences (3.14) and (3.15) which are additional information necessary to close the estimates;

– the following cancellation manufactured in the definition (3.9) from (2.33) and (3.13):

$$\begin{aligned}\left|(\mathcal{J}_{i,j})_s-(2(j-1)+i)(1+\mathcal{J}_{i,j})\frac{\lambda_s}{\lambda}\right| &= \frac{4(j-1)+2i}{(1-J_1)^{4(j-1)+2i+1}}\left|(J_1)_s-\frac{1}{2}\frac{\lambda_s}{\lambda}(1-J_1)\right| \\ &\lesssim |b|+\mathcal{N}_{i,\mathrm{loc}}.\end{aligned} \tag{3.27}$$

Remark 3.6. Note that the gain in (3.27) with respect to (2.29) motivates the presence of the factor $1+\mathcal{J}_{i,j}$ in (3.8).

The estimates (3.27), (3.14) and (3.15) together with the bootstrap bounds (3.5) and (3.6) and the control (3.23) imply that

$$\begin{aligned}|(f_{3,2}^{(i,j)})^>| &\lesssim (|b|+\mathcal{N}_{i,\mathrm{loc}})\left(1+\frac{1}{\lambda^{10/9}}\right)\mathcal{N}_{i,\mathrm{loc}}^{8/9} \\ &\lesssim |b|(1+\delta(\varkappa^*)|b|^{-5/9})\mathcal{N}_{i,\mathrm{loc}}^{8/9}+\mathcal{N}_{i,\mathrm{loc}}(1+\delta(\varkappa^*)\mathcal{N}_{i,\mathrm{loc}}^{-5/9})\mathcal{N}_{i,\mathrm{loc}}^{8/9} \\ &\lesssim \delta(\varkappa^*)(\mathcal{N}_{i,\mathrm{loc}}+|b|^4).\end{aligned}$$

The collection of the above estimates yields the bound

$$|f_3^{(i,j)}| \lesssim \delta(\varkappa^*)\left(\int(\varepsilon_y^2+\varepsilon^2)\varphi'_{i,B}+|b|^4\right).$$

Step 6. $f_4^{(i)}$ term.

First,

$$|(Q_b)_s| = \left|b_s P(\chi(|b|^\gamma y)+\gamma|b|^\gamma y\chi'(|b|^\gamma y))\right| \lesssim |b_s|.$$

We use the Sobolev bound

$$\big\|\varepsilon^2\sqrt{\psi_B}\,\big\|_{L^\infty}^2 \lesssim \delta(\boldsymbol{\varkappa}^*)\int(\varepsilon_y^2+\varepsilon^2)\psi_B \tag{3.28}$$

to obtain

$$\int \psi_B|\varepsilon|^5 \lesssim \|\psi_B^{1/2}\varepsilon^2\|_{L^\infty}^{3/2}\int\psi_B^{1/4}\varepsilon^2 \lesssim \biggl(\int\varepsilon^2\biggr)^{3/4}\int(\varepsilon_y^2+\varepsilon^2)\psi_B \lesssim \delta(\boldsymbol{\varkappa}^*)\int(\varepsilon_y^2+\varepsilon^2)\psi_B,$$

and thus, from (2.30), $|Q_b|\leqslant C$ and (3.20),

$$|f_4^{(i)}| \lesssim |b_s|\int\psi_B(\varepsilon^2|Q_b|^3+|\varepsilon|^5) \lesssim (b^2+\mathcal{N}_{i,\mathrm{loc}})\int(\varepsilon_y^2+\varepsilon^2)\psi_B \lesssim \delta(\boldsymbol{\varkappa}^*)\int(\varepsilon_y^2+\varepsilon^2)\varphi'_{i,B}.$$

Step 7. Proof of (3.12).

First, we estimate from the homogeneity of the non-linearity and the Sobolev bound (3.28),

$$\int\psi_B|(\varepsilon+Q_b)^6-Q_b^6-6\varepsilon Q_b^5| \lesssim \int\psi_B(|Q_b|^4\varepsilon^2+|\varepsilon^6|) \lesssim \delta(\boldsymbol{\varkappa}^*)\int(\varepsilon_y^2+\varepsilon^2)\psi_B.$$

The upper bound follows immediately.

The lower bound follows from the structure (3.8) of $\mathcal{F}_{i,j}$ which is a localization of the linearized Hamiltonian close to Q. Indeed, we rewrite

$$\begin{aligned}\mathcal{F}_{i,j} = \int\psi_B\varepsilon_y^2+\varphi_{i,B}\varepsilon^2-5\int Q^4\varepsilon^2+\mathcal{J}_{i,j}\int\varphi_{i,B}\varepsilon^2 \\ -\frac{1}{3}\int\psi_B[(\varepsilon+Q_b)^6-Q_b^6-6\varepsilon Q_b^5-15Q_b^4\varepsilon^2]\,dy-5\int\psi_B(Q_b^4-Q^4)\varepsilon^2.\end{aligned}$$

The small L^2 term is estimated from (3.9) and (3.13):

$$|\mathcal{J}_{i,j}|\int\varphi_{i,B}\varepsilon^2 \lesssim \delta(\boldsymbol{\varkappa}^*)\int\varphi_{i,B}\varepsilon^2.$$

The non-linear term is estimated using the homogeneity of the non-linearity and the Sobolev bound (3.28):

$$\int\psi_B|(\varepsilon+Q_b)^6-Q_b^6-6\varepsilon Q_b^5-15Q_b^4\varepsilon^2| \lesssim \int\psi_B(|Q_b|^3|\varepsilon|^3+|\varepsilon|^6) \lesssim \delta(\boldsymbol{\varkappa}^*)\int(\varepsilon_y^2+\varepsilon^2)\psi_B.$$

The coercivity of the linearized energy (2.3) together with the choice of orthogonality conditions (2.20) and a standard localization argument([15]) now ensure the coercivity for B large enough:

$$\int(\psi_B\varepsilon_y^2+\varphi_{i,B}\varepsilon^2-5\psi_BQ^4\varepsilon^2) \geqslant \mu\mathcal{N}_i,$$

and the lower bound (3.12) follows.

This concludes the proof of Proposition 3.1. □

([15]) See, for example, [20, Appendix] for more details.

3.2. Dynamical control of the tail

We now provide an elementary dynamical control of the L^2 tail on the right of the soliton which will allow us to close the bootstrap bound (H3) of Proposition 3.1 in the setting of Theorem 1.2. Consider a smooth function

$$\varphi_{10}(y)=\begin{cases} 0 & \text{for } y\leqslant 0,\\ y^{10} & \text{for } y\geqslant 1,\end{cases} \quad \text{with } \varphi_{10}'\geqslant 0.$$

LEMMA 3.7. (Dynamical control of the tail on the right) *Under the assumptions of Proposition* 3.1,

$$\frac{1}{\lambda^{10}}\frac{d}{ds}\left(\lambda^{10}\int\varphi_{10}\varepsilon^2\right)\lesssim\mathcal{N}_{1,\mathrm{loc}}+b^2. \tag{3.29}$$

Proof. We compute, from (3.24),

$$\frac{1}{2}\frac{d}{ds}\int\varphi_{10}\varepsilon^2=\int\varepsilon_s\varepsilon\varphi_{10}=\int\varphi_{10}\varepsilon\bigg[\frac{\lambda_s}{\lambda}\Lambda\varepsilon+(-\varepsilon_{yy}+\varepsilon-(\varepsilon+Q_b)^5+Q_b^5)_y \\ +\left(\frac{\lambda_s}{\lambda}+b\right)\Lambda Q_b+\left(\frac{x_s}{\lambda}-1\right)(\varepsilon+Q_b)_y+\Phi_b+\Psi_b\bigg].$$

We integrate the linear term by parts and use that $y\varphi_{10}'=10\varphi_{10}$ for $y\geqslant 1$ and $\varphi_{10}'''\ll\varphi_{10}'$ for y large enough, to derive the bound

$$\int\varphi_{10}\varepsilon\left[\frac{\lambda_s}{\lambda}\Lambda\varepsilon+(-\varepsilon_{yy}+\varepsilon)_y\right]=-\frac{1}{2}\frac{\lambda_s}{\lambda}\int y\varphi_{10}'\varepsilon^2-\frac{3}{2}\int\varphi_{10}'\varepsilon_y^2-\frac{1}{2}\int\varphi_{10}'\varepsilon^2+\frac{1}{2}\int\varphi_{10}'''\varepsilon^2 \\ \leqslant-\frac{10}{2}\frac{\lambda_s}{\lambda}\int\varphi_{10}\varepsilon^2-\frac{1}{4}\int\varphi_{10}'(\varepsilon_y^2+\varepsilon^2)+C\mathcal{N}_{1,\mathrm{loc}}.$$

The terms involving the geometrical parameters are controlled from the exponential localization of Q_b on the right and (2.29) and (2.30):

$$\left|\frac{\lambda_s}{\lambda}+b\right|\left|\int\varphi_{10}\varepsilon(\Lambda Q_b)\right|\lesssim(b^2+\mathcal{N}_{1,\mathrm{loc}}^{1/2})\mathcal{N}_{i,\mathrm{loc}}^{1/2}\lesssim\mathcal{N}_{1,\mathrm{loc}}+b^2,$$

$$\left|\frac{x_s}{\lambda}-1\right|\left|\int\varphi_{10}\varepsilon(\varepsilon+Q_b)_y\right|\lesssim(b^2+\mathcal{N}_{1,\mathrm{loc}}^{1/2})\left[\mathcal{N}_{1,\mathrm{loc}}^{1/2}+\int\varphi_{10}'\varepsilon^2\right] \\ \lesssim\mathcal{N}_{1,\mathrm{loc}}+b^2+\delta(\varkappa^*)\int\varphi_{10}'\varepsilon^2,$$

$$\int|\varphi_{10}\varepsilon\Phi_b|\lesssim|b_s|\mathcal{N}_{1,\mathrm{loc}}^{1/2}\lesssim b^2+\mathcal{N}_{1,\mathrm{loc}}.$$

We control similarly the interaction with the error from (2.12):

$$\int|\varphi_{10}\varepsilon\Psi_b|\lesssim b^2\mathcal{N}_{1,\mathrm{loc}}^{1/2}\lesssim b^2+\mathcal{N}_{1,\mathrm{loc}}.$$

By integration by parts in the non-linear term, we can remove all derivatives on ε to obtain (using $|Q_b|+|(Q_b)_y|\leqslant Ce^{-y/2}$ for $y>0$)

$$\left|\int \varphi_{10}\varepsilon[(\varepsilon+Q_b)^5-Q_b^5]_y\right| \lesssim \int_{y>0}\varphi_{10}e^{-y/2}\varepsilon^2(|\varepsilon|^3+1)+\int\varphi_{10}'\varepsilon^6$$
$$\lesssim \int_{y>0} e^{-y/4}\varepsilon^2(|\varepsilon|^3+1)+\int\varphi_{10}'\varepsilon^6.$$

Thus, by standard Sobolev estimates,

$$\left|\int \varphi_{10}\varepsilon[(\varepsilon+Q_b)^5-Q_b^5]_y\right| \lesssim \mathcal{N}_{1,\mathrm{loc}}+\delta(\varkappa^*)\int\varphi_{10}'(\varepsilon_y^2+\varepsilon^2).$$

The collection of the above estimates yields the bound

$$\frac{d}{ds}\int\varphi_{10}\varepsilon^2+10\frac{\lambda_s}{\lambda}\int\varphi_{10}\varepsilon^2\lesssim\mathcal{N}_{1,\mathrm{loc}}+b^2,$$

and (3.29) is proved. □

4. Rigidity near the soliton. Proof of Theorem 1.2

This section is devoted to the proof of the following proposition which classifies the behavior of any solution close to Q and directly implies Theorem 1.2. Let $u_0\in H^1$ be such that

$$u_0=Q+\varepsilon_0,\quad \|\varepsilon_0\|_{H^1}<\alpha_0\quad\text{and}\quad\int_{y>0}y^{10}\varepsilon_0^2(y)\,dy<1, \tag{4.1}$$

and let $u(t)$ be the corresponding solution of (1.1) on $[0,T)$. Let $\mathcal{T}_{\alpha^*}$ be the L^2 modulated tube around the manifold of solitary waves given by (1.13) and define the *exit time*

$$t^*=\sup\{0<t<T:u(t')\in\mathcal{T}_{\alpha^*}\text{for all } t'\in[0,t]\},$$

which satisfies $t^*>0$ by assumption on the data. We claim the following result.

PROPOSITION 4.1. (Rigidity/Dynamical version) *There exist universal constants*

$$0<\alpha_0^*\ll\alpha^*\ll\varkappa^*\quad\textit{and}\quad C^*>1$$

such that the following holds. If u_0 satisfy (4.1) with $0<\alpha_0<\alpha_0^*$, then $u(t)$ satisfies the assumptions (H1)–(H3) *of Proposition 3.1 on* $[0,t^*)$.

Moreover, let t_1^* be the separation time defined by

$$t_1^*=\begin{cases}0, & \textit{if } |b(0)|\geqslant C^*\mathcal{N}_1(0),\\ \sup\{0<t<t^*:|b(t')|<C^*\mathcal{N}_1(t')\textit{ for all } t'\in[0,t]\}, & \textit{otherwise.}\end{cases} \tag{4.2}$$

Then the following trichotomy holds:

(Soliton) *If* $t_1^*=t^*$, *then* $t_1^*=t^*=T=\infty$. *In addition,*

$$\mathcal{N}_2(t)\to 0 \ \textit{ and } \ b(t)\to 0 \quad \textit{as } t\to\infty, \tag{4.3}$$

and

$$\lambda(t)=\lambda_\infty(1+o(1)) \ \textit{ and } \ x(t)=\frac{t}{\lambda_\infty^2}(1+o(1)) \quad \textit{as } t\to\infty, \tag{4.4}$$

for some λ_∞ *satisfying* $|\lambda_\infty-1|\leqslant\delta(\alpha_0)$.

(Exit) *If* $t_1^*<t^*$ *with* $b(t_1^*)\leqslant -C^*\mathcal{N}_1(t_1^*)$, *then* $t^*<T$. *In particular,*

$$\inf_{\substack{\lambda_0>0\\ x_0\in\mathbb{R}}}\left\|u(t^*)-\frac{1}{\lambda_0^{1/2}}Q\Big(\frac{\cdot-x_0}{\lambda_0}\Big)\right\|_{L^2}=\alpha^*. \tag{4.5}$$

In addition,

$$\lambda(t^*)\geqslant\frac{C(\alpha^*)}{\delta(\alpha_0)}. \tag{4.6}$$

(Blow up) *If* $t_1^*<t^*$ *with* $b(t_1^*)\geqslant C^*\mathcal{N}_1(t_1^*)$, *then* $t^*=T$. *In addition,* $T<\infty$ *and there exists* $0<\ell_0<\delta(\alpha_0)$ *such that*

$$\lim_{t\to T}\frac{\lambda(t)}{(T-t)}=\ell_0,\quad \lim_{t\to T}\frac{b(t)}{(T-t)^2}=\ell_0^3,\quad \lim_{t\to T}(T-t)x(t)=\frac{1}{\ell_0^2}, \tag{4.7}$$

and the following bounds hold:

$$\|\varepsilon_x(t)\|_{L^2}\lesssim\lambda^2(t)[|E_0|+\delta(\alpha_0)] \quad \textit{and} \quad \|\varepsilon(t)\|_{L^2}\lesssim\delta(\alpha_0). \tag{4.8}$$

Remark 4.2. Note that $u(t)$ belongs to the tube $\mathcal{T}_{\alpha^*}$ as long as $\frac{1}{3}\leqslant\lambda(t)\leqslant 3$ and that the three cases are equivalently characterized by

(Soliton) for all t, $\lambda(t)\in\left[\frac{1}{2},2\right]$;

(Exit) there exists $t_0>0$ such that $\lambda(t_0)>2$;

(Blow up) there exists $t_0>0$ such that $\lambda(t_0)<\frac{1}{2}$.

A continuity argument thus ensures that the cases (Exit) and (Blow up) are open in $\mathcal{A}$.

Also, note that on (t_1^*,t^*), $\lambda(t)$ is almost monotonic and the separation time t_1^* defines a trapped regime, i.e.

$$|b(t)|\gtrsim C^*\mathcal{N}_1(t) \quad \text{for } t_1^*<t<t^*,$$

and hence the scenario is chosen at this point.

The rest of this section is devoted to the proof of Proposition 4.1. First, note that, by Lemma 2.5, u admits the following decomposition on $[0,t^*]$:

$$u(t,x)=\frac{1}{\lambda^{1/2}(t)}(Q_{b(t)}+\varepsilon)\left(t,\frac{x-x(t)}{\lambda(t)}\right),$$

with, due to (4.1),

$$\|\varepsilon(0)\|_{H^1}+|b(0)|+|1-\lambda(0)|\lesssim\delta(\alpha_0)\quad\text{and}\quad\int_{y>0}y^{10}\varepsilon^2(0,y)\,dy\leqslant 2. \tag{4.9}$$

In particular, arguing as in the proof of (3.14), we have

$$\mathcal{N}_2(0)\lesssim\delta(\alpha_0). \tag{4.10}$$

For $\varkappa^*$ as in Proposition 3.1, define

$$t^{**}=\sup\{0<t<t^*: u \text{ satisfies (H1)–(H3) on } [0,t]\}.$$

Note that $t^{**}>0$ is well defined by (4.9), (4.10) and a straightforward continuity argument. Recall that $s=s(t)$ is the rescaled time (2.22). We let $s^{**}=s(t^{**})$ and $s^*=s(t^*)$. One important step in the proof is to obtain $t^{**}=t^*$ by improving (H1)–(H3) on $[0,t^{**}]$.

4.1. Consequences of the monotonicity formula

We start with *coupling* the dispersive bounds (3.10) and (3.11) with the modulation equation for b given by (2.37) to derive the key rigidity property at the heart of our analysis.

LEMMA 4.3. *The following bounds hold*:

(1) (Dispersive bounds) *For* $i=1,2$, *for all* $0\leqslant s_1\leqslant s_2<s^{**}$,

$$\mathcal{N}_i(s_2)+\int_{s_1}^{s_2}\int(\varepsilon_y^2+\varepsilon^2)(s)\varphi'_{i,B}\,ds\lesssim\mathcal{N}_i(s_1)+|b^3(s_2)|+|b^3(s_1)|, \tag{4.11}$$

$$\frac{\mathcal{N}_i(s_2)}{\lambda^2(s_2)}+\int_{s_1}^{s_2}\frac{\int(\varepsilon_y^2+\varepsilon^2)(s)\varphi'_{i,B}+|b|^4}{\lambda^2(s)}\,ds\lesssim\frac{\mathcal{N}_i(s_1)}{\lambda^2(s_1)}+\left[\frac{|b^3(s_1)|}{\lambda^2(s_1)}+\frac{|b^3(s_2)|}{\lambda^2(s_2)}\right]. \tag{4.12}$$

(2) (Control of the dynamics for b) *For all* $0\leqslant s_1\leqslant s_2<s^{**}$,

$$\int_{s_1}^{s_2}b^2(s)\,ds\lesssim\mathcal{N}_1(s_1)+|b(s_2)|+|b(s_1)|, \tag{4.13}$$

and, for a universal constant $K_0>1$,

$$\left|\frac{b(s_2)}{\lambda^2(s_2)}-\frac{b(s_1)}{\lambda^2(s_1)}\right|\leqslant K_0\left[\frac{b^2(s_1)}{\lambda^2(s_1)}+\frac{b^2(s_2)}{\lambda^2(s_2)}+\frac{\mathcal{N}_1(s_1)}{\lambda^2(s_1)}\right]. \tag{4.14}$$

(3) (Control of the scaling dynamics) *Let* $\lambda_0(s)=\lambda(s)(1-J_1(s))^2$. *Then, on* $[0,s^{**})$,

$$\left|\frac{(\lambda_0)_s}{\lambda_0}+b\right|\lesssim\int\varepsilon^2e^{-|y|/10}+|b|(\mathcal{N}_2^{1/2}+|b|). \tag{4.15}$$

Proof. We start proving (4.11) and (4.12). We first observe, from (2.42), the bound

$$b^2 \leqslant -b_s + C\mathcal{N}_{1,\mathrm{loc}}. \tag{4.16}$$

By the monotonicity formula (3.10), with (3.12),

$$\begin{aligned}\mathcal{N}_i(s_2) + \int_{s_1}^{s_2}\int (\varepsilon_y^2+\varepsilon^2)(s)\varphi'_{i,B}\,ds &\lesssim \mathcal{F}_{i,1}(s_2) + \mu\int_{s_1}^{s_2}\int(\varepsilon_y^2+\varepsilon^2)(s)\varphi'_{i,B}\,ds \\ &\leqslant \mathcal{F}_{i,1}(s_1) + \int_{s_1}^{s_2} b^4(s)\,ds \lesssim \mathcal{N}_i(s_1) + \int_{s_1}^{s_2} b^4(s)\,ds,\end{aligned}$$

and thus, using (4.16), (3.4) and the fact that $|b|$ is small,

$$\mathcal{N}_i(s_2) + \int_{s_1}^{s_2}\int(\varepsilon_y^2+\varepsilon^2)(s)\varphi'_{i,B}\,ds \lesssim \mathcal{N}_i(s_1) + |b^3(s_2)| + |b^3(s_1)|.$$

Similarly, from (3.11) and (3.12),

$$\begin{aligned}\frac{\mathcal{N}_i(s_2)}{\lambda^2(s_2)} + \int_{s_1}^{s_2}\frac{1}{\lambda^2(s)}\int(\varepsilon_y^2+\varepsilon^2)(s)\varphi'_{i,B}\,ds &\lesssim \frac{\mathcal{F}_{i,2}(s_2)}{\lambda^2(s_2)} + \mu\int_{s_1}^{s_2}\frac{1}{\lambda^2(s)}\int(\varepsilon_y^2+\varepsilon^2)(s)\varphi'_{i,B}\,ds \\ &\lesssim \frac{\mathcal{F}_{i,2}(s_1)}{\lambda^2(s_1)} + \int_{s_1}^{s_2}\frac{b^4(s)}{\lambda^2(s)}\,ds \\ &\lesssim \frac{\mathcal{N}_i(s_1)}{\lambda^2(s_1)} + \int_{s_1}^{s_2}\frac{b^4(s)}{\lambda^2(s)}\,ds.\end{aligned} \tag{4.17}$$

We now integrate by parts in time, using (4.16) and (2.29), to estimate

$$\begin{aligned}\int_{s_1}^{s_2}\frac{b^4(s)}{\lambda^2(s)}\,ds &\leqslant \int_{s_1}^{s_2}\frac{-b^2b_s}{\lambda^2}\,ds + \delta(\varkappa^*)\int_{s_1}^{s_2}\frac{\mathcal{N}_{1,\mathrm{loc}}(s)}{\lambda^2(s)}\,ds \\ &= -\frac{1}{3}\left[\frac{b^3}{\lambda^2}\right]_{s_1}^{s_2} - \frac{2}{3}\int_{s_1}^{s_2} b^3\frac{\lambda_s}{\lambda^3}\,ds + \delta(\varkappa^*)\int_{s_1}^{s_2}\frac{\mathcal{N}_{1,\mathrm{loc}}(s)}{\lambda^2(s)}\,ds \\ &\leqslant \left[\frac{|b^3(s_1)|}{\lambda^2(s_1)} + \frac{|b^3(s_2)|}{\lambda^2(s_2)} + \delta(\varkappa^*)\int_{s_1}^{s_2}\frac{\mathcal{N}_{1,\mathrm{loc}}(s)}{\lambda^2(s)}\,ds\right] + \frac{2}{3}\int_{s_1}^{s_2}\frac{b^4(s)}{\lambda^2(s)}\,ds \\ &\quad + C\int_{s_1}^{s_2}\frac{|b|^3}{\lambda^2}[b^2+\mathcal{N}_{1,\mathrm{loc}}^{1/2}]\,ds \\ &\leqslant \left[\frac{|b^3(s_1)|}{\lambda^2(s_1)} + \frac{|b^3(s_2)|}{\lambda^2(s_2)}\right] + \delta(\varkappa^*)\int_{s_1}^{s_2}\frac{\mathcal{N}_{1,\mathrm{loc}}(s)}{\lambda^2(s)}\,ds + \left[\frac{2}{3}+\delta(\varkappa^*)\right]\int_{s_1}^{s_2}\frac{b^4(s)}{\lambda^2(s)}\,ds,\end{aligned}$$

and thus, for $\varkappa^*$ small,

$$\int_{s_1}^{s_2}\frac{b^4(s)}{\lambda^2(s)}\,ds \lesssim \left[\frac{|b^3(s_1)|}{\lambda^2(s_1)} + \frac{|b^3(s_2)|}{\lambda^2(s_2)}\right] + \delta(\varkappa^*)\int_{s_1}^{s_2}\frac{\mathcal{N}_{1,\mathrm{loc}}(s)}{\lambda^2(s)}\,ds. \tag{4.18}$$

Inserting this bound into (4.17) concludes the proof of (4.12).

The virtue of (4.11) and (4.12) is to reduce the control of the full problem to the sole control of the parameter b which is driven by the sharp ODE (2.37).

We now prove (4.13) and (4.14). The estimate (4.13) is derived by integrating (4.16) in time using (4.11). We then compute, from (2.37), (2.29) and the a-priori bound([16]) $|J|\lesssim\mathcal{N}_{1,\mathrm{loc}}^{1/2}$:

$$\begin{aligned}\left|\frac{d}{ds}\left(\frac{b}{\lambda^2}e^J\right)\right| &= \left|\frac{d}{ds}\left(\frac{b}{\lambda^2}\right)+\frac{b}{\lambda^2}J_s\right|e^J\\ &\lesssim \left|\frac{\lambda_s}{\lambda}\frac{b}{\lambda^2}J\right|+\frac{1}{\lambda^2}\left(\int\varepsilon^2e^{-|y|/10}+|b|^3\right)\\ &\lesssim \frac{b^2}{\lambda^2}|J|+\frac{1}{\lambda^2}(\mathcal{N}_{1,\mathrm{loc}}+|b|^3)\\ &\lesssim \frac{b^2}{\lambda^2}\mathcal{N}_{1,\mathrm{loc}}^{1/2}+\frac{1}{\lambda^2}(\mathcal{N}_{1,\mathrm{loc}}+|b|^3)\\ &\lesssim \frac{1}{\lambda^2}(\mathcal{N}_{1,\mathrm{loc}}+|b|^3).\end{aligned}\tag{4.19}$$

Integrating in time and using (4.16) and (4.12), we obtain, for all $s,s'\in[s_1,s_2]$,

$$\left|\left[\frac{b}{\lambda^2}e^J\right]_s^{s'}\right|\lesssim\frac{\mathcal{N}_1(s_1)}{\lambda^2(s_1)}+(|b(s')|+|b(s)|)\sup_{[s_1,s_2]}\frac{|b|}{\lambda^2}.\tag{4.20}$$

From (4.11),

$$|e^{J(s)}-1|\lesssim|J(s)|\lesssim\mathcal{N}_1^{1/2}(s)\lesssim(\mathcal{N}_1(s_1)+|b^3(s)|+|b^3(s_1)|)^{1/2}.\tag{4.21}$$

First, from (4.21) we obtain

$$\sup_{[s_1,s_2]}\frac{|b|}{\lambda^2}\lesssim\min_{[s_1,s_2]}\frac{|b|}{\lambda^2}+\frac{\mathcal{N}_1(s_1)}{\lambda^2(s_1)}+\Big(\sup_{[s_1,s_2]}|b|\Big)\sup_{[s_1,s_2]}\frac{|b|}{\lambda^2},$$

so that

$$\sup_{[s_1,s_2]}\frac{|b|}{\lambda^2}\lesssim\min_{[s_1,s_2]}\frac{|b|}{\lambda^2}+\frac{\mathcal{N}_1(s_1)}{\lambda^2(s_1)}.$$

In particular, by (4.21), we obtain

$$\begin{aligned}\left|\frac{b(s)}{\lambda^2(s)}(e^{J(s)}-1)\right| &\lesssim\left(\min_{[s_1,s_2]}\frac{|b|}{\lambda^2}+\frac{\mathcal{N}_1(s_1)}{\lambda^2(s_1)}\right)(\mathcal{N}_1(s_1)+|b^3(s)|+|b^3(s_1)|)^{1/2}\\ &\lesssim\frac{\mathcal{N}_1(s_1)}{\lambda^2(s_1)}+\frac{b^2(s_1)}{\lambda^2(s_1)}+\frac{b^2(s)}{\lambda^2(s)}.\end{aligned}$$

([16]) Recall that J given by (2.36) is a well-localized L^2 scalar product.

Second, combining these estimates with (4.20) taken at $s=s_1$ and $s'=s_2$, we obtain

$$\begin{aligned}\left|\frac{b(s_2)}{\lambda^2(s_2)}-\frac{b(s_1)}{\lambda^2(s_1)}\right| &\lesssim \left|\frac{b(s_1)}{\lambda^2(s_1)}(e^{J(s_1)}-1)\right|+\left|\frac{b(s_2)}{\lambda^2(s_2)}(e^{J(s_2)}-1)\right|+\frac{\mathcal{N}_1(s_1)}{\lambda^2(s_1)}\\ &\quad +(|b(s_1)|+|b(s_2)|)\min_{[s_1,s_2]}\frac{|b|}{\lambda^2}\\ &\lesssim \frac{\mathcal{N}_1(s_1)}{\lambda^2(s_1)}+\left(\frac{b^2(s_1)}{\lambda^2(s_1)}+|\frac{b^2(s_2)|}{\lambda^2(s_2)}\right).\end{aligned}$$

We finally prove (4.15). We integrate the scaling law using the sharp modulation equation (2.33). From (3.13),

$$\left|\frac{\lambda}{\lambda_0}-1\right| \lesssim |J_1| \lesssim \delta(\varkappa^*), \tag{4.22}$$

and thus, from (2.33), we get

$$\begin{aligned}\left|\frac{(\lambda_0)_s}{\lambda_0}+b-c_1b^2\right| &= \left|\frac{1}{1-J_1}\left[(1-J_1)\frac{\lambda_s}{\lambda}+b-2(J_1)_s\right]-\frac{J_1}{1-J_1}b\right|\\ &\lesssim \int \varepsilon^2 e^{-|y|/10}+|b|(\mathcal{N}_2^{1/2}+|b|^2).\end{aligned}$$

This concludes the proof of Lemma 4.3. □

We are now in position to prove the trichotomy of Proposition 4.1. Let

$$C^* = 10K_0, \tag{4.23}$$

where K_0 is the universal constant in (4.14) and let the separation time t_1^* be given by (4.2).

4.2. The soliton case

Assume that

$$t_1^* = t^*, \quad \text{i.e. } |b(t)| \leqslant C^*\mathcal{N}_1(t) \text{ for all } t\in[0,t^*]. \tag{4.24}$$

We first prove that in this case $t^{**}=t^*$, which means that the bootstrap estimates (H1)–(H3) of Proposition 3.1 hold on $[0,t^*]$. Indeed, we claim that, for all $s\in[0,s^{**})$,

$$|b(s)|+\mathcal{N}_2(s)+\|\varepsilon(s)\|_{L^2}+|1-\lambda(s)| \lesssim \delta(\alpha_0), \tag{4.25}$$

$$\frac{|b(s)|+\mathcal{N}_2(s)}{\lambda^2(s)} \lesssim \delta(\alpha_0), \tag{4.26}$$

$$\int_{y>0} y^{10}\varepsilon^2(s,y)\,dy \leqslant 5. \tag{4.27}$$

Taking $\alpha^*>0$ small enough (compared to $\varkappa^*$), this guarantees by a standard continuity argument that $t^{**}=t^*$.

We now prove (4.25)–(4.27). First, observe that, by (3.22) and the definition of t^{**}, on $[0,s^{**}]$,

$$\mathcal{N}_1 \lesssim \int (\varepsilon_y^2+\varepsilon^2)\varphi'_{2,B} \quad \text{and} \quad \mathcal{N}_1 \lesssim \mathcal{N}_2 \lesssim \delta(\varkappa^*). \tag{4.28}$$

Therefore, from (4.24), (4.11) and (2.30), for all $s\in[0,s^{**})$,

$$\begin{aligned}|b(s)-b(0)| &\leqslant \int_0^s |b_s(s')|\,ds' \lesssim \int_0^s (b^2+\mathcal{N}_{1,\mathrm{loc}})(s')\,ds' \lesssim \int_0^s (\delta(\varkappa^*)(C^*)^2+1)\mathcal{N}_1(s')\,ds' \\ &\lesssim \int_0^s \int (\varepsilon_y^2+\varepsilon^2)(s')\varphi'_{2,B}\,dy\,ds' \lesssim \mathcal{N}_2(0)+\delta(\varkappa^*)(|b(s)|+|b(0)|).\end{aligned}$$

We thus conclude from (4.9) that, for all $s\in[0,s^{**})$,

$$|b(s)| \lesssim |b(0)|+\mathcal{N}_2(0) \lesssim \delta(\alpha_0).$$

Then, from (4.11) and (4.13),

$$\mathcal{N}_2(s)+\int_0^s \left(b^2+\int (\varepsilon_y^2+\varepsilon^2)(s')\varphi'_{2,B}\,dy\right)ds' \lesssim \delta(\alpha_0). \tag{4.29}$$

Inserting this into the conservation of the L^2 norm (2.27) using (2.15) ensures that

$$\int |\varepsilon|^2 \lesssim \delta(\alpha_0).$$

Note that we also have, from (3.13),

$$|J_1|+|J_2| \leqslant \delta(\alpha_0). \tag{4.30}$$

We now compute the variation of scaling from (4.15), which together with (4.24) implies that

$$\left|\frac{(\lambda_0)_s}{\lambda_0}\right| \lesssim |b|+\mathcal{N}_{1,\mathrm{loc}} \lesssim \mathcal{N}_1 \lesssim \int (\varepsilon_y^2+\varepsilon^2)(s)\varphi'_{2,B},$$

and thus from (4.29), for all $0\leqslant s<s^{**}$,

$$\left|\log\frac{\lambda_0(s)}{\lambda_0(0)}\right| \lesssim \mathcal{N}_2(0)+\delta(\alpha_0) \lesssim \delta(\alpha_0).$$

Hence, from (4.22) and (4.30),

$$\left|\frac{\lambda(s)}{\lambda(0)}-1\right| \lesssim \delta(\alpha_0),$$

which, with (4.9), implies that

$$|1-\lambda(s)| \lesssim \delta(\alpha_0) \quad \text{for all } s\in[0,s^{**}). \tag{4.31}$$

Together with (4.25), this implies (4.26). We now integrate (3.29) using (4.9), (4.31) and (4.29) and obtain

$$\begin{aligned}\int y^{10}\varepsilon^2(s)\,dy &\leqslant \frac{\lambda^{10}(0)}{\lambda^{10}(s)}\int y^{10}\varepsilon^2(0)\,dy+\frac{C}{\lambda^{10}(s)}\int_0^s \lambda^{10}(s')(b^2+\mathcal{N}_{1,\mathrm{loc}})(s')\,ds' \\ &\leqslant 2+\delta(\alpha_0)\leqslant 3,\end{aligned}$$

and (4.27) is proved.

We therefore conclude that $t^*=T$ and $u(t)$ remains in the tube $\mathcal{T}_{\alpha^*}$ for all $t\in[0,T)$ from (4.25). Moreover, inserting (4.25) in the conservation of the energy (2.28), we get

$$\|\varepsilon_y(t)\|_{L^2}\lesssim C \quad \text{for all } t\in[0,T).$$

Hence the solution $u(t)$ is uniformly bounded in H^1 and thus global: $T=\infty$.

It remains to show the convergence (4.3)–(4.4). From (2.30), (4.29) and (4.31) we get

$$\int_0^\infty |b_t|\,dt \lesssim \int_0^\infty |b_s|\,ds \lesssim \int_0^\infty \left(b^2+\int(\varepsilon_y^2+\varepsilon^2)(s)\varphi_{2,B}'\right)ds \lesssim \delta(\alpha_0), \tag{4.32}$$

which implies

$$\lim_{t\to\infty} b(t)=0 \tag{4.33}$$

and the existence of a sequence $t_n\to\infty$ such that

$$\int(\varepsilon_y^2+\varepsilon^2)(t_n)\varphi_{2,B}'\to 0 \quad \text{as } t_n\to\infty.$$

By (4.28), $\mathcal{N}_1(t_n)\to 0$ as $n\to\infty$ and thus, using the monotonicity (4.11),

$$\mathcal{N}_1(t)\to 0 \quad \text{as } t\to\infty.$$

Together with the uniform bound (4.27), we also obtain

$$\mathcal{N}_2(t)\to 0 \quad \text{as } t\to\infty. \tag{4.34}$$

Finally, from (4.24), (4.15) and (4.32),

$$\int_0^\infty \left|\frac{d}{dt}\log\lambda_0\right|dt \lesssim \int_0^\infty \left|\frac{d}{ds}\log\lambda_0\right|ds \lesssim \delta(\alpha_0),$$

and thus

$$\lim_{t\to\infty}\lambda_0(t)=\lambda_0^\infty \quad \text{with } |\lambda_0^\infty-1|\lesssim\delta(\alpha_0).$$

Now, from (4.34),

$$|J_1|\lesssim\mathcal{N}_2^{1/2}\to 0 \quad \text{as } t\to\infty$$

and thus, from (4.13),

$$\lim_{t\to\infty}\lambda(t)=\lambda^\infty \quad \text{with } |\lambda^\infty-1|\lesssim\delta(\alpha_0). \tag{4.35}$$

The translation parameter is controlled using (2.29), (4.34) and (4.35), which imply

$$x_t=\frac{1}{\lambda^2}\frac{x_s}{\lambda}=\frac{1+o(1)}{\lambda_\infty^2} \quad \text{as } t\to\infty.$$

This concludes the proof of (4.3) and (4.4).

4.3. Exit case

We now assume that $t_1^*<t^*$ and

$$b(s_1^*)\leqslant -C^*\mathcal{N}_1(s_1^*). \tag{4.36}$$

Observe first that arguing on $[0,s_1^*]$ as in the soliton case, where the parameter b is controlled by $\mathcal{N}_1$, we obtain, for all $s\in[0,s_1^*]$,

$$|\lambda(s)-1|+|b(s)|+\mathcal{N}_2(s)+\int_0^s\!\!\int(\varepsilon_y^2+\varepsilon^2)\varphi_{2,B}'\,ds'\lesssim\delta(\alpha_0) \tag{4.37}$$

and

$$\int_{y>0}y^{10}\varepsilon^2(s,y)\,dy\leqslant 5. \tag{4.38}$$

In particular, $t_1^*<t^{**}\leqslant t^*$. Now, we claim that

$$t^{**}=t^* \quad \text{and} \quad t^*<T,$$

which means that the solution leaves the tube $\mathcal{T}_{\alpha^*/2}$ in finite time.

We first prove that $t^{**}=t^*$. We improve (H1)–(H3) on $[t_1^*,t^{**}]$ to obtain $t^{**}=t^*$. The proof is different from the one for the soliton case, since now b is not controlled by $\mathcal{N}_1$. The fundamental observation is that (4.14), (4.23) and (4.36) imply the rigidity

$$-2|\ell^*|\leqslant\frac{b(s)}{\lambda^2(s)}\leqslant-\frac{|\ell^*|}{2} \quad \text{for all } s\in[s_1^*,s^{**}), \tag{4.39}$$

where we have set, from (4.37),

$$\ell^* = \frac{b(s_1^*)}{\lambda^2(s_1^*)} \leqslant -C^* \frac{\mathcal{N}_1(s_1^*)}{\lambda^2(s_1^*)} < 0, \quad |\ell^*| \lesssim \delta(\alpha_0). \tag{4.40}$$

Together with (4.12) and (4.37), this implies the bound

$$\frac{|b(s)|+\mathcal{N}_2(s)}{\lambda^2(s)} \lesssim \delta(\alpha_0) \quad \text{for all } s\in[0,s^{**}],$$

and (H2) is improved for α^* small compared to $\varkappa^*$. We now observe that, using $b<0$ from (4.39) and (4.15), for all $s\in[s_1^*,s^{**})$,

$$\frac{(\lambda_0)_s(s)}{\lambda_0(s)} \gtrsim -\mathcal{N}_{1,\mathrm{loc}}.$$

Together with (4.11) and the definition of λ_0, this yields the almost monotonicity property of λ:

$$\lambda(\sigma_2) \geqslant \tfrac{1}{2}\lambda(\sigma_1) \quad \text{for all } s_1^* \leqslant \sigma_1 \leqslant \sigma_2 < s^{**}. \tag{4.41}$$

We integrate (3.29) using (4.11), (4.41), (4.37), (4.38) and (4.13) to get, for all $s_1^*\leqslant s<s^{**}$,

$$\begin{aligned}\int \varphi_{10}\varepsilon^2(s)\,dy &\leqslant \frac{\lambda^{10}(s_1^*)}{\lambda^{10}(s)} \int \varphi_{10}\varepsilon^2(s_1^*)\,dy + \frac{C}{\lambda^{10}(s)} \int_{s_1^*}^{s} \lambda^{10}(s')(\mathcal{N}_{1,\mathrm{loc}}(s')+b^2(s'))\,ds' \\ &\leqslant 3+C\int_{s_1^*}^{s} (\mathcal{N}_{1,\mathrm{loc}}(s')+b^2(s'))\,ds' \\ &\leqslant 3+C(|b(s_1^*)|+|b(s)|+\mathcal{N}_1(s_1^*)) \\ &\leqslant 3+\delta(\varkappa^*),\end{aligned}$$

and (H3) is improved. We now improve (H1). Since $u(t)\in\mathcal{T}_{\alpha^*}$ on $[0,t^*)$, we have, by (2.21), that $|b(s)|\leqslant\delta(\alpha^*)\ll\varkappa^*$ for all $s\in[0,s^*)$. By (4.11), it follows that $N_2(s)\ll\varkappa^*$ for all $s\in[0,s^{**})$. By (2.27), $\|\varepsilon(s)\|_{L^2}\ll\varkappa^*$ for all $s\in[0,s^{**})$, and (H1) is improved. In conclusion, we have proved $t^{**}=t^*$ again in this case.

We now prove that $t^*<T$. Let us show that (Exit) occurs in finite time. We divide (4.15) by λ_0^2 and use (4.39) and (4.22) to estimate, on $[t_1^*,t^*)$,

$$\frac{|\ell^*|}{3} - C\frac{\mathcal{N}_{1,\mathrm{loc}}}{\lambda^2} \leqslant (\lambda_0)_t \leqslant 3|\ell^*| + C\frac{\mathcal{N}_{1,\mathrm{loc}}}{\lambda^2}.$$

Integrating in time, for all $t\in[t_1^*,t^*)$, we get

$$\frac{|\ell^*|(t-t_1^*)}{3} - C_1\int_{t_1^*}^{t} \frac{\mathcal{N}_{1,\mathrm{loc}}}{\lambda^2}\,d\tau \leqslant \lambda_0(t)-\lambda_0(t_1^*) \leqslant 3|\ell^*|(t-t_1^*) + C_2\int_{t_1^*}^{t_2} \frac{\mathcal{N}_{1,\mathrm{loc}}}{\lambda^2}\,d\tau.$$

From the monotonicity (4.41) and then (4.11),

$$\int_{t_1^*}^t \frac{\mathcal{N}_{1,\mathrm{loc}}}{\lambda^2}\,d\tau = \int_{s_1^*}^s \lambda \mathcal{N}_{1,\mathrm{loc}}\,d\sigma \lesssim \lambda(s)\int_{s_1^*}^s \mathcal{N}_{1,\mathrm{loc}}\,d\sigma \lesssim \delta(\varkappa^*)\lambda(t),$$

and we therefore obtain the bound, for all $t\in[t_1^*,t^*)$,

$$\tfrac{1}{4}(|\ell^*|(t-t_1^*)+\lambda_0(t_1^*)) \leqslant \lambda(t) \leqslant 4(|\ell^*|(t-t_1^*)+\lambda_0(t_1^*)).$$

This yields the following estimates on b from (4.39), for all $t\in[t_1^*,t^*)$,

$$-40|\ell^*|(|\ell^*|(t-t_1^*)+\lambda_0(t_1^*))^2 \leqslant b(t) \leqslant -\tfrac{1}{40}|\ell^*|(|\ell^*|(t-t_1^*)+\lambda_0(t_1^*))^2. \tag{4.42}$$

Inserting this bound into (4.11) yields the control

$$\mathcal{N}_2(t) \lesssim C(t),$$

which, inserted into the energy and mass conservation laws (2.27) and (2.28), yields the H^1 bound

$$\|\varepsilon(t)\|_{H^1} \lesssim C(t).$$

It follows that $t^*=T<\infty$ is not possible. On the other hand, $t^*=T=\infty$ is also impossible since then, by (4.42), $b(t)\to-\infty$ as $t\to\infty$, which contradicts the definition of t^*. Thus, $t^*<T\leqslant\infty$.

Finally, we observe that the scaling parameter is large at the exit time for α small compared to α^*. Indeed, $|b(t^*)|\gtrsim(\alpha^*)^4$ from (2.27), and thus, from (4.39) and (4.40),

$$\lambda^2(t^*) \geqslant \frac{1}{2}\frac{|b(t^*)|}{|\ell^*|} \geqslant \frac{C(\alpha^*)}{\delta(\alpha_0)}.$$

4.4. Blow up case

We now assume that $t_1^*<t^*$ and

$$b(s_1^*) \geqslant C^*\mathcal{N}_1(s_1^*) > 0. \tag{4.43}$$

As before we have, for all $s\in[0,s_1^*]$,

$$|\lambda(s)-1|+|b(s)|+\mathcal{N}_2(s)+\int_0^s\int(\varepsilon_y^2+\varepsilon^2)\varphi_{2,B}'\,ds' \lesssim \delta(\alpha) \tag{4.44}$$

and

$$\int_{y>0} y^{10}\varepsilon^2(s,y)\,dy \leqslant 5. \tag{4.45}$$

In particular, $t_1^*<t^{**}\leqslant t^*$. In this case, we claim that $t^{**}=t^*=T$ and $T<\infty$.

We first prove that $t^{**}=t^*=T$. First, we improve the bounds (H1)–(H3) of Proposition 3.1. From (4.14), (4.23) and (4.43), we recover the rigidity

$$\frac{\ell^*}{2}\leqslant\frac{b(s)}{\lambda^2(s)}\leqslant 2\ell^*\quad\text{for all } s\in[s_1^*,s^{**}),\tag{4.46}$$

where we set, from (4.44),

$$\ell^*=\frac{b(s_1^*)}{\lambda^2(s_1^*)}>0,\quad |\ell^*|\lesssim\delta(\alpha_0).\tag{4.47}$$

Together with (4.12) and (4.44), this implies the bound

$$\frac{|b(s)|+\mathcal{N}_2(s)}{\lambda^2(s)}\lesssim\delta(\alpha_0)\quad\text{for all } s\in[0,s^*],$$

and (H2) is improved provided α^* is small compared to $\varkappa^*$. We now observe, from $b>0$ and (4.15), that, on $[s_1^*,s^{**})$,

$$-\frac{(\lambda_0)_s}{\lambda_0}\gtrsim-\mathcal{N}_{1,\mathrm{loc}},$$

which, together with (4.11) and the definition of λ_0, yields the almost monotonicity

$$\lambda(\sigma_2)\leqslant\tfrac{3}{2}\lambda(\sigma_1)\quad\text{for all } s_1^*\leqslant\sigma_1\leqslant\sigma_2<s^{**}.\tag{4.48}$$

In particular, from (4.44),

$$\lambda(s)\leqslant 2\quad\text{for all } s\in[0,s^{**}).\tag{4.49}$$

This yields, with (4.44), (4.46), (4.43) and (4.11), that, for all $0\leqslant s\leqslant s^{**}$,

$$|b(s)|\lesssim\lambda^2(s)\ell^*\lesssim\delta(\alpha_0)\quad\text{and}\quad\mathcal{N}_2(s)+\int_0^s\!\!\int(\varepsilon_y^2+\varepsilon^2)\varphi_{2,B}'\,ds'\lesssim\delta(\alpha_0).$$

The conservation of the L^2 norm (2.27) implies that

$$\|\varepsilon\|_{L^2}^2\lesssim\delta(\alpha_0),\tag{4.50}$$

and (H1) is improved. We now integrate (3.29) using (4.11), (4.49), (4.44) and (4.13) and obtain, for all $0\leqslant s<s^{**}$,

$$\begin{aligned}\int\varphi_{10}\varepsilon^2(s)\,dy&\leqslant\frac{\lambda^{10}(0)}{\lambda^{10}(s)}\int\varphi_{10}\varepsilon^2(0)\,dy+\frac{C}{\lambda^{10}(s)}\int_0^s\lambda^{10}(\mathcal{N}_{1,\mathrm{loc}}+b^2)\,ds'\\&\leqslant\frac{1}{\lambda^{10}(s)}\biggl[5+C\int_0^s(\mathcal{N}_{1,\mathrm{loc}}+b^2)\,ds'\biggr]\leqslant\frac{5+\delta(\varkappa^*)}{\lambda^{10}(s)},\end{aligned}$$

and (H3) is improved. We conclude that $t^{**}=t^*$. Moreover, by (4.50), for α_0 small enough compared to α^*, we get $t^*=T$, since the condition in the definition of $\mathcal{T}_{\alpha^*}$ is also improved by this estimate.

We now prove that $T<\infty$. We divide (4.15) by λ_0^2 and use (4.46) and (4.22) to estimate, on $[t_1^*,T)$,

$$\frac{|\ell^*|}{3}-C\frac{\mathcal{N}_{1,\mathrm{loc}}}{\lambda^2}\leqslant-(\lambda_0)_t\leqslant 3|\ell^*|+C\frac{\mathcal{N}_{1,\mathrm{loc}}}{\lambda^2}.$$

We integrate in time and obtain in particular, for all $t\in[t_1^*,T)$,

$$0\leqslant\lambda_0(t)\leqslant\lambda_0(t_1^*)-\frac{|\ell^*|(t-t_1^*)}{3}+C_1\int_{t_1^*}^t\frac{\mathcal{N}_{1,\mathrm{loc}}}{\lambda^2}\,d\tau. \tag{4.51}$$

Now, from the bound (4.49) again and (4.11),

$$\int_{t_1^*}^t\frac{\mathcal{N}_{1,\mathrm{loc}}}{\lambda^2}\,d\tau=\int_{s_1^*}^s\lambda(\sigma)\mathcal{N}_{1,\mathrm{loc}}\,d\sigma\lesssim 2\int_{s_1^*}^s\mathcal{N}_{1,\mathrm{loc}}\,d\sigma\lesssim 1,$$

and thus (4.51) implies that

$$T<\infty\quad\text{and in particular}\quad\lambda(t)\to 0\text{ as }t\to T.$$

The conservation of energy (2.28) implies that

$$\|\varepsilon_y(t)\|_{L^2}^2\lesssim\lambda^2(t)|E_0|+\mathcal{N}_2(t), \tag{4.52}$$

and thus, from (H2),

$$\|\varepsilon_y(t)\|_{L^2}+b(t)+\mathcal{N}_2(t)\to 0\quad\text{as }t\to T. \tag{4.53}$$

We now prove (4.7)–(4.8). We estimate from (4.46), (4.19) and (4.12), using that $T<\infty$,

$$\int_0^\infty\left|\frac{d}{ds}\left(\frac{b}{\lambda^2}e^J\right)\right|ds\lesssim\int_0^\infty\frac{1}{\lambda^2}(\mathcal{N}_{1,\mathrm{loc}}+|b|^3)\,ds<\infty,$$

and thus be^J/λ^2 has a limit as $t\to T$. Moreover,

$$|J(t)|\lesssim\mathcal{N}_2^{1/2}(t)\to 0\quad\text{as }t\to T$$

from (H2), and thus, from (4.46) and (4.47),

$$\frac{b(t)}{\lambda^2(t)}\to\ell_0>0\text{ as }t\to T,\quad\text{with }|\ell_0|\lesssim\delta(\alpha_0). \tag{4.54}$$

The time integration of (4.15) using (4.54), (4.48) and (4.11) yields

$$\left|\lambda_0(t)-\int_t^T \frac{b}{\lambda^2}\,dt'\right| \lesssim \int_t^T \frac{\mathcal{N}_{1,\mathrm{loc}}+o(b)}{\lambda^2}\,dt' \lesssim \int_s^\infty \lambda\mathcal{N}_{1,\mathrm{loc}}\,ds'+o(T-t)$$
$$\lesssim o(T-t)+\lambda(s)\int_s^\infty \mathcal{N}_{1,\mathrm{loc}}\,ds' = o(|T-t|+\lambda(t)),$$

and thus, using (4.54) again,

$$\lim_{t\to T}\frac{\lambda_0(t)}{T-t}=\ell_0.$$

Moreover, from (4.22),

$$\left|\frac{\lambda(t)}{\lambda_0(t)}-1\right| \lesssim |J_1(t)| \to 0 \quad \text{as } t\to T.$$

The control of the translation parameter follows from (2.29) and (H2), which yield

$$x_t=\frac{1}{\lambda^2}\frac{x_s}{\lambda}=\frac{1}{\lambda^2}(1+o(1)),$$

and (4.7) follows. Finally, the L^2 bound in (4.8) follows from (4.50), and the rest of (4.8) follows from (H2) and the conservation of energy (2.28):

$$\|\varepsilon_y(t)\|_{L^2}^2 \lesssim \lambda^2(t)|E_0|+|b(t)|+\mathcal{N}_2(t) \lesssim (|E_0|+\delta(\alpha_0))\lambda^2(t).$$

This concludes the proof of Proposition 4.1.

5. Blow up for $E_0 \leqslant 0$

In this section, we let an initial data

$$u_0\in\mathcal{A} \quad \text{with } E_0\leqslant 0.$$

We moreover assume that u_0 is not a solitary wave up to symmetries. We claim that the corresponding solution $u(t)$ to gKdV blows up in finite time in the (Blow up) regime described by Proposition 4.1.

Let us first recall the following standard orbital stability statement which follows from the variational characterization of the ground state and a standard concentration/compactness argument.

Lemma 5.1. (Orbital stability) *Let $\alpha>0$ small enough and a function $v\in H^1$ be such that*

$$\left|\int v^2-\int Q^2\right| \leqslant \alpha \quad \textit{and} \quad E(v)\leqslant \alpha\int v_x^2.$$

Then there exist $(\lambda_v,x_v)\in\mathbb{R}_+^*\times\mathbb{R}$ such that

$$\|Q-\varepsilon_0\lambda_v^{1/2}v(\lambda_v x+x_v)\|_{H^1}\leqslant\delta(\alpha), \quad \varepsilon_0\in\{-1,1\}.$$

For $\alpha>0$ small enough compared to α^*, it follows from the conservation of mass and energy that u remains in the tube $\mathcal{T}_{\alpha^*}$ on $[0,T)$. Therefore, only the case (Blow up) and (Soliton) can occur in Proposition 4.1. We argue by contradiction and assume that (Soliton) occurs.

Case $E_0<0$. This case is particularly simple to treat using the estimates of Proposition 4.1. Indeed, (2.28) (consequence of the conservation of energy), combined with the asusmption $E_0<0$ and the asymptotic stability statements (4.3) and (4.4), implies

$$\lambda^2(t)|E_0|+\int|\varepsilon_y|^2\lesssim|b(t)|+\mathcal{N}_1(t)\to 0\quad\text{as } t\to\infty,$$

and thus

$$\lambda(t)\to 0\quad\text{as } t\to\infty,$$

which hence contradicts the soliton dynamics displayed in (4.4).

Case $E_0=0$. This case is substantially more subtle and in particular there is no obvious obstruction to the (Soliton) dynamics. In fact, the conservation of energy (2.28) yields, with (4.3) and (4.4),

$$\int|\varepsilon_y|^2\lesssim|b(t)|+\mathcal{N}_1(t)\to 0\quad\text{as } t\to\infty,\tag{5.1}$$

but there is no further simple information on $\lambda(t)$. Our aim is to show that this $\dot{H}^1$ convergence implies global L^2 dispersion, and hence the solution has minimal mass which for $E_0=0$ is possible only for the solitary wave itself.

By rescaling, we may without loss of generality assume that $\lambda_\infty=1$ in (4.4). We claim the following result.

LEMMA 5.2. (L^2 compactness) *Assume that $E_0=0$ and that $u(t)$ satisfies the* (*Soliton*) *case. Then*

$$\int_{x-x(t)<-x_0}u_x^2(t,x)\,dx\lesssim\frac{1}{x_0^3}\qquad\textit{for all } t\geqslant 0 \textit{ and } x_0>1,\tag{5.2}$$

$$\int_{x-x(t)<-x_0}u^2(t,x)\,dx\lesssim\frac{1}{\sqrt{x_0}}\qquad\textit{for all } t\geqslant 0 \textit{ and } x_0>1.\tag{5.3}$$

Assume Lemma 5.2. Then, from (4.3), for all $x_0>1$,

$$|b(t)|+\int_{y>-x_0}|\varepsilon(t,y)|^2\,dy\to 0\quad\text{as } t\to\infty,$$

and thus, from (5.3) and (4.4),

$$\int|u_0|^2=\int u^2(t)=\int|Q_{b(t)}+\varepsilon(t)|^2\to\int Q^2\quad\text{as } t\to\infty.$$

Hence u_0 has critical mass and a contradiction follows.

Proof. Without loss of generality, by translation invariance, we assume that, for all $t \geqslant 0$,

$$|\lambda(t)-1| \leqslant \tfrac{1}{100}, \quad |x_t(t)-1| \leqslant \tfrac{1}{100} \quad \text{and} \quad \|\varepsilon(t)\|_{H^1}+|b(t)| \leqslant \tfrac{1}{100}. \tag{5.4}$$

From the decomposition of $u(t)$, there exists $a_0>1$ such that, for α small enough, for all $t\in[0,T)$,

$$\int_{x<-a_0/2} u^2(t,x+x(t))\,dx \leqslant \int_{y<-a_0/8} (\varepsilon(t)+Q_{b(t)})^2(y)\,dy \leqslant \frac{1}{100}. \tag{5.5}$$

Such an $a_0>1$ is now fixed.

Step 1. First decay property of u_x using almost monotonicity of a localized energy. We claim that there exists $C>0$ such that, for all $t_0\geqslant 0$ and $x_0>a_0$,

$$\int_{x-x(t_0)<-x_0} u_x^2(t_0,x)\,dx \leqslant \frac{C}{x_0^2}. \tag{5.6}$$

Proof of (5.6). Let ψ be a C^3 function such that, for $c>0$,

$$\begin{gathered}\psi \equiv \begin{cases} 1 & \text{on } (-\infty,-3], \\ 0 & \text{on } \left[-\frac{1}{2},\infty\right), \end{cases} \\ \psi' = -\tfrac{1}{2} \text{ on } [-2,-1], \quad \psi' \leqslant 0 \text{ on } \mathbb{R}, \quad (\psi'')^2 \leqslant -c\psi' \quad \text{and} \quad (\psi')^2 \leqslant c\psi \text{ on } \mathbb{R}.\end{gathered} \tag{5.7}$$

Let $x_0>a_0$. Define, for all $t>0$,

$$E_{x_0}(t) = \int \left(u_x^2 - \frac{1}{3}u^6\right)(t,x)\psi(\tilde{x})\,dx, \tag{5.8}$$

where

$$\tilde{x} = \frac{x-x(t)}{\xi(t)} \quad \text{and} \quad \xi(t) = x_0 + \frac{x(t)-x(t_0)}{4}.$$

First, observe that $\lim_{t\to\infty} E_{x_0}(t)=0$ by (5.1), (4.4) and the Gagliardo–Nirenberg inequality. Then, we control the variation of $E_{x_0}(t)$ on $[t_0,\infty)$. By (2.50),

$$\begin{aligned}\frac{d}{dt}E_{x_0}(t) = -\frac{1}{\xi(t)}\int (u_{xx}+u^5)^2\psi'(\tilde{x}) - \frac{2}{\xi(t)}\int u_{xx}^2\psi'(\tilde{x}) + \frac{10}{\xi(t)}\int u^4u_x^2\psi'(\tilde{x}) \\ + \frac{1}{\xi^3(t')}\int u_x^2\psi'''(\tilde{x}) - \frac{x_t(t)}{\xi(t)}\int \left(u_x^2 - \frac{1}{3}u^6\right)\left(1+\frac{1}{4}\tilde{x}\right)\psi'(\tilde{x}).\end{aligned} \tag{5.9}$$

All the integrals above are restricted to $\tilde{x}\in\left[-3,-\frac{1}{2}\right]$, since $\psi'(\tilde{x})=0$ for $\tilde{x}\notin\left[-3,-\frac{1}{2}\right]$. In particular, we have

$$-\frac{x_t(t)}{\xi(t)}\int u_x^2\left(1+\frac{1}{4}\tilde{x}\right)\psi'(\tilde{x}) \geqslant -\frac{1}{4}\frac{1}{\xi(t)}\int u_x^2\psi'(\tilde{x}).$$

By (5.4) and $\|u\|_{L^\infty}^4 \lesssim \|u_x\|_{L^2}^2 \|u\|_{L^2}^2 \lesssim 1$,

$$\frac{10}{\xi(t)} \int u^4 u_x^2 |\psi'(\tilde{x})| \lesssim \frac{1}{\xi(t)} \|u\|_{L^\infty}^4 \int u_x^2 |\psi'(\tilde{x})| \leqslant \frac{1}{100} \frac{1}{\xi(t)} \int u_x^2 |\psi'(\tilde{x})|.$$

Moreover,

$$\left| \frac{1}{\xi^3(t)} \int u_x^2 \psi'''(\tilde{x}) \right| \lesssim \frac{1}{\xi^3(t)} \int u_x^2(t') \lesssim \frac{1}{\xi^3(t)}.$$

Now, we treat the u^6 term. Recall the following standard computation (see, e.g., the proof of [26, Lemma 6]), for a C^1 positive function ϕ such that $\phi'/\sqrt{\phi} \lesssim 1$, for all $v \in H^1(\mathbb{R})$,

$$\begin{aligned} \|v^2 \sqrt{\phi}\,\|_{L^\infty} &\leqslant \sup_{x\in\mathbb{R}} \left| \int_{-\infty}^x \left(2v'v\sqrt{\phi} + \frac{1}{2} v^2 \frac{\phi'}{\sqrt{\phi}} \right) \right| \\ &\lesssim \left(\int v^2 \right)^{1/2} \left(\int (v')^2 \phi + \int v^2 \frac{(\phi')^2}{\phi} \right)^{1/2}. \end{aligned} \tag{5.10}$$

Using this estimate, and the fact that $\psi''(\tilde{x})^2/|\psi'(\tilde{x})| \lesssim 1$, we obtain

$$\begin{aligned} \|u^2 \sqrt{-\psi'(\tilde{x})}\,\|_{L^\infty}^2 &\lesssim \left(\int_{\operatorname{supp}\phi} u^2 \right) \left(\int u_x^2 |\psi'(\tilde{x})| + \frac{1}{\xi^2(t)} \int u^2 \frac{\psi''(\tilde{x})^2}{|\psi'(\tilde{x})|} \right) \\ &\lesssim \left(\int_{\operatorname{supp}\phi} u^2 \right) \left(\int u_x^2 |\psi'(\tilde{x})| + \frac{C}{\xi^2(t)} \int u^2 \right). \end{aligned} \tag{5.11}$$

Since $x_0 > a_0$, by (5.5), we have

$$\int_{\tilde{x}\in[-3,-1/2]} u^2(t) \leqslant \int_{x<-x_0/2} u^2(t, x+x(t))\,dx \leqslant \frac{1}{100}.$$

Thus, we get

$$\begin{aligned} \left| \int_{\tilde{x}\in[-3,-1/2]} u^6 \psi'(\tilde{x}) \right| &\lesssim \left(\int_{\tilde{x}\in[-3,-1/2]} u^2 \right)^2 \left(\int u_x^2 |\psi'(\tilde{x})| + \frac{C}{\xi^2} \int u^2 \right) \\ &\leqslant \frac{1}{100} \int u_x^2 |\psi'(\tilde{x})| + \frac{C}{\xi^2} \int u^2. \end{aligned} \tag{5.12}$$

Combining these estimates, we get

$$\frac{d}{dt} E_{x_0}(t) \gtrsim \frac{1}{\xi(t)} \int u_{xx}^2(t) |\psi'(\tilde{x})| + \frac{1}{\xi(t)} \int u_x^2 |\psi'(\tilde{x})| - C\xi^{-3}(t). \tag{5.13}$$

Integrating between t_0 and ∞, using that $\lim_{t\to\infty} E_{x_0}(t) = 0$ and (5.4), we get

$$E_{x_0}(t_0) = \int \left(u_x^2 - \frac{1}{3} u^6 \right)(t_0) \psi\left(\frac{x - x(t_0)}{x_0} \right) dx \lesssim \frac{1}{x_0^2}, \tag{5.14}$$

$$\int_0^\infty \left[\int u_{xx}^2(t) |\psi'(\tilde{x})| + \int u_x^2 |\psi'(\tilde{x})| \right] \frac{dt}{\xi(t)} \lesssim \frac{1}{x_0^2}. \tag{5.15}$$

Using (5.10) and (5.5), we have

$$\int u^6(t_0)\psi\left(\frac{x-x(t_0)}{x_0}\right)dx \lesssim \left(\int_{\tilde{x}\leqslant -1/2} u^2(t_0)\right)^2\left(\int u_x^2(t_0)\psi\left(\frac{x-x(t_0)}{x_0}\right)+\frac{1}{x_0^2}\int u^2(t_0)\right)$$
$$\leqslant \frac{1}{100}\int u_x^2(t_0)\psi\left(\frac{x-x(t_0)}{x_0}\right)+\frac{C}{x_0^2}\int u^2(t_0).$$

Therefore, for all $t_0\in[0,T)$ and $x_0>a_0$, we have obtained

$$\int_{x-x(t_0)<-x_0}(u_x^2(t_0,x)+u^6(t_0,x))\,dx\lesssim\frac{1}{x_0^2}. \tag{5.16}$$

Since $\psi'(\tilde{x})=0$ for $\tilde{x}<-3$ and $\tilde{x}>-\frac{1}{2}$, using (5.15), we have

$$\int_0^\infty\!\!\int u_{xx}^2(t')\psi(\tilde{x})\,dt'<\infty.$$

Moreover,

$$\frac{d}{dx_0}\left(\int_0^\infty\!\!\int u_{xx}^2(t)\psi(\tilde{x})\,dt\right)=\int_0^\infty\!\!\int u_{xx}^2(t)\frac{-\tilde{x}}{\xi(t)}\psi'(\tilde{x})\,dt$$
$$\lesssim\int_0^\infty\frac{1}{\xi(t)}\int u_{xx}^2(t)\psi'(\tilde{x})\,dt\lesssim\frac{1}{x_0^2}.$$

Integrating in x_0, we get

$$\int_0^\infty\!\!\int u_{xx}^2\psi(\tilde{x})\,dt'\leqslant\frac{C}{x_0}$$

and arguing in a similar way for u_x, we obtain

$$\int_0^\infty\!\!\int[u_{xx}^2(t)\psi(\tilde{x})+u_x^2(t)\psi(\tilde{x})]\,dt\leqslant\frac{1}{x_0}. \tag{5.17}$$

Step 2. Refined decay property of u_x.

We claim the improved decay

$$\int_{x<-x_0+x(t_0)}u_x^2(t_0,x)\,dx\lesssim\frac{1}{x_0^3}\quad\text{for all } x_0>2a_0. \tag{5.18}$$

To obtain this improved estimate, we introduce

$$G_{x_0}(t)=\int u_x^2(t)\psi(\tilde{x}).$$

By direct computation,

$$\frac{d}{dt}G_{x_0}(t)=-\frac{3}{\xi(t)}\int u_{xx}^2\psi'(\tilde{x})-\frac{x_t(t)}{\xi(t)}\int u_x^2\left(1+\frac{1}{4}\tilde{x}\right)\psi'(\tilde{x})+\frac{1}{\xi^3(t)}\int u_x^2\psi'''(\tilde{x})$$
$$-20\int u_x^3u^3\psi(\tilde{x})+\frac{5}{\xi(t)}\int u_x^2u^4\psi'(\tilde{x}).$$

The second and the last terms in the right-hand side are treated as before. For the third term, we use (5.16) and $\psi'''=0$ for $\tilde{x}\geqslant -\frac{1}{2}$, which gives

$$\frac{1}{\xi^3(t)}\int u_x^2(t)\psi'''(\tilde{x}) \lesssim \frac{1}{\xi^3(t)}\int_{x\leqslant -\xi(t)/2} u_x^2 \lesssim \frac{1}{\xi^5(t)} \lesssim \frac{\xi_t(t)}{\xi^5(t)}.$$

Finally, the term $\int u_x^3u^3\psi(\tilde{x})$ is controlled as follows, using (5.10) with $\phi=\psi(\tilde{x})$:

$$\begin{aligned}
\left|\int u_x^3u^3\psi(\tilde{x})\right| &\leqslant \big\|u_x^2\sqrt{\psi(\tilde{x})}\,\big\|_{L^\infty}\int \big|u_xu^3\sqrt{\psi(\tilde{x})}\,\big| \\
&\leqslant \big\|u_x^2\sqrt{\psi(\tilde{x})}\,\big\|_{L^\infty}\left(\int u_x^2\psi(\tilde{x})\right)^{1/2}\left(\int_{x<-x_0/2+x} u^6\right)^{1/2} \\
&\lesssim \left(\int_{x<-x_0/2+x} u_x^2\right)^{1/2}\left(\left(\int u_{xx}^2\psi(\tilde{x})\right)^{1/2}+\left(\frac{1}{\xi^2}\int u_x^2\psi(\tilde{x})\right)^{1/2}\right) \\
&\qquad\times\left(\int u_x^2\psi(\tilde{x})\right)^{1/2}\left(\int_{x<-x_0/2+x} u^6\right)^{1/2} \\
&\lesssim \left(\int_{x<-x_0/2+x} u_x^2\right)^{1/2}\left(\int_{x<-x_0/2+x} u^6\right)^{1/2}\left(\int u_{xx}^2\psi(\tilde{x})+\int u_x^2\psi(\tilde{x})\right) \\
&\lesssim \frac{1}{x_0^2}\left(\int u_{xx}^2\psi(\tilde{x})+\int u_x^2\psi(\tilde{x})\right).
\end{aligned}$$

In conclusion of these estimates, we have obtained

$$\frac{d}{dt}G_{x_0}(t) \gtrsim -\frac{\xi_t}{\xi^5}-\frac{1}{x_0^2}\left(\int (u_{xx}^2+u_x^2)\psi(\tilde{x})\right).$$

Therefore, by integration over $[t_0,\infty)$, using (5.17) and $\lim_{t\to\infty} G_{x_0}(t)=0$, we obtain

$$G_{x_0}(t_0) \lesssim \frac{1}{x_0^3},$$

which proves (5.2).

Step 3. L^2 estimate.

We deduce from (5.2) some L^2 tightness for u. Indeed, for $x_0>1$,

$$\|u(t,\cdot)\|^2_{L^\infty(x-x(t)<-x_0)} \lesssim \int_{x-x(t)<-x_0} |u_xu|\,dx \lesssim \left(\int_{x-x(t)<-x_0} u_x^2\right)^{1/2}\left(\int u^2\right)^{1/2} \lesssim \frac{1}{x_0^{3/2}},$$

from which

$$\int_{x-x(t)\leqslant -x_0} |u(t,x)|^2\,dx \lesssim \int_{y>x_0}\frac{dy}{|y|^{3/2}} \lesssim \frac{1}{\sqrt{x_0}},$$

and (5.3) follows. □

6. Sharp description of the blow-up regime

We now finish the proof of Theorem 1.1 by proving (1.16) and (1.18) in the framework of a blow-up solution in $\mathcal{T}_{\alpha^*}$. We further use L^2 and H^1 monotonicity properties *away* from the soliton to propagate the dispersive information in larger regions to the left than the norm $\mathcal{N}_i$ controlled by Proposition 4.1, and this will yield the sharp behavior (1.18).

We let

$$\tilde{u}(t,x)=u(t,x)-\frac{1}{\lambda^{1/2}(t)}Q_{b(t)}\biggl(\frac{x-x(t)}{\lambda(t)}\biggr).$$

PROPOSITION 6.1. (Improved dispersive bounds away from the soliton) *Let $u_0\in\mathcal{A}$ be such that $u(t)$ blows up in finite time T and*

$$u(t)\in\mathcal{T}_{\alpha^*}\quad \textit{for all } t\in[0,T).$$

Then, the following properties hold:

(i) (H^1 estimates around the soliton)

$$\sup_{R>1}\sup_{t\in[T-1/\ell_0^2R,T)} R^2\int_{x>R}\tilde{u}^2(t,x)\,dx<\infty, \tag{6.1}$$

$$\lim_{R\to\infty}\sup_{t\in[T-1/\ell_0^2R,T)}\int_{x>R}\tilde{u}_x^2(t,x)\,dx=0, \tag{6.2}$$

$$\lim_{t\to T}\frac{1}{(T-t)^2}\int_{x-x(t)\geqslant -x(t)/\log(T-t)}\tilde{u}^2(t,x)\,dx=0. \tag{6.3}$$

(ii) (Existence and asymptotic of the dispersed remainder) *there exists $u^*\in H^1$ such that*

$$\tilde{u}\to u^*\quad \textit{in } L^2 \textit{ as } t\to T \tag{6.4}$$

and

$$\int_{x>R}(u^\star)^2(x)\,dx\sim\frac{\|Q\|_{L^1}^2}{8\ell_0R^2}\quad \textit{as } R\to\infty. \tag{6.5}$$

The rest of this section is devoted to the proof of Proposition 6.1.

6.1. H^1 monotonicity away from the soliton

We aim at refining the dispersive estimate (4.12) by propagating it to the left of the solitary wave, since $\mathcal{N}_2$ involves an exponentially well-localized norm at the left of the soliton. For this, we use H^1 monotonicity tools in the spirit of [26] and [20].

LEMMA 6.2. (Monotonicity away from the soliton core) *There exist universal constants $a_0 \gg 1$ and $0<\delta_0 \ll 1$ such that the following holds. Let $0 \leqslant t_0 < T$, close enough to T, and $0<\nu<\frac{1}{10}$ be such that*

$$\frac{\lambda^2(t_0)}{\nu} < \delta_0. \tag{6.6}$$

Let

$$\phi(x) = \frac{2}{\pi}\arctan\left(\exp\left(\frac{\sqrt{\nu}}{5}x\right)\right)$$

be such that

$$\begin{gathered} \lim_{x\to\infty}\phi(x)=1, \quad \lim_{x\to-\infty}\phi(x)=0, \\ \phi'''(x) \leqslant \frac{\nu}{25}\phi'(x) \quad \text{and} \quad |\phi''(x)| \lesssim \sqrt{\nu}\phi'(x) \quad \text{for all } x \in \mathbb{R}. \end{gathered} \tag{6.7}$$

Then, for all $y_0 > a_0$ and $t_0 \leqslant t < T$, we have the L^2 monotonicity bound

$$\begin{aligned} &\int \tilde{u}^2(t,x)\phi\left(\frac{x-x(t_0)}{\lambda(t_0)} - \nu\frac{t-t_0}{\lambda^3(t_0)} + y_0\right)dx + 2(b(t)-b(t_0))(P,Q) \\ &\qquad \lesssim \int \tilde{u}^2(t_0,x)\phi\left(\frac{x-x(t_0)}{\lambda(t_0)} + y_0\right)dx + \frac{1}{\sqrt{\nu}}e^{-\sqrt{\nu}y_0/10} + \lambda^{2+1/4}(t_0) \end{aligned} \tag{6.8}$$

and the H^1 monotonicity bound

$$\begin{aligned} &\int\left(\tilde{u}_x^2 - \frac{1}{3}\tilde{u}^6\right)(t,x)\phi\left(\frac{5}{4}\left(\frac{x-x(t_0)}{\lambda(t_0)} - \nu\frac{t-t_0}{\lambda^3(t_0)} + y_0\right)\right)dx - 2\left(\frac{b(t)}{\lambda^2(t)} - \frac{b(t_0)}{\lambda^2(t_0)}\right)(P,Q) \\ &\qquad \lesssim \int\left(\tilde{u}_x^2(t_0,x) + \frac{\tilde{u}^2(t_0,x)}{\lambda^2(t_0)}\right)\phi\left(\frac{x-x(t_0)}{\lambda(t_0)} + y_0\right)dx + \frac{1}{\sqrt{\nu}}\frac{e^{-\sqrt{\nu}y_0/10}}{\lambda^2(t_0)} + \lambda^{1/2}(t_0). \end{aligned} \tag{6.9}$$

The proof of Lemma 6.2 is postponed to Appendix A.

6.2. Proof of Proposition 6.1

Step 1. Proof of (6.1).

The estimate (6.1) is a direct consequence of (6.8) and the space-time control of local terms (4.12) which implies that

$$\frac{\mathcal{N}_i(t_2)}{\lambda^2(t_2)} + \int_{t_1}^{t_2} \frac{\int(\varepsilon_y^2+\varepsilon^2)(s)\varphi'_{i,B}}{\lambda^5(\tau)}\,d\tau \lesssim \delta(\alpha_0). \tag{6.10}$$

Indeed, if we fix $\nu = \frac{1}{16}$ in Lemma 6.2 (note that $B > 40 = 10/\sqrt{\nu}$), then (6.6) is satisfied from the blow-up assumption for t close enough to T, and we estimate the right-hand side of (6.8) as

$$\begin{aligned}
&\int \tilde{u}^2(t_0,x)\phi\bigg(\frac{x-x(t_0)}{\lambda(t_0)}+y_0\bigg)\,dx\\
&\qquad=\int \varepsilon^2(t_0)\phi(y+y_0)\\
&\qquad\lesssim \int_{y<-y_0}\varepsilon^2(t_0)e^{\nu(y+y_0)/10}+\int_{y>-y_0}\varepsilon^2(t_0)\\
&\qquad\lesssim \int_{y<-y_0}\varepsilon^2(t_0)e^{(y+y_0)/B}+e^{y_0/B}\int_{-y_0<y<0}\varepsilon^2(t_0)e^{y/B}+\int_{y>0}\varepsilon^2(t_0)\\
&\qquad\lesssim e^{y_0/B}\mathcal{N}_{1,\mathrm{loc}}(t_0).
\end{aligned}\tag{6.11}$$

Let $R\gg1$ be large enough and t_R be such that $x(t_R)=R$, so that

$$T-t_R=\frac{1}{\ell_0^2R}(1+o_R(1))=\frac{\lambda(t_R)}{\ell_0}(1+o_R(1))\quad\text{as } R\to\infty.\tag{6.12}$$

We now make essential use of the fact that the space-time estimate (6.10) is better for local L^2 terms than the pointwise bound given by (H2). Indeed, the law (4.7) and (6.12) ensure that, for R large,

$$\lambda(\tau)=\ell_0\bigg[T-t_R+\frac{1}{R}o_R(1)\bigg]\geqslant\frac12\lambda(t_R)\quad\text{for all } \tau\in[t_R-(R\ell_0)^{-5/2},t_R],$$

and thus (6.10) implies that

$$(R\ell_0)^5\int_{t_R-(R\ell_0)^{-5/2}}^{t_R}\int(\varepsilon_y^2+\varepsilon^2)(t)\varphi_{1,B}'\,dt\lesssim\int_0^T\frac{\int(\varepsilon_y^2+\varepsilon^2)(t)\varphi_{1,B}'}{\lambda^5(t)}\,dt\lesssim\delta(\alpha_0).$$

Thus, there exists $\bar t_R\in[t_R-(R\ell_0)^{-5/2},t_R]$ such that

$$\int(\varepsilon_y^2+\varepsilon^2)(\bar t_R)\varphi_{1,B}'\lesssim\delta(\alpha_0)(\ell_0R)^{-5/2}\sim\delta(\alpha_0)\lambda(\bar t_R)^{5/2}\tag{6.13}$$

which is a strict gain on the pointwise bound (H2). Note also the relations

$$b(\bar t_R)=\ell_0^3(T-\bar t_R)^2(1+o_R(1))=\frac{1}{\ell_0R^2}(1+o_R(1))\quad\text{and}\quad x(\bar t_R)=R+o_R(1).\tag{6.14}$$

We now apply (6.8) to $u(t)$ with

$$\nu=\tfrac{1}{16},\quad t_0=\bar t_R\quad\text{and}\quad y_0=y_R=40\log\ell_0R^3.$$

We obtain, from (6.11), (6.13), (6.14) and $B \gg 1$, that, for all $t \in [\bar t_R, T)$,

$$\begin{aligned}\int \tilde u^2(t,x)\phi\biggl(\frac{x-x(\bar t_R)}{\lambda(\bar t_R)}-\frac{1}{16}\frac{t-\bar t_R}{\lambda^3(\bar t_R)}+y_R\biggr)-2b(\bar t_R)(P,Q)&\\ \lesssim e^{y_R/B}\mathcal{N}_{1,\mathrm{loc}}(\bar t_R)+e^{-y_R/40}+(T-t_R)^{2+1/4}=o_R\biggl(\frac{1}{R^2}\biggr)&\end{aligned} \tag{6.15}$$

and

$$2b(\bar t_R)(P,Q)=\frac{\|Q\|_{L^2}^2}{8\ell_0}\frac{1}{R^2}(1+o_R(1)).$$

Moreover, using (6.14), we estimate, for all $x>2R$ and all $t \geqslant \bar t_R$,

$$\frac{x-x(\bar t_R)}{\lambda(\bar t_R)}-\frac{1}{16}\frac{t-\bar t_R}{\lambda^3(\bar t_R)}+y_R \geqslant \frac{2R-R}{\lambda(\bar t_R)}-\frac{1}{16}\frac{t-\bar t_R}{\lambda^3(\bar t_R)} \geqslant \frac{1}{\ell_0\lambda^2(t_R)}>0.$$

Thus, from (6.14) and (6.15), and also using that $\phi(y) \geqslant \frac12$ for $y>0$, we obtain

$$\int_{x>2R}\tilde u^2(t,x)\,dx \lesssim \frac{1}{\ell_0 R^2}\quad \text{for all } t\in\biggl[T-\frac{1}{2\ell_0^2 R},T\biggr),$$

and (6.1) follows.

Step 2. Proof of (6.2).

We now apply (6.9) to $u(t)$ with the same choice as before

$$\nu=\tfrac{1}{16},\quad t_0=\bar t_R\quad\text{and}\quad y_0=y_R=40\log \ell_0 R^3.$$

We estimate, as in the proof of (6.11) and using (6.13),

$$\int \tilde u_x^2(\bar t_R,x)\phi\biggl(\frac{x-x(\bar t_R)}{\lambda(\bar t_R)}+y_R\biggr)\,dx \lesssim e^{y_R/B}\frac{\int \varepsilon_y^2(\bar t_R)\varphi_{1,B}'}{\lambda^2(\bar t_R)}=o_R(1).$$

Using (6.11), we obtain, for all $t\in[\bar t_R,T)$,

$$\begin{aligned}\int\biggl(\tilde u_x^2(t,x)-\frac13\tilde u^6(t,x)\biggr)\phi\biggl(\frac54\biggl(\frac{x-x(\bar t_R)}{\lambda(\bar t_R)}-\frac{1}{16}\frac{t-\bar t_R}{\lambda^3(\bar t_R)}+y_R\biggr)\biggr)\,dx&\\ \lesssim\biggl|\frac{b(t)}{\lambda^2(t)}-\frac{b(\bar t_R)}{\lambda^2(\bar t_R)}\biggr|+R^2e^{-y_R/40}+o_R(1)=o_R(1),&\end{aligned}$$

where, in the last step, we used the fact that

$$\lim_{t\to T}\frac{b(t)}{\lambda^2(t)}=\ell_0.$$

Observe now the bound from Sobolev, (4.8) and (6.15):

$$\begin{aligned}\int \tilde u^6(t,x)\phi\biggl(\frac54\biggl(\frac{x-x(\bar t_R)}{\lambda(\bar t_R)}-\frac{1}{16}\frac{t-\bar t_R}{\lambda^3(\bar t_R)}+y_R\biggr)\biggr)\,dx&\\ \leqslant C\|\tilde u\|_{L^\infty}^4\int \tilde u^2(t,x)\phi\biggl(\frac{x-x(\bar t_R)}{\lambda(\bar t_R)}-\frac{1}{16}\frac{t-\bar t_R}{\lambda^3(\bar t_R)}+y_R\biggr)\,dx \lesssim \frac{1}{R^2},&\end{aligned}$$

and (6.2) follows.

Step 3. Proof of (6.3).

Let t be close to T. The space time estimate (6.10) and (4.7) ensure that

$$\frac{1}{\lambda^5(t)}\int_{t-20(T-t)/|\log(T-t)|}^{t-10(T-t)/|\log(T-t)|}\mathcal{N}_{1,\mathrm{loc}}(\tau)\,d\tau\lesssim\int_0^T\frac{\mathcal{N}_{1,\mathrm{loc}}(\tau)}{\lambda^5(\tau)}\,d\tau\lesssim\delta(\alpha),$$

and thus there exists

$$\bar t\in\left[t-\frac{20(T-t)}{|\log(T-t)|},t-\frac{10(T-t)}{|\log(T-t)|}\right]$$

such that

$$\mathcal{N}_{1,\mathrm{loc}}(\bar t)\leqslant\ell_0^5(T-t)^4|\log(T-t)|.$$

Moreover, from (4.7), we have

$$x(t)-x(\bar t)\geqslant(t-\bar t)\min_{[t,\bar t]}x_t\geqslant\frac{9}{\ell_0^2(T-t)|\log(T-t)|}\geqslant\frac{8x(t)}{|\log(T-t)|},\tag{6.16}$$

and

$$b(t)-b(\bar t)=o((T-t)^2)\quad\text{as } t\to T.$$

We now apply (6.8) with

$$\nu=\tfrac{1}{16},\quad y_0=\bar y=40|\log(T-t)|\quad\text{and}\quad t_0=\bar t.$$

The right-hand side of(6.8) is estimated using (6.11) and we obtain

$$\int\tilde u^2(t,x)\phi\left(\frac{x-x(\bar t)}{\lambda(\bar t)}-\frac{1}{10}\frac{t-\bar t}{\lambda^3(\bar t)}+\bar y\right)dx=o((T-t)^2)\quad\text{as } t\to T.$$

Moreover, if x is such that

$$x-x(t)\geqslant-\frac{x(t)}{|\log(T-t)|},$$

then, from (6.16) and (4.7),

$$\frac{x-x(\bar t)}{\lambda(\bar t)}-\frac{1}{10}\frac{t-\bar t}{\lambda^3(\bar t)}\geqslant\frac{1}{\lambda(\bar t)|\log(T-t)|}\left[8x(t)-\frac{1}{10}\frac{10(T-t)}{\lambda^2(t)}\right]>0,$$

and then $\phi(y)\geqslant\frac12$ for $y>0$ yields (6.3).

Remark 6.3. Observe that (6.1) and (6.2) imply that, for all $R>1$,

$$\int_{x>R}\tilde u^2\left(T-\frac{1}{200\ell_0^2R},x\right)dx\lesssim\frac{1}{R^2}\quad\text{and}\quad\lim_{R\to\infty}\int_{x>R}\tilde u_x^2\left(T-\frac{1}{200\ell_0^2R},x\right)dx=0.$$

In particular, given t close enough to T, we chose $R=(200\ell_0^2(T-t))^{-1}<\frac{1}{100}x(t)$ and conclude that

$$\int_{x>x(t)/100}\tilde u^2(t,x)\,dx\lesssim(T-t)^2\quad\text{and}\quad\lim_{t\to T}\int_{x>x(t)/100}\tilde u_x^2(t,x)\,dx=0.\tag{6.17}$$

Step 4. L^2 tightness.

First observe, by a direct check using (4.7), that

$$\frac{1}{\lambda^{1/2}(t)}Q_{b(t)}\left(\frac{x-x(t)}{\lambda(t)}\right)-\frac{1}{\lambda^{1/2}(t)}Q\left(\frac{x-x(t)}{\lambda(t)}\right)\to 0 \quad \text{in } L^2 \text{ as } t\to T,$$

and hence (1.16) is equivalent to showing the existence of a strong limit

$$\tilde{u}(t)\to u^* \quad \text{in } L^2 \text{ as } t\to T. \tag{6.18}$$

We first claim that the sequence is *tight*, i.e. for all $\varepsilon>0$ there exists $A_\varepsilon>1$ such that for all $t\in[0,T)$ one has

$$\int_{|x|>A_\varepsilon}\tilde{u}^2(t,x)\,dx<\varepsilon. \tag{6.19}$$

On the right $x>A_\varepsilon$, where non-linear interactions take place, the claim directly follows from (6.1). On the left, this is a simple linear claim which follows from the finiteness of the time interval $[0,T)$, the H^1 bound (4.8) and a Kato L^2 localization argument. Indeed, let t_ε be close enough to T such that

$$\int_{t_\varepsilon}^T\int_{x<0}(u_x^2+u^2)\,dx\,dt<\varepsilon. \tag{6.20}$$

Let ψ be a C^3 function such that

$$\psi\equiv\begin{cases}1 & \text{on } (-\infty,-2],\\ 0 & \text{on } [-1,\infty),\end{cases} \quad \text{and} \quad \psi'\leqslant 0 \quad \text{on } \mathbb{R}. \tag{6.21}$$

Pick $A_\varepsilon>1$ large enough so that $\int u^2(t_\varepsilon)\psi(x+A)\leqslant\varepsilon$. Then, by (2.49),

$$\frac{d}{dt}\int u^2(t)\psi(x+A)=-3\int u_x^2(t)\psi'(x+A)+\int u^2(t)\psi'''(x+A)+\frac{5}{3}\int u^6(t)\psi'(x+A),$$

and thus, from (6.20), for all $t\in[t_\varepsilon,T)$,

$$\left|\int u^2(t)\psi(x+A)-\int u^2(t_\varepsilon)\psi(x+A)\right|\leqslant C\varepsilon$$

and (6.19) follows. Now the uniform H^1 bound (4.8) ensures that for any sequence $t_n\to T$ there exists a subsequence $t_{\phi(n)}\to T$ and $u^\star\in H^1$ such that $\tilde{u}(t_{\phi(n)})\rightharpoonup u^\star$ in H^1 weakly and $\tilde{u}(t_{\phi(n)})\to u^\star$ in L^2 strongly from (6.19) and the local compactness of the Sobolev embedding. By a weak convergence argument, the limit $u^\star$ does not depend on the sequence $\{t_n\}_{n=1}^\infty$. Indeed, let θ be a C^∞ function with support in $[-K,K]$. Then

$$\left|\frac{d}{dt}\int u\theta\right|=\left|\int u^5\theta_x+\int u\theta_{xxx}\right|\leqslant C_\theta\int_{-K}^K(|u|^5+|u|)\leqslant C_{\theta,K},$$

and thus $\int u(t)\theta$ has a limit as $t\to T$, and (6.18) follows. Note that the regularity $u^*\in H^1$ follows from (6.18) and (4.8).

Step 5. Universal behavior of u^* at the singularity.

We now turn to the proof of the universal behavior of u^* (6.5) at the singularity, which follows from lower and upper bounds.

(i) (Upper bound) Let $R\gg 1$ be large enough. Let t_R be such that

$$x(t_R)=R,$$

so that, from (4.7),

$$\begin{aligned}\frac{\lambda(t_R)}{\ell_0}&=(T-t_R)(1+o_R(1))=\frac{1}{\ell_0^2 R}(1+o_R(1)),\\ b(t_R)&=\ell_0^3(T-t_R)^2(1+o_R(1))=\frac{1}{\ell_0 R^2}(1+o_R(1)).\end{aligned}$$

We apply (6.8) to $u(t)$ with

$$\nu=\nu_R=\frac{1}{\log^2 R},\quad y_0=y_R=10\log^2 R^3\quad\text{and}\quad t_0=t_R$$

which satisfy the condition (6.6) for R large enough, and obtain, for all $t\in[t_R,T)$,

$$\begin{aligned}\int \tilde u^2(t,x)\phi\biggl(\frac{x-x(t_R)}{\lambda(t_R)}-\nu_R\frac{t-t_R}{\lambda^3(t_R)}+y_R\biggr)\,dx-2b(t_R)\int PQ\\ \lesssim\int \tilde u^2(t_R,x)\phi\biggl(\frac{x-x(t_R)}{\lambda(t_R)}+y_R\biggr)\,dx+\frac{1}{\nu_R}e^{-\sqrt{\nu_R}y_R/10}+(T-t_R)^{2+1/4}\\ \lesssim\int \tilde u^2(t_R,x)\phi\biggl(\frac{x-x(t_R)}{\lambda(t_R)}+y_R\biggr)\,dx+o\biggl(\frac{1}{R^2}\biggr).\end{aligned}$$

Note that

$$\frac{-x(t_R)}{|\log(T-t_R)|}=-\frac{R}{\log R}(1+o_R(1))\ll\lambda(t_R)y_R,$$

so that, by (6.3),

$$\begin{aligned}&\int \tilde u^2(t_R)\phi\biggl(\frac{x-x(t_R)}{\lambda(t_R)}+y_R\biggr)\,dx\\ &\quad\lesssim e^{-\sqrt{\nu_R}y_R/10}\int \tilde u^2(t_R,x)\,dx+\int_{x-x(t_R)>-2\lambda(t_R)y_R}\tilde u^2(t_R,x)\,dx=\frac{1}{R^2}o_R(1).\end{aligned}$$

We thus conclude, from (2.5), that

$$\int \tilde u^2(t,x)\phi\biggl(\frac{x-x(t_R)}{\lambda(t_R)}-\nu_R\frac{t-t_R}{\lambda^3(t_R)}+y_R\biggr)\,dx\leqslant\frac{2\int PQ}{\ell_0R^2}(1+o_R(1))=\frac{\|Q\|_{L^1}^2}{8\ell_0R^2}(1+o_R(1)).$$

Passing to the limit $t\to T$, we find that

$$R^2\int (u^\star)^2(x)\phi\biggl(\frac{x-x(t_R)}{\lambda(t_R)}-\nu_R\frac{T-t_R}{\lambda^3(t_R)}+y_R\biggr)\,dx\leqslant\frac{\|Q\|_{L^1}^2}{8\ell_0}(1+o_R(1)).$$

Using that $x(t_R)=R$ and $\lambda(t_R)=(1+o_R(1))/R\ell_0$, and passing to the limit $R\to\infty$ yields

$$\limsup_{R\to\infty}R^2\int_{x>(1+\nu_R)R}(u^\star)^2(x)\,dx\leqslant\frac{\|Q\|_{L^1}^2}{8\ell_0},$$

which now easily implies that

$$\limsup_{R\to\infty}R^2\int_{x>R}(u^\star)^2(x)\,dx\leqslant\frac{\|Q\|_{L^1}^2}{8\ell_0}. \tag{6.22}$$

(ii) (Lower bound) Let ω be a smooth function satisfying

$$\omega\equiv\begin{cases}0 & \text{on } (-\infty,-1],\\ 1 & \text{on } [0,\infty),\end{cases}\qquad\text{and}\quad \omega'\geqslant 0\quad\text{on } \mathbb{R}.$$

Let $0<\nu<\frac{1}{10}$ be arbitrary and let ω_ν be defined by $\omega_\nu(x)=\omega(x/\nu)$. For $R>1$ large, we define t_R such that $x(t_R)=R$ as before. Using the identity (2.49), we have, for all $t_R\leqslant t<T$,

$$\begin{aligned}\frac{d}{dt}\int u^2\omega_\nu\biggl(\frac{x-R+4\log R}{R}\biggr)&\\ &\geqslant-\frac{3}{R}\int u_x^2\omega_\nu'\biggl(\frac{x-R+4\log R}{R}\biggr)+\frac{1}{R^3}\int u^2\omega_\nu'''\biggl(\frac{x-R+4\log R}{R}\biggr)\\ &\geqslant-\frac{C_\nu}{R}\int_{(1-\nu)R<x+4\log R<R}u_x^2-\frac{C_\nu}{R^3}\int_{(1-\nu)R<x+4\log R<R}u^2.\end{aligned}$$

By (6.2) and the properties of Q_b (see in particular (2.9) and (2.11)) we have

$$\sup_{t\in[t_R,T)}\int_{(1-\nu)R<x+4\log R<R}u_x^2(t,x)\,dx=o_R(1)\quad\text{as } R\to\infty.$$

Since $T-t_R\lesssim 1/\ell_0^2R$, by integrating over $[t_R,t]$ we obtain, for all $t\in[t_R,T)$,

$$\int u^2(t)\omega_\nu\biggl(\frac{x-R+4\log R}{R}\biggr)\geqslant\int u^2(t_R)\omega_\nu\biggl(\frac{x-R+4\log R}{R}\biggr)+o_R\biggl(\frac{1}{R^2}\biggr). \tag{6.23}$$

We now develop u in terms of Q_b and $\tilde u$. On the one hand, a simple computation ensures

$$\begin{aligned}\int u^2(t)\omega_\nu\biggl(\frac{x-R+4\log R}{R}\biggr)&=\int Q^2+\int\tilde u^2(t)\omega_\nu\biggl(\frac{x-R+4\log R}{R}\biggr)+o_{t\to T}(1)\\ &\to\int Q^2+\int (u^*)^2(t)\omega_\nu\biggl(\frac{x-R+4\log R}{R}\biggr)\quad\text{as } t\to T.\end{aligned}$$

Next,

$$\begin{aligned}\int u^2(t_R)\omega_\nu\left(\frac{x-R+4\log R}{R}\right)\\ &=\int Q^2+2(P,Q)b(t_R)+\int \tilde{u}^2(t_R)\omega_\nu\left(\frac{x-R+4\log R}{R}\right)+o_R\left(\frac{1}{R^2}\right)\\ &\geqslant \int Q^2+\frac{1}{R^2}\left(\frac{\|Q\|_{L^1}^2}{8\ell_0}+o_R(1)\right),\end{aligned}$$

where we used (6.5) to treat the cross term. We therefore conclude, from (6.23),

$$\liminf_{R\to\infty} R^2\int_{x>(1-\nu)R-4\log R}(u^\star)^2(x)\,dx\geqslant \frac{\|Q\|_{L^1}^2}{8\ell_0}$$

and, since ν is arbitrary,

$$\liminf_{R\to\infty} R^2\int_{x>R}(u^\star)^2(x)\,dx\geqslant \frac{\|Q\|_{L^1}^2}{8\ell_0}.$$

This concludes the proof of Proposition 6.1.

Appendix A.

A.1. Proof of Lemma 6.2

Let $a_0\gg1$ and $0<\delta_0\ll1$ be two constants to be chosen.

For $t_0\in[0,T)$, we consider the renormalized solution

$$z(t',x')=\lambda^{1/2}(t_0)u(\lambda^3(t_0)t'+t_0,\lambda(t_0)x'+x(t_0)),\quad t'\in[0,T_z),\quad T_z=\frac{T-t_0}{\lambda^3(t_0)}. \tag{A.1}$$

The function z admits the decomposition

$$z(t',x')=\frac{1}{\lambda_z^{1/2}(t')}(Q_{b_z}+\varepsilon_z)\left(t',\frac{x-x_z(t')}{\lambda_z(t')}\right)=\frac{1}{\lambda_z^{1/2}(t')}Q_{b_z(t')}\left(\frac{x-x_z(t')}{\lambda_z(t')}\right)+\tilde{z}(t',x'), \tag{A.2}$$

with, explicitely,

$$\begin{aligned}\varepsilon_z(t')&=\varepsilon(\lambda^3(t_0)t'+t_0), & \lambda_z(t')&=\frac{\lambda(\lambda^3(t_0)t'+t_0)}{\lambda(t_0)},\\ x_z(t')&=\frac{x(\lambda^3(t_0)t'+t_0)-x(t_0)}{\lambda(t_0)}, & b_z(t')&=b(\lambda^3(t_0)t'+t_0).\end{aligned}$$

In particular,

$$\lambda_z(0)=1,\quad x_z(0)=0\quad\text{and}\quad b_z(0)=b(t_0). \tag{A.3}$$

The monotonicity bound (4.48) and (4.8) ensure that, for all $t'\in[0,T_z)$,

$$\|(\varepsilon_z)_x(t')\|_{L^2}^2 \lesssim \lambda_z^2(t')\lambda^2(t_0)(|E_0|+\delta(\alpha)), \qquad \|\varepsilon_z(t')\|_{L^2}^2 \lesssim \delta(\alpha), \tag{A.4}$$

$$\|z(t')\|_{H^1} \lesssim \lambda^2(t_0)(|E_0|+\delta(\alpha)) \leqslant \delta_0, \qquad \lambda_z(t') \leqslant \tfrac{3}{2}, \tag{A.5}$$

provided t_0 is close enough to T and α is small enough.

We denote by $\mathcal{N}_2(t')$ the quantity defined in (3.3) for $z(t')$. From (H2), and then (6.6), we have

$$\theta_z = \sup_{t'\in[0,T_z]} \left|\frac{b_z(t')+\mathcal{N}_{2,z}(t')}{\lambda_z^2(t')}\right| = \sup_{t\in[t_0,T)} \lambda^2(t_0)\left|\frac{b(t)+\mathcal{N}_2(t)}{\lambda^2(t)}\right| \lesssim \lambda^2(t_0)\delta(\alpha). \tag{A.6}$$

Lemma 6.2 follows directly from the following monotonicity result on $\tilde{z}$ and (A.6).

Lemma A.1. (Monotonicity in renormalized variables) *Assume* (A.3)–(A.6). *Then, for all* $y_0>a_0$ *and all* $t'\in[0,T_z)$, *the following hold*:

(i) (L^2 monotonicity)

$$\begin{aligned} &\int \tilde{z}^2(t')\phi(x'-\nu t'+y_0)\,dx' + 2(P,Q)(b_z(t')-b_z(0)) \\ &\qquad + \frac{1}{4}\int_0^{t'}\int (z_x^2+\nu z^2)(t'')\phi'(x'-\nu t'+y_0)\,dx'\,dt'' \\ &\qquad\qquad \lesssim \theta_z^{9/8} + \int \tilde{z}^2(0)\phi(x'+y_0)\,dx' + \frac{1}{\sqrt{\nu}}e^{-\sqrt{\nu}y_0/10}. \end{aligned} \tag{A.7}$$

(ii) (H^1 monotonicity)

$$\begin{aligned} &\int\left(\tilde{z}_x^2-\frac{1}{3}\tilde{z}^6\right)(t')\phi\left(\frac{5}{4}(x'-\nu t'+y_0)\right)dx' - 2(P,Q)\left(\frac{b_z(t')}{\lambda_z^2(t')}-\frac{b_z(0)}{\lambda_z^2(0)}\right) \\ &\quad + \frac{1}{4}\int_0^{t'}\int (z_{xx}^2+\nu z_x^2)(t'')\phi\left(\frac{5}{4}(x'-\nu t''+y_0)\right)dx'\,dt'' \\ &\qquad \lesssim \lambda^2(t_0)\theta_z^{1/4} + \int [\tilde{z}_x^2(t_0)+\tilde{z}^2(t_0)]\phi(x'+y_0)\,dx' + \frac{1}{\sqrt{\nu}}e^{-\sqrt{\nu}y_0/10}. \end{aligned} \tag{A.8}$$

Undoing the transformation (A.1) and applying Lemma A.1 yields Lemma 6.2.

Proof. The proof is closely related to the argument in [26] and [20].
We define, for $y_0>1$ and $0<\nu<\frac{1}{10}$, the following localized mass and energy quantities:

$$M_0(t') = \int z^2(t',x')\phi(x'-\nu t'+y_0)\,dx',$$

$$E_0(t') = \frac{1}{2}\int\left(z_x^2-\frac{1}{3}z^6\right)(t',x')\phi\left(\frac{5}{4}(x'-\nu t'+y_0)\right)dx'.$$

Step 1. Monotonicity in L^2 for z.

We claim that

$$M_0(t')-M_0(0)+\frac{1}{4}\int_0^{t'}\int(z_x^2+\nu z^2)(t'')\phi'(x'-\nu t''+y_0)\,dx'\,dt''\lesssim\frac{1}{\sqrt{\nu}}e^{-\sqrt{\nu}y_0/10}. \quad (A.9)$$

Indeed, we use formula (2.49) and (6.7) to estimate

$$\frac{d}{dt'}M_0(t')\leqslant\int\biggl(-3z_x^2-\frac{24}{25}\nu z^2+\frac{5}{3}z^6\biggr)\phi'(x'-\nu t'+y_0).$$

We claim that the non-linear term([17]) is controllable up to an exponentially small term after integration in time. Indeed, first recall from [26, Lemma 6] and (6.7) that, for all $v\in H^1$, $a>0$ and $b\in\mathbb{R}$,

$$\|v^2(\phi')^{1/2}\|^2_{L^\infty(|x-b|>a)}\lesssim\|v\|^2_{L^2(|x-b|>a)}\biggl(\int v_x^2\phi'+\int v^2\frac{(\phi'')^2}{\phi'}\biggr) \quad (A.10)$$

$$\lesssim\|v\|^2_{L^2(|x-b|>a)}\biggl(\int v_x^2\phi'+\nu\int v^2\phi'\biggr). \quad (A.11)$$

Fix $a_0\gg1$ such that

$$\biggl(\int_{2|y|>a_0}Q^2\biggr)^2\leqslant\delta_0.$$

On the one hand, by (A.11),

$$\begin{aligned}\int_{|x'-x_z(t')|>a_0}&z^6\phi'(x'-\nu t'+y_0)\\ &\leqslant\|z\|^2_{L^2(|x'-x_z(t')|>a_0)}\|z^2\phi'(x'-\nu t'+y_0)^{1/2}\|^2_{L^\infty(|x'-x_z(t')|>a_0)}\\ &\lesssim\|z\|^4_{L^2(|x'-x_z(t')|>a_0)}\int(z_x^2+\nu z^2)\phi'(x'-\nu t'+y_0).\end{aligned}$$

Since

$$\|z\|^2_{L^2(|x'-x_z(t')|>a_0)}\lesssim\int_{\lambda_z(t')|y|>a_0}Q_b^2(y)\,dy+\int\varepsilon_z^2\lesssim\delta_0+\delta(\alpha),$$

we obtain, for δ_0 small enough and α small enough,

$$\begin{aligned}\int_{|x'-x_z(t')|>a_0}z^6\phi'(x'-\nu t'+y_0)&\lesssim(\delta_0+\delta(\alpha))\int(z_x^2+\nu z^2)\phi'(x'-\nu t'+y_0)\\ &\leqslant\frac{1}{4}\int(z_x^2+\nu z^2)\phi'(x'-\nu t'+y_0).\end{aligned}$$

([17]) Which has the wrong sign.

On the other hand, the modulation equation (2.29) and the upper bound on scaling (A.4) ensure that

$$(x_z)_t = \frac{1}{\lambda_z^2}\frac{(x_z)_s}{\lambda_z} \geqslant \frac{1+\delta(\alpha_0)}{\lambda_z^2} \geqslant \frac{1}{5}, \tag{A.12}$$

and thus in particular

$$x_z(t') \geqslant x_z(0)+\tfrac{1}{5}t' \geqslant \tfrac{1}{10}t'+\nu t'. \tag{A.13}$$

We then estimate, from Sobolev's inequality,

$$\|z\|_{L^6}^6 \lesssim \|z\|_{H^1}^2\|z\|_{L^2}^4 \lesssim \frac{1}{\lambda_z^2} \lesssim (x_z)_t(t'),$$

and obtain, for all $y_0>a_0$,

$$\begin{aligned}\int_{|x'-x_z(t')|<a_0} z^6\phi'(x'-\nu t'+y_0) &\lesssim (x_z)_t(t')\|\phi'(x'-\nu t'+y_0)\|_{L^\infty(|x'-x_z(t')|<a_0)}\\ &\lesssim (x_z)_t(t')e^{-\sqrt{\nu}(x_z(t')-a_0-\nu t'+y_0)/10}\\ &\lesssim (x_z)_t(t')e^{-\sqrt{\nu}x_z(t')/100-\sqrt{\nu}y_0/10}.\end{aligned}$$

In conclusion, we have the L^2 motonicity formula: for all $t'\in[0,t_0)$,

$$\frac{d}{dt'}M_0(t')+\frac{1}{4}\int(z_x^2+\nu z^2)(t')\phi'(x'-\nu t'+y_0)\,dx' \lesssim (x_z)_t(t')e^{-\sqrt{\nu}x_z(t')/100}e^{-\sqrt{\nu}y_0/10},$$

and by integration between 0 and t' using that $x_z(0)=0$, for all $t'\in[0,T_z)$,

$$M_0(t')+\frac{1}{4}\int_0^{t'}\int(z_x^2+\nu z^2)(t'')\phi'(x'-\nu t'+y_0)\,dx'\,dt'' \leqslant M_0(0)+\frac{C}{\sqrt{\nu}}e^{-\sqrt{\nu}y_0/10}.$$

Step 2. Monotonicity in L^2 for $\tilde{z}$. Proof of (A.7).

We now rewrite the monotonicity (A.9) using the decomposition (A.2). We compute

$$\begin{aligned}M_0(t') &= \int z^2(t',x')\phi(x'-\nu t'+y_0)\,dx'\\ &= \int (Q_{b_z(t')}(y)+\varepsilon_z(y,t'))^2\phi(\lambda_z(t')y+x_z(t')-\nu t'+y_0)\,dy\,dt'\\ &= \int Q_{b_z(t')}^2\phi(\lambda_z(t')y+x_z(t')-\nu t'+y_0)+\int \tilde{z}^2(t',x')\phi(x'-\nu t'+y_0)\,dx'\\ &\quad +2\int Q_{b_z(t')}\varepsilon_z(t')\phi(\lambda_z(t')y+x_z(t')-\nu t'+y_0)\,dy\,dt.\end{aligned}$$

We estimate, using the lower bound (A.13),

$$\begin{aligned}&\int Q^2_{b_z(t')}\phi(\lambda_z(t')y+x_z(t')-\nu t'+y_0)\\&\quad=\int Q^2+2b_z(t')(P,Q)+b_z^2(t')\int \chi^2_{b_x(t')}P^2\phi(\lambda_z(t')y+x_z(t')-\nu t'+y_0)+O(e^{-\sqrt{\nu}y_0/10})\\&\quad=\int Q^2+2b_z(t')(P,Q)+O(e^{-\sqrt{\nu}y_0/10})+O(b_z^{2-\gamma}(t')),\end{aligned}$$

where we have used that $b^2\int P^2\chi_b^2=O(b^{2-\gamma})$. Now, by Hölder's inequality,

$$\begin{aligned}&2b_z(t')\left|\int \varepsilon_z(t')\chi_{b_z}P\phi(\lambda_z(t')y+x_z(t')-\nu t'+y_0)\right|\\&\quad\lesssim b_z(t')^{(1-\gamma)/2}\int \varepsilon_z^2(t')\phi(\lambda_z(t')y+x_z(t')-\nu t'+y_0)+b_z(t')^{(3+\gamma)/2}\int P^2\chi^2_{b_z}\\&\quad\leqslant b_z(t')^{(1-\gamma)/2}\int \tilde z^2(t',x')\phi(x'-\nu t'+y_0)\,dx'+b_z(t')^{(3-\gamma)/2}.\end{aligned}\tag{A.14}$$

We now insert these estimates into (A.9) and use, from (A.4) and the definition of θ_z,

$$|b_z(t')|\lesssim\theta_z,\tag{A.15}$$

and thus derive from the initialization (A.3) the bound (note that $\gamma=\frac34$), for all $t'\in[0,T_z)$,

$$\begin{aligned}&\int \tilde z^2(t',x')\phi(x'-\nu t'+y_0)\,dx+\int_0^{t'}\int (z_x^2+\nu z^2)(t'')\phi'(x'-\nu t''+y_0)\,dx'\,dt''\\&\qquad\lesssim\theta_z^{9/8}+\int \tilde z^2(0,x')\phi(x'+y_0)\,dx'+\frac{1}{\sqrt{\nu}}e^{-\sqrt{\nu}y_0/10}.\end{aligned}\tag{A.16}$$

Reinserting this bound into (A.14) and (A.9), keeping track of the b_z powers now yields (A.7).

Step 3. Energy monotonicity for z.

We claim the energy monotonicity

$$\begin{aligned}&E_0(t')-E_0(0)+\frac14\int_0^{t'}\int (z_{xx}^2+\nu z_x^2)(t'')\phi\left(\frac54(x'-\nu t''+y_0)\right)dx'\,dt''\\&\qquad\lesssim\left(\theta_z^{9/8}+\int \tilde z^2(t_0)\phi(x'+y_0)\,dx'+\frac{1}{\sqrt{\nu}}e^{-\sqrt{\nu}y_0/10}\right)^{5/4}.\end{aligned}\tag{A.17}$$

Indeed we estimate, from formula (2.50) and (6.7),

$$\begin{aligned}\frac{d}{dt'}E_0(t')&=-\frac54\int\left((z_{xx}+z^5)^2+2z_{xx}^2-10z^4z_x^2+\nu\left(z_x^2-\frac13z^6\right)\right)\phi'\left(\frac54(x'-\nu t'+y_0)\right)\\&\quad+\left(\frac54\right)^3\int z_x^2\phi'''\left(\frac54(x'-\nu t'+y_0)\right)\\&\leqslant-\frac54\int\left(2z_{xx}^2+\frac{\nu}{2}z_x^2-\frac{\nu}{3}z^6-10z^4z_x^2\right)\phi'\left(\frac54(x'-\nu t'+y_0)\right).\end{aligned}\tag{A.18}$$

We need to treat the non-linear terms. We claim that

$$\begin{aligned}\int_0^{T_z}\int z^4 z_x^2\phi'\left(\frac{5}{4}(x'-\nu t'+y_0)\right)dx'\,dt'\\ \lesssim \delta_0\int_0^{T_z}\int (z_{xx}^2+\nu z_x^2)\phi'\left(\frac{5}{4}(x'-\nu t'+y_0)\right)dx'\,dt'+\frac{1}{\sqrt{\nu}}e^{-\sqrt{\nu}y_0/8}\\ +\int_0^{T_z}\int z^6\phi'\left(\frac{5}{4}(x'-\nu t'+y_0)\right)dx'\,dt'\end{aligned} \tag{A.19}$$

for some small enough $\delta_0>0$, and

$$\int_0^{T_z}\int z^6(t')\phi'\left(\frac{5}{4}(x'-\nu t'+y_0)\right)dt' \lesssim \left(\theta_z^{9/8}+\int \tilde{z}^2(t_0)\phi(x'+y_0)dx'+\frac{1}{\sqrt{\nu}}e^{-\sqrt{\nu}y_0/10}\right)^{5/4}. \tag{A.20}$$

Integrating (A.18) in time and inserting (A.19) and (A.20) yields (A.17).

Proof of (A.19). For $a_1>0$ large enough, we have[18]

$$\frac{1}{\lambda_z^2(t')}\int_{|x|>a_1}(Q')^2\left(\frac{x}{\lambda_z(t')}\right)dx \lesssim \frac{1}{\lambda_z^2(t')}e^{-2a_1/\lambda_z(t')}\lesssim\frac{1}{a_1^2}\leqslant\delta_0,$$

and thus

$$\int_{|x-x_z(t')|>a_1} z_x^2 \lesssim \int_{|x-x_z(t')|>a_1}\tilde{z}_x^2(x)+\frac{1}{\lambda_z^2(t')}\int_{|x|>a_1}(Q')^2\left(\frac{x}{\lambda_z(t')}\right)\lesssim\delta_0, \tag{A.21}$$

where we used the smallness in the H^1 bound (A.5).

We now write

$$\int_{|x-x_z(t)|>a_1} z^4 z_x^2\phi'\left(\frac{5}{4}(x'-\nu t'+y_0)\right)\lesssim\int_{|x-x_z(t)|>a_1}(z^2z_x^4+z^6)\phi'\left(\frac{5}{4}(x'-\nu t'+y_0)\right),$$

and need only treat the first term according to the expected bound (A.19). We estimate the outer integral by using the localized Gagliardo–Nirenberg inequality (A.11) and the outer smallness by (A.21):

$$\begin{aligned}\int_{|x-x_z(t)|>a_1} z^2 z_x^4\phi'\left(\frac{5}{4}(x'-\nu t'+y_0)\right)\\ \lesssim \left\|z_x^2(\phi')^{1/2}\left(\frac{5}{4}(x'-\nu t'+y_0)\right)\right\|^2_{L^\infty(|x-x_z(t')|>a_1)}\|z\|_{L^2}^2\\ \lesssim \|z_x\|^2_{L^2(|x-x_z(t')|>a_1)}\int (z_{xx}^2+\nu z_x^2)\phi'\left(\frac{5}{4}(x'-\nu t'+y_0)\right)\\ \lesssim \delta_0\int (z_{xx}^2+\nu z_x^2)\phi'\left(\frac{5}{4}(x'-\nu t'+y_0)\right).\end{aligned}$$

[18] Using that $x^2e^{-x}\lesssim 1$ for $x\geqslant 0$.

The inner integral is estimated by Sobolev's inequality,

$$\int z^4 z_x^2 \lesssim \|z\|_{L^\infty}^4 \|z_x\|_{L^2}^2 \lesssim \|z\|_{L^2}^2 \|z_x\|_{L^2}^4 \lesssim \frac{1}{\lambda_z^4},$$

and hence, using the structure of ϕ and (A.13),

$$\begin{aligned}\int_{|x-x_z(t)|<a_1} z^4 z_x^2 \phi'\Big(\frac{5}{4}(x'-\nu t'+y_0)\Big) &\lesssim \frac{1}{\lambda_z^4(t')}\Big\|\phi'\Big(\frac{5}{4}(x'-\nu t'+y_0)\Big)\Big\|_{L^\infty(|x-x_z(t')|<a_1)}\\ &\lesssim \frac{1}{\lambda_z^4(t')} e^{-\sqrt{\nu}x_z(t')/100-\sqrt{\nu}y_0/8}.\end{aligned}$$

We now claim that

$$\frac{1}{c_0\lambda_z^2(t')} e^{-c_0 x_z(t')} + \int_0^{T_z} \frac{1}{\lambda_z^4(t')} e^{-c_0 x_z(t')}\, dt' \lesssim \frac{1}{c_0}, \tag{A.22}$$

with $c_0 = C\sqrt{\nu}$, which completes the proof of (A.19). □

Indeed, first observe, from the definition of θ_z and the rough modulation equation (2.29), that

$$|(\lambda_z)_t| = \left|\frac{1}{\lambda_z^2}\frac{-(\lambda_z)_s}{\lambda_z}\right| \lesssim \frac{1}{\lambda_z^2}(|b_z| + \sqrt{\theta_z}\lambda_z) \lesssim \frac{\sqrt{\theta_z}}{\lambda_z},$$

and thus, from (A.12) and integration by parts in time,

$$\begin{aligned}\int_0^{t'} \frac{1}{\lambda_z^4} e^{-c_0 x_z}\, d\tau \lesssim \int_0^{t'} \frac{(x_z)_t}{\lambda_z^2} e^{-c_0 x_z}\, d\tau &= \left[-\frac{1}{c_0\lambda_z^2} e^{-c_0 x_z}\right]_0^{t'} - \frac{1}{c_0}\int_0^{t'} \frac{2(\lambda_z)_t}{\lambda_z^3} e^{-c_0 x_z}\, d\tau\\ &\leqslant \frac{1}{c_0}\left[1 - \frac{1}{\lambda_z^2(t')} e^{-c_0 x_z(t')}\right] + \frac{2\sqrt{\theta_z}}{c_0}\int_0^{t'} \frac{1}{\lambda_z^4} e^{-c_0 x_z}\, d\tau,\end{aligned}$$

and (A.22) now follows from the a-priori smallness (A.6) and (6.6).

Proof of (A.20). Since $\phi'\big(\frac{5}{4}x\big) \lesssim (\phi')^{5/4}(x)$, (A.11) yields

$$\begin{aligned}&\int z^6 \phi'\Big(\frac{5}{4}(x'-\nu t'+y_0)\Big)\\ &\quad\leqslant \|z^2(\phi')^{1/2}(x'-\nu t'+y_0)\|_{L^\infty}^2 \int z^2 (\phi')^{1/4}(x'-\nu t'+y_0)\\ &\quad\lesssim \Big(\int z^2\Big)^{7/4}\Big(\int z^2\phi'(x'-\nu t'+y_0)\Big)^{1/4}\int (z_x^2+\nu z^2)\phi'(x'-\nu t'+y_0)\\ &\quad\lesssim \Big(\int z^2\phi'(x'-\nu t'+y_0)\Big)^{1/4}\int (z_x^2+\nu z^2)\phi'(x'-\nu t'+y_0).\end{aligned}$$

We now estimate

$$\int z^2\phi'(x'-\nu t'+y_0)\lesssim\int \tilde z^2\phi'(x'-\nu t'+y_0)+\int Q_{b_z}^2(y)\phi'(\lambda_z(t')y+x_z(t)-\nu t'+y_0).$$

On the one hand, by (A.16) and $\phi'\lesssim\phi$,

$$\int \tilde z^2\phi'(x'-\nu t'+y_0)\lesssim \theta_z^{9/8}+\int \tilde z^2(0)\phi(x'+y_0)\,dx'+\frac{1}{\sqrt{\nu}}e^{-\sqrt{\nu}y_0/10}.$$

On the other hand, from the space decoupling (A.13),

$$\begin{aligned}\int Q_b^2(y)\phi'(\lambda_z(t')y+x_z(t)-\nu t'+y_0)&\lesssim |b|^{2-\gamma}(t')+\int Q^2(y)\phi'(\lambda_z(t')y+x_z(t)-\nu t'+y_0)\\&\lesssim \theta_z^{5/4}+\frac{1}{\sqrt{\nu}}e^{-\sqrt{\nu}y_0/10}.\end{aligned}$$

The space-time estimate (A.20) now follows from (A.16). □

Step 4. Energy monotonicity for $\tilde z$. Proof of (A.8).

We now rewrite the monotonicity (A.17) using the decomposition (A.2). We compute

$$2\lambda_z^2(t')E_0(t')=\int\left[(Q_{b_z}+\varepsilon_z)_y^2-\frac13(Q_{b_z}+\varepsilon_z)^6\right](t',y)\phi\left(\frac54(\lambda_z(t')y+x_z(t')-\nu t'+y_0)\right)dy$$

and develop this expression. The contribution of the Q_b term is estimated using $E(Q)=0$ and the separation in space (A.13), which implies that

$$\begin{aligned}\int[(Q_b)_y^2+Q_b^6]\left[1-\phi\left(\frac54(\lambda_z(t')y+x_z(t')-\nu t'+y_0)\right)\right]dy&\\ \lesssim |b_z|^{1+\gamma}+\frac{1}{\sqrt{\nu}}e^{-\sqrt{\nu}x_z(t')/20}e^{-\sqrt{\nu}y_0/10}.&\end{aligned}$$

The cross terms are treated using the orthogonality condition (2.20) and we obtain, similarly to the proof of (2.28),

$$\begin{aligned}&2\lambda_z^2(t')E_0(t')\\&\quad=-2b_z(t')(P,Q)+\int\left[(\varepsilon_z)_y^2-\frac13\varepsilon_z^6\right](t',y)\phi\left(\frac54(\lambda_z(t')y+x_z(t')-\nu t'+y_0)\right)dy\\&\qquad+O\left[\frac{1}{\sqrt{\nu}}e^{-\sqrt{\nu}(x_z(t')+y_0)/10}+|b_z(t')|^2+|b_z(t')|^{1-\gamma}\left(\int(\varepsilon_z)_y^2+\int\varepsilon_z^2e^{-|y|}\right)\right].\end{aligned}\tag{A.23}$$

We now divide by $\lambda_z(t')$ and estimate, from (A.4),

$$\frac{1}{\lambda_z^2(t')}\left[|b_z(t')|^2+|b_z(t')|^{1-\gamma}\left(\int(\varepsilon_z)_y^2+\int\varepsilon_z^2e^{-|y|}\right)\right]\lesssim\lambda^2(t_0)\theta_z^{1/4},$$

and conclude, using (A.22) and (A.23), that

$$\begin{aligned}2E_0(t')=-\frac{2b_z(t')}{\lambda_z^2(t')}(P,Q)+\int\left[\tilde z_x^2-\frac13\tilde z^6\right]\phi\left(\frac54(x'-\nu t'+y_0)\right)dx'&\\+O\left(\lambda^2(t_0)\theta_z^{1/4}+\frac{1}{\sqrt{\nu}}e^{-\sqrt{\nu}y_0/10}\right),&\end{aligned}$$

which, together with the monotonicity (A.17) and L^2 smallness of $\tilde z$, yields (A.8). □

A.2. Proof of Lemma 3.4

The proof of Lemma 3.4 is based on coercivity properties of the virial quadratic form under suitable repulsivity properties. We recall this property in the following lemma.

LEMMA A.2. ([16, Proposition 4]) *There exists* $\mu>0$ *such that, for all* $v\in H^1(\mathbb{R})$,

$$3\int v_y^2+\int v^2-5\int Q^4v^2+20\int yQ'Q^3v^2\geqslant \mu\int v_y^2+v^2-\frac{1}{\mu}\left(\int vy\Lambda Q\right)^2-\frac{1}{\mu}\left(\int vQ\right)^2.$$

We now turn to the proof of Lemma 3.4, which is a simple consequence of Lemma A.2 using a standard localization argument (see for example the proof of [20, Proposition 9]). Indeed, let ζ be a smooth function such that

$$\zeta(y)=\begin{cases} 0 & \text{for } |y|>\frac{1}{4},\\ 1 & \text{for } |y|<\frac{1}{8},\end{cases} \quad \text{and} \quad 0\leqslant\zeta\leqslant 1 \text{ on } \mathbb{R}.$$

Set

$$\tilde\varepsilon(y)=\varepsilon(y)\zeta_B(y), \quad \text{where } \zeta_B(y)=\zeta\left(\frac{y}{B}\right).$$

Lemma A.2 applied to $\tilde\varepsilon$ gives

$$(3-\mu)\int\tilde\varepsilon_y^2+(1-\mu)\int\tilde\varepsilon^2-5\int Q^4\tilde\varepsilon^2+20\int yQ'Q^3\tilde\varepsilon^2\geqslant-\frac{1}{\mu}\left(\int\tilde\varepsilon y\Lambda Q\right)^2-\frac{1}{\mu}\left(\int\tilde\varepsilon Q\right)^2. \tag{A.24}$$

On the one hand,

$$\begin{aligned}\int\tilde\varepsilon_y^2&=\int\varepsilon_y^2\zeta_B^2+\int\varepsilon^2(\zeta_B')^2-\frac{1}{2}\int\varepsilon^2(\zeta_B^2)''\leqslant\int_{|y|<B/4}\varepsilon_y^2+\frac{C}{B^2}\int_{|y|<B/4}\varepsilon^2,\\ \int\tilde\varepsilon^2&=\int\varepsilon^2\zeta_B^2\leqslant\int_{|y|<B/4}\varepsilon^2,\end{aligned}$$

and, by $yQ'<0$ and then by the exponential decay of Q and Q',

$$\begin{aligned}&-5\int Q^4\tilde\varepsilon^2+20\int yQ'Q^3\tilde\varepsilon^2\\ &\qquad\leqslant-5\int_{|y|<B/4}Q^4\tilde\varepsilon^2+20\int_{|y|<B/4}yQ'Q^3\tilde\varepsilon^2\\ &\qquad\leqslant-5\int_{|y|<B/2}Q^4\tilde\varepsilon^2+20\int_{|y|<B/2}yQ'Q^3\tilde\varepsilon^2+Ce^{-B/16}\int_{B/4<|y|<B/2}\varepsilon^2.\end{aligned}$$

Thus, for B large,

$$\begin{aligned}
&(3-\mu)\int \tilde{\varepsilon}_y^2+(1-\mu)\int \tilde{\varepsilon}^2-5\int Q^4\tilde{\varepsilon}^2+20\int yQ'Q^3\tilde{\varepsilon}^2\\
&\quad\leqslant (3-\mu)\int_{|y|<B/4}\varepsilon_y^2\\
&\quad+(1-\mu)\int_{|y|<B/4}\varepsilon^2-5\int_{|y|<B/2}Q^4\tilde{\varepsilon}^2+20\int_{|y|<B/2}yQ'Q^3\tilde{\varepsilon}^2+\frac{C}{B^2}\int_{|y|<B/4}\varepsilon^2\\
&\quad\leqslant (3-\mu)\int_{|y|<B/2}\varepsilon_y^2+\Big(1-\frac{\mu}{2}\Big)\int_{|y|<B/2}\varepsilon^2-5\int_{|y|<B/2}Q^4\tilde{\varepsilon}^2+20\int_{|y|<B/2}yQ'Q^3\tilde{\varepsilon}^2.
\end{aligned}$$

On the other hand, by (2.20),

$$\left|\int \tilde{\varepsilon}y\Lambda Q\right|=\left|\int \varepsilon\zeta_B y\Lambda Q\right|=\left|\int \varepsilon(1-\zeta_B)y\Lambda Q\right|\lesssim e^{-B/16}\left(\int \varepsilon^2 e^{-|y|/2}\right)^{1/2},$$

and similarly for $\int \tilde{\varepsilon}Q$. Inserted in (A.24), these estimates finish the proof of Lemma A.2.

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