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Fusion systems and localities

by

Andrew Chermak

Kansas State University
Manhattan, KS, U.S.A.

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Introduction

Let S be a finite p-group, with p being a prime. A *fusion system* on S is a category whose objects are the subgroups of S, and whose morphisms are injective group homomorphisms, among which are all of those which are induced by conjugation by elements of S. A fusion system $\mathcal{F}$ on S is *saturated* if it satisfies some further conditions, such as would be found to hold if S were a Sylow subgroup of a finite group G and if the morphisms in $\mathcal{F}$ were the homomorphisms between subgroups of S induced by conjugation within G.

Saturated fusion systems were introduced by Lluis Puig (as "Frobenius categories") in notes which, although widely influential, remained unpublished for some years. Puig's formalism provided a setting for the Brauer theory of blocks of characters of finite groups, in which no ambient finite group need be assumed. Somewhat later, David Benson [5] suggested the possibility of associating a "classifying space" to each Frobenius category. The notion of such a classifying space was then formulated in a rigorous way by Carles

Broto, Ran Levi, and Bob Oliver in [7], thereby providing a generalized setting for the homotopy theory of p-completed classifying spaces of finite groups. Here also, as in Puig's setup, no ambient finite group is required. Instead, what is required is a "linking system" (or "p-local finite group") attached to a given saturated fusion system, and which has a richer and, in many respects, a more "group-like" structure than the fusion system alone.

More recently, linking systems and their homotopy-theoretic correlatives have been further generalized by Bob Oliver and Joana Ventura [16] to "transporter systems". More recently still, the notions of linking system and transporter system have been treated by Puig in his book [18], where they are called "$\mathcal{F}$-localities"—but where the homotopical context is absent (as will also be the case in the present work).

This paper is intended, in part, as a step toward providing a setting for the methods of the so-called "p-local analysis" from finite group theory, in which no ambient finite group is required. The formalism developed here turns out to be equivalent in a technical sense to that of [7] and [16], but it is pitched in a completely different language—one which involves nothing of categories and functors—and it has a more recognizably finite group-like flavor. Partly for this reason, and partly because the "p-local" in "p-local finite groups" already has a meaning for finite group theorists, we have chosen to adopt Puig's terminology; and so this paper involves the study of what we call *localities*. We retain the terminology from [7] for the special sort of locality known as a linking system. The aim is to establish some basic structural properties of localities in general, and to prove the following result.

THEOREM. (Main theorem) *Let $\mathcal{F}$ be a saturated fusion system on the finite p-group S, with p being a prime. Then there exists a centric linking system $\mathcal{L}$ such that $\mathcal{F}$ is the fusion system generated by the conjugation maps in $\mathcal{L}$ between subgroups of S. Further, $\mathcal{L}$ is uniquely determined by $\mathcal{F}$, up to an an isomorphism which restricts to the identity map on S.*

We remark that if $\mathcal{L}$ is a centric linking system on S, then it is straightforward to show that the fusion system $\mathcal{F}_S(\mathcal{L})$ generated by the conjugation maps in $\mathcal{L}$ between subgroups of S is saturated (see Proposition 2.18 (a) below). Thus, the effect of the main theorem is that there is a one-to-one correspondence, up to a rigid notion of isomorphism, between saturated fusion systems and centric linking systems.

In this introduction we shall outline our proof of the main theorem, and point out the indirect way in which it relies on the classification of the finite simple groups (hereinafter referred to as the CFSG).

A group may be regarded as a set G together with an "inversion map" and a multivariable "product" $\Pi\colon \mathbf{W}\to G$, where $\mathbf{W}=\mathbf{W}(G)$ is the free monoid on G. The usual definition of a group is easily formulated in terms of Π instead of the binary multiplication. To obtain the notion of "partial group", one drops the requirement that Π be defined on all words in $\mathbf{W}$, and one places certain conditions on the subset $\mathbf{D}$ of $\mathbf{W}$ on which Π is defined, while retaining the essential properties that one expects from a product.

Once Definition 2.1 is given, partial analogs of basic group-theoretic notions immediately suggest themselves, including the notions of homomorphism and subgroup. A partial subgroup of a partial group may in fact be a group. Moreover, it may be the case for a given partial group $\mathcal{M}$ that there is a collection Δ of subgroups which determines the domain $\mathbf{D}$ of the product Π. Namely, it may happen that a word $w=(f_1, ..., f_n)$ is in the domain $\mathbf{D}$ if and only if there exists a sequence $(X_0, ..., X_n)$ of "objects" (i.e. members of Δ) such that X_{i-1} is conjugated by f_i to X_i for all i, $1 \leqslant i \leqslant n$. If such is the case, and if moreover any subgroup of an object containing a conjugate of an object is again an object, then the pair $(\mathcal{M}, \Delta)$ is an *objective partial group*.

Our interest is in objective partial groups $\mathcal{L}=(\mathcal{M}, \Delta)$ such that the set Δ of objects has a unique maximal member S with respect to inclusion, and where S is a finite p-group which is maximal (though not necessarily uniquely so) in the poset of all p-subgroups of $\mathcal{M}$. When these conditions are met, and $\mathcal{M}$ is finite, then the triple $\mathcal{L}=(\mathcal{M}, \Delta, S)$ is a *locality*. A locality $\mathcal{L}$ is a Δ-*linking system* if, for any object $P \in \Delta$, the centralizer subgroup $C_{\mathcal{L}}(P)$ is just the center $Z(P)$ of P. If, moreover, Δ is the set of all subgroups P of S such that $C_S(Q)=Z(Q)$ for every $\mathcal{L}$-conjugate Q of P with $Q \leqslant S$, then $\mathcal{L}$ is a *centric linking system*.

With any locality $\mathcal{L}=(\mathcal{M}, \Delta, S)$ is associated a fusion system $\mathcal{F}:=\mathcal{F}_S(\mathcal{L})$ on S, whose morphisms are those maps ϕ from one subgroup of S into another, such that ϕ is a composition of restrictions of $\mathcal{L}$-conjugation maps between objects. We find that for any locality $\mathcal{L}$, the pair $(\mathcal{L}, \mathcal{F}_S(\mathcal{L}))$ is essentially the same thing as a "transporter system" in the sense of Oliver and Ventura [16], and we show that all transporter systems arise from localities in this way. The proof is given in an appendix, so as not to interrupt the flow of the development. The appendix includes also a proof that the main theorem implies the corresponding result for "centric linking systems" taken in the sense of [7].

In §1 we introduce saturated fusion systems (and the notion of "fully normalized" subgroup) in an unconventional way, in analogy to the way in which one defines a scheme as a gluing-together of affine schemes. In this analogy, the "affine" things are the fusion systems $\mathcal{F}_R(H)$ of finite groups H at a Sylow p-subgroup R, where H has the property that $C_H(O_p(H)) \leqslant O_p(H)$. A fusion system $\mathcal{F}$ on a finite p-group S is saturated provided that $\mathcal{F}$ is locally affine, $\mathcal{F}$ is "generated" by its affine subsystems and every $\mathcal{F}$-centric subgroup of S has a fully normalized $\mathcal{F}$-conjugate. Another way to say this is that our definition of saturation is based on the notion, due to Aschbacher [2], of a "model" of a "constrained" fusion system. In any case, our formulation is equivalent to those found in [7] and elsewhere. Readers who are already familiar with fusion systems and with models will find little new after Definition 1.4. Indeed, the purpose of §1 is primarily to fix terminology and notation—and to state a result (Proposition 1.10) due independently to Oliver and Puig—which lies at the foundation of this work.

The definitions pertaining to partial groups, objective partial groups, and localities are introduced, and a few of their elementary consequences are derived, in §2. Among these consequences is the basic one (Proposition 2.11) that for any locality $\mathcal{L}=(\mathcal{M},\Delta,S)$ and any $f\in\mathcal{M}$ the *set* S_f of elements $x\in S$ such that the product $x^f:=f^{-1}xf$ is defined and is an element of S, is in fact an object, and hence a sub*group* of S. As a corollary, we obtain the result (Proposition 2.22) that every subgroup of $\mathcal{M}$ is contained in the normalizer of an object, and that all p-subgroups of $\mathcal{M}$ are conjugate to subgroups of S.

In §3 we introduce homomorphisms of partial groups and of partial normal subgroups. A few very basic consequences of the definitions are derived, but it is not until the focus is restricted to localities, in §4, that these concepts begin to bear fruit. We make no attempt in this paper to formulate a general notion of homomorphism of localities beyond an obvious notion of isomorphism.

In §4 we provide some basic computational tools for working with a locality $\mathcal{L}=(\mathcal{L},\Delta,S)$ and a partial normal subgroup $\mathcal{N}\trianglelefteq\mathcal{L}$. The "Frattini lemma" (Corollary 4.8) says that every element $f\in\mathcal{L}$ can be written as a product xh (or $hy:=hx^h$) with $x,y\in\mathcal{N}$ and with $h\in N_{\mathcal{L}}(S\cap\mathcal{N})$. The section ends with a result (Lemma 4.9) on extending an automorphism of a sub-locality of a finite group to an automorphism of the group itself.

It is in §5 that the proof of the main theorem begins to take shape. The main results (Theorems 5.14 and 5.15) yield a concrete procedure for constructing a locality $\mathcal{L}^+$ from a locality $\mathcal{L}$ having a smaller set of objects. Thus, suppose that one is given a locality $\mathcal{L}$ with the set Δ of objects, and maximal object S; and suppose that one is given also a fusion system $\mathcal{F}$ on S such that $\mathcal{L}$ is "$\mathcal{F}$-natural", in the sense that for any object P, the set of $\mathcal{L}$-conjugation maps from P into S is equal to the set of $\mathcal{F}$-homomorphisms of P into S. Now suppose further that one is given a subgroup T of S, such that T is not in Δ, but with the property that every pair of distinct $\mathcal{F}$-conjugates of T in S generates a member of Δ. One may assume (upon replacing T by a suitable $\mathcal{L}$-conjugate) that T is fully normalized in $\mathcal{F}$, in the sense of Definition 1.2. There are then two questions to consider. First: under what conditions is it possible to regard $\mathcal{L}$ as the "restriction" to Δ of an $\mathcal{F}$-natural locality $\mathcal{L}^+$ whose set Δ^+ of objects is the union of Δ and the set of overgroups in S of $\mathcal{F}$-conjugates of T? Second: under what conditions are two such "extensions" of $\mathcal{L}$ to Δ^+ "rigidly isomorphic" (i.e. isomorphic via an isomorphism which restricts to the identity map on S)? Theorems 5.14 and 5.15 provide a complete answer to these questions; and in doing so they provide a blueprint for the proof of the main theorem.

In brief, Theorem 5.14 says that there exists an $\mathcal{F}$-natural locality $\mathcal{L}^+$ extending $\mathcal{L}$ in the prescribed manner, provided that there exists

(1) a finite group M containing $N_S(T)$ as a Sylow p-subgroup, and with fusion system $\mathcal{F}_{N_S(T)}(M)$ equal to $N_{\mathcal{F}}(T)$, and

(2) a rigid isomorphism λ from the normalizer locality $N_{\mathcal{L}}(T)$ to a locality $\mathcal{L}_{\Delta_T}(M)$ contained in M,

where Δ_T is the set of objects $Q\in\Delta$ such that $T\leqslant Q\leqslant N_S(T)$, and $\mathcal{L}_{\Delta_T}(M)$ is the locality obtained by restricting the group M (itself viewed as a locality) to Δ_T. Further, Theorem 5.15 says that if λ and λ' are two isomorphisms as in (2), then the resulting localities $\mathcal{L}^+(\lambda)$ and $\mathcal{L}^+(\lambda')$ are rigidly isomorphic if and only if the composition λ^{-1} followed by λ' extends to an automorphism of M.

Lemma 5.18 establishes that every locality $\mathcal{L}=(\mathcal{M},\Delta,S)$ can be constructed in the above way, by an iterative procedure. For example, one may begin with the group $N_{\mathcal{L}}(S)$, regarded as the restriction of $\mathcal{L}$ to a locality with a single object. One then proceeds (via the "+-operation" outlined above) to construct the restriction of $\mathcal{L}$ to larger and larger sets of objects, until the set Δ has been exhausted. At that point $\mathcal{L}$ itself will have been recovered as a "filtration" of its restrictions to an increasing sequence of subsets of Δ.

In §6 we provide a proof of the main theorem modulo a technical condition on localities in finite groups which is proved in §7 as Proposition 7.1. In somewhat more detail: the proof of the main theorem depends on being able to produce an iterative procedure, of the kind described in the preceding paragraph, by which to create a linking system rather than to recover one, starting only with a saturated fusion system $\mathcal{F}$ on S and with the set $\Delta=\mathcal{F}^c$ of $\mathcal{F}$-centric subgroups of S. The procedure begins with the linking system $\mathcal{L}_0$ of $N_{\mathcal{F}}(R)$ for some suitably chosen $R\in\Delta$; and where the existence and uniqueness of $\mathcal{L}_0$ is given by a result (see Proposition 1.10 below), obtained independently by Oliver and Puig, which lies at the foundation of this paper. The difficulty, in going from one step to the next via the +-construction, lies in showing that what has already been constructed (and constructed uniquely, up to rigid isomorphism) yields an essentially unique rigid isomorphism λ at the local level required for the next step. This requires finding a good way to descend, step by step, through Δ—and this is what is achieved in §6. The argument focuses on properties of one version of the Thompson J-subgroup $J(R)$ of a finite p-group R, and on properties of finite groups G such that R is a Sylow p-subgroup of G, $F^*(G)=O_p(G)$ and $J(R)$ is not a normal subgroup of G. Thus, §6 provides a method of "descent", while Proposition 7.1 enables the argument in §6 and completes the proof of the main theorem. By ordering things in this way, all of the non-elementary finite group theory involved in the proof of the main theorem is pushed to the very end.

Proposition 7.1 concerns so-called FF-*pairs* (G, V), where G is a finite group such that $O_p(G)=1$, and where V is a faithful G-module over the field of p elements, having the following property:

$$\text{There exists a non-identity abelian } p\text{-subgroup } A \text{ of } G \text{ (called a "best offender" in } G \text{ on } V) \text{ such that } |A|\,|C_V(A)| \geqslant |B|\,|C_V(B)| \text{ for every subgroup } B \text{ of } A, \tag{$*$}$$

and where G is generated by the set of such best offenders. The classification of such pairs (G, V) has been carried out piecemeal, over a period of many years, by many authors. It has only very recently been given a complete treatment (including the determination of the best offenders in the case where V is irreducible and G is almost simple) by Meierfrankenfeld and Stellmacher [15], as part of the project initiated by Meierfrankenfeld to provide an alternative approach to the classification of finite simple groups of local characteristic p. Parts of the classification of FF-pairs (for example the decomposition into "J-components") are elementary, but as things stand at this date, the determination of the possible J-components themselves relies on the CFSG. Though we have attempted to organize the arguments on the basis of general principles where possible (see for example Definition 7.9), any proof based on the CFSG, by its very nature, is opportunistic to some degree, and not entirely principled.

We should alert those readers who are familiar with arguments involving the Thompson J-subgroup $J(P)$ of a finite p-group P, that in this paper $J(P)$ is not defined in the way that has gained currency over the course of the decades since Thompson first introduced his version of $J(P)$. That is, we define $J(P)$ to be the subgroup of P that is generated by the abelian subgroups of P of maximal order (as in [20]), rather than the elementary abelian subgroups of maximal order. This is the definition which is needed here, for reasons that will become clear from the arguments in §6 and §7.

Remark. Our tendency is toward right-hand notation for mappings, in any discussion which may involve composition of mappings. In particular, if $\mathcal{C}$ is a category, and X, Y, and Z are objects of $\mathcal{C}$, then composition defines a mapping

$$\mathrm{Mor}_{\mathcal{C}}(X,Y)\times\mathrm{Mor}_{\mathcal{C}}(Y,Z)\longrightarrow\mathrm{Mor}_{\mathcal{C}}(X,Z).$$

Consistent with this policy, conjugation within any group G is taken in the right-handed sense, so that $x^g=g^{-1}xg$ for any $x, g\in G$.

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1. Fusion systems, saturation, and models

This section is, in part, a review of the basic notions pertaining to fusion systems and saturation; but the definitions of "fully normalized subgroup" and of saturation that turn out to be most convenient for the task at hand are not the standard ones. Still, the ideas are due to Puig [17], while the terminology that we employ is that of [7], which has gained broad currency.

Let p be a prime, G be a finite group, and S be a Sylow p-subgroup of G. For subgroups P and Q of S, set

$$N_G(P,Q)=\{g\in G \mid P^g\leqslant Q\}.$$

Here P^g is the set of elements $x^g:=g^{-1}xg$, for $x\in P$. Set

$$\mathrm{Hom}_G(P,Q)=\{c_g\colon P\to Q \mid g\in N_G(P,Q)\},$$

where $c_g\colon P\to Q$ is the conjugation map $x\mapsto x^g$ induced by g. The *fusion system* $\mathcal{F}_S(G)$ *induced on* S *by* G is the category whose objects are the subgroups of S, and where the set of morphisms $P\to Q$ is $\mathrm{Hom}_G(P,Q)$. More generally, we give the following definition.

Definition 1.1. Let S be a finite p-group. A *fusion system* on S is a category $\mathcal{F}$, whose objects are the subgroups of S, and whose morphisms satisfy the following two conditions:

(a) $\mathrm{Hom}_S(P,Q)\subseteq\mathrm{Hom}_{\mathcal{F}}(P,Q)$ for all subgroups P and Q of S;

(b) every $\mathcal{F}$-homomorphism can be factored in $\mathcal{F}$ as an $\mathcal{F}$-isomorphism followed by an inclusion map, and every $\mathcal{F}$-isomorphism is an isomorphism of groups.

Example. For any finite p-group S there is the *total fusion system* $\overline{\mathcal{F}}(S)$, characterized by

$$\mathrm{Hom}_{\overline{\mathcal{F}}(S)}(P,Q)=\mathrm{Inj}(P,Q),$$

where $\mathrm{Inj}(P,Q)$ is the set of all injective group homomorphisms $P\to Q$.

Let $\mathcal{F}$ be a fusion system on S and let $P\leqslant S$ be a subgroup of S. A subgroup $Q\leqslant S$ is an *$\mathcal{F}$-conjugate* of P if $Q=P\phi$ for some $\mathcal{F}$-isomorphism ϕ.

Definition 1.2. Let $\mathcal{F}$ be a fusion system on S. A subgroup P of S is *fully normalized* in $\mathcal{F}$ provided that, for each $\mathcal{F}$-conjugate Q of P, there exists an $\mathcal{F}$-homomorphism $\psi: N_S(Q)\to N_S(P)$ such that $Q\psi=P$.

Example. If $\mathcal{F}=\mathcal{F}_S(G)$, with G being a finite group, and $S\in\mathrm{Syl}_p(G)$, then every subgroup of S has a fully normalized $\mathcal{F}$-conjugate, by Sylow's theorem.

Definition 1.3. Let S be a finite p-group and let $\mathbf{F}$ be a subset of $\mathrm{Hom}(\overline{\mathcal{F}}(S))$ (i.e. a subset of the set of morphisms of the total fusion system on S) such that $\mathbf{F}$ contains $\mathrm{Hom}(\mathcal{F}_S(S))$. The fusion system on S *generated by* $\mathbf{F}$ is the category whose objects are the subgroups of S, and whose morphisms are the homomorphisms $\phi: P\to Q$ such that ϕ is a composition of restrictions of members of $\mathbf{F}$.

We note that it is immediate from Definition 1.1 that the "fusion system generated by $\mathbf{F}$" is in fact a fusion system on S.

Example. Let $\mathcal{F}$ be a fusion system on S and let $T\leqslant S$ be a subgroup of S, with T fully normalized in $\mathcal{F}$. Define $N_{\mathcal{F}}(T)$ to be the fusion system on $N_S(T)$ generated by the set of all $\mathcal{F}$-homomorphisms $\phi: P\to N_S(T)$ such that $T\trianglelefteq P$ and such that $T\phi=T$.

A collection Δ of subgroups of S is *closed under $\mathcal{F}$-conjugation* (or, is *$\mathcal{F}$-invariant*) if $P\phi\in\Delta$ whenever $P\in\Delta$ and $\phi\in\mathrm{Hom}_{\mathcal{F}}(P,S)$. We say that Δ is *overgroup closed* if $Q\in\Delta$ whenever Q is a subgroup of S which contains a member of Δ.

Example. For any fusion system $\mathcal{F}$ on S, let $\mathcal{F}^c$ be the largest $\mathcal{F}$-invariant collection Δ of subgroups P of S such that $C_S(P)\leqslant P$ for all $P\in\Delta$. Then $S\in\mathcal{F}^c$, and $\mathcal{F}^c$ is overgroup closed in S. The members of $\mathcal{F}^c$ are the *$\mathcal{F}$-centric* subgroups of S.

Definition 1.4. Let $\mathcal{F}$ be a fusion system on S and let Δ be a non-empty collection of subgroups of S, such that Δ is both overgroup closed and closed under $\mathcal{F}$-conjugation. Then $\mathcal{F}$ is *Δ-saturated* if the following two conditions hold:

(A) every member of Δ has a fully normalized $\mathcal{F}$-conjugate;

(B) for each $P\in\Delta\cap\mathcal{F}^c$ such that P is fully normalized in $\mathcal{F}$, there exists a finite group M such that $N_S(P)\in\mathrm{Syl}_p(M)$, and with $N_{\mathcal{F}}(P)=\mathcal{F}_{N_S(P)}(M)$.

If $\mathcal{F}$ is $\mathcal{F}^c$-saturated, and $\mathcal{F}$ is generated by the union of its subsystems $N_{\mathcal{F}}(P)$ as P ranges over the fully normalized members of $\mathcal{F}^c$, then $\mathcal{F}$ is *saturated*.

Remark 1.5. (a) The above definition of saturation is equivalent to the (by now) standard one given in [7], and hence also to the various equivalent formulations found in [6] and [19]. Actually, in view of the main theorem, one may be satisfied to know that a fusion system satisfying the standard definition of saturation satisfies the conditions of Definition 1.4. That the standard definition implies (A) is an easy exercise, while (B) follows from [2, statements 2.4 and 2.5]. The reverse implication (that Definition 1.4 really does define saturation in the standard sense) is given by [6, Theorem A].

(b) For any finite group G with Sylow p-subgroup S, the fusion system $\mathcal{F}_S(G)$ is Δ-saturated, for any non-empty, overgroup closed, $\mathcal{F}_S(G)$-invariant collection Δ of subgroups of S.

Definition 1.6. Let $\mathcal{F}$ be a fusion system on S, and let $T \leqslant S$ be a subgroup of S. Then T is *normal* in $\mathcal{F}$ if $\mathcal{F}=N_{\mathcal{F}}(T)$. The (unique) largest subgroup of S which is normal in $\mathcal{F}$ is denoted $O_p(\mathcal{F})$. More generally, T is *strongly closed* in $\mathcal{F}$ if $P\phi \leqslant T$ whenever $P \leqslant T$ and $\phi \in \mathrm{Hom}_{\mathcal{F}}(P,S)$. More generally still, T is *weakly closed* in $\mathcal{F}$ if $T\phi = T$ for all $\phi \in \mathrm{Hom}_{\mathcal{F}}(T,S)$.

LEMMA 1.7. *Let $\mathcal{F}$ be a saturated fusion system on S, let $P \leqslant S$ be a subgroup of S such that P is fully normalized in $\mathcal{F}$, and let U be a subgroup of P such that $N_S(P) \leqslant N_S(U)$. Then there exists $\phi \in \mathrm{Hom}_{\mathcal{F}}(P,S)$ such that both $P\phi$ and $U\phi$ are fully normalized in $\mathcal{F}$.*

Proof. By condition (A) in Definition 1.4, there exists $\phi \in \mathrm{Hom}_{\mathcal{F}}(N_S(U),S)$ such that $V:=U\phi$ is fully normalized in $\mathcal{F}$. Set $Q=P\phi$. As $N_S(P) \leqslant N_S(U)$, and P is fully normalized, ϕ restricts to an isomorphism $N_S(P) \to N_S(Q)$. Now let $\psi \in \mathrm{Hom}_{\mathcal{F}}(Q,S)$ and set $R=Q\psi$. Then R is an $\mathcal{F}$-conjugate of P, and so there exists $\eta \in \mathrm{Hom}_{\mathcal{F}}(N_S(R), N_S(P))$ with $R\eta = P$. Composing η with ϕ yields an $\mathcal{F}$-homomorphism $N_S(R) \to N_S(Q)$, so Q is fully normalized in $\mathcal{F}$. □

Definition 1.8. Let $\mathcal{F}$ be a saturated fusion system over S. Then $\mathcal{F}$ is *constrained* if $O_p(\mathcal{F})$ is $\mathcal{F}$-centric.

The following terminology is taken from [2].

Definition 1.9. Let $\mathcal{F}$ be a constrained fusion system over S, and let M be a finite group. Then M is a *model* for $\mathcal{F}$ provided that

(1) S is a Sylow p-subgroup of M,
(2) $\mathcal{F}=\mathcal{F}_S(M)$, and
(3) $C_M(O_p(M)) \leqslant O_p(M)$.

Notice that if M is a model for $\mathcal{F}$ then $O_p(M)=O_p(\mathcal{F})$.

The definition of model in [2] (or, equivalently, of "localizer" in [18]) is somewhat more flexible than the one we have given here; but Definition 1.9 will suffice for our purposes. The following quoted result may be interpreted as saying that the main theorem holds in the case where $\mathcal{F}$ is constrained. This special result lies at the foundation of our proof of the main theorem.

Proposition 1.10. *Let $\mathcal{F}$ be a constrained fusion system over the finite p-group S. Then the following hold*:

(a) *There exists a model M for $\mathcal{F}$.*

(b) *Let M_1 and M_2 be models for $\mathcal{F}$. Then there exists an isomorphism*

$$\beta: M_1 \longrightarrow M_2$$

such that β restricts to the identity map on S. Moreover, if β' is any other such isomorphism, then the automorphism $\beta^{-1}\circ\beta'$ of M_2 is an inner automorphism c_z, given by conjugation by an element $z\in Z(S)$. In particular,

(c) *if M is a model for $\mathcal{F}$ then $\{c_z \mid z\in Z(S)\}$ is the set of automorphisms of M which restrict to the identity map on S.*

Proof. Statement (a), and the uniqueness of M up to isomorphism, appear as Proposition 4.3 in [6]. A different treatment, along with the "strong uniqueness" of M in statement (b), is due to Puig [18, Theorem 18.6]. There is also a subsequent (and independent) proof by Oliver—including the important statement (b) [3, §III, Theorem 5.10]. □

Lemma 1.11. *Let M be a model of the saturated, constrained fusion system $\mathcal{F}$ over S, and let $\mathcal{E}$ be a saturated fusion system on S such that the set* $\mathrm{Hom}(\mathcal{E})$ *of $\mathcal{E}$-homomorphisms is contained in* $\mathrm{Hom}(\mathcal{F})$. *Then M contains a unique model H for $\mathcal{E}$.*

Proof. Set $T=O_p(M)$, and let H be the set of all $g\in M$ such the conjugation automorphism c_g of T is in $\mathcal{E}$. The set of all such c_g with $g\in H$ is equal to $\mathrm{Aut}_{\mathcal{E}}(T)$, so H is a subgroup of M. Moreover, $S\leqslant H$ as $\mathcal{F}_S(S)\subseteq\mathcal{E}$.

Let $\mathcal{E}'$ be the fusion system $\mathcal{F}_S(H)$. Then

$$\Lambda := \mathrm{Aut}_{\mathcal{E}'}(T) = \mathrm{Aut}_H(T) = \mathrm{Aut}_{\mathcal{E}}(T).$$

Fix $\lambda\in\Lambda$, let $h\in H$ with $c_h=\lambda$, and let P_λ be the largest subgroup P of S such that $\mathrm{Aut}_P(T)^\lambda\leqslant\mathrm{Aut}_S(T)$. Set $P=P_\lambda$ and let Q be the preimage in S of $\mathrm{Aut}_P(T)^\lambda$. As conjugation by h induces an automorphism of $\mathrm{Aut}_H(T)$, the natural isomorphism of $\mathrm{Aut}_H(T)\to H/Z(T)$ yields $P^h=Q$. That is, α extends to an $\mathcal{E}'$-isomorphism $\phi: P\to Q$. Since $\mathcal{E}$ is constrained, also $\mathcal{E}$ has a model, and so α extends also to an $\mathcal{E}$-isomorphism

$\psi: P \to Q$. Then $\phi \circ \psi^{-1}$ is an $\mathcal{F}$-automorphism which restricts to the identity on T, and so $\psi = \phi \circ c_z$ for some $z \in Z(T)$. Since $\mathcal{F}_S(S) \subseteq \mathcal{E} \cap \mathcal{E}'$, we conclude that $\mathrm{Iso}(\mathcal{E}) = \mathrm{Iso}(\mathcal{E}')$. Then $\mathrm{Hom}(\mathcal{E}) = \mathrm{Hom}(\mathcal{E}')$ by Definition 1.1 (b). Thus, $\mathcal{E} = \mathcal{E}'$, and H is a model for $\mathcal{E}$.

Now suppose that there is another subgroup K of M which is a model for $\mathcal{E}$. Let $c: M \to \mathrm{Aut}(T)$ be the map which sends $g \in M$ to the automorphism c_g of T. Then $\mathrm{Ker}(c) = Z(T) \leqslant H \cap K$, and $Kc = \mathrm{Aut}_K(T) = \mathrm{Aut}_H(T) = Hc$, so $K = H$. □

By a *group of Lie type in characteristic p* we mean a finite group $O^{p'}(C_{\overline{K}}(\sigma))$, where $\overline{K}$ is a semisimple algebraic group over the algebraic closure $\bar{\mathbb{F}}_p$ of the field of p elements, and where σ is a Steinberg endomorphism of $\overline{K}$. The following well-known result will play an important role in §7.

LEMMA 1.12. *Let G be a group of Lie type in characteristic p, let $S \in \mathrm{Syl}_p(G)$ be a Sylow p-subgroup of G, and let X be a parabolic subgroup of G containing S. Then $O_p(X)$ is weakly closed in $\mathcal{F}_S(G)$.*

Proof. Let Φ be the root system (or twisted root system) associated with G, and let Φ^+ be the set of positive roots, taken so that S is generated by the set of root subgroups U_α for $\alpha \in \Phi^+$. Set $Q = O_p(X)$, set $B = N_G(S)$, and let H be a complement to S in B. For any subset Δ of Φ^+ let Δ' be the set of roots $-\alpha$ such that $\alpha \in \Phi^+$ and $\alpha \notin \Delta$. Standard results concerning the structure of parabolic subgroups (see [12, Theorem 2.6.5]) yield the existence of a subset $\Delta := \Delta(X)$ of Φ^+, such that

$$Q = \langle U_\delta \mid \delta \in \Delta \rangle, \quad O^{p'}(X) = \langle U_\gamma \mid \gamma \in \Phi^+ \cup \Delta' \rangle, \quad \text{and} \quad X = O^{p'}(X)H. \tag{$*$}$$

Let $g \in N_G(Q, S)$. By Alperin's fusion theorem there is a sequence $(R_1, ..., R_n)$ of subgroups of S, and elements $g_i \in N_G(R_i)$, such that $g_i \in N_G(R_i)$, $Q \leqslant R_1$, $Q^{g_1 \cdots g_i} \leqslant R_i$ for all i, and such that $g = hg_1 \ldots g_n$ for some $h \in C_G(Q)$. Moreover, the groups R_i may be chosen so that $R_i = O_p(N_G(R_i))$ and $N_S(R_i) \in \mathrm{Syl}_p(N_G(R_i))$, and then a theorem of Borel and Tits [12, Theorem 3.1.3] yields the result that each $N_G(R_i)$ is a parabolic subgroup of G over S. Thus, in order to prove that $Q' = Q$, and hence that Q is weakly closed in $\mathcal{F}_S(G)$, it suffices to consider the case where $g = g_1 \in Y$ for some parabolic subgroup $Y = N_G(R)$ of G over S, with $Q \leqslant R = O_p(Y)$.

Set $\Gamma = \Delta(N_G(R))$. Then $\Delta \subseteq \Gamma$ and $\Gamma' \subseteq \Delta'$. Applying $(*)$ to both $N_G(R)$ and X, we obtain $N_G(R) \leqslant X$. Thus $g \in N_G(Q)$, as required. □

2. Partial groups, objective partial groups, and localities

For any set X we write $\mathbf{W}(X)$ for the free monoid on X. Thus, an element of $\mathbf{W}(X)$ is a finite sequence of (or *word in*) the elements of X, and the multiplication in $\mathbf{W}(X)$

consists of concatenation of sequences (denoted $u\circ v$). The use of the same symbol "$\circ$" for concatenation of sequences and for composition of functions should cause no confusion.

The *length* $\ell(w)$ of the word $w=(x_1,...,x_n)$ is n. The "empty word" is the word $(\varnothing)$ of length 0. We shall make no careful distinction between the set X and the set of words of length 1. That is to say, we regard X as a subset of $\mathbf{W}(X)$ via the identification $x\mapsto(x)$.

Definition 2.1. Let $\mathcal{M}$ be a non-empty set, and let $\mathbf{W}=\mathbf{W}(\mathcal{M})$ be the free monoid on $\mathcal{M}$. Let $\mathbf{D}$ be a subset of $\mathbf{W}$ such that

(1) $\mathcal{M}\subseteq\mathbf{D}$, and

$$u\circ v\in\mathbf{D} \quad\Longrightarrow\quad u,v\in\mathbf{D}.$$

(Notice that (1) implies that also the empty word is in $\mathbf{D}$.) A mapping $\Pi:\mathbf{D}\to\mathcal{M}$ is a *product* if

(2) Π restricts to the identity map on $\mathcal{M}$, and

(3) $u\circ v\circ w\in\mathbf{D}\Rightarrow u\circ(\Pi(v))\circ w\in\mathbf{D}$ and $\Pi(u\circ v\circ w)=\Pi(u\circ(\Pi(v))\circ w)$.

An *inversion* on $\mathcal{M}$ consists of an involutory bijection $f\mapsto f^{-1}$ on $\mathcal{M}$, together with the mapping $u\mapsto u^{-1}$ on $\mathbf{W}$ given by

$$(f_1,...,f_n)\longmapsto(f_n^{-1},...\,f_1^{-1}).$$

A *partial group* consists of a product $\Pi:\mathbf{D}\to\mathcal{M}$, together with an inversion $(\cdot)^{-1}$ on $\mathcal{M}$, such that

(4) $u\in\mathbf{D}\Rightarrow u^{-1}\circ u\in\mathbf{D}$ and $\Pi(u^{-1}\circ u)=\mathbf{1}$,

where $\mathbf{1}$ denotes the image of the empty word under Π.

We list some elementary consequences of the definition, as follows.

LEMMA 2.2. *Let $\mathcal{M}$ (with $\mathbf{D}$, Π and inversion) be a partial group.*

(a) Π *is* $\mathbf{D}$-multiplicative. *That is, if $u\circ v$ is in $\mathbf{D}$ then the word $(\Pi(u),\Pi(v))$ of length 2 is in $\mathbf{D}$, and*

$$\Pi(u\circ v)=\Pi(u)\Pi(v),$$

where $\Pi(u)\Pi(v)$ is an abbreviation for $\Pi((\Pi(u),\Pi(v))$.

(b) Π *is* $\mathbf{D}$-associative. *That is,*

$$u\circ v\circ w\in\mathbf{D} \quad\Longrightarrow\quad \Pi(u\circ v)\Pi(w)=\Pi(u)\Pi(v\circ w).$$

(c) *If $u\circ v\in\mathbf{D}$ then $u\circ(\mathbf{1})\circ v\in\mathbf{D}$ and $\Pi(u\circ(\mathbf{1})\circ v)=\Pi(u\circ v)$.*

(d) *If $u\circ v\in\mathbf{D}$ then both $u^{-1}\circ u\circ v$ and $u\circ v\circ v^{-1}$ are in $\mathbf{D}$, $\Pi(u^{-1}\circ u\circ v)=\Pi(v)$, and $\Pi(u\circ v\circ v^{-1})=\Pi(u)$.*

(e) (Cancellation rule) *If* $u\circ v, u\circ w\in\mathbf{D}$, *and* $\Pi(u\circ v)=\Pi(u\circ w)$, *then* $\Pi(v)=\Pi(w)$ (*and similarly for right cancellation*).

(f) *If* $u\in\mathbf{D}$ *then* $u^{-1}\in\mathbf{D}$, *and* $\Pi(u^{-1})=\Pi(u)^{-1}$. *In particular,* $\mathbf{1}^{-1}=\mathbf{1}$.

(g) (Uncancellation rule) *Let* $u,v,w\in\mathbf{W}$, *and suppose that both* $u\circ v$ *and* $u\circ w$ *are in* $\mathbf{D}$ *and that* $\Pi(v)=\Pi(w)$. *Then* $\Pi(u\circ v)=\Pi(u\circ w)$. (*Similarly for right uncancellation.*)

Proof. Let $u\circ v\in\mathbf{D}$. Then condition (3) in Definition 2.1 applies to $(\varnothing)\circ u\circ v$ and yields the result that $(\Pi(u))\circ v\in\mathbf{D}$ with $\Pi(u\circ v)=\Pi((\Pi(u))\circ v)$. Now apply again (3) to $(\Pi(u))\circ v\circ(\varnothing)$, to obtain (a).

Let $u\circ v\circ w\in\mathbf{D}$. Then $u\circ v$ and w are in $\mathbf{D}$ by condition (1) in Definition 2.1, and $\mathbf{D}$-multiplicativity yields $\Pi(u\circ v\circ w)=\Pi(u\circ v)\Pi(w)$. Similarly, $\Pi(u\circ v\circ w)=\Pi(u)\Pi(v\circ w)$, and (b) holds.

Notice that statement (c) is immediate from condition (3) in Definition 2.1.

Assume that $u\circ v\in\mathbf{D}$. Then $v^{-1}\circ u^{-1}\circ u\circ v\in\mathbf{D}$ by condition (4) in Definition 2.1, and then also $u^{-1}\circ u\circ v\in\mathbf{D}$. Multiplicativity then yields

$$\Pi(u^{-1}\circ u\circ v)=\Pi(u^{-1}\circ u)\Pi(v)=\mathbf{1}\Pi(v)=\Pi(\varnothing)\Pi(v)=\Pi((\varnothing)\circ v)=\Pi(v).$$

As $(w^{-1})^{-1}=w$ for any $w\in\mathbf{W}$, one obtains $w\circ w^{-1}\in\mathbf{D}$ for any $w\in\mathbf{D}$, and $\Pi(w\circ w^{-1})=\mathbf{1}$. From this one easily completes the proof of (d).

Now let $u\circ v$ and $u\circ w$ be in $\mathbf{D}$, with $\Pi(u\circ v)=\Pi(u\circ w)$. Then (d) (together with multiplicativity and associativity, which will not be explicitly mentioned hereafter) yield

$$\Pi(v)=\Pi(u^{-1}\circ u\circ v)=\Pi(u^{-1})\Pi(u)\Pi(v)=\Pi(u^{-1})\Pi(u)\Pi(w)=\Pi(u^{-1}\circ u\circ w)=\Pi(w),$$

and (e) holds.

Let $u\in\mathbf{D}$. Then $u\circ u^{-1}\in\mathbf{D}$, and then $\Pi(u)\Pi(u^{-1})=\mathbf{1}$. But also $(\Pi(u),\Pi(u)^{-1})\in\mathbf{D}$, and $\Pi(u)\Pi(u)^{-1}=\mathbf{1}$. Now (f) follows by cancellation.

Let u, v and w be as in (g). Then $u^{-1}\circ u\circ v$ and $u^{-1}\circ u\circ w$ are in $\mathbf{D}$ by (d). By two applications of (d), $\Pi(u^{-1}\circ u\circ v)=\Pi(v)=\Pi(w)=\Pi(u^{-1}\circ u\circ w)$, so $\Pi(u\circ v)=\Pi(u\circ w)$ by (e). That is, $\Pi(u)\Pi(v)=\Pi(u)\Pi(w)$, and (g) holds. □

LEMMA 2.3. *Let* $\mathcal{M}$ *be a partial group, and write* xy *for* $\Pi(x,y)$ *when* $(x,y)\in\mathbf{D}$.

(a) *For each* $x\in\mathcal{M}$, *both* $(x,\mathbf{1})$ *and* $(\mathbf{1},x)$ *are in* $\mathbf{D}$, *and* $\mathbf{1}x=x\mathbf{1}$.

(b) *For each* $x\in\mathcal{M}$, *both* (x^{-1},x) *and* (x,x^{-1}) *are in* $\mathbf{D}$, *and* $x^{-1}x=\mathbf{1}=xx^{-1}$.

(c) *If* $\mathbf{W}(\mathcal{M})=\mathbf{D}$ *then* $\mathcal{M}$ *is a group via the binary operation* $(x,y)\mapsto xy$.

Proof. As $x=\varnothing\circ x=x\circ\varnothing$, and as $\Pi(x)=x$ by condition (2) in Definition 2.1, and since $\Pi(\varnothing)=\mathbf{1}$, statement (a) follows from Lemma 2.2 (a). Point (b) is immediate from condition (4) in Definition 2.1. Thus, $\mathbf{1}$ is an identity element for $\mathcal{M}$ by (a), and x^{-1} is an inverse for x by (b). Finally, if $\mathcal{M}\times\mathcal{M}\times\mathcal{M}\subseteq\mathbf{D}$ then the operation $(x,y)\mapsto xy$ is associative by Lemma 2.2 (b). In particular, (c) holds. □

Examples 2.4. (1) The first example is the basic one, in which $\mathcal{M}$ is a group G, $\mathbf{1}$ is the identity element of G, g^{-1} is the inverse of g in G, $\mathbf{D}=\mathbf{W}(G)$, and Π is the (multi-variable) product in G. Let "$\cdot$" be the binary operation given by restricting Π to $\mathcal{M}\times\mathcal{M}$. Then $(\mathcal{M},\cdot)$ is a group by Lemma 2.3 (c), and visibly that group is equal to G. Conversely, if $(\mathcal{M},\mathbf{D},\Pi)$ is a partial group in which $\mathbf{D}=\mathbf{W}$ then $(\mathcal{M},\cdot)$ is a group, again by Lemma 2.3 (c).

(2) Let G be a group and let Δ be a collection of subgroups of G. For $X\in\Delta$ and $g\in G$ write X^g for the subgroup $g^{-1}Xg\leqslant G$. One then obtains a partial group $\mathcal{M}=\mathcal{M}(G,\Delta)$, for which $\mathbf{D}$ is the set of all words $w=(g_1,...,g_n)\in\mathbf{W}(G)$ such that there exists $X\in\Delta$ with $X^{g_1\cdots g_i}\in\Delta$ for all i $(1\leqslant i\leqslant n)$. Take Π to be the restriction to $\mathbf{D}$ of the multivariable product in G, inversion as the restriction to $\mathcal{M}$ of inversion in G, and $\mathbf{1}$ as the identity element of G. Notice that if there exists $X\in\Delta$ with $X^g\in\Delta$ for all $g\in G$, then all products are defined, and one then recovers G as a bona fide group.

(3) Here is a special case of example (2). Let G be the group $O_4^+(2)$ (or equivalently, the wreath product $S_3\wr C_2$). Thus, G is a group of order 72, with a normal elementary abelian subgroup A of order 9, and with a dihedral Sylow 2-subgroup S acting faithfully on A. Let Δ be the set of subgroups of S of order 2. Then, as a *set*, the partial group $\mathcal{M}$ (defined as in example (2)) is equal to G, since every element of G fuses some involution of S into S. But $\mathbf{D}(\mathcal{M})$ is a proper subset of $\mathbf{W}(\mathcal{M})$, so $\mathcal{M}$ is not a group.

It is often convenient to eliminate the symbol "Π" and to speak of "the product $f_1 ... f_n$". More generally, if $\{X_i\}_{i=1}^n$ is a collection of subsets of $\mathcal{M}$ then the "product set $X_1 ... X_n$" is by definition the image under Π of the set of words $(f_1,...,f_n)\in\mathbf{D}$ such that $f_i\in X_i$ for all i. If $X_i=\{f_i\}$ is a singleton, then we may write f_i in place of X_i in such a product. Thus, for example, the product Xfg stands for the set of all $\Pi(x,f,g)$ with $(x,f,g)\in\mathbf{D}$, and with $x\in X$.

A word of urgent warning: in writing products in the above way one may be led, mistakenly, into imagining that "associativity" holds in a stronger sense than that which is given by Lemma 2.2 (b). For example, one should not suppose, if $(f,g,h)\in\mathbf{W}$, and both (f,g) and (fg,h) are in $\mathbf{D}$, that (f,g,h) is in $\mathbf{D}$. That is, it may be that "the product fgh" is undefined, even though the product $(fg)h$ is defined. Of course, one is tempted to simply extend the domain $\mathbf{D}$ to include such triples (f,g,h), and to "define" the product fgh to be $(fg)h$. The trouble is that it may also be the case that gh and $f(gh)$ are defined (via $\mathbf{D}$), but that $(fg)h\neq f(gh)$.

Let $\mathcal{M}$ be a partial group and let $\mathcal{H}$ be a non-empty subset of $\mathcal{M}$. Then $\mathcal{H}$ is a *partial subgroup* of $\mathcal{M}$ if $\mathcal{H}$ is closed under inversion ($f\in\mathcal{H}$ implies $f^{-1}\in\mathcal{H}$) and with respect to products. The latter condition means that $\Pi(w)\in\mathcal{H}$ whenever $w\in\mathbf{W}(\mathcal{H})\cap\mathbf{D}$. If in fact $\mathbf{W}(\mathcal{H})$ is contained in $\mathbf{D}$, then $\mathcal{H}$ is a *subgroup* of $\mathcal{M}$ (i.e. a partial subgroup which is a group) by Lemma 2.3.

For a partial group $\mathcal{M}$ and $f\in\mathcal{M}$, write $\mathbf{D}(f)$ for the set of all $x\in\mathcal{M}$ such that the product $f^{-1}xf$ is defined. There is then a mapping

$$c_f\colon \mathbf{D}(f)\longrightarrow \mathcal{M}$$

given by $x\mapsto f^{-1}xf$ (and called *conjugation by* f). Since our preference is for "right-hand" notation, we write

$$x\longmapsto (x)c_f \quad \text{or} \quad x\longmapsto x^f$$

for conjugation by f.

At this early point, and in the context of arbitrary partial groups, one can say very little about the maps c_f. The cancellation rule (Lemma 2.2 (e)) implies that each c_f is injective, but beyond that, the following lemma may be the best that can be obtained.

LEMMA 2.5. *Let* $\mathcal{M}$ *be a partial group and let* $f\in\mathcal{M}$. *Then the following hold*:

(a) $\mathbf{1}\in\mathbf{D}(f)$ *and* $\mathbf{1}^f=\mathbf{1}$;

(b) $\mathbf{D}(f)$ *is closed under inversion, and* $(x^{-1})^f=(x^f)^{-1}$ *for all* $x\in\mathbf{D}(f)$;

(c) c_f *is a bijection* $\mathbf{D}(f)\to\mathbf{D}(f^{-1})$, *and* $c_{f^{-1}}=(c_f)^{-1}$;

(d) $\mathcal{M}=\mathbf{D}(\mathbf{1})$, *and* $x^{\mathbf{1}}=x$ *for each* $x\in\mathcal{M}$.

Proof. By condition (4) in Definition 2.1, $f\circ\varnothing\circ f^{-1}=f\circ f^{-1}\in\mathbf{D}$, so $\mathbf{1}\in\mathbf{D}(f)$ and then $\mathbf{1}^f=\mathbf{1}$ by Lemma 2.3 (a). Thus (a) holds. Now let $x\in\mathbf{D}(f)$ and set $w=(f^{-1},x,f)$. Then $w\in\mathbf{D}$, and $w^{-1}=(f^{-1},x^{-1},f)$ by Definition 2.1. Then condition (4) in Definition 2.1 yields $w^{-1}\circ w\in\mathbf{D}$, and so $w^{-1}\in\mathbf{D}$ by condition (1). This shows that $\mathbf{D}(f)$ is closed under inversion. Also, condition (4) yields $\mathbf{1}=\Pi(w^{-1}\circ w)=(x^{-1})^f x^f$, and then $(x^{-1})^f=(x^f)^{-1}$ by Lemma 2.2 (f). This completes the proof of (b).

As $w\in\mathbf{D}$, Lemma 2.2 (d) implies that $f\circ w$ and then $f\circ w\circ f^{-1}$ are in $\mathbf{D}$. Now condition (3) in Definition 2.1 and two applications of Lemma 2.2 (d) yield

$$fx^ff^{-1}=\Pi(f,f^{-1},x,f,f^{-1})=\Pi((f,f^{-1},x)\circ f\circ f^{-1})=\Pi(f,f^{-1},x)=x.$$

Thus $x^f\in\mathbf{D}(f^{-1})$ with $(x^f)^{f^{-1}}=x$, and hence (c) holds.

Finally, $\mathbf{1}=\mathbf{1}^{-1}$ by Lemma 2.2 (f), and $\varnothing\circ x\circ\varnothing=x\in\mathbf{D}$ for any $x\in\mathcal{M}$, proving (d). □

If X is a subgroup of $\mathcal{M}$ with $X\subseteq\mathbf{D}(f)$, write X^f for $\{x^f\,|\,x\in X\}$. Example 2.4 (3), with X a fours group contained in S, and with f a suitable element of order 3, shows that X^f need not be a group with respect to the product Π.

For subgroups X and Y of $\mathcal{M}$, set

$$N_{\mathcal{M}}(X,Y)=\{f\in\mathcal{M}\,|\,X\subseteq\mathbf{D}(f)\text{ and }X^f\leqslant Y\},$$

and set

$$N_{\mathcal{M}}(X)=\{f\in\mathcal{M}\mid X\subseteq\mathbf{D}(f)\text{ and }X^f=X\}.$$

In practice, all of the objective partial groups that will be encountered in this paper have the property that their objects are finite, so we will always have $N_{\mathcal{M}}(X,X)=N_{\mathcal{M}}(X)$ by Lemma 2.5 (c). Write $C_{\mathcal{M}}(X)$ for the set of all $f\in N_{\mathcal{M}}(X)$ such that $x^f=x$ for all $x\in X$.

Henceforth, if X is a subgroup of a partial group $\mathcal{M}$, any statement involving the expression "X^f" should be understood as being based on the tacit hypothesis $X\subseteq\mathbf{D}(f)$.

Example 2.4 (2) may be formalized and generalized as follows.

Definition 2.6. Let $\mathcal{M}$ be a partial group and let Δ be a collection of subgroups of $\mathcal{M}$. Let $\mathbf{D}_\Delta$ be the set of all $w=(f_1,...,f_n)\in\mathbf{W}(\mathcal{M})$ such that

$$\text{there exists }(X_0,...,X_n)\in\mathbf{W}(\Delta)\text{ with }(X_{i-1})^{f_i}=X_i\text{ for all }i\ (1\leqslant i\leqslant n).\qquad(*)$$

Then $(\mathcal{M},\Delta)$ is an *objective partial group* (in which Δ is the set of *objects*), if the following two conditions hold:

(O1) $\mathbf{D}=\mathbf{D}_\Delta$;

(O2) whenever $X,Z\in\Delta$, Y is a subgroup of Z, and $f\in\mathcal{M}$ is such that X^f is a subgroup of Y, then $Y\in\Delta$.

We say that a word $w=(f_1,...,f_n)$ is *in* $\mathbf{D}$ *via* $(X_0,...,X_n)$ if the condition $(*)$ in Definition 2.6 applies specifically to w and $(X_0,...,X_n)$. We may also say, more simply, that w *is in* $\mathbf{D}$ *via* X_0, since the sequence $(X_0,...,X_n)$ is determined by w and X_0.

Remark. Notice that in the preceding definition, one needs to already have $\mathbf{D}$ in order to know what $\mathbf{D}_\Delta$ is, since $\mathbf{D}_\Delta$ is defined in terms of conjugation in the partial group defined by $\mathbf{D}$. In practice, when one tries to construct an objective partial group, it is often easy to decide on a suitable $\mathbf{D}$ which yields the partial group that one wants, and which has the property that $\mathbf{D}\subseteq\mathbf{D}_\Delta$. But it can then be very difficult to establish the reverse inclusion $\mathbf{D}\supseteq\mathbf{D}_\Delta$. In fact, much of this paper is built around three such exercises: one of them in the appendix (in order to establish that Oliver–Ventura "transporter systems" give rise to localities), and one each in §4 and §5.

Remark. Condition (O2) in Definition 2.6 has been stated in the form appropriate for this paper, where objects will always be finite p-groups, from Definition 2.9 on. A more general formulation would be:

(O2)′ whenever $X,Z\in\Delta$, $Y\leqslant Z$ is a subgroup of Z, and $f\in\mathcal{M}$ with $X^f\leqslant Y$, then $N_Y(X^f)\in\Delta$.

Lemma 2.7. *Let* $(\mathcal{M},\Delta)$ *be an objective partial group.*

(a) $N_{\mathcal{M}}(X)$ *is a subgroup of* $\mathcal{M}$ *for each* $X\in\Delta$.

(b) *Let* $f \in \mathcal{M}$ *and let* $X \in \Delta$ *with* $X^f \in \Delta$. *Then* $N_{\mathcal{M}}(X) \subseteq \mathbf{D}(f)$, *and*

$$c_f \colon N_{\mathcal{M}}(X) \longrightarrow N_{\mathcal{M}}(X^f)$$

is an isomorphism of groups.

(c) *Let* $w = (f_1, ..., f_n) \in \mathbf{D}$ *via* $(X_0, ..., X_n)$. *Then*

$$c_{f_1} \circ ... \circ c_{f_n} = c_{\Pi(w)}$$

as maps from X_0 *to* X_n.

Proof. We start by proving (c). Let w and $(X_0, ..., X_n)$ be as in (c). For any $x \in X_0$ set $u_x = w^{-1} \circ (x) \circ w$. Then $u_x \in \mathbf{D}_\Delta$ via X_n. Setting $f = \Pi(w)$, and recalling that $\Pi(w^{-1}) = f^{-1}$ (by Lemma 2.2 (f)), we get

$$\Pi(f^{-1}, x, f) = \Pi(u_x) = ((... (x)^{f_1}) ...)^{f_n},$$

by repeated application of Lemma 2.2 (a). This yields (c).

Let $X \in \Delta$ and set $L = N_{\mathcal{M}}(X)$. Then $\mathcal{L}$ is non-empty since $\mathbf{1} \in L$ by Lemma 2.5 (b). Further, L is closed with respect to inversion by Lemma 2.5. For any $w \in \mathbf{W}(L)$, the condition (O1) in Definition 2.6 implies that $w \in \mathbf{D}$ via X, and then $\Pi(w) \in L$ by (c). Now (a) follows from Lemma 2.3 (c).

Let $f \in \mathcal{M}$ with $X \subseteq \mathbf{D}(f)$ and $X^f \in \Delta$. Let $x, y \in L$ and set $u = (f^{-1}, x, f, f^{-1}, y, f)$. Then $u \in \mathbf{D}_\Delta$ via X^f. Thus $u \in \mathbf{D}$ by (O1), and then Lemma 2.2 (a) yields $\Pi(u) = x^f y^f$. We note also that Definition 2.1 (3) and Lemma 2.3 (b) yield

$$\Pi(x, f, f^{-1}, y) = \Pi(x \circ \mathbf{1} \circ y)$$

and so $\Pi(x, f, f^{-1}, y) = xy$ by Lemma 2.3 (b). Then

$$\Pi(u) = \Pi(f^{-1} \circ (x, f, f^{-1}, y) \circ f) = (xy)^f$$

by Definition 2.1 (3), and thus $c_f \colon L \to L^f$ is a homomorphism of groups. Hence c_f is an isomorphism by Lemma 2.5 (c), proving (b). □

Remark 2.8. We mention two structures associated with a given objective partial group $(\mathcal{M}, \Delta)$.

(1) There is a category $\mathcal{C} = \mathrm{Cat}(\mathcal{M}, \Delta)$ whose set of objects is Δ, whose morphisms are triples (f, X, Y) with $X, Y \in \Delta$ and with $f \in N_{\mathcal{M}}(X, Y)$, and where composition of morphisms is given by the product in $\mathcal{M}$:

$$(f, X, Y) \circ (g, Y, Z) = (fg, X, Z),$$

(in right-hand notation). The morphisms $(\mathbf{1}, X, Y)$ with $X \leqslant Y$ are called *inclusion morphisms*. Notice that every morphism in $\mathcal{C}$ can be factored in a unique way as an isomorphism followed by an inclusion morphism.

(2) There is a category $\mathcal{F}=\mathcal{F}(\mathcal{M}, \Delta)$, to be called the *fusion system* of $(\mathcal{M}, \Delta)$, and defined as follows. First, the objects of $\mathcal{F}$ are the groups U such that $U \leqslant X$ for some $X \in \Delta$. Then, the morphisms in $\mathcal{F}$ from U to V are taken to be the group homomorphisms $\phi: U \rightarrow V$ such that ϕ can be factored as a composition of restrictions of conjugation homomorphisms $c_f: X \rightarrow Y$ between objects.

(3) There is another category $\mathcal{F}^*=\mathcal{F}^*(\mathcal{M}, \Delta)$, in which $\mathrm{Ob}(\mathcal{F}^*)=\mathrm{Ob}(\mathcal{F}(\mathcal{M}, \Delta))$, but where $\mathrm{Hom}_{\mathcal{F}^*}(U, V)$ is the set (containing $\mathrm{Hom}_{\mathcal{F}}(U, V)$) of all homomorphisms $\phi: U \rightarrow V$ such that ϕ is a composition of restrictions of conjugation homomorphisms $c_f: X \rightarrow Y$ between subgroups X and Y of $\mathcal{M}$. Here X and Y are not assumed to be objects. It appears to be a highly non-trivial question, as to whether the fusion systems $\mathcal{F}$ and $\mathcal{F}^*$ are necessarily equal—even in the case of localities or linking systems (defined below).

Here is the main definition.

Definition 2.9. Let p be a prime, let $\mathcal{L}$ be a partial group, and let S be a finite p-subgroup of $\mathcal{L}$. Then $(\mathcal{L}, S)$ is a *locality* if $\mathcal{L}$ is finite, and provided that there exists a set Δ of subgroups of S such that $S \in \Delta$ and such that the following two conditions hold:

(L1) $(\mathcal{L}, \Delta)$ is objective;

(L2) S is maximal in the poset (ordered by inclusion) of finite p-subgroups of $\mathcal{L}$.

We also say that $\mathcal{L}$ is a *locality on S via Δ*.

There are a number of special sorts of localities that deserve special names. In order to assign names to them in a way that is consistent with established usage, we define $\mathcal{F}:=\mathcal{F}_S(\mathcal{L})$ to be the fusion system on S whose homomorphisms are the compositions of restrictions of conjugation maps in $\mathcal{L}$ from one object to another. That is, $\mathcal{F}$ is the fusion system $\mathcal{F}(\mathcal{L}, \Delta)$ defined in Remark 2.8 (2).

A locality $\mathcal{L}$ is a Δ-*linking system* if $C_{\mathcal{L}}(P) \leqslant P$ for each $P \in \Delta$. If moreover Δ is the set of all $\mathcal{F}$-centric subgroups of S then $\mathcal{L}$ is a *centric linking system*.

Example/Lemma 2.10. *Let M be a finite group, let S be a Sylow p-subgroup of M, set $\mathcal{F}=\mathcal{F}_S(M)$, and let Γ be a non-empty $\mathcal{F}$-invariant collection of subgroups of S such that Γ is overgroup closed in S. Let $\mathcal{L}$ be the set of all $g \in M$ such that $S \cap S^g \in \Gamma$, and set $\mathbf{D}=\mathbf{D}_\Gamma$ (as defined in Definition 2.6). Then $\mathcal{L}$ is a partial group via the restriction of the multivariable product in M to $\mathbf{D}$. Moreover, $(\mathcal{L}, S)$ is a locality via Γ, to be denoted $\mathcal{L}_\Gamma(M)$.*

Proof. If $g\in\mathcal{L}$ then $(S\cap S^{g^{-1}})^g=S\cap S^g\in\Gamma$, and then $(S\cap S^{g^{-1}})\in\Gamma$ since Γ is $\mathcal{F}$-invariant. Thus $\mathcal{L}\subseteq\mathbf{D}$, and $\mathcal{L}$ is contained in the partial group $\mathcal{M}=\mathcal{M}(M,\Gamma)$ given by Example 2.4 (2). In that example, $\mathcal{M}$ is the set of all $g\in M$ such that there exists $P\in\Gamma$ with $P^g\in\Gamma$. Such an element g has the property that $S\cap S^g\in\Gamma$ since Γ is overgroup closed, and so $\mathcal{L}=\mathcal{M}$. Example 2.4 (2) now shows that $\mathcal{L}$ is a partial group with respect to the multivariable product and the inversion in G. The condition (O1) for objectivity is given by the definition of $\mathbf{D}$, while (O2) is immediate from the assumption that Γ is overgroup closed and $\mathcal{F}$-invariant. Thus, $(\mathcal{L},\Gamma)$ is objective. All members of Γ are subgroups of S, and S is maximal in the poset of p-subgroups of G, so $(\mathcal{L},S)$ is a locality via Γ. □

For any locality $(\mathcal{L},S)$, let $\Omega(\mathcal{L},S)$ be the set of all collections Δ of subgroups of S, such that $S\in\Delta$ and such that $(\mathcal{L},\Delta)$ is objective. We say that $(\mathcal{L},S)$ is *complete* if it satisfies the following condition:

$$\text{For each } \Delta\in\Omega(\mathcal{L},S) \text{ and each } f\in\mathcal{L}, \text{ the set } S_f=\{s\in S\mid s^f\in S\} \\ \text{is a member of } \Delta. \text{ In particular, } S_f \text{ is a sub}\textit{group} \text{ of } S. \qquad (*)$$

PROPOSITION 2.11. *Every locality is complete.*

Proof. Let $\mathcal{L}$ be a locality on S, let $\Delta\in\Omega(\mathcal{L},S)$, and let $f\in\mathcal{L}$. The word (f) of length 1 is in $\mathbf{D}:=\mathbf{D}(\mathcal{L})$, so there exists $P\in\Delta$ with $Q:=P^f\in\Delta$. Let $a\in S_f$, and set $b=a^f$. Then $\{a,a^{-1},f\}\subseteq N_{\mathcal{L}}(P,S)$, while $b\in N_{\mathcal{L}}(Q,S)$. Thus $(a^{-1},f,b)\in\mathbf{D}$ via P^a. Then also $(f,b)\in\mathbf{D}$, while $(a,f)\in\mathbf{D}$ via $P^{a^{-1}}$. From $f^{-1}af=b$ we get $af=fb$ by Lemma 2.2 (e), and hence

$$a^{-1}fb=a^{-1}(fb)=a^{-1}(af)=f,$$

by $\mathbf{D}$-associativity. Since $a^{-1}fb$ conjugates P^a into S, we conclude that

(1) $P^a\leqslant S_f$ for all $a\in S_f$ and for all $P\in\Delta$ for which $P^f\leqslant S$.

In order to show that S_f is a subgroup of S it suffices to show that $xy\in S_f$ for all $x,y\in S_f$, since by Lemma 2.5 (b) S_f is closed under inversion. From (1), both P^x and $(P^x)^y$ are subgroups of S, and hence in Δ by (O2). Further, (1) yields P^{xf} and $(P^{xy})^f$ in Δ. Thus

$$w:=(f^{-1},x,f,f^{-1},y,f)\in\mathbf{D} \quad \text{via } (P^f,P,P^x,P^{xf},P^x,P^{xy},(P^{xy})^f).$$

Then $(f^{-1}xf)(f^{-1}yf)=f^{-1}(xy)f$, and since $x^fy^f\in S$ we get $(xy)^f\in S$. That is, $xy\in S_f$, and S_f is a subgroup of S. As S_f contains a member of Δ, (O2) then yields $S_f\in\Delta$. □

COROLLARY 2.12. *Let $\mathcal{L}$ be a locality on S. There is then a unique smallest collection Γ of subgroups of S such that $(\mathcal{L},\Gamma)$ is objective.*

Proof. Set $\Omega=\Omega(\mathcal{L},S)$ and set $\Gamma=\bigcap\Omega$. That is, Γ is the set of all P such that $P\in\Delta$ for all $\Delta\in\Omega$. Let $w=(f_1,...,f_n)\in\mathbf{D}$. Then for each $\Delta\in\Omega$ there exists $P_\Delta\in\Delta$ such that $w\in\mathbf{D}$ via Δ, by (O1). Set $Q_0=\langle P_\Delta\,|\,\Delta\in\Omega\rangle$. Then Proposition 2.11 shows that $Q_0\leqslant S_{f_1}$ and that there is a well-defined sequence $(Q_0,...,Q_n)$ of subgroups of S such that $Q_i=(Q_{i-1})^{f_i}$ for all i with $1\leqslant i\leqslant n$. Each Q_i is in Γ by (O2), so $(\mathcal{L},\Gamma)$ satisfies (O1). Now let $X,Y\in\Gamma$ and let $f\in\mathcal{L}$ with $X^f\leqslant Y$. As $Y\in\Delta$ for all $\Delta\in\Omega$, (O2) implies that the same holds for X^f, and so $X^f\in\Gamma$. That is, $(\mathcal{L},\Gamma)$ satisfies (O2), and $(\mathcal{L},\Gamma)$ is objective. □

Lemma 2.13. *Let $\mathcal{L}$ be a locality on S. Then there is a unique largest set Γ of subgroups of S such that $(\mathcal{L},\Gamma)$ is objective.*

Proof. Let $\Delta_1,\Delta_2\in\Omega:=\Omega(\mathcal{L},S)$ and set $\Delta=\Delta_1\cup\Delta_2$. It will suffice to show that $\Delta\in\Omega$.

Let $(P_0,...,P_n)\in\mathbf{W}(\Delta)$, and let $(f_1,...,f_n)\in\mathbf{W}(\mathcal{L})$ be such that $P_{k-1}^{f_k}=P_k$ for all k from 1 to n. Since all objects are subgroups of S, and $S\in\Delta_i$ for all i, the condition (O2) on $(\mathcal{L},\Delta_i)$ implies that if $P_0\in\Delta_i$ then also each P_k is in Δ_i. Thus,

$$\mathbf{D}_\Delta\subseteq\mathbf{D}_{\Delta_1}\cup\mathbf{D}_{\Delta_2}=\mathbf{D}(\mathcal{L})\subseteq\mathbf{D}_\Delta,$$

and so $\mathbf{D}_\Delta=\mathbf{D}(\mathcal{L})$. That is, $(\mathcal{L},\Delta)$ satisfies condition (O1).

It remains to show that $(\mathcal{L},\Delta)$ satisfies (O2). So, let $X,Y\in\Delta$ and let $f\in\mathcal{L}$ with $X^f\leqslant Y$. Then $X^f\leqslant S\in\Delta_1\cap\Delta_2$. If $X\in\Delta_i$ then (O2) applied to $(\mathcal{L},\Delta_i)$ yields $X^f\in\Delta_i$, so $X^f\in\Delta$ and the proof is complete. □

For any word w in $\mathbf{W}(\mathcal{L})$, $\mathcal{L}$ being a locality on S, we have also the notion of S_w, treated in the following lemma.

Lemma 2.14. *Let $\mathcal{L}$ be a locality on S, set $\mathbf{D}=\mathbf{D}(\mathcal{L})$, and let $\Delta\in\Omega(\mathcal{L},S)$. Let $w=(f_1,...,f_n)\in\mathbf{W}(\mathcal{L})$, and define S_w to be the set of all elements $s_0\in S$ such that there is a sequence $(s_0,s_1,...,s_n)$ of elements of S given by $(s_{i-1})^{f_i}=s_i$, $1\leqslant i\leqslant n$. Then the following properties hold*:

(a) *S_w is a subgroup of S, and $S_w\in\Delta$ if and only if $w\in\mathbf{D}$.*

(b) *Let $w,w'\in\mathbf{D}$ with $\Pi(w)=\Pi(w')$, and with $S_w=S_{w'}$. Let $u,v\in\mathbf{W}$. Then*

$$u\circ w\circ v\in\mathbf{D}\quad\Longleftrightarrow\quad u\circ w'\circ v\in\mathbf{D}.$$

Proof. (a) Let $x_0,y_0\in S_w$, and define x_i recursively by $x_i=(x_{i-1})^{f_i}$, $1\leqslant i\leqslant n$. Similarly define y_i. Then $x_{i-1}y_{i-1}\in S_{f_i}$ by Proposition 2.11, and $(x_{i-1}y_{i-1})^{f_i}=x_iy_i$ by Lemma 2.7 (b). Thus S_w is closed under multiplication. Since Lemma 2.5 (b) shows that S_w is closed under inversion, and since $\mathbf{1}\in S_w$, S_w is then a subgroup of S. If $S_w\in\Delta$ then $w\in\mathbf{D}$ by (O1). Conversely, if $w\in\mathbf{D}$ via some $P\in\Delta$, then $P\leqslant S_w$ and (O2) yields $S_w\in\Delta$.

(b) Set $a=u\circ w\circ v$ and $b=u\circ w'\circ v$, and assume that $a\in\mathbf{D}$. Then $(S_a)^{\Pi(u)}\leqslant S_{w\circ v}$ and

$$S_{w\circ v}=\{s\in S_w\mid s^{\Pi(w)}\in S_v\}=\{s\in S_{w'}\mid s^{\Pi(w')}\in S_v\}=S_{w'\circ v}.$$

Thus $(S_a)^{\Pi(u)}\leqslant S_{w'\circ v}$ and $b\in\mathbf{D}$ via S_a. □

For any locality $(\mathcal{L},S)$, we write $\mathcal{F}_S(\mathcal{L})$ for the fusion system on S generated by the conjugation maps in $\mathcal{L}$ between objects. Notice that $\mathcal{F}_S(\mathcal{L})$ does not depend on the choice Δ of the set of objects, since Proposition 2.11 shows that S_f is independent of Δ for $f\in\mathcal{L}$.

Definition 2.15. Let $\mathcal{L}=(\mathcal{L},\Delta,S)$ be a locality and let $P\in\Delta$ be an object. Then P is *centric* in $\mathcal{L}$ if $C_{\mathcal{L}}(P)/Z(P)$ is a p'-group; *radical* in $\mathcal{L}$ if $P=O_p(N_{\mathcal{L}}(P))$; and *essential* in $\mathcal{L}$ provided that

(i) P is centric in $\mathcal{L}$,

(ii) $N_S(P)\in\mathrm{Syl}_p(N_{\mathcal{L}}(P))$, and

(iii) $N_{\mathcal{L}}(P)/P$ has a strongly p-embedded subgroup.

Notice that condition (iii) implies that P is radical in $\mathcal{L}$.

Definition 2.16. Let $\mathcal{L}=(\mathcal{L},\Delta,S)$ be a locality, let Δ^e be the set of objects $Q\in\Delta$ such that Q is essential in $\mathcal{L}$, and set $\mathbf{A}=\mathbf{A}(\mathcal{L})=\Delta^e\cup\{S\}$. Let $f\in\mathcal{L}$. Then f is $\mathbf{A}$-*decomposable* if there exists $w=(g_1,\dots g_n)\in\mathbf{D}(\mathcal{L})$ such that the following hold:

(i) $S_f\leqslant S_w$ and $f=\Pi(w)$;

(ii) for all i, S_{g_i} is in $\mathbf{A}$, and either $g_i\in O^{p'}(N_{\mathcal{L}}(S_{g_i}))$ or $S_{g_i}=S$.

The Alperin–Goldschmidt fusion theorem [10] implies that in a locality $\mathcal{L}=\mathcal{L}_\Gamma(M)$ of a finite group G, an element $f\in\mathcal{L}$ is $\mathbf{A}$-decomposable provided that $C_G(S_f)\leqslant S_f$. In particular, each $f\in\mathcal{L}$ is $\mathbf{A}$-decomposable if $C_M(O_p(M))\leqslant O_p(M)$. The following result provides a generalization to linking systems. Recall the definition of Δ-linking system preceding Example/Lemma 2.10.

Proposition 2.17. *Let $\mathcal{L}=(\mathcal{L},\Delta,S)$ be a Δ-linking system and define $\mathbf{A}(\mathcal{L})$ as above. Let $f\in\mathcal{L}$. Then f is $\mathbf{A}(\mathcal{L})$-decomposable.*

Proof. Among all f for which the lemma fails to hold, choose f with $P:=S_f$ as large as possible. Then $P\neq S$. Set $P'=P^f$ and set $\mathbf{A}=\mathbf{A}(\mathcal{L})$.

Let Q be a fully normalized $\mathcal{L}$-conjugate of P (and hence also of P'), and let $g,h\in\mathcal{L}$ with $Q=P^g=(P')^h$. Thus, $N_S(Q)\in\mathrm{Syl}_p(N_{\mathcal{L}}(Q))$. By Lemma 2.7 (b) and Sylow's theorem, g and h may be chosen so that $N_S(Q)$ contains both $N_S(P)^g$ and $N_S(P')^h$. The maximality of P then implies that g and h are $\mathbf{A}$-decomposable. Then g^{-1} and h^{-1} are $\mathbf{A}$-decomposable via the inverses of words which yield $\mathbf{A}$-composability for g and h.

Set $f'=g^{-1}fh$, $M=N_{\mathcal{L}}(Q)$, and $R=N_S(Q)$. Then $f'\in M$, and $u:=(g,f',h^{-1})\in \mathbf{D}$ via P, and $\Pi(u)=f$. If f' is $\mathbf{A}$-decomposable then so is f, and therefore we may assume that $f=f'$ and $P=Q$. That is, we are reduced to establishing the proposition for the finite group M rather than the locality $\mathcal{L}$. If $P\in\mathbf{A}$ then f is $\mathbf{A}$-decomposable by definition, so we may assume that this is not the case. Applying the Alperin–Goldschmidt theorem to M, $c_f\in\operatorname{Aut}(P)$ is a composition $c_f=c_{g_1}\circ...\circ c_{g_n}$ with $g_i\in N_M(E_i)$ for some $E_i\in\mathbf{A}(M)$. As $P\notin\mathbf{A}$, $|P|<|E_i|$ for all i. The maximality of P then implies that each g_i is $\mathbf{A}$-decomposable, and hence also $g:=g_1\,...\,g_n$ is $\mathbf{A}$-decomposable. Finally, $z:=fg^{-1}\in C_M(P)=Z(P)$, so $f=zg$ is $\mathbf{A}$-decomposable. □

PROPOSITION 2.18. *Let $(\mathcal{L},S)$ be a locality via Δ, and let $\mathcal{F}:=\mathcal{F}_S(\mathcal{L})$ be the associated fusion system on S. Then the following properties hold*:

(a) *$\mathcal{F}$ is Δ-saturated, and if $\mathcal{F}^c\subseteq\Delta$ then $\mathcal{F}$ is saturated*;

(b) *for each $P\in\Delta$, the map $N_{\mathcal{L}}(P)\to\operatorname{Aut}_{\mathcal{F}}(P)$ given by $f\mapsto c_f$ is a surjective homomorphism with kernel $C_{\mathcal{L}}(P)$*;

(c) *if $P\in\Delta$ and P is fully normalized in $\mathcal{F}$ then $N_S(P)$ is a Sylow p-subgroup of $N_{\mathcal{L}}(P)$*;

(d) *if $P\in\Delta$ and $\phi\in\operatorname{Hom}_{\mathcal{F}}(P,S)$ then $\phi=c_f$ for some $f\in N_{\mathcal{L}}(P,S)$.*

Proof. We first show the following:

(1) For each $P\in\Delta$ there exists $f\in N_{\mathcal{L}}(P,S)$ such that $N_S(P^f)$ is a Sylow p-subgroup of $N_{\mathcal{L}}(P^f)$.

By (L2), (1) holds for $P=S$ (and for $f=\mathbf{1}$). Among all $P\in\Delta$ for which (1) fails, choose P so that $|S{:}P|$ is as small as possible. We are free to replace P with any $\mathcal{L}$-conjugate of P in S, so we may assume that $|N_S(P)|$ is maximal among all such conjugates. Set $R=N_S(P)$, and let R^* be a Sylow p-subgroup of $N_{\mathcal{L}}(P)$ containing R. Then R is a proper subgroup of R^*, and hence also a proper subgroup of $N_{R^*}(R)$. By the minimality of $|S{:}P|$, there exists an $\mathcal{L}$-conjugate $Q:=R^f$ of R such that $N_S(Q)$ is a Sylow p-subgroup of $N_{\mathcal{L}}(Q)$. Without loss, we may replace f with fg for any $g\in N_{\mathcal{L}}(Q)$ since any such product fg is defined via (R,Q,Q). By Sylow's theorem, we may therefore assume that $N_{R^*}(R)^f\leqslant N_S(Q)$. But $N_{R^*}(R)^f$ normalizes P^f, and we thereby contradict the maximality of $|N_S(P)|$. Thus, (1) is proved.

Next, let $P\in\Delta$ and $\phi\in\operatorname{Hom}_{\mathcal{F}}(P,S)$. By definition, ϕ is a composition $\phi=\phi_1\circ...\circ\phi_n$, where ϕ_i is given by conjugation by an element h_i of $\mathcal{L}$, and where

$$P(\phi_1\circ...\circ\phi_i)\leqslant S \tag{2}$$

for all i with $1\leqslant i\leqslant n$. Then the word $w=(h_1,...,h_n)$ is in $\mathbf{D}$ via P, and Lemma 2.7 (c) yields $P\phi=P^h$, where $h=\Pi(w)$. Thus (d) holds, and one observes that point (b) follows immediately from (d).

We may now complete the proof of (a) and (c). Namely, let $P\in\Delta$ and let $Q=P^f$ be an $\mathcal{L}$-conjugate of P, as in (1), so that $N_S(Q)\in\mathrm{Syl}_p(N_{\mathcal{L}}(Q))$. As $N_S(P)^f\leqslant N_{\mathcal{L}}(Q)$, there then exists $g\in N_{\mathcal{L}}(Q)$ such that $(N_S(P)^f)^g\leqslant N_S(Q)$. As $c_f\circ c_g\in\mathcal{F}$, we conclude that Q is fully normalized in $\mathcal{F}$, in the sense of Definition 1.2. Thus, $\mathcal{F}$ satisfies condition (A) in Definition 1.4 for Δ-saturation. On the other hand, suppose that P itself is fully normalized in $\mathcal{F}$. Then, by (2), there exists $h\in\mathcal{L}$ such that $N_S(Q)^h=N_S(P)$ and with $Q^h=P$. This shows that $N_S(P)\in\mathrm{Syl}_p(N_{\mathcal{L}}(P))$ (and thus (c) holds).

Set $M=N_{\mathcal{L}}(P)$. By definition, each ϕ in $N_{\mathcal{F}}(P)$ extends to an $\mathcal{F}$-homomorphism which maps P to P. Then (d) implies that $\phi=c_f$ for some $f\in M$. Thus $\mathcal{F}_{N_S(P)}(M)=N_{\mathcal{F}}(P)$, so that $\mathcal{F}$ satisfies condition (B) in Definition 1.4 for Δ-saturation. This completes the proof that $\mathcal{F}$ is Δ-saturated.

Suppose that $\mathcal{L}$ is a centric linking system. Then Δ is the set of $\mathcal{F}$-centric subgroups of S, by definition. By Proposition 2.17, $\mathcal{F}$ is generated by the fusion systems $N_{\mathcal{F}}(P)$ for $P\in\Delta$ in this case, so by Definition 1.4, $\mathcal{F}$ is saturated. This completes the proof of (a), and of the lemma. ☐

Recall the notion of normalizer from Lemma 2.7.

LEMMA 2.19. *Let $(\mathcal{L},S)$ be a locality via the set Δ of objects, let T be a subgroup of S, and set $\Delta_T=\{N_P(T)\,|\,T\leqslant P\in\Delta\}$.*

(a) *$N_{\mathcal{L}}(T)$ is a partial subgroup of $\mathcal{L}$.*

(b) *If $\Delta_T\subseteq\Delta$, then $(N_{\mathcal{L}}(T),\Delta_T)$ is an objective partial group.*

(c) *If $\Delta_T\subseteq\Delta$, and $|N_S(T)|\geqslant|N_S(U)|$ for every $\mathcal{L}$-conjugate U of T in S, then $(N_{\mathcal{L}}(T),N_S(T))$ is a locality via Δ_T.*

Proof. Let $w=(f_1,...,f_n)\in\mathbf{W}(N_{\mathcal{L}}(T))$, and suppose that $w\in\mathbf{D}:=\mathbf{D}(\mathcal{L})$ via a sequence $(P_0,...,P_n)$ of objects. Then $\langle P_{i-1},T\rangle\leqslant S_{f_i}$ for all i, by completeness, and then

$$\langle P_{i-1},T\rangle^{f_i}=\langle P_i,T\rangle.$$

Thus, $T\leqslant S_w$, and we may assume for the sake of simplicity that $T\leqslant P_i$ for all i. Set $f=\Pi(w)$. Then Lemma 2.7 (c) yields $T^f=T$, and so $N_{\mathcal{L}}(T)$ is closed under products. One observes that if $f\in N_{\mathcal{L}}(T)$ and $x\in T$, with $(f^{-1},x,f)\in\mathbf{D}$ via $P\in\Delta$, then $(f,x^{-1},f^{-1})\in\mathbf{D}$ via P^{x^f}. Since an analogous statement holds when x is replaced by x^{-1}, it follows that $N_{\mathcal{L}}(T)$ is closed under inversion, and so (a) is proved.

For the remainder of the proof, we may assume that $\Delta_T\subseteq\Delta$. Set

$$\mathbf{D}_T=\mathbf{D}_\Delta\cap\mathbf{W}(N_{\mathcal{L}}(T))$$

(where $\mathbf{D}_\Delta$ is defined as in Definition 2.6). With w and $(P_0, ..., P_n)$ as in the proof of (a), we may then replace P_i with $N_{P_i}(T)$, and this shows that $\mathbf{D}_T$ is contained in the subset $\mathbf{D}_{\Delta_T}$ of $\mathbf{W}(N_\mathcal{L}(T))$. The reverse inclusion is obvious, so $(N_\mathcal{L}(T), N_S(T))$ satisfies the condition (O1) for objectivity. Any overgroup in $N_S(T)$ of an element of Δ_T is again in Δ_T, so the condition (O2) is satisfied, and $(N_\mathcal{L}(T), \Delta_T)$ is an objective partial group. Thus, (b) holds.

Now assume further that T has been chosen so that $|N_S(T)| \geqslant |N_S(U)|$ for each $\mathcal{L}$-conjugate U of T in S. In order to show that $(N_\mathcal{L}(T), N_S(T))$ is a locality via Δ_T, it suffices to show that $N_S(T)$ is maximal in the poset of p-subgroups of $N_\mathcal{L}(T)$. Set $R = N_S(T)$, let R_1 be a p-subgroup of $N_\mathcal{L}(T)$ containing R, and set $R_2 = N_{R_1}(R)$. As $R \in \Delta$, there exists $f \in \mathcal{L}$ such that $Q := R^f$ is fully normalized in $\mathcal{F}_S(\mathcal{L})$, by Proposition 2.18 (a). Then $N_S(Q)$ is a Sylow p-subgroup of $N_\mathcal{L}(Q)$, and so there exists $g \in N_\mathcal{L}(Q)$ such that $(R_2)^{fg} \leqslant N_S(Q)$. But $(R_2)^{fg} \leqslant N_S(T^{fg})$, and the maximality condition on R then yields $R = R_2$ and $R = R_1$. This completes the proof of (c). □

Definition 2.20. Let $(\mathcal{L}, \Delta, S)$ be a locality, and let $\Gamma \subseteq \Delta$ be a non-empty subset such that Γ is both overgroup-closed in S and $\mathcal{F}_S(\mathcal{L})$-invariant. Set $\mathbf{D} = \mathbf{D}(\mathcal{L})$, set

$$\mathbf{D}|_\Gamma := \{w \in \mathbf{D} \mid S_w \in \Gamma\},$$

and let $\mathcal{L}|_\Gamma$ be the set of words of length 1 in $\mathbf{D}|_\Gamma$, regarded as a subset of $\mathcal{L}$. The *restriction of* $\mathcal{L}$ *to* Γ consists of $\mathcal{L}|_\Gamma$ together with the restriction to $\mathbf{D}|_\Gamma$ of the product in $\mathcal{L}$, and the restriction to $\mathcal{L}|_\Gamma$ of the inversion in $\mathcal{L}$.

LEMMA 2.21. *Let $(\mathcal{L}, \Delta, S)$ be a locality, and let Γ be a non-empty subset of Δ such that Γ is both overgroup-closed in S and $\mathcal{F}_S(\mathcal{L})$-invariant.*

(a) *$\mathbf{D}|_\Gamma$ is the set $\mathbf{D}_\Gamma$ of Definition 2.6, and $(\mathcal{L}|_\Gamma, \Gamma, S)$ is a locality.*

(b) *If $\mathcal{L}$ is a group M, then $\mathcal{L}|_\Gamma$ is the locality $\mathcal{L}_\Gamma(M)$ in Example/Lemma 2.10.*

Proof. Set $\mathcal{M} = \mathcal{L}|_\Gamma$. For any $w \in \mathbf{W}$, the condition that S_w be in Γ is the defining condition for $\mathbf{D}|_\Gamma$, and in view of Lemma 2.14 (a) it is also the defining condition for $\mathbf{D}_\Gamma$. These subsets of $\mathbf{W}$ are therefore identical, and $(\mathcal{M}, \Gamma)$ satisfies the condition (O1) for objectivity. Condition (O2) is given by the assumption that Γ is closed in $\mathcal{F}_S(\mathcal{L})$, so $(\mathcal{M}, \Gamma)$ is objective. All members of Γ are subgroups of S, and S is maximal in the poset of p-subgroups of $\mathcal{M}$ since the corresponding statement holds in $\mathcal{L}$. As $\mathcal{L}$ is finite, so is $\mathcal{M}$, thus $\mathcal{M}$ is a locality, and (a) holds.

Suppose that $\mathcal{L}$ is in fact a group M, and set $\mathcal{K} = \mathcal{L}_\Gamma(M)$. By definition, an element g of M is in $\mathcal{K}$ if and only if $S \cap S^g \in \Gamma$. The latter condition means that $S_g = S \cap S^{g^{-1}}$, so $g \in \mathcal{K}$ if and only if $S_g \in \Gamma$. Similarly, $w \in \mathbf{D}(\mathcal{K})$ if and only if $S_w \in \Gamma$. This shows that $\mathbf{D}(\mathcal{K}) = \mathbf{D}_\Gamma$, and then (b) follows from (a). □

We shall refer to the locality $(\mathcal{L}|_\Gamma, \Gamma, S)$ as the restriction of $\mathcal{L}$ to Γ.

The following proposition gives two applications of completeness to localities.

PROPOSITION 2.22. *Let $\mathcal{L}$ be a locality on S and let $\Delta \in \Omega(\mathcal{L}, S)$. Then the following hold*:

(a) *Every subgroup of $\mathcal{L}$ is a Δ*-local subgroup. *That is*, *for any subgroup H of $\mathcal{L}$*, *there exists $U \in \Delta$ such that $H \leqslant N_{\mathcal{L}}(U)$.*

(b) *Every p-subgroup of $\mathcal{L}$ is conjugate to a subgroup of S.*

Proof. (a) Let $w=(h_1, ..., h_n) \in \mathbf{W}(H)$ be chosen so that the sequence $(g_1, ..., g_n)$, in which $g_i = h_1 ... h_i$, includes all of the elements of H. As H is a subgroup of $\mathcal{L}$, we have $\mathbf{W}(H) \subseteq \mathbf{D}$ (all products in H are defined), and so $w \in \mathbf{D}$. Thus, there exists $P \in \Delta$ such that $P^{g_i} \in \Delta$ for all i. Set $U = \langle P^{g_i} \mid 1 \leqslant i \leqslant n \rangle$, which is a subgroup of S. As $H = \{g_i\}_{i=1}^n$, $U = \langle P^H \rangle$, and so $H \leqslant N_{\mathcal{L}}(U)$. Here $U \in \Delta$ as Δ is overgroup closed in S.

(b) Let Q be a p-subgroup of $\mathcal{L}$. Then Q is finite, as $\mathcal{L}$ is. By (a) there exists $U \in \Delta$ with $Q \leqslant N_{\mathcal{L}}(U)$. By Proposition 2.18 (a) there is an $\mathcal{L}$-conjugate $V = U^f$ of U such that $N_S(V)$ is a Sylow p-subgroup of $N_{\mathcal{L}}(V)$. By Sylow's theorem, there then exists $g \in N_{\mathcal{L}}(V)$ such that $Q^{fg} \leqslant N_S(V)$. □

3. Homomorphisms and partial normal subgroups

Whenever $\mathcal{M}$ and $\mathcal{M}'$ are partial groups, we write $\mathbf{W}$ for $\mathbf{W}(\mathcal{M})$ and $\mathbf{W}'$ for $\mathbf{W}(\mathcal{M}')$. Similarly for $\mathbf{D}$ and $\mathbf{D}'$, for Π and Π', and for $\mathbf{1}$ and $\mathbf{1}'$. We shall make no such careful distinction regarding the inversion maps for $\mathcal{M}$ and $\mathcal{M}'$.

Definition 3.1. Let $\mathcal{M}$ and $\mathcal{M}'$ be partial groups, let $\beta: \mathcal{M} \to \mathcal{M}'$ be a mapping, and let $\beta^*: \mathbf{W} \to \mathbf{W}'$ be the induced mapping. Then β is a *homomorphism* (*of partial groups*) if

(H1) $\mathbf{D}\beta^* \subseteq \mathbf{D}'$, and

(H2) $(\Pi(w))\beta = \Pi'(w\beta^*)$ for all $w \in \mathbf{D}$.

The *kernel* of β is the set $\mathrm{Ker}(\beta)$ of all $g \in \mathcal{M}$ such that $g\beta = \mathbf{1}'$. We say that β is an *isomorphism* if there exists a homomorphism $\beta': \mathcal{M}' \to \mathcal{M}$ such that $\beta \circ \beta'$ and $\beta' \circ \beta$ are identity mappings.

LEMMA 3.2. *Let $\beta: \mathcal{M} \to \mathcal{M}'$ be a homomorphism of partial groups. Then $\mathbf{1}\beta = \mathbf{1}'$ and $(f^{-1})\beta = (f\beta)^{-1}$ for all $f \in \mathcal{M}$.*

Proof. Since $\mathbf{1}\mathbf{1} = \mathbf{1}$, (H1) and (H2) yield $\mathbf{1}\beta = (\mathbf{1}\mathbf{1})\beta = (\mathbf{1}\beta)(\mathbf{1}\beta)$, and then $\mathbf{1}\beta = \mathbf{1}'$ by left or right cancellation. Since $(f, f^{-1}) \in \mathbf{D}$ for any $f \in \mathcal{M}$, by Lemma 2.3 (b), (H1) yields $(f\beta, (f^{-1})\beta) \in \mathbf{D}'$, and then $\mathbf{1}\beta = (ff^{-1})\beta = (f\beta)((f^{-1})\beta)$ by (H2). Finally, since $\mathbf{1}\beta = \mathbf{1}' = (f\beta)(f\beta)^{-1}$, left cancellation yields $(f^{-1})\beta = (f\beta)^{-1}$. □

LEMMA 3.3. *Let $\beta:\mathcal{M}\to\mathcal{M}'$ be a homomorphism of partial groups, and set $\mathcal{N}=\mathrm{Ker}(\beta)$. Then $\mathcal{N}$ is a partial subgroup of $\mathcal{M}$, and $f^{-1}\mathcal{N}f\subseteq\mathcal{N}$ for all $f\in\mathcal{M}$. That is, $g^f\in\mathcal{N}$ whenever $g\in\mathcal{N}\cap\mathbf{D}(f)$.*

Proof. By Lemma 3.2, $\mathcal{N}$ is closed under inversion. If w is in $\mathbf{W}(\mathcal{N})\cap\mathbf{D}$ then the map $\beta^*:\mathbf{W}\to\mathbf{W}'$ induced by β sends w to a word of the form $(\mathbf{1}',...,\mathbf{1}')$. Then $\Pi'(w\beta^*)=\mathbf{1}'$, and thus $\Pi(w)\in\mathcal{N}$. This shows that $\mathcal{N}$ is a partial subgroup of $\mathcal{M}$. Now let $f\in\mathcal{M}$ and let $g\in\mathcal{N}\cap\mathbf{D}(f)$. Then

$$(f^{-1},g,f)\beta^*=((f\beta)^{-1},\mathbf{1}',f\beta)\quad\text{(by Lemma 3.2)},$$

so that

$$(g^f)\beta=\Pi'((f^{-1},g,f)\beta^*)=\Pi'(f\beta)^{-1},\mathbf{1}',f\beta)=\mathbf{1}'.$$ □

Definition 3.4. Let $\mathcal{M}$ be a partial group and let $\mathcal{N}$ be a partial subgroup of $\mathcal{M}$. Then $\mathcal{N}$ is a *partial normal subgroup* of $\mathcal{M}$ if $f^{-1}\mathcal{N}f\subseteq\mathcal{N}$ for all $f\in\mathcal{M}$. (That is, $x^f\in\mathcal{N}$ whenever $x\in\mathcal{N}\cap\mathbf{D}(f)$.)

We may write $\mathcal{N}\trianglelefteq\mathcal{M}$ to indicate that $\mathcal{N}$ is a partial normal subgroup of $\mathcal{M}$.

Definition 3.5. Let $\mathcal{L}=(\mathcal{L},\Delta,S)$ and $\mathcal{L}'=(\mathcal{L}',\Delta,S)$ be localities having the same set of objects. An isomorphism $\beta:\mathcal{L}\to\mathcal{L}'$ of partial groups is *rigid* (over S) if β restricts to the identity map $S\to S$.

LEMMA 3.6. *Let $(\mathcal{L},\Delta,S)$ and $(\mathcal{L}',\Delta,S)$ be localities having the same set Δ of objects, and let $\beta:\mathcal{L}\to\mathcal{L}'$ be a surjective homomorphism of partial groups. Suppose that*

(1) *$S_f=S_{f\beta}$ for all $f\in\mathcal{L}$, and*

(2) $\mathrm{Ker}(\beta)=\mathbf{1}$.

Then β is an isomorphism.

Proof. Let $h\in\mathcal{L}'$ and let $f,g\in\mathcal{L}$ with $f\beta=g\beta=h$. Then $S_f=S_g$ by (1), so $(f^{-1},g)\in\mathbf{D}$ via $(S_f)^f$, and $(f^{-1}g)\beta=\mathbf{1}$. Thus $f=g$ by (2), and β is a bijection.

Let $w'=(h_1,...,h_n)\in\mathbf{D}(\mathcal{L}')$, set $g_i=h_i\beta^{-1}$, and set $w=(g_1,...,g_n)$. Then $w\in\mathbf{D}$ via $S_{w'}$ by (1), and $\Pi(w)\beta=\Pi'(w')$ as β is a homomorphism. Thus

$$\Pi'(w')\beta^{-1}=\Pi(w'(\beta^{-1})^*))$$

and β^{-1} is a homomorphism. □

LEMMA 3.7. *Let $\mathcal{L}=(\mathcal{L},\Delta,S)$ be a locality and let $\mathcal{N}$ be a partial normal subgroup of $\mathcal{L}$. Set $\Gamma=\{P\cap\mathcal{N}\,|\,P\in\Delta\}$ and suppose that $\Gamma\subseteq\Delta$. Then $(\mathcal{N},\Gamma,S\cap\mathcal{N})$ is a locality.*

Proof. Let $w=(f_1,...,f_n)\in\mathbf{D}$ via $P\in\Delta$. Then also $w\in\mathbf{D}$ via $Q:=P\cap\mathcal{N}$, and hence $(\mathcal{N},\Gamma)$ is objective. Each member of Γ is a subgroup of $T:=S\cap\mathcal{N}$, and $\mathcal{N}$ is finite, so it only remains to show that T is maximal in the poset of p-subgroups of $\mathcal{N}$, in order to conclude that $(\mathcal{N},\Gamma,T)$ is a locality.

Let R be a p-subgroup of $\mathcal{N}$ containing T. As $T\leqslant S_x$ for each $x\in S$, it follows from the definition of partial normal subgroup (Definition 3.4) that $T\trianglelefteq S$. As $T\in\Gamma$, S is then a Sylow p-subgroup of the group $N_{\mathcal{L}}(T)$, and hence $N_R(T)^g\leqslant S$ for some $g\in N_{\mathcal{L}}(T)$. The definition of partial normal subgroup then yields $N_R(T)^g\leqslant\mathcal{N}$, so $N_R(T)^g=T$ and $N_R(T)=T$. Thus $R=T$, as required. □

LEMMA 3.8. *Let $(\mathcal{L},\Delta,S)$ be a Δ-linking system, and let β be a rigid automorphism of $\mathcal{L}$. Let $(\mathcal{K},\Gamma,R)$ be a locality such that $\mathcal{K}$ is a partial subgroup of $\mathcal{L}$, and such that $\Gamma\subseteq\Delta$. Then β restricts to a rigid automorphism of $(\mathcal{K},\Gamma,R)$.*

Proof. Let $f\in\mathcal{K}$ and set $P=R_f$. That is, P is the largest subgroup of R which is conjugated by f into R (obtained by applying Proposition 2.11 to the locality $\mathcal{K}$). Then $P\in\Gamma$ by Proposition 2.11, and hence $P\in\Delta$. Since $\mathcal{L}$ is a Δ-linking system, one has $C_{\mathcal{K}}(P)=C_{\mathcal{L}}(P)=Z(P)$. Now Proposition 2.17 implies that f is $\mathbf{A}$-decomposable, where $\mathbf{A}$ is the union of $\{R\}$ and the set of $\mathcal{K}$-essential objects in Γ. Thus $f=\Pi(w)$, where $w=(g_1,...,g_n)\in\mathbf{D}(\mathcal{K})$, each g_i normalizes some $Q_i\in\mathbf{A}$, $Q_i=R_{g_i}$, and $P\leqslant S_w$.

Since β is rigid, each Q_i is β-invariant, and β then restricts to an automorphism γ_i on each $N_{\mathcal{L}}(Q_i)$ by Lemma 2.7 (b). But also β centralizes Q_i, and $C_{\mathcal{L}}(Q_i)=Z(Q_i)$. Taking commutators in the subgroup $\mathrm{Aut}_{\mathcal{L}}(Q_i)\langle\beta\rangle$ of $\mathrm{Aut}(Q_i)$, we then obtain

$$[N_{\mathcal{L}}(Q_i),\beta]\leqslant Z(Q_i)\leqslant N_{\mathcal{K}}(Q_i).$$

We have thus shown that each of the groups $N_{\mathcal{K}}(Q_i)$ is β-invariant. Then

$$f\beta=(\Pi(w))\beta=\Pi(w\beta^*)=\Pi(g_1\beta,...,g_n\beta)\in\mathcal{K},$$

and so $\mathcal{K}$ is β-invariant. The same holds for β^{-1}, so β restricts to an automorphism β_K of $\mathcal{K}$. As $R\leqslant S$, we have that β_K is rigid. □

LEMMA 3.9. *Let M be a finite group, and let $K\leqslant M$ be a subgroup. Let S be a Sylow p-subgroup of M, set $\mathcal{F}=\mathcal{F}_S(M)$, and let Γ be a non-empty $\mathcal{F}$-invariant set of subgroups of S, such that Γ is overgroup closed in S. Let $\mathcal{L}:=\mathcal{L}_\Gamma(M)$ be the locality given by Example/Lemma 2.10, and set $\mathcal{K}=K\cap\mathcal{L}$. Then $\mathcal{K}$ is a partial subgroup of $\mathcal{L}$, and is a partial normal subgroup if $K\trianglelefteq M$.*

Proof. One observes first of all that $\mathcal{K}$ is closed under the inversion in M, which is the inversion in $\mathcal{L}$. Let $w=(x_1,...,x_n)\in\mathbf{D}(\mathcal{L})\cap\mathbf{W}(\mathcal{K})$. Then $\Pi(w)\in\mathcal{L}\cap K$, and so $\mathcal{K}$ is a partial subgroup of $\mathcal{L}$.

Now assume that $K \trianglelefteq M$, let $f \in \mathcal{L}$ and let $x \in \mathcal{K} \cap \mathbf{D}(f)$. Then $x^f \in \mathcal{L}$ and $x^f \in K$, so $x^f \in \mathcal{K}$. Thus $\mathcal{K}$ is a partial normal subgroup of $\mathcal{M}$. □

Recall from Definition 2.16 the notion of $\mathcal{L}$-essential subgroup.

Lemma 3.10. *Let $(\mathcal{L}, \Delta, S)$ be a Δ-linking system and let β be an endomorphism of the partial group $\mathcal{L}$ such that β restricts to the identity automorphism on $O^{p'}(N_{\mathcal{L}}(R))$ for each $\mathcal{L}$-essential subgroup $R \leqslant S$, and restricts to the identity automorphism also on $N_{\mathcal{L}}(S)$. Then β is the identity automorphism of $\mathcal{L}$.*

Proof. Let $f \in \mathcal{L}$ and set $Q = S_f$. Let $\mathbf{A}$ be the union of $\{S\}$ and the set of all $\mathcal{L}$-essential subgroups of S. Then f is $\mathbf{A}$-decomposable by Proposition 2.17. In particular, $f = \Pi(w)$ for some $w = (g_1, ..., g_n) \in \mathbf{D}$ such that $g_i \in N_{\mathcal{L}}(R_i)$ for some $R_i \in \mathbf{A}$. It is then immediate from Definition 3.1 and from the hypothesis concerning β, that $f\beta = f$. □

4. The Frattini lemma

This section develops two of the main computational tools that will enable the later arguments. We obtain an analog of the Frattini lemma in Corollary 4.8, which shows that if $\mathcal{N}$ is a partial normal subgroup of a locality $\mathcal{L}$, then each element of $\mathcal{L}$ may be written as a product of an element $f \in \mathcal{N}$ and an element $g \in N_{\mathcal{L}}(T)$, where $T = S \cap \mathcal{N}$. The "splitting lemma" (Lemma **??**) refines the choice of f and g. We end with an important application (Lemma 4.9) which provides a criterion for extending an automorphism of a linking system in a finite group to an automorphism of the group itself.

The notation S_f and S_w, defined in Proposition 2.11 and Lemma 2.14, will be employed without further comment.

The following hypothesis (and notation) will be assumed throughout this section.

Hypothesis 4.1. There is given a locality $\mathcal{L} = (\mathcal{L}, \Delta, S)$ and a partial normal subgroup $\mathcal{N}$ of $\mathcal{L}$. Set $T = S \cap \mathcal{N}$.

Lemma 4.2. *The following hold*:

(a) *T is strongly closed in $\mathcal{F}_S(\mathcal{L})$ and T is maximal in the poset of all p-subgroups of $\mathcal{N}$*;

(b) *If $P \in \Delta$ and $x \in \mathcal{N}$ with $P \leqslant S_x$, then $PT = P^x T$*;

(c) *If $T = 1$, then $N_{\mathcal{N}}(P, S) = C_{\mathcal{N}}(P)$ for all $P \in \Delta$.*

Proof. Let $x \in T$ and let $\phi \in \mathcal{F} := \mathcal{F}_S(\mathcal{L})$ such that x lies in the domain of ϕ. As ϕ is a composition of restrictions of conjugation maps between objects, it suffices, in proving (a), to consider only the case where $x\phi = x^f$ for some $f \in \mathcal{L}$; and in that case we have $x\phi \in \mathcal{N}$. Thus $x\phi \in S \cap \mathcal{N} = T$, and so T is strongly closed in $\mathcal{F}$. Now let R be a

p-subgroup of $\mathcal{N}$ containing T. By Proposition 2.22 (b) we may choose R with $R \leqslant S$, and then $R = T$. Thus (a) holds.

Next, let $g \in P \in \Delta$ and let $x \in N_{\mathcal{N}}(P, S)$. Then the word $w = (x^{-1}, g, x, g^{-1})$ is in $\mathbf{D}$ via P^x, and then $\Pi(w) = x^{-1} x^{g^{-1}} = g^x g^{-1}$. Thus $\Pi(w) \in \mathcal{N} \cap S = T$, and $g^x \in gT$. In particular, this proves (c), and it shows that $P^x \leqslant PT$. Upon replacing (P, x) with (P^x, x^{-1}), the same argument shows that $P \leqslant P^x T$, and this yields (b). □

Definition 4.3. Let $\mathcal{L} \circ \Delta$ be the set of all pairs $(f, P) \in \mathcal{L} \times \Delta$ such that $P \leqslant S_f$. Define a relation $\uparrow$ on $\mathcal{L} \circ \Delta$ by $(f, P) \uparrow (g, Q)$ if there exist elements $x \in N_{\mathcal{N}}(P, Q)$ and $y \in N_{\mathcal{N}}(P^f, Q^g)$ such that $xg = fy$.

This relation may be indicated by means of a commutative diagram

$$\begin{array}{ccc} Q & \xrightarrow{\;g\;} & Q^g \\ {\scriptstyle x}\uparrow & & \uparrow{\scriptstyle y} \\ P & \xrightarrow{\;f\;} & P^f \end{array} \qquad (*)$$

of conjugation maps, labeled by the conjugating elements, and in which the horizontal arrows are isomorphisms and the vertical arrows are injective homomorphisms. The relation $(f, P) \uparrow (g, Q)$ may also be expressed by

$$w := (x, g, y^{-1}, f^{-1}) \in \mathbf{D} \text{ via } P \quad \text{and} \quad \Pi(w) = \mathbf{1}.$$

It is easy to see that $\uparrow$ is a reflexive and transitive relation on $\mathcal{L} \circ \Delta$. We say that (f, P) is *maximal* in $\mathcal{L} \circ \Delta$ if $(f, P) \uparrow (g, Q)$ implies that $|P| = |Q|$. As S is finite there exist maximal elements in $\mathcal{L} \circ \Delta$. Since $(f, P) \uparrow (f, S_f)$ for $(f, P) \in \mathcal{L} \circ \Delta$, we have $P = S_f$ for every maximal element (f, P). For this reason, we introduce the following terminology.

Definition 4.4. Let $f \in \mathcal{L}$. Then f is $\uparrow$-*maximal* in $\mathcal{L}$ if (f, S_f) is maximal in $\mathcal{L} \circ \Delta$.

The following is the first main result of this section.

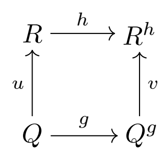
PROPOSITION 4.5. *Let $f \in \mathcal{L}$ and suppose that f is $\uparrow$-maximal. Then $T \leqslant S_f$.*

The proof requires two preliminary lemmas.

LEMMA 4.6. *Let $(g, Q), (h, R) \in \mathcal{L} \circ \Delta$ with $(g, Q) \uparrow (h, R)$ and suppose that $T \leqslant R$. Then there exists a unique $y \in \mathcal{N}$ with $g = yh$. Moreover,*

(a) *$y \in N_{\mathcal{N}}(Q, R)$ and $Q \leqslant S_{(y,h)}$;*
(b) *if $N_T(Q^g) \in \mathrm{Syl}_p(N_{\mathcal{N}}(Q^g))$, then $N_T(Q^y) \in \mathrm{Syl}_p(N_{\mathcal{N}}(Q^y))$.*

Proof. By the definition of $\uparrow$, there exist elements $u\in N_{\mathcal{N}}(Q,R)$ and $v\in N_{\mathcal{N}}(Q^g,R^h)$ such that $(u,h,v^{-1},g^{-1})\in\mathbf{D}$ via Q, and such that $\Pi(w)=\mathbf{1}$, as indicated in the diagram



In particular, $uh=gv$. Since $T\leqslant R$, points (a) and (b) of Lemma 4.2 yield

$$T=T^h, \quad Q^uT=QT\leqslant R, \quad \text{and} \quad Q^gT=Q^{gv}T\leqslant R^h.$$

Then

$$w:=(u,h,v^{-1},h^{-1})\in\mathbf{D} \quad \text{via} \quad (Q,Q^u,Q^{uh},Q^{uhv^{-1}}=Q^g,Q^{gh^{-1}}).$$

Set $y=\Pi(w)$. Then $y=u(v^{-1})^{h^{-1}}\in N_{\mathcal{N}}(Q,R)$. Since (u,h,v^{-1},h^{-1},h) and (g,v,v^{-1}) are in $\mathbf{D}$ (as $\mathcal{L}$ is a partial group), we get $yh=uhv^{-1}=g$. This yields (a), and the uniqueness of y is given by right cancellation.

Suppose now that $N_T(Q^g)\in\mathrm{Syl}_p(N_{\mathcal{N}}(Q^g))$. As $N_T(Q^y)^h=N_T(Q^g)$, it follows from Lemma 2.7 (b) that $N_T(Q^y)\in\mathrm{Syl}_p(N_{\mathcal{N}}(Q^y))$. □

LEMMA 4.7. *Suppose that f is $\uparrow$-maximal and let $y\in N_{\mathcal{N}}(S_f,S)$. Then*

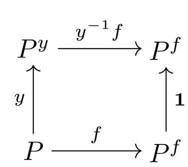
$$|T\cap S_f|=|T\cap(S_f)^y| \quad \textit{and} \quad (f,S_f)\uparrow(y^{-1}f,(S_f)^y).$$

In particular, $y^{-1}f$ is $\uparrow$-maximal.

Proof. Set $P=S_f$. Then $P^yT=PT$, by Lemma 4.2 (b). Thus

$$|P^y:P^y\cap T|=|P^yT:T|=|PT:T|=|P:P\cap T|,$$

and so $|T\cap P|=|T\cap P^y|$. The diagram



shows that $(f,P)\uparrow(y^{-1}f,P^y)$. □

Proof of Proposition 4.5. Let f be $\uparrow$-maximal. Set $P=S_f$ and $Q=P^f$, and suppose first that $N_T(P)\in\mathrm{Syl}_p(N_{\mathcal{N}}(P))$. Then $N_T(P)^f\in\mathrm{Syl}_p(N_{\mathcal{N}}(Q))$, by Lemma 2.7 (b), and there exists $x\in N_{\mathcal{N}}(Q)$ such that $N_T(Q)\leqslant(N_T(P)^f)^x$. Here $(f,x)\in\mathbf{D}$ via P, so $(N_T(P)^f)^x=N_T(P)^{fx}$, and then $(f,P)\uparrow(fx,N_T(P)P)$. As f is $\uparrow$-maximal, we conclude that $N_T(P)\leqslant P$, and hence $T\leqslant P$. Thus $T\leqslant S_f$ if $N_T(P)\in\mathrm{Syl}_p(N_{\mathcal{N}}(P))$. Assuming that f provides a counterexample to Proposition 4.5, we conclude that

(1) $N_T(P)\notin\mathrm{Syl}_p(N_{\mathcal{N}}(P))$.

Among all counterexamples to Proposition 4.5, choose f so that first $|P\cap T|$ and then $|P|$ are as large as possible. Choose $g\in N_{\mathcal{L}}(Q,S)$ so that Q^g is fully normalized in $\mathcal{F}_S(\mathcal{L})$, and set $h=fg$ and $R=P^h$. As $R=Q^g$ is fully normalized we have $N_S(R)\in\mathrm{Syl}_p(N_{\mathcal{L}}(R))$, and then $N_T(R)\in\mathrm{Syl}_p(N_{\mathcal{N}}(R))$. Let $(h^{-1},R)\uparrow(h',S_{h'})$, where h' is $\uparrow$-maximal, and set $R'=S_{h'}$ and $P'=(R')^{h'}$. Thus, there exist $y,z\in\mathcal{N}$ such that $yh'=h^{-1}z$, $R^y\leqslant R'$, and $P^z\leqslant P'$, as indicated in the diagram

$$\begin{array}{ccc} R' & \xrightarrow{h'} & P' \\ {\scriptstyle y}\uparrow & & \uparrow{\scriptstyle z} \\ R & \xrightarrow{h^{-1}} & P. \end{array}$$

Then $(T\cap R)^y\leqslant T\cap R'$.

Suppose that $T\nleqslant R'$. The conditions on the choice of f then yield $(T\cap R)^y=T\cap R'$ and $R^y=R'$. Since $N_T(R)\in\mathrm{Syl}_p(N_{\mathcal{N}}(R))$, we get $N_T(R)^y\in\mathrm{Syl}_p(N_{\mathcal{N}}(R'))$, and so there exists $x\in N_{\mathcal{N}}(R')$ such that $(N_T(R)^y)^x=N_T(R')$. Replacing y and h' with yx and $x^{-1}h'$, we then obtain $N_T(R)^y=N_T(R')$. But then $T\leqslant R'$ by (1), in any case, and then also $T\leqslant P'$.

Evidently $(h,P)\uparrow((h')^{-1},P')$, so by Lemma 4.6 there exists $\tilde{y}\in N_{\mathcal{N}}(P,S)$ such that $h=\tilde{y}(h')^{-1}$, $P\leqslant S_{(\tilde{y},(h')^{-1})}$, and $N_T(P^{\tilde{y}})\in\mathrm{Syl}_p(N_{\mathcal{N}}(P^{\tilde{y}}))$. Then Lemma 4.7 applies to (f,P) and $\tilde{y}$, and yields the result that $\tilde{y}^{-1}f$ is $\uparrow$-maximal and $S_{\tilde{y}^{-1}f}=P^{\tilde{y}}$. Thus (1) implies that $T\leqslant P^{\tilde{y}}$, and then also $T\leqslant P$. □

Recall from Definition 2.18 (c) that the partial group $N_{\mathcal{L}}(T)$ is a locality via the set Δ_T of objects $Q\in\Delta$ with $Q\leqslant T$.

COROLLARY 4.8. (Frattini lemma) *Let $\mathcal{L}=(\mathcal{L},\Delta,S)$ be a locality, let $\mathcal{N}$ be a partial normal subgroup of $\mathcal{L}$, and set $T=S\cap\mathcal{N}$. Then $\mathcal{L}=\mathcal{N}N_{\mathcal{L}}(T)$ as a product of partial subgroups of $\mathcal{L}$.*

Proof. Let $f\in\mathcal{L}$, set $P=S_f$, and choose $(g,Q)\in\mathcal{L}\circ\Delta$ so that $(f,P)\uparrow(g,Q)$ and so that g is $\uparrow$-maximal. By transitivity of $\uparrow$, we may take $Q=S_g$. Then $T\leqslant Q$ by Proposition 4.5, and then by Lemma 4.6 there exists $y\in N_{\mathcal{N}}(P,Q)$ with $f=yg$. Here $g\in N_{\mathcal{L}}(T)$ by Lemma 4.2 (a). □

LEMMA 4.9. *Let M be a finite group, let S be a Sylow p-subgroup of M, and let K be a normal subgroup of M. Set $\mathcal{F}=\mathcal{F}_S(M)$ and let Γ be a non-empty, overgroup closed, $\mathcal{F}$-invariant collection of subgroups of S. Let $\mathcal{L}:=\mathcal{L}_\Gamma(M)$ be the locality given by Example/Lemma 2.10, and let β be a rigid automorphism of $\mathcal{L}$. Assume that the following three conditions hold*:

(1) *$Q\cap K\in\Gamma$ for all $Q\in\Gamma$*;

(2) $C_M(O_p(M)) \leqslant O_p(M) \leqslant K$;

(3) Γ *is a set of* $\mathcal{F}$*-centric subgroups of* S.

Set $\Phi=\{Q\cap K \mid Q\in\Gamma\}$, *and set* $\mathcal{K}=\mathcal{L}_\Phi(K)$. *Then*

(a) β *restricts to a rigid automorphism* $\varkappa$ *of* $\mathcal{K}$, *and*

(b) β *extends to an automorphism of* M *if and only if* $\varkappa$ *extends to an automorphism of* K.

Proof. Set $Y=O_p(M)$ and let $P\in\Gamma$. Since $C_M(Y)\leqslant Y$ by (2), the Thompson $A\times B$ lemma [11, Theorem 5.3.4] implies that $C_M(P)$ is a p-group. Then $C_M(P)=C_{\mathcal{L}}(P)=Z(P)$, as P is $\mathcal{F}$-centric by (3). Point (a) then follows from Lemma 3.8. Further, Proposition 2.17 yields

(4) every $f\in\mathcal{L}$ is a product $\Pi(f_1, ..., f_n)$, where f_i is in a normalizer $N_{\mathcal{L}}(R_i)$ for some $R_i\in\Gamma$.

Let (M,β) be a counterexample to (b) with $|M|$ as small as possible. Let K_0 be the subgroup of K generated by the subset $\mathcal{K}$ of K. Let $g\in K\cap\mathcal{L}$. Then $S_g\in\Gamma$ and $S_g\cap K\in\Phi$, so $g\in\mathcal{K}$. Thus $K\cap\mathcal{L}\subseteq\mathcal{K}$. The reverse inclusion is given by the definition of $\mathcal{K}$, so $\mathcal{K}=K\cap\mathcal{L}$. Then $\mathcal{K}$ is a partial normal subgroup of $\mathcal{L}$ by Lemma 3.10. Set $T=S\cap K$ and observe that for any $h\in N_M(T)$ we have $(h^{-1}, g, h)\in\mathbf{D}$ via $(S_g\cap T)^h$, and hence $g^h\in\mathcal{K}$. Thus, $\mathcal{K}$ is $N_M(T)$-invariant, so also K_0 is $N_M(T)$-invariant.

Set $M_0=N_M(T)K_0$ and set $\mathcal{L}_0=\mathcal{L}_\Gamma(M_0)$. We next show

(5) $N_M(P)\leqslant M_0$ for all $P\in\Gamma$.

Among all P for which (5) fails to hold, choose P so that $|P|$ is as large as possible. Suppose that P is not fully normalized in $\mathcal{F}$, and let P' be a fully normalized $\mathcal{F}$-conjugate of P. Then Alperin's theorem yields a sequence $w=(g_1, ..., g_n)$ of elements of M and a sequence $(R_1, ..., R_n)$ of fully normalized $\mathcal{F}$-centric subgroups of S, such that $P_0:=P\leqslant R_1$, $P_i:=P^{g_1\cdots g_i}\leqslant R_i$ for all i, and $P'=P^{\Pi(w)}$. One may assume that n is minimal for these conditions, and hence $P_i\neq R_i$ for any i. The maximality of $|P|$ in the choice of P then yields $N_M(R_i)\leqslant M_0$ for all i, and hence $\Pi(w)\in M_0$. Without loss, then, we may assume that $P=P'$.

With P fully normalized in $\mathcal{F}$ we obtain $N_T(P)\in\mathrm{Syl}_p(N_K(P))$. As $N_K(P)\trianglelefteq N_M(P)$, the Frattini lemma yields

$$N_M(P)=N_K(P)(N_M(N_T(P))\cap N_M(P)). \qquad (*)$$

If $T\leqslant P$ then $(*)$ yields $N_M(P)\leqslant K_0N_M(T)=M_0$, contrary to the choice of P. Thus $T\nleqslant P$, and hence $N_T(P)\nleqslant P$. Set $Q=N_T(P)P$. Then $N_M(Q)\leqslant M_0$ by the maximality of $|P|$, and then $(*)$ again implies that $N_M(P)\leqslant M_0$. This completes the proof of (5). Now (4) yields $\mathcal{L}\subseteq M_0$. Thus,

(6) $\mathcal{L}_0=\mathcal{L}$.

Suppose next that K_0 is a proper subgroup of K. Then $K\cap M_0=N_K(T)K_0=K_0$, and so M_0 is a proper subgroup of M. Since $\mathcal{L}_0=\mathcal{L}$, the minimality of $|M|$ then yields an extension of β to an automorphism γ of M_0. The condition (2), together with Proposition 1.10 (c) then implies that $\gamma=c_z$ is conjugation by some $z\in Z(S)$. Since c_z is also an automorphism of M, we have an extension of β to an automorphism of M in this case, so we conclude that $K_0=K$.

Let $h,\bar{h}\in K$, let $x,\bar{x}\in N_M(T)$, and suppose that (h,x) and $(\bar{h},\bar{x})$ are in $\mathbf{D}(\mathcal{L})$ with $hx=\bar{h}\bar{x}$. Set $P=S_h\cap K$ and $\overline{P}=S_{\bar{h}}\cap K$, and set $Q=P^h$ and $\overline{Q}=\overline{P}^{\bar{h}}$. Then $(h,x,\bar{x}^{-1})\in\mathbf{D}$ via P, and $\Pi(h,x,\bar{x}^{-1})=\bar{h}$. It follows that $P=\overline{P}$ and that $(h^{-1},\bar{h})\in\mathbf{D}$ via Q. Then

$$\Pi(h^{-1}\beta,\bar{h}\beta)=(h^{-1}\bar{h})\beta=(x\bar{x}^{-1})\varkappa. \qquad (**)$$

By hypothesis, there exists an extension η of $\varkappa$ to an automorphism of K. It follows from $(**)$ that there is a well-defined mapping $\gamma\colon M\to M$ given by $\gamma\colon hx\mapsto(h\eta)(x\beta)$ for $h\in K$ and $x\in N_M(T)$.

In order to show that γ is a homomorphism, it suffices to show that $(h\eta)^{x\beta}=(h^x)\eta$ for all $h\in K$ and $x\in N_M(T)$. As $K_0=K$ we may write h as a product $\Pi_K(h_1,...,h_n)$ with $h_i\in\mathcal{K}$. Then

$$(h^x)\eta=(h_1^x\dots h_n^x)\eta=(h_1^x)\eta\dots(h_n^x)\eta=(h_1\eta)^{x\beta}\dots(h_n\eta)^{x\beta}=(h_1\eta\dots h_n\eta)^{x\beta}=(h\eta)^{x\beta},$$

as required.

We check that $\mathrm{Ker}(\gamma)=1$. Namely, if $(h\eta)(x\beta)=1$ with h and x as above, then $x\in N_K(T)$ and $x\beta=x\eta$, and then $hx=1$ as η is injective. Thus γ is injective, and is therefore an automorphism of M. □

5. Filtrations

Recall that for any partial group $\mathcal{M}$ and subgroups X and Y of $\mathcal{M}$, $N_{\mathcal{M}}(X,Y)$ is the set of all $f\in\mathcal{M}$ such that $X\subseteq\mathbf{D}(f)$ and $X^f\subseteq Y$. Write $\mathrm{Hom}_{\mathcal{M}}(X,Y)$ for the set of all conjugation maps $c_f\colon X\to Y$ with $f\in N_{\mathcal{M}}(X,Y)$.

Definition 5.1. Let S be a finite p-group, let $\mathcal{F}$ be a fusion system on S, and let Δ be a non-empty, $\mathcal{F}$-invariant collection of subgroups of S, closed with respect to overgroups in S. Let $\mathcal{L}$ be a partial group such that Δ is a set of subgroups of $\mathcal{L}$, and such that $\mathbf{D}(\mathcal{L})=\mathbf{D}_\Delta$ in the sense of (O1) in Definition 2.6. Then $\mathcal{L}$ is *$\mathcal{F}$-natural* if $\mathrm{Hom}_{\mathcal{L}}(P,Q)=\mathrm{Hom}_{\mathcal{F}}(P,Q)$ for all $P,Q\in\Delta$.

Definition 5.2. Let M be a finite group, let S be a Sylow p-subgroup of M, and set $\mathcal{F}=\mathcal{F}_S(M)$. Let Γ be an $\mathcal{F}$-invariant, overgroup-closed collection of subgroups of S. Let $\mathcal{L}=\mathcal{L}_\Gamma(M)$ be the locality given by Example/Lemma 2.10. (Equivalently, by Lemma 2.20 (b), $\mathcal{L}$ is the locality obtained by restricting the group M, itself viewed as a locality on the set of all subgroups of S, to the set Γ.) Let $\mathrm{Aut}_0(\mathcal{L})$ be the set of rigid automorphisms of $\mathcal{L}$, and let $\gamma,\gamma'\in\mathrm{Aut}_0(\mathcal{L})$. Then γ and γ' are *M-equivalent* if $\gamma^{-1}\circ\gamma'$ extends to an automorphism of the group M.

Notice that M-equivalence is in fact an equivalence relation on $\mathrm{Aut}_0(\mathcal{L})$.

Hypothesis 5.3. Assume given:

(1) a fusion system $\mathcal{F}$ on the finite p-group S;

(2) an $\mathcal{F}$-natural locality $(\mathcal{L},\Delta,S)$;

(3) a subgroup T of S, fully normalized in $\mathcal{F}$, and having the property that $\langle U,V\rangle\in\Delta$ for every pair of distinct $\mathcal{F}$-conjugates U and V of T;

(4) a finite group M such that $T\trianglelefteq M$, $N_S(T)\in\mathrm{Syl}_p(M)$, and $N_\mathcal{F}(T)=\mathcal{F}_{N_S(T)}(M)$; and

(5) a rigid isomorphism $\lambda\colon N_\mathcal{L}(T)\to\mathcal{L}_{\Delta_T}(M)$, where Δ_T is the set of all $P\in\Delta$ such that $T\trianglelefteq P$ (and where $\mathcal{L}_{\Delta_T}(M)$ is the locality given by Example/Lemma 2.10).

Hypothesis 5.3 will be assumed throughout the remainder of this section. The symbols $\mathbf{W}$, $\mathbf{D}$, and Π will always refer to $\mathcal{L}$, while Π_M denotes the multivariable product in the group M.

LEMMA 5.4. *Let U be an $\mathcal{F}$-conjugate of T. Then the following hold*:

(a) *$N_P(U)\in\Delta$ for every object $P\in\Delta$ such that $U\leqslant P$;*

(b) *There exists $x\in\mathcal{L}$ such that $T^x=U$ and such that $N_{S_x}(T)^x=N_S(U)$.*

Proof. (a) Let $P\in\Delta$ with $U\leqslant P$. If $U\trianglelefteq P$ then there is nothing to prove, while if U is not normal in P then $N_P(U)$ contains an $\mathcal{L}$-conjugate of $U^g\neq U$ of U, where $g\in N_P(N_P(U))$. As $\mathrm{Hom}_S(U,S)\subseteq\mathrm{Hom}_\mathcal{F}(U,S)$, U^g is an $\mathcal{F}$-conjugate of T, and then Hypothesis 5.3 (3) yields $\langle U,U^g\rangle\in\Delta$. As Δ is overgroup closed in S, we obtain (a).

As T is fully normalized in $\mathcal{F}$, there exists $\psi\in\mathrm{Hom}_\mathcal{F}(N_S(U),S)$ such that $U\psi=T$. Here $N_S(U)\in\Delta$ by (a). As $\mathcal{L}$ is $\mathcal{F}$-natural, ψ is given by conjugation by an element $x'\in\mathcal{L}$. Setting $x=(x')^{-1}$, (b) follows. □

Let Θ be the set of all triples

$$\theta=(x^{-1},g,y)\in\mathcal{L}\times M\times\mathcal{L}$$

which satisfy

(1) $T\leqslant S_x\cap S_y$, $N_{S_x}(T)^x=N_S(T^x)$, and $N_{S_y}(T)^y=N_S(T^y)$.

Define a relation $\sim_0$ on Θ as follows:

(2) $(x^{-1}, g, y) \sim_0 (\bar{x}^{-1}, \bar{g}, \bar{y})$ if

(i) $T^x = T^{\bar{x}}$, $T^y = T^{\bar{y}}$, and

(ii) $(\bar{x}x^{-1})\lambda \cdot g = \bar{g} \cdot (\bar{y}y^{-1})\lambda$ (as elements of M).

Notice that (2) (ii) makes sense. Namely, taking $U := T^x = T^{\bar{x}}$ (by (i)), we get $(\bar{x}, x^{-1}) \in \mathbf{D}$ via $(N_{S_{\bar{x}}}(T), N_S(U), N_{S_x}(T))$ by (1) and Lemma 5.4 (a), and hence we have $\bar{x}x^{-1} \in N_{\mathcal{L}}(T)$. Similarly, $\bar{y}y^{-1} \in N_{\mathcal{L}}(T)$.

One may depict the relation $\sim_0$ by means of a diagram, as follows:

$$\begin{array}{ccccccc}
U & \xrightarrow{x^{-1}} & T & \xrightarrow{g} & T & \xrightarrow{y} & V \\
\Big\| & & \Big\uparrow {\scriptstyle (\bar{x}x^{-1})\lambda} & & \Big\uparrow {\scriptstyle (\bar{y}y^{-1})\lambda} & & \Big\| \\
U & \xrightarrow[\bar{x}^{-1}]{} & T & \xrightarrow[\bar{g}]{} & T & \xrightarrow[\bar{y}]{} & V,
\end{array}$$

where $V = T^y = T^{\bar{y}}$. As $\mathcal{L}$ is $\mathcal{F}$-natural, the conjugation maps

$$c_{x^{-1}}: S_{x^{-1}} \longrightarrow S \quad \text{and} \quad c_{y^{-1}}: S_{y^{-1}} \longrightarrow S$$

are in $\mathcal{F}$, and thus U and V are $\mathcal{F}$-conjugates of T.

Lemma 5.5. *$\sim_0$ is an equivalence relation on Θ.*

Proof. It is evident that $\sim_0$ is reflexive and symmetric. Let $\theta_i = (x_i^{-1}, g_i, y_i) \in \Theta$, $1 \leqslant i \leqslant 3$, with $\theta_1 \sim_0 \theta_2 \sim_0 \theta_3$. Then $T^{x_1} = T^{x_3}$ and $T^{y_1} = T^{y_3}$. Notice that

$$(x_3, x_2^{-1}, x_2, x_1^{-1}) \in \mathbf{D} \quad \text{via } N_S(U)^{x_3^{-1}}$$

and

$$(y_3, y_2^{-1}, y_2, y_1^{-1}) \in \mathbf{D} \quad \text{via } N_S(V)^{y_3^{-1}}.$$

Then

$$\begin{aligned}
(x_3x_1^{-1})\lambda \cdot g_1 &= (x_3x_2^{-1})\lambda \cdot (x_2x_1^{-1})\lambda \cdot g_1 = (x_3x_2^{-1})\lambda \cdot g_2 \cdot (y_2y_1^{-1})\lambda \\
&= g_3 \cdot (y_3y_2^{-1})\lambda \cdot (y_2y_1^{-1})\lambda = g_3 \cdot (y_3y_1^{-1})\lambda,
\end{aligned}$$

which completes the proof. □

Define a relation $\vdash$ from $\mathcal{L}$ to Θ, by taking $f \vdash (x^{-1}, g, y)$ if

$$g \in \mathrm{Im}(\lambda), \quad (x^{-1}, g\lambda^{-1}, y) \in \mathbf{D}, \quad \text{and } f = \Pi(x^{-1}, g\lambda^{-1}, y).$$

Let $\sim_1$ be the symmetrization of $\vdash$, and let $\approx$ be the weakest equivalence relation on $\mathcal{L} \cup \Theta$ containing the union of $\sim_0$ and $\sim_1$. The $\approx$-class C of an element $\theta = (x^{-1}, g, y)$ of Θ may be denoted $[x^{-1}, g, y]$.

The following lemma is immediate from the definition of $\sim_0$.

Lemma 5.6. *Let Σ be a $\sim_0$-equivalence class in Θ, let $\theta=(x^{-1},g,y)\in\Sigma$, and set $U=T^x$ and $V=T^y$. Then the pair (U,V) depends only on Σ, and not on the choice of representative θ.*

Lemma 5.7. *Let $f\in\mathcal{L}$, and suppose that S_f contains an $\mathcal{F}$-conjugate U of T. Set $V=U^f$, and let $\Xi=\Xi(f,U,V)$ be the set of all $\theta\in\Theta$ such that $U=T^x$, $V=T^y$ and $f\sim_1\theta$. Then the following hold*:

(a) Ξ *is a* $\sim_0$*-class of* Θ;

(b) *If* $\bar{f}\in\mathcal{L}$ *and* $\bar{f}\sim_1\theta$ *for some* $\theta\in\Xi$, *then* $f=\bar{f}$.

Proof. As T is fully normalized in $\mathcal{F}$, and since $\mathcal{L}$ is $\mathcal{F}$-natural, there exist elements $x,y\in\mathcal{L}$ such that both $N_S(U)^{x^{-1}}$ and $N_S(V)^{y^{-1}}$ are contained in $N_S(T)$, and such that $T^x=U$ and $T^y=V$. Then $(x,f,y^{-1})\in\mathbf{D}$ via $N_{S_f}(U)^{x^{-1}}$, and the product $h=xfy^{-1}$ is an element of $N_{\mathcal{L}}(T)$. Set $g=h\lambda$. Then $(x^{-1},g,y)\in\Gamma$.

Set $E=N_{S_f}(U)$, and set $A=E^{x^{-1}}$, $B=A^g$, and $F=E^f$. Then $B=F^{y^{-1}}$, and each of E, A, B and F is in Δ by Lemma 5.4 (a). Let Σ be the $\sim_0$-class containing θ, and let $\bar{\theta}=(\bar{x}^{-1},\bar{g},\bar{y})\in\Delta$. Then $U=T^{\bar{x}}$ and $V=T^{\overline{Y}}$, by Lemma 5.6. By the definition of Θ, we have $E\leqslant S_{\bar{x}^{-1}}$ and the group $\bar{A}:=E^{\bar{x}^{-1}}$ is contained in $N_S(T)$. Similarly, $F\leqslant S_{\bar{y}^{-1}}$ and $\overline{B}:=F^{\bar{y}^{-1}}\leqslant N_S(T)$. These facts, together with the rigidity of λ, result in a sequence of conjugation maps between objects in Δ, in which the conjugating elements are as indicated in the following diagram:

$$E\xrightarrow{x^{-1}}A\xrightarrow{x\bar{x}^{-1}}\bar{A}\xrightarrow{(\bar{x}x^{-1})\lambda}A\xrightarrow{g}B\xrightarrow{(y\bar{y}^{-1})\lambda}\overline{B}\xrightarrow{\bar{y}y^{-1}}B\xrightarrow{y}F. \qquad (*)$$

As $\theta\sim_0\bar{\theta}$, we have

$$(\bar{x}x^{-1})\lambda\cdot g\cdot(y\bar{y}^{-1})\lambda=\bar{g},$$

and thus $(*)$ yields

$$E\xrightarrow{\bar{x}^{-1}}\bar{A}\xrightarrow{\bar{g}}\overline{B}\xrightarrow{\bar{y}}F.$$

As $\bar{A}$ and $\overline{B}$ are in Δ, it follows that $\bar{g}=\bar{h}\lambda$ for some $\bar{h}\in N_{\mathcal{L}}(T)$. But also

$$\begin{aligned}f&=\Pi(x^{-1},h,y)=\Pi(x^{-1},x,\bar{x}^{-1},\bar{x}x^{-1},h,y\bar{y}^{-1},\bar{y},y^{-1},y)=\Pi(\bar{x}^{-1},\bar{x}x^{-1},h,y\bar{y}^{-1},\bar{y})\\&=\Pi(\bar{x}^{-1},((\bar{x}x^{-1})\lambda\cdot g\cdot(y\bar{y}^{-1})\lambda)\lambda^{-1},\bar{y})=\Pi(\bar{x}^{-1},(\bar{g})\lambda^{-1},\bar{y})=\Pi(\bar{x}^{-1},\bar{h},\bar{y}).\end{aligned}$$

Thus $f\sim_1\bar{\theta}$, and (a) holds. If $\bar{f}\in\mathcal{L}$ with $\bar{f}\sim_1\theta$, then $f=\Pi(x^{-1},h,y)=\bar{f}$, and we have (b). □

Let $\mathcal{L}^+$ be the set $(\mathcal{L}\cup\Theta)/\approx$ of equivalence classes, and let $\mathcal{L}_0^+$ be the set of all $C\in\mathcal{L}^+$ such that $C\cap\Theta\neq\varnothing$.

LEMMA 5.8. *Let* $C \in \mathcal{L}^+$.
(a) *If* $C \cap \mathcal{L} = \varnothing$ *then* C *is a* $\sim_0$*-class in* Θ.
(b) *If* $C \cap \Theta = \varnothing$ *then* $C = \{f\}$ *for some* $f \in \mathcal{L}$.

Proof. This is immediate from the definitions of $\sim_0$ and $\sim_1$. □

LEMMA 5.9. (a) *Let* $f, g \in \mathcal{L}$. *Then* $f \approx g$ *if and only if* $f = g$.
(b) *If* $\theta \in \Theta$ *and* $f \in \mathcal{L}$ *with* $f \approx \theta$, *then* $f \sim_1 \theta$.

Proof. (a) As $f \approx g$, there is a sequence

$$(f = f_0, \theta_1, \bar{\theta}_1, f_1, ..., f_{n-1}, \theta_n, \bar{\theta}_n, f_n = g)$$

such that θ_i and $\bar{\theta}_i$ are in Θ, f_i is in $\mathcal{L}$, and with $f_{i-1} \sim_1 \theta_i \sim_0 \bar{\theta}_i \sim_1 f_i$. Then $f_{i-1} = f_i$ by Lemma 5.7 (b), and so $f = g$.

(b) Let $\theta \in \Theta$ and $f \in \mathcal{L}$ with $f \approx \theta$. Statement (a), together with Lemma 5.7 (a) and Lemma 5.6, then yields a sequence

$$f \sim_1 \theta_1 \sim_1 f \sim_1 ... \sim_1 f \sim_1 \theta_n = \theta,$$

and this proves (b). □

Define $\mathbf{D}_0^+$ to be the set of words $w = (C_1, ..., C_n) \in \mathbf{W}(\mathcal{L}_0^+)$ such that, for some choice of representatives (x_i^{-1}, g_i, y_i) of the classes C_i, the products $y_i x_{i+1}^{-1}$ are defined in $\mathcal{L}$ and lie in $N_{\mathcal{L}}(T)$, $1 \leqslant i < n$. For such a word w, and such a choice of representatives, set

$$w_0 = (g_1, (y_1 x_2^{-1})\lambda, g_2, ..., (y_{n-1} x_n^{-1})\lambda, g_n).$$

We now wish to define a mapping $\Pi_0^+ : \mathbf{D}_0^+ \to \mathcal{L}_0^+$ by taking

$$\Pi_0^+(w) = [x_1^{-1}, \Pi_M(w_0), y_n]. \qquad (*)$$

Of course, we will define $\Pi_0^+(\varnothing)$ to be $[\mathbf{1}, 1_M, \mathbf{1}]$.

LEMMA 5.10. *There is a well-defined mapping* $\Pi_0^+ : \mathbf{D}_0^+ \to \mathcal{L}_0^+$, *given by* $(*)$.

Proof. By induction on word-length, we need only show that Π_0^+ is well defined on words $w = (C_1, C_2) \in \mathbf{D}_0^+$ of length 2. Let $\mathcal{D}(w)$ be the set of all pairs $(\theta_1, \theta_2) \in C_1 \times C_2$, where $\theta_i = (x_i^{-1}, g_i, y_i)$, such that $(y_1, x_2^{-1}) \in \mathbf{D}$ and $T^{y_1} = T^{x_2}$. That is, $\mathcal{D}(w)$ is the set of all (θ_1, θ_2) for which it is possible to form a "product" as in $(*)$. The problem is to show that $[x_1^{-1}, g_1 \cdot (y_1 x_2^{-1})\lambda \cdot g_2, y_2]$ is independent of the choice of representatives $\theta_i \in C_i$.

Fix $(\theta_1, \theta_2) \in \mathcal{D}(w)$ and set $U_0 = T^{x_1}$, $U_1 = T^{y_1} = T^{x_2}$, and $U_2 = T^{y_2}$. Suppose first that $C_i \cap \mathcal{L} \neq \varnothing$ for both $i = 1$ and $i = 2$, and that $(f_1, f_2) \in \mathbf{D}$, where f_i is the unique element (see Lemma 5.8 (b)) of $C_i\cap\mathcal{L}$. Set $v=(f_1,f_2)$, set $E_1=N_{(S_v)^{f_1}}(U_1)$, and set $E_0=(E_1)^{f_1^{-1}}$ and $E_2=(E_1)^{f_2}$. Then $v\in\mathbf{D}$ via E_0. Using the definition of Θ, one observes that

$$A_1:=(E_0)^{x_1^{-1}}\leqslant N_S(T)\quad\text{and}\quad B_1:=(A_1)^{g_1}=(E_1)^{y_1^{-1}}.$$

Similarly, one has

$$A_2:=(E_1)^{x_2^{-1}}\leqslant N_S(T)\quad\text{and}\quad B_2:=(A_1)^{g_2}=(E_2)^{y_2^{-1}}.$$

Thus, the groups A_i and B_i are in Δ, and $g_1\cdot(y_1x_2^{-1})\lambda\cdot g_2\in\operatorname{Im}(\lambda)$. Setting $h_i=g_i\lambda^{-1}$, one then has $(h_1,y_1x_2^{-1},h_2)\in\mathbf{D}$ via A_1, and

$$(x_1^{-1},g_1\cdot(y_1x_2^{-1})\lambda\cdot g_2,y_2)\sim_1 f_1f_2.$$

The result is independent of the choice of $(\theta_1,\theta_2)\in\mathcal{D}(w)$, so the lemma holds in this case. Thus, we may assume that no such pair (f_1,f_2) exists.

We now aim to show that (U_0,U_1,U_2) is independent of the choice of $(\theta_1,\theta_2)\in\mathcal{D}(w)$. This is given by Lemmas 5.6 and 5.8 (a) if $C_i\cap\mathcal{L}=\varnothing$ for both $i=1$ and $i=2$. Suppose next that $C_1\cap\mathcal{L}=\varnothing\neq C_2\cap\mathcal{L}$. Here C_1 uniquely determines (U_0,U_1), and then $U_1=T^{x_2}$ since $y_1x_2^{-1}\in N_{\mathcal{L}}(T)$. Setting $U_2=(U_1)^{f_2}$ for $f_2\in C_2\cap\mathcal{L}$, it follows from Lemma 5.9 (a) that, again, (U_0,U_1,U_2) depends only on (C_1,C_2) and not on the choice of representatives. The next case, where $C_1\cap\mathcal{L}\neq\varnothing=C_2\cap\mathcal{L}$, evidently yields the same result. By assumption, we have $v=(f_1,f_2)\notin\mathbf{D}$, and so $S_v\notin\Delta$. There is then a unique $\mathcal{L}$-conjugate U_0 of T with $U_0\leqslant S_v$. Set $U_1=(U_0)^{f_1}$ and $U_2=(U_1)^{f_2}$. Then each U_i is an $\mathcal{L}$-conjugate of T; and (U_0,U_1,U_2) is uniquely determined by (C_1,C_2), as desired.

Let $(\bar\theta_1,\bar\theta_2)\in\mathcal{D}(w)$, with $\bar\theta_i=(\bar x_i^{-1},\bar g_i,\bar y_i)$. The result of the preceding paragraph, taken with Lemmas 5.7 (a) and 5.8 (a) then yields $\theta_i\sim_0\bar\theta_i$. The definition of $\sim_0$ then yields the commutative diagram

$$\begin{array}{ccccccccccccc}
U_0 & \xrightarrow{x_1^{-1}} & T & \xrightarrow{g_1} & T & \xrightarrow{y_1} & U_1 & \xrightarrow{x_2^{-1}} & T & \xrightarrow{g_2} & T & \xrightarrow{y_2} & U_2\\
\Vert & & \downarrow{\scriptstyle r_1} & & \downarrow{\scriptstyle h_1} & & \Vert & & \downarrow{\scriptstyle r_2} & & \downarrow{\scriptstyle h_2} & & \Vert\\
U_0 & \xrightarrow[\bar x_1^{-1}]{} & T & \xrightarrow[\bar g_1]{} & T & \xrightarrow[\bar y_1]{} & U_1 & \xrightarrow[\bar x_2^{-1}]{} & T & \xrightarrow[\bar g_2]{} & T & \xrightarrow[\bar y_2]{} & U_2.
\end{array}$$

(Here $r_i=(x_i\bar x_i^{-1})\lambda$ and $h_i=(y_i\bar y_i^{-1})\lambda$.) The "middle" portion of this diagram leads at once to a commutative diagram in M as follows:

$$\begin{array}{ccccccc}
T & \xrightarrow{g_1} & T & \xrightarrow{(y_1x_2^{-1})\lambda} & T & \xrightarrow{g_2} & T\\
\downarrow{\scriptstyle r_1} & & \downarrow{\scriptstyle h_1} & & \downarrow{\scriptstyle g_2} & & \downarrow{\scriptstyle h_2}\\
T & \xrightarrow[\bar g_1]{} & T & \xrightarrow[(\bar y_1\bar x_2^{-1})\lambda]{} & T & \xrightarrow[\bar g_2]{} & T.
\end{array}$$

The result is a diagram

$$\begin{array}{ccccccc}
U_0 & \xrightarrow{x_1^{-1}} & T & \xrightarrow{\Pi_M(w_0)} & T & \xrightarrow{y_2} & U_2 \\
\Vert & & \downarrow{\scriptstyle r_1} & & \downarrow{\scriptstyle h_2} & & \Vert \\
U_0 & \xrightarrow[\bar{x}_1^{-1}]{} & T & \xrightarrow[\Pi_M(\bar{w}_0)]{} & T & \xrightarrow[\bar{y}_2]{} & U_2
\end{array}$$

which establishes that Π_0^+ is well defined. □

Let $u=(f_1, ..., f_n)\in\mathbf{W}$ and let $v=(C_1, ..., C_n)\in\mathbf{W}(\mathcal{L}_0^+)$. We shall write $u\approx v$ to indicate that $f_i\in C_i$ for all i. Write $\mathbf{D}_0^+\cap\mathbf{D}$ for the set of all $v\in\mathbf{D}_0^+$ such that there exists $u\in\mathbf{D}$ with $u\approx v$.

LEMMA 5.11. *Π_0^+ and Π agree on $\mathbf{D}_0^+\cap\mathbf{D}$.*

Proof. Let $w=(C_1, ..., C_n)\approx(f_1, ..., f_n)$ be in $\mathbf{D}_0^+\cap\mathbf{D}$, and let $\theta_i=(x_i^{-1}, g_i, y_i)\in C_i$ be chosen so that $f_i\sim_1\theta_i$. Let $(U_0, ..., U_n)$ be the sequence of $\mathcal{L}$-conjugates of T given by $T^{x_i}=U_{i-1}$ and $T^{y_i}=U_i$. As $C_i\approx f_i$, we have $g_i\in\mathrm{Im}(\lambda)$, and $(x_i^{-1}, (g_i)\lambda^{-1}, y_i)$ is in $\mathbf{D}$ via a subgroup of $N_{S_{g_i}}(U_{i-1})$. This shows that when w is viewed as an element of $\mathbf{D}$, one has $U_0\leqslant S_w$. Setting $P_0=N_{S_w}(U_0)$, we get $P_0\in\Delta$ by Lemma 5.4 (a), and $w\in\mathbf{D}$ via P_0.

Set $h_i=(g_i)\lambda^{-1}$, and set

$$v=(x_1^{-1}, h_1, y_1, ..., x_n^{-1}, h_n, y_n) \quad\text{and}\quad v_0=(h_1, y_1x_2^{-1}, ..., y_{n-1}x_n^{-1}, h_n).$$

Set $P_i=P_0^{f_1\cdots f_i}$. Then Lemma 5.4 (a) implies that $v\in\mathbf{D}$ via P_0, since each P_{i-1} is a member of Δ contained in $S_{f_i}\cap N_S(T)$. Then also $v_0\in\mathbf{D}$ via $(P_0)^{x_1^{-1}}$, and since also $v_0\in\mathbf{W}(N_{\mathcal{L}}(T))$, the isomorphism $\lambda: N_{\mathcal{L}}(T)\to\mathcal{L}_{\Delta_T}(M)$ sends $\Pi(v_0)$ to $\Pi_M(w_0)$, where

$$w_0=(g_1, \lambda(y_1x_2^{-1}), ..., \lambda(y_{n-1}x_n^{-1}), g_n).$$

We now obtain

$$\Pi_0^+(C_1, ..., C_n)=[x_1^{-1}, \Pi_M(w_0), y_n]\approx\Pi(f_1, ..., f_n)$$

since $(x_1^{-1}, \Pi(v_0), y_n)\in\mathbf{D}$ (via P_0). This yields the lemma. □

By Lemmas 5.8 and 5.9 (a), we may identify $\mathcal{L}_0^+$ and $\mathcal{L}$ with their images in $\mathcal{L}^+$ via $\approx$. Set $\mathcal{L}_0=\mathcal{L}_0^+\cap\mathcal{L}$ and $\mathcal{L}_1=\mathcal{L}\setminus\mathcal{L}_0$. Thus, $\mathcal{L}^+$ is the disjoint union of $\mathcal{L}_0^+$ and $\mathcal{L}_1$. Set $\mathbf{D}^+=\mathbf{D}_0^+\cup\mathbf{D}$. By Lemma 5.11, there is a "product"

$$\Pi^+=\Pi_0^+\cup\Pi: \mathbf{D}^+\longrightarrow\mathcal{L}^+$$

whose restriction to $\mathbf{D}_0^+$ is Π_0^+, and whose restriction to $\mathbf{D}$ is Π. Set $\mathbf{1}^+ = [\mathbf{1}, 1_M, \mathbf{1}]$ (or equivalently, via $\approx$, $\mathbf{1}^+ = \mathbf{1}$). We now define an "inversion map" on $\mathcal{L}_0^+$ by

$$[x^{-1}, f, y] \longmapsto [y^{-1}, f^{-1}, x].$$

(That this is indeed a well-defined involutory bijection of $\mathcal{L}_0^+$ will be shown in the proof of Lemma 5.12, immediately below.) We extend the inversion on $\mathcal{L}_0^+$ to all of $\mathcal{L}^+$ in the obvious way, by forming the union with the inversion on $\mathcal{L}$.

LEMMA 5.12. *$\mathcal{L}^+$, with the above product, identity element and inversion, is a partial group.*

Proof. That $\mathbf{D}^+$ contains $\mathcal{L}^+$ as words of length 1 is immediate from the definition, as is the fact that $u \circ v \in \mathbf{D}^+$ implies $u, v \in \mathbf{D}^+$. Thus Definition 2.1 (1) holds for $\mathcal{L}^+$. That Π^+ restricts to the identity map on $\mathcal{L}^+$, and that Π^+ is multiplicative is immediate from the definition of Π^+, and so points (2) and (3) of Definition 2.1 hold for $\mathcal{L}^+$. It remains to check that inversion is well defined and that $\mathcal{L}^+$ satisfies Definition 2.1 (4).

Let $\theta = (x^{-1}, g, y) \in \Theta$ and set $\theta^{-1} = (y^{-1}, g^{-1}, x)$. The conditions on x (immediately following Lemma 5.4) which define θ as being in Θ are that $T \leqslant S_x$ and that $T^x \leqslant S$ and $N_{S_x}(T)^x = N_S(T^x)$; and these are the same conditions on x that are required in order that θ^{-1} be in Θ. The analogous set of conditions applies to y, by symmetry, so we obtain $\theta^{-1} \in \Theta$. Now let $\bar{\theta} = (\bar{x}^{-1}, \bar{f}, \bar{y}) \in \Theta$, with $\theta \sim_0 \bar{\theta}$. Thus $T^x = T^{\bar{x}}$, $T^y = T^{\bar{y}}$, and

$$(\bar{x}x^{-1})\lambda \cdot g = \bar{g} \cdot (\bar{y}y^{-1})\lambda. \tag{$*$}$$

The condition $(*)$ concerns multiplication in the group M, where inversion and straightforward manipulation yields

$$(\bar{y}y^{-1})\lambda \cdot g^{-1} = \bar{g}^{-1} \cdot (\bar{x}x^{-1})\lambda.$$

This shows that $\theta^{-1} \sim_0 \bar{\theta}^{-1}$. Now suppose that $f \in \mathcal{L}$ and that $\theta \sim_1 f$. This means that $g \in \mathrm{Im}(\lambda)$, $(x^{-1}, g\lambda^{-1}, y) \in \mathbf{D}$, and $f = \Pi(x^{-1}, g\lambda^{-1}, y)$. Then $f^{-1} = \Pi(y^{-1}, g^{-1}\lambda^{-1}, x)$ by Lemma 2.2 (f), and so $\theta^{-1} \sim_1 f^{-1}$. This completes the verification that there is a well-defined mapping $[x^{-1}, g, y] \mapsto [y^{-1}, g^{-1}, x]$ on $\approx$-classes, and which agrees with the inversion map from $\mathcal{L}$ on $\mathcal{L}_0^+ \cap \mathcal{L}$. Evidently, this inversion map on $\mathcal{L}^+$ is then an involutory bijection. One readily verifies that $u^{-1} \circ u \in \mathbf{D}^+$ if $u \in \mathbf{D}^+$, and that then $\Pi^+(u^{-1} \circ u) = \mathbf{1}^+$. Thus Definition 2.1 (4) holds for $\mathcal{L}^+$, and $\mathcal{L}^+$ is a partial group. □

Let Δ^+ be the union of Δ and the set of subgroups P of S such that P contains an $\mathcal{F}$-conjugate of T. We now have a candidate, in the partial group $\mathcal{L}^+$, for a locality whose set of objects is Δ^+. In order to establish the conditions (O1) and (O2) for objectivity in Lemma 2.5, it must be shown that if A is an object in Δ^+, and $C\in\mathcal{L}^+$ with $A^C\leqslant S$, then either $C\in\mathcal{L}$ or else C is of the form $[x^{-1},g,y]$ with $T^x=A$ and $T^y=A^C$. This is not immediate, since the statement "$A^C\leqslant S$" merely says that for each $a\in A$, we have $(C^{-1},a,C)\in\mathbf{D}^+$ and $a^C\in S$. The following lemma addresses this issue.

LEMMA 5.13. *Let $\theta=(x^{-1},g,y)\in\Theta$, let C be the $\approx$-class of θ, and let A be an $\mathcal{F}$-conjugate of T such that $A^C\leqslant S$.*

(a) *If $\theta\approx f\in\mathcal{L}$, then $A\leqslant S_f$.*

(b) *If $C\cap\mathcal{L}=\varnothing$, then $A=T^x$ and $A^C=T^y$.*

Proof. (a) Let $a\in A$ and set $b=a^C$. Set $P=S_f$ and $Q=P^f$. Then $(a^{-1},f,b)\in\mathbf{D}$ via (P^a,P,Q,Q^b), since a and b are elements of S, and similarly (a,f) and (f,b) are in $\mathbf{D}$.

Set $w=(C^{-1},a,C)$. As $w\in\mathbf{D}^+$, we have also $(a,C)\in\mathbf{D}^+$ and the axioms for a partial group (Definition 2.1) yield

$$\Pi^+(a,C)=\Pi^+((C)\circ w)=\Pi^+(C,\Pi^+(w))=\Pi^+(C,b).$$

(In particular, $(C,b)\in\mathbf{D}^+$.) Since

$$\Pi^+(C^{-1},a,C)=b=\Pi(f^{-1},f,b)=\Pi^+(C^{-1},C,b)$$

by Lemma 5.11, left cancellation and Lemma 5.11 again yield $\Pi(a,f)=\Pi(f,b)$. Then

$$f=\Pi(a^{-1},a,f)=\Pi(a^{-1},f,b),$$

by Lemma 2.2 (e). Since conjugation by $a^{-1}fb$ carries P^a to Q^b, we conclude that $P^a\leqslant S_f$. That is, $P^a=P$, and then $(f^{-1},a,f)\in\mathbf{D}$ (via Q). Apply Lemma 5.11 to get $a^f=b$ in the partial group $\mathcal{L}$. Thus $a\in S_f$, and (a) holds.

(b) Here C is a $\sim_0$-class by Lemma 5.8 (a). Set $U=T^x$ and $V=T^y$, and recall that U and V depend only on C, by Lemma 5.6. Again let $a\in A$ and set $b=a^C$. As $(C^{-1},a,C)\in\mathbf{D}^+$, it follows that $(C^{-1},a,C)\in\mathbf{D}^+$ via (V,U,U,V), and thus $a\in N_S(U)$. Similarly, $b\in N_S(V)$. Then, by (1) in the definition of Θ, both $a^{x^{-1}}$ and $b^{y^{-1}}$ are in $N_S(T)$.

Set $w=(a^{-1}x^{-1},\mathbf{1},xa^2)\in\mathbf{W}(\mathcal{L})$, and observe that $w\in\mathbf{D}$ via $N_S(U)$, and that also $w\in\mathbf{D}^+$ via (U,T,T,U). In particular, we get $a\sim_1 D:=[a^{-1}x^{-1},1_M,xa^2]$. The representatives (y^{-1},g^{-1},x) of C^{-1}, $(a^{-1}x^{-1},1_M,xa^2)$ of D, and (x^{-1},g,y) of C have the property that $\Pi^+(C^{-1},D,C)$ may be given in terms of these representatives. That is, we have

$$\Pi^+(C^{-1},a,C)=\Pi^+(C^{-1},D,C)=[y^{-1},g^{-1}(xax^{-1})g,y]=[y^{-1},(a^{x^{-1}})^g,y],$$

since the products $x(a^{-1}x^{-1})$ and $(xa^2)x^{-1}$ are defined in $\mathcal{L}$ and lie in $N_{\mathcal{L}}(T)$. As $a^C = b$, we then have

$$b \approx (y^{-1}, (a^{x^{-1}})^g, y).$$

The reader may verify that $y \sim_1 (y, 1_M, \mathbf{1})$ and that

$$v := ((\mathbf{1}, 1_M, y), (y^{-1}, (a^{x^{-1}})^g, y), (y^{-1}, 1_M, \mathbf{1})) \in \mathbf{D}^+,$$

and further that

$$\Pi^+(v) = [\mathbf{1}, (a^{x^{-1}})^g, \mathbf{1}].$$

Then, appealing to Lemma 5.9 (b), we get

$$(\mathbf{1}, b^{y^{-1}}, \mathbf{1}) \sim_1 b^{y^{-1}} \sim_1 (\mathbf{1}, (a^{x^{-1}})^g, \mathbf{1}).$$

Now $(\mathbf{1}, b^{y^{-1}}, \mathbf{1}) \sim_0 (\mathbf{1}, (a^{x^{-1}})^g, \mathbf{1})$ by Lemma 5.7 (a). A glance at the diagram following the definition of $\sim_0$ will now convince the reader that $b^{y^{-1}} = (a^{x^{-1}})^g$. Thus $(A^{x^{-1}})^g = B^{y^{-1}}$, and we obtain a sequence of conjugation maps (between subgroups of S) as follows:

$$UA \xrightarrow{x^{-1}} TA^{x^{-1}} \xrightarrow{g} TB \xrightarrow{y\bar{y}^1} VB.$$

Since $C \notin \mathcal{L}$, it follows that $UA \notin \Delta$, whence $A = U$ and $B = V$. This completes the proof of (b). □

The following two theorems, along with proposition Proposition 1.10 above, form the foundation for our proof of the main theorem.

Theorem 5.14. *Assume Hypothesis 5.3 and let Δ^+ be the union of Δ and the set of subgroups P of S such that P contains an $\mathcal{F}$-conjugate of T. Then $(\mathcal{L}^+, \Delta^+, S)$ is an $\mathcal{F}$-natural locality. Moreover,*

(a) *the isomorphism $\lambda: N_{\mathcal{L}}(T) \to \mathcal{L}_{\Delta_T}(M)$ extends in a unique way to an isomorphism $\lambda^+: N_{\mathcal{L}^+}(T) \to M$ of groups, such that*

$$[x^{-1}, g, y]\lambda^+ = \Pi_M(x^{-1}\lambda, g, y\lambda) \tag{$*$}$$

for any $x, y \in N_{\mathcal{L}}(T)$;

(b) *if $\mathcal{L}$ is a Δ-linking system and $C_M(T) \leqslant T$, then $\mathcal{L}^+$ is a Δ^+-linking system.*

Proof. Let $w = (C_1, ..., C_n) \in \mathbf{W}(\mathcal{L}^+)$ and $(U_0, ..., U_n) \in \mathbf{W}(\Delta^+)$, with $U_{i-1}^{C_i} = U_i$ for all i, $1 \leqslant i \leqslant n$. Suppose first that U_0 (and hence each U_i) is an $\mathcal{L}$-conjugate of T. Then Lemma 5.13 implies that $w \in \mathbf{D}_0^+$. On the other hand, if $U_0 \in \Delta$ then each U_i is in Δ, and

$w \in \mathbf{D}$. Thus, either way, we get $w \in \mathbf{D}^+$, so that $(\mathcal{L}^+, \Delta^+)$ satisfies condition (O1) in the Definition 2.6 of objectivity.

We next check that $\mathcal{L}^+$ is $\mathcal{F}$-natural. For P and Q in Δ we have $\mathrm{Hom}_{\mathcal{L}^+}(P,Q)=\mathrm{Hom}_{\mathcal{L}}(P,Q)$ by construction, and therefore $\mathrm{Hom}_{\mathcal{L}^+}(P,Q)=\mathrm{Hom}_{\mathcal{F}}(P,Q)$ in this case. Now let U and V be $\mathcal{L}$-conjugates of T, and let $C \in N_{\mathcal{L}^+}(U,V)$. If $C \approx f$ for some $f \in \mathcal{L}$ then $U^f=V$; and since $c_f\colon S_f \to S$ is an $\mathcal{F}$-homomorphism we conclude that $c_C\colon U \to V$ is an $\mathcal{F}$-homomorphism. On the other hand, suppose that $C \in \mathcal{L}^+ \setminus \mathcal{L}$, and let $(x^{-1},g,y) \in C$. Then Lemma 5.13 yields $T^x=U$ and $T^y=V$. Here $\mathcal{F}_{N_S(T)}(M)=N_{\mathcal{F}}(T)$ by hypothesis, so each of $c_x, c_{g\lambda^{-1}}$, and c_y is an $\mathcal{F}$-homomorphism, and then so is c_C. Thus $\mathrm{Hom}_{\mathcal{L}^+}(U,V) \subseteq \mathcal{F}$, and $\mathcal{L}^+$ is $\mathcal{F}$-natural. Then also $\mathcal{L}^+$ satisfies condition (O2) for objectivity, and so $(\mathcal{L}^+, \Delta^+)$ is an objective partial group.

By definition, S is the unique maximal member of Δ^+. Since $N_{\mathcal{L}^+}(S)=N_{\mathcal{L}}(S)$, it follows that also S is maximal in the poset of finite p-subgroups of $\mathcal{L}^+$, and so $\mathcal{L}^+$ satisfies conditions (L1) and (L2) in Definition 2.9. As $\mathcal{L}$ and M are finite, so is the set Θ of triples (x^{-1},g,y), and thus $\mathcal{L}^+$ is a locality.

Set $H=N_{\mathcal{L}^+}(T)$ and $K=N_{\mathcal{L}}(T)$. Let $C=[x^{-1},g,y] \in H$. Then $x,y \in K$, and we have $C=[\mathbf{1},(x^{-1}\lambda)g(y\lambda),\mathbf{1}]$. Because of this, it is now readily verified that there is a well-defined mapping $\lambda^+\colon H \to M$ given by $C \mapsto (x^{-1}\lambda)g(y\lambda)$, that λ^+ coincides with λ on K, and that λ^+ is a homomorphism of groups. Suppose that $C \in \mathrm{Ker}(\lambda^+)$. Then $(x^{-1}\lambda)g(y\lambda)=1_M$, so $g=(xy^{-1})\lambda \in \mathrm{Im}(\lambda)$, and then $(x^{-1},g^{-1},y) \sim_1 \mathbf{1}$. Thus $\mathrm{Ker}(\lambda^+)=\mathbf{1}$, and λ^+ is then an isomorphism since M is finite. The uniqueness of λ^+ is immediate from condition $(*)$, so (a) holds.

Suppose next that $\mathcal{L}$ is a Δ-linking system and that $C_M(T) \leqslant T$. Let $U \in \Delta^+$ with $U \notin \Delta$. Then there is a unique $\mathcal{F}$-conjugate U_0 of T contained in U, and hence we have $N_{\mathcal{L}^+}(U) \leqslant N_{\mathcal{L}^+}(U_0)$. As $\mathcal{L}^+$ is $\mathcal{F}$-natural, there exists $C \in \mathcal{L}^+$ such that $(U_0)^C=T$. Conjugation by C induces an isomorphism of $N_{\mathcal{L}^+}(U_0)$ with $N_{\mathcal{L}^+}(T)$, and hence with M. Since $C_M(X) \leqslant X$ for any p-subgroup X of M containing T, it follows that $C_{\mathcal{L}^+}(U) \leqslant U$. By construction, $N_{\mathcal{L}^+}(P)=N_{\mathcal{L}}(P)$ for $P \in \Delta$, so we conclude that $\mathcal{L}^+$ is a Δ^+-linking system. Thus (b) holds. □

We also write $\mathcal{L}^+(\lambda)$ for the locality constructed by Theorem 5.14, in order to emphasize its dependence on the isomorphism $\lambda\colon N_{\mathcal{L}}(T) \to \mathcal{L}_{\Delta_T}(M)$. In the same vein, we may write $\mathcal{L}_0^+(\lambda)$ for the partial subgroup $\mathcal{L}_0^+$ of $\mathcal{L}^+$.

THEOREM 5.15. *Assume Hypothesis 5.3, and let Δ^+ be the union of Δ and the set of all subgroups of S which contain an $\mathcal{F}$-conjugate of T.*

(a) *Let $\mathcal{L}^*$ be a locality via the set Δ^+ of objects, let $\mathcal{L}'$ be the restriction of $\mathcal{L}^*$ to Δ, let $\beta\colon \mathcal{L} \to \mathcal{L}'$ be a rigid isomorphism, and let $\beta_T\colon N_{\mathcal{L}}(T) \to N_{\mathcal{L}'}(T)$ be the isomorphism* *induced by restriction of* β. *Assume that there is given an isomorphism* $\mu\colon M\to N_{\mathcal{L}^*}(T)$ *of groups, such that* μ *restricts to the identity map on* $N_S(T)$. *Let* μ_0 *be the restriction of* μ *to* $\mathcal{L}_{\Delta_T}(M)$, *and set* $\lambda=\beta_T\circ\mu_0^{-1}$. *Then there exists a unique isomorphism* $\beta^+\colon\mathcal{L}^+(\lambda)\to\mathcal{L}^*$ *such that* β^+ *restricts to* β *on* $\mathcal{L}$ *and to* $\lambda^+\circ\mu$ *on* $N_{\mathcal{L}^+(\lambda)}(T)$.

(b) *Let* $(\mathcal{L},\Delta,S)$ *and* $(\mathcal{L}',\Delta,S)$ *be localities having the same set* Δ *of objects, let* $\beta\colon\mathcal{L}\to\mathcal{L}'$ *be a rigid isomorphism, and let* $\beta_T\colon N_{\mathcal{L}}(T)\to N_{\mathcal{L}'}(T)$ *be the rigid isomorphism given by restriction of* β. *Further, let* $\lambda\colon N_{\mathcal{L}}(T)\to\mathcal{L}_{\Delta_T}(M)$ *and* $\lambda'\colon N_{\mathcal{L}'}(T)\to\mathcal{L}_{\Delta_T}(M)$ *be rigid isomorphisms, and let* μ_0 *be the automorphism* $\lambda^{-1}\circ\beta_T\circ\lambda'$ *of* $\mathcal{L}_{\Delta_T}(M)$.

(i) *There exists an isomorphism* $\beta^+\colon\mathcal{L}^+(\lambda)\to(\mathcal{L}')^+(\lambda')$ *extending* β *if and only if* μ_0 *extends to an automorphism* μ *of* M.

(ii) *Let* μ *be an extension of* μ_0 *to* M. *Then there is a unique isomorphism*

$$\beta^+\colon\mathcal{L}^+(\lambda)\longrightarrow(\mathcal{L}')^+(\lambda')$$

having the property that β^+ *restricts to* β *on* $\mathcal{L}$ *and to* $\lambda^+\circ\mu\circ((\lambda')^+)^{-1}$ *on* $N_{\mathcal{L}^+}(T)$. *Moreover,* β^+ *is then given explicitly on* $\mathcal{L}_0^+(\lambda)$ *by*

$$[x^{-1},g,y]\longmapsto[x^{-1}\beta,g\mu,y\beta]$$

for $(x^{-1},g,y)\in\Theta$.

(c) *Suppose that there exists a rigid isomorphism* $N_{\mathcal{L}}(T)\to\mathcal{L}_{\Delta_T}(M)$, *and that all rigid automorphisms of* $\mathcal{L}_{\Delta_T}(M)$ *are* M*-equivalent* (*as in Definition* 5.2). *Then, up to rigid isomorphism, there exists a unique locality* $(\mathcal{L}^*,\Delta^+,S)$ *whose restriction to* Δ *is* $\mathcal{L}$, *and having the property that* $N_{\mathcal{L}^*}(T)$ *is equal to* M.

We now prove the points (b), (a) and (c) of Theorem 5.15, in this order.

Proof of Theorem 5.15 (b). Set $N=N_{\mathcal{L}}(T)$ and $N'=N_{\mathcal{L}'}(T)$. Let Θ' be the subset of $\mathcal{L}'\times M\times\mathcal{L}'$ defined by the conditions immediately following Lemma 5.4, but now with respect to λ'. In order to avoid confusion, we shall distinguish the relations $\sim_0$, $\sim_1$ and $\approx$ in the two cases, by the following sort of notational device. Thus, for example, if $\phi=(x^{-1},g,y)$ and $\bar\phi=(\bar x^{-1},\bar g,\bar y)$ are in Θ', then we shall write

$$\phi\sim_0\bar\phi\quad(\text{rel }\lambda')$$

to indicate that the products $\bar x x^{-1}$ and $\bar y y^{-1}$ exist, and are elements of N', and satisfy the condition

$$(\bar x x^{-1})\lambda'\cdot g=\bar g\cdot(\bar y y^{-1})\lambda'.$$

The expressions "$f\sim_1\theta$ (rel λ')", and "$f\approx\theta$ (rel λ')" should be understood similarly. The relations $\sim_0$, $\sim_1$, and $\approx$ on Θ will be provided with a corresponding "rel λ" in order to lend emphasis to this distinction.

Set $\mu_0=\lambda^{-1}\circ\beta_T\circ\lambda'$ and assume that μ_0 extends to an automorphism μ of M. Let α be the mapping on Θ given by

$$\alpha\colon(x^{-1},g,y)\longmapsto((x\beta)^{-1},g\mu,y\beta).$$

Since β is rigid, $T\beta=T$, and from this it is easily verified that $\mathrm{Im}(\alpha)\subseteq\Theta'$. There is an obvious inverse to α, so in fact α is a bijection $\Theta\to\Theta'$.

We claim that α sends $\sim_0$-classes (rel λ) to $\sim_0$-classes (rel λ'). Namely, let θ and $\bar\theta$ be elements of Θ, written in the usual way, and assume that $\theta\sim_0\bar\theta$ (rel λ). Setting $a=\bar x x^{-1}$ and $b=\bar y y^{-1}$, we then have $(a\lambda)\cdot g=\bar g\cdot(b\lambda)$ in M. Applying μ, we obtain

$$(a\beta)\lambda'\cdot g\mu=\bar g\mu\cdot(b\beta)\lambda',$$

and thus $((x\beta)^{-1},g\mu,y\beta)\sim_0((\bar x\beta)^{-1},\bar g\mu,\bar y\beta)$ (rel λ'). This proves the claim.

Now let $f\in\mathcal{L}$, and suppose that $f\sim_1\theta$ (rel λ). That is, suppose that there exists $h\in N$ with $g=h\lambda$, that $(x^{-1},h,y)\in\mathbf{D}$, and that $f=\Pi(x^{-1},h,y)$. Then $((x\beta)^{-1},h\beta,y\beta)\in\mathbf{D}'$ and $f\beta=\Pi'((x\beta)^{-1},h\beta,y\beta)$. Since $(h\beta)\lambda'=g\mu$, we conclude that α sends $\sim_1$-classes (rel λ) into $\sim_1$-classes (rel λ'). By Lemmas 5.8 and 5.9, any $\approx$-class C relative to λ is either a $\sim_0$-class or is of the form $\{f\}\cup X$, where X is the set of all $\theta\in\Theta$ such that $f\sim_1\theta$. Since the same is true for $\approx$-classes relative to λ', we conclude that α respects the $\approx$-relations relative to λ and λ'.

Set $\mathcal{L}^+=\mathcal{L}^+(\lambda)$ and $(\mathcal{L}')^+=(\mathcal{L}')^+(\lambda')$, and similarly define $\mathcal{L}_0^+$ and $(\mathcal{L}_0')^+$. Thus α induces a mapping

$$\begin{aligned}\gamma\colon\mathcal{L}_0^+&\longrightarrow(\mathcal{L}_0')^+,\\ [x^{-1},g,y]&\longmapsto[(x\beta)^{-1},g\mu,y\beta],\end{aligned}\tag{$*$}$$

and $f\gamma=f\beta$ for all $f\in\mathcal{L}$ such that $f\sim_1\theta$ for some $\theta\in\Theta$. Since $\mathcal{L}^+$ is the union of $\mathcal{L}$ and $\mathcal{L}_0^+$, α extends to a mapping

$$\gamma\colon\mathcal{L}^+\longrightarrow(\mathcal{L}')^+,$$

which restricts to the identity map on $\mathcal{L}$. Further, since all of the above arguments can be carried out with α^{-1} in place of α, γ is a bijection.

We next show that γ is a homomorphism of partial groups. Set $\mathbf{D}(\lambda)=\mathbf{D}((\mathcal{L}')^+(\lambda))$, and similarly define $\mathbf{D}(\lambda')$. Let Π^+ and $(\Pi')^+$ be the corresponding products. Let $w=(C_1,...,C_n)\in\mathbf{D}(\lambda)$ and set $Q=S_w$. If Q contains no $\mathcal{L}$-conjugate of T, then $C_i\in\mathcal{L}$ for all i, $w\gamma^*=w\beta^*$, and hence $(\Pi')^+(w\gamma^*)=(\Pi^+(w))\gamma$. On the other hand, suppose that Q contains an $\mathcal{L}$-conjugate of T. Then $C_i=[x_i^{-1},g_i,y_i]$ for some $(x_i^{-1},g_i,y_i)\in\Theta$, and $\Pi_{\lambda'}^+(w\gamma^*)=[(x_1\beta)^{-1},\Pi_M(u_0),y_n\beta]$, where

$$u_0=(g_1\mu,(y_1x_2^{-1})\beta\lambda',...,(y_{n-1}x_n^{-1})\beta\lambda',g_n\mu).$$

One observes that $u_0 = w_0\mu^*$, where

$$w_0 = (g_1, (y_1 x_2^{-1})\lambda, ..., (y_{n-1}x_n^{-1})\lambda, g_n),$$

and hence $(\Pi')^+(w\gamma^*) = (\Pi^+(w))\gamma$, and γ is a homomorphism. Since the roles of $(\mathcal{L}, \lambda)$ and $(\mathcal{L}', \lambda')$ can be reversed, γ is then an isomorphism, and γ is rigid since β is rigid.

Set $N^+ = N_{\mathcal{L}^+}(T)$ and $(N')^+ = N_{(\mathcal{L}')^+(\lambda')}(T)$, and suppose that there is given an isomorphism $\sigma: \mathcal{L}^+(\lambda) \to (\mathcal{L}')^+(\lambda')$ such that σ restricts to β on $\mathcal{L}$. Let $\sigma_T: N^+ \to (N')^+$ be the isomorphism induced by σ. Then μ_0 extends to the automorphism

$$\nu = (\lambda^+)^{-1} \circ \sigma_T \circ (\lambda')^+$$

of M, and this completes the proof of (b) (i). In order to obtain (b) (ii), one need only observe that the formula $(*)$ defines the unique mapping $\mathcal{L}^+(\lambda)_0 \to \mathcal{L}^+(\lambda')_0$ which, in union with β, defines a homomorphism $\beta^+: \mathcal{L}^+ \to (\mathcal{L}')^+$ which restricts to β on $\mathcal{L}$ and to $\lambda^+ \circ \mu \circ ((\lambda')^+)^{-1}$ on N^+. □

Proof of Theorem 5.15 (a). Let $\lambda': N_{\mathcal{L}'}(T) \to \mathcal{L}_{\Delta_T}(M)$ be the restriction of μ^{-1} to $N_{\mathcal{L}'}(T)$. To simplify the notation, we shall write $\mathcal{L}(\lambda)$ for $\mathcal{L}^+(\lambda)$, and $\mathcal{L}'(\lambda')$ for $(\mathcal{L}')^+(\lambda')$. Set $\mathcal{L}_T = N_{\mathcal{L}}(T)$ and $\mathcal{L}(\lambda)_T = N_{\mathcal{L}(\lambda)}(T)$, and similarly define $\mathcal{L}'_T$ and $\mathcal{L}'(\lambda')_T$. Also, set $\mathcal{L}^*_T = N_{\mathcal{L}^*}(T)$, and let $\mu_0: \mathcal{L}_{\Delta_T}(M) \to \mathcal{L}'_T$ be the restriction of μ.

Suppose first that Theorem 5.15 (a) holds in the case where $\mathcal{L} = \mathcal{L}'$ and where β is the identity map on $\mathcal{L}$. We shall show that Theorem 5.15 (a) then holds in generality. Thus, taking $(\mathcal{L}', \lambda')$ in the role of $(\mathcal{L}, \lambda)$, the assumed special case of Theorem 5.15 (a) yields an isomorphism $\phi: \mathcal{L}'(\lambda') \to \mathcal{L}^*$ such that ϕ restricts to the identity on $\mathcal{L}'$ and to $(\lambda')^+ \circ \mu$ on $\mathcal{L}'(\lambda)_T$.

Since $\lambda = \beta_T \circ \mu_0^{-1}$ by hypothesis, we have

$$\lambda^{-1} \circ \beta_T \circ \lambda' = \mu_0 \circ \beta_T^{-1} \circ \beta_T \circ \mu_0^{-1} = \mathrm{id}_{\mathcal{L}_{\Delta_T}}(M).$$

Thus $\lambda^{-1} \circ \beta_T \circ \lambda'$ extends to the identity automorphism of M. By Theorem 5.15 (b) (proved above) there then exists an isomorphism $\gamma: \mathcal{L}(\lambda) \to \mathcal{L}'(\lambda')$ whose restriction to $\mathcal{L}$ is β, and whose restriction to $\mathcal{L}(\lambda)_T \to \mathcal{L}'(\lambda')$ is $\lambda^+ \circ ((\lambda')^+)^{-1}$. Now set $\beta^+ = \gamma \circ \phi$. Then β^+ restricts to $\beta: \mathcal{L} \to \mathcal{L}'$, and on $\mathcal{L}(\lambda)_T$ to

$$(\lambda^+ \circ ((\lambda')^+)^{-1}) \circ ((\lambda')^+ \circ \mu) = \lambda^+ \circ \mu.$$

Thus, β^+ fulfills the requirements of the statement of Theorem 5.15 (a). The uniqueness of β^+ subject to the given conditions follows in the usual way (for example, as in the proof of uniqueness in Theorem 5.15 (b)), and is omitted.

We assume for the remainder of the proof that $\mathcal{L}=\mathcal{L}'$ and that β is the identity automorphism of $\mathcal{L}$. Then also β_T is the identity automorphism of $\mathcal{L}_T$, and $\lambda^{-1}=\mu_0$.

Let $\theta=(x^{-1},g,y)\in\Theta$. Then $(x^{-1},g\mu,y)\in\mathbf{D}^*$ via the object T^x, and one checks that the product $\Pi^*(x^{-1},g\mu,y)$ in $\mathcal{L}^*$ remains unchanged when θ is replaced by any $\bar{\theta}$ such that $\theta\sim_0\bar{\theta}$. In some detail: let $\bar{\theta}=(\bar{x}^{-1},\bar{g},\bar{y})$ with $\theta\sim_0\bar{\theta}$. Then $T^x=T^{\bar{x}}$ by Lemma 5.6. Since $\mu_0=\lambda^{-1}$, we get

$$\Pi^*(x^{-1},g\mu,y)=\Pi^*(\bar{x}^{-1},\bar{x}x^{-1},g\mu,y\bar{y}^{-1},\bar{y})=\Pi^*(\bar{x}^{-1},\bar{g},\bar{y}).$$

Suppose next that $f\in\mathcal{L}$ with $f\sim_1\theta$. Then $g=h\lambda$ for some $h\in N_{\mathcal{L}}(T)$, $(x^{-1},h,y)\in\mathbf{D}$, and f is equal to the product $x^{-1}hy$ in $\mathcal{L}$. This yields

$$\Pi^*(x^{-1},g\mu,y)=\Pi^*(x^{-1},h,y)=f,$$

since β is the identity automorphism of $\mathcal{L}$. Now Lemma 5.9 (b) implies that $\Pi^*(x^{-1},g\mu,y)$ depends only on the $\approx$-class of θ, and we have a well-defined mapping $\gamma_0\colon\mathcal{L}_0^+\to\mathcal{L}^*$ given by

$$\gamma_0\colon[x^{-1},g,y]\longrightarrow\Pi^*(x^{-1},g\mu,y)$$

and which restricts to the identity map on $\mathcal{L}_0^+\cap\mathcal{L}$. One also observes that

$$[x^{-1},g,y]\lambda^+=\Pi_M(x^{-1}\lambda,g,y\lambda)$$

for $[x^{-1},g,y]\in N_{\mathcal{L}^+}(T)$.

We now define the mapping $\gamma\colon\mathcal{L}^+\to\mathcal{L}^*$ to be the union of the identity map on $\mathcal{L}$ and γ_0. Notice that if $C=[x^{-1},g,y]\in N_{\mathcal{L}^+}(T)$, then

$$C\lambda^+=(x^{-1}\lambda\cdot g\cdot y\lambda)\quad\text{and}\quad(x^{-1}\lambda\cdot g\cdot y\lambda)\mu=\Pi^*(x^{-1},g\mu,y).$$

Thus the restriction of γ to $N_{\mathcal{L}^+}(T)$ is $\lambda^+\circ\mu$.

We next show the following fact:

(1) Let $C\in\mathcal{L}^+$, set $f^*=C\gamma$, and let $a\in S$. Then

$$a^C\in S\quad\Longleftrightarrow\quad a^{f^*}\in S\ \text{and}\ a^C=a^{f^*}.$$

The proof is as follows. First, let $C=[x^{-1},g,y]\in\mathcal{L}_0^+\setminus\mathcal{L}$, and let (U,V) be the pair of $\mathcal{F}$-conjugates of T, uniquely determined by C, such that $U=T^x$ and $V=T^y$. Then $U^{x^{-1}}=T=T^g=V^{y^{-1}}$. For any $a\in U$ we then get

$$a^C=((a^{x^{-1}})^g)^y=(((a^{x^{-1}})^g)\mu)^y=((a^{x^{-1}})^{g\mu})^y=a^{C\gamma}\tag{$*$}$$

as $C\gamma=\Pi^*(x^{-1}, g\mu, y)$, where the second and third equalities follow since μ is rigid.

More generally, we find that $(*)$ holds for $C\in\mathcal{L}_0^+$ and any $a\in S_C$. Namely, if $C\cap\mathcal{L}=\{f\}$ and $a\in S_f$, then $a=a\gamma$ and $f=f\gamma$, and hence $a^C=a^f=(a^f)\gamma=a^{f\gamma}=a^{C\gamma}$.

One observes that (1) and $(*)$ may be read "in reverse". Namely, if $f^*\in\mathcal{L}^*$, and $f^*=C\gamma$ with $C\in\mathcal{L}_0^+\setminus\mathcal{L}$, then $a^{f^*}=a^C$ for any $a\in S$ such that $a^{f^*}\in S$. Finally, in the case where $C\in\mathcal{L}^+$ with $C\cap\mathcal{L}\neq\varnothing$, there is really nothing to show since $S\leqslant\mathcal{L}$. Thus (1) holds.

We may now show that γ is a homomorphism of partial groups. Namely, let γ^* be the map $\mathbf{W}(\mathcal{L}^+)\to\mathbf{W}(\mathcal{L}^*)$ induced by γ. Let $w\in\mathbf{D}^+$, set $X=S_w$, and suppose first that $X\in\Delta$. Then $w\in\mathbf{D}(\mathcal{L})$ via X, and $w=w\gamma^*$. On the other hand, suppose that $X\notin\Delta$. Then $w=(C_1, ..., C_n)$, where $C_i\in\mathcal{L}_0^+$, and there is a uniquely determined sequence $(U_0, ..., U_n)$ of $\mathcal{F}$-conjugates of T such that $U_{i-1}^{C_i}=U_i$ for all i from 1 to n. Set $f_i^*=C_i\gamma$ and set $w^*=(f_1^*, ..., f_n^*)$. Then (1) yields $U_{i-1}^{f_i^*}=U_i$ for all i, and thus $w^*\in\mathbf{D}^*$. The verification that

$$\Pi^*(w^*)=(\Pi^+(w))\gamma,$$

and thus that γ is a homomorphism, is now a formality. We treat the case $n=2$ in detail; and for this it suffices to consider the case where $w=(C_1, C_2)\in\mathbf{D}_0^+$, since the restriction of γ to $\mathcal{L}$ is the identity map. Let $\theta_i=(x_i^{-1}, g_i, y_i)\in C_i$, with $(y_1, x_2^{-1})\in\mathbf{D}$ and with $y_1x_2^{-1}\in N_{\mathcal{L}}(T)$. Then $\Pi^+(w)=[x_1^{-1}, g_1\cdot(y_1x_2^{-1})\lambda\cdot g_2, y_2]$, and

$$\begin{aligned}\Pi^+(w)\gamma&=\Pi^*(x_1^{-1}, (g_1\cdot(y_1x_2^{-1})\lambda\cdot g_2)\mu, y_2)=\Pi^*(x_1^{-1}, g_1\mu\cdot(y_1x_2^{-1})\lambda\mu\cdot g_2\mu, y_2)\\&=\Pi^*(x_1^{-1}, g_1\mu\cdot y_1x_2^{-1}\cdot g_2\mu, y_2)=\Pi^*(x_1^{-1}, g_1\mu, y_1, x_2^{-1}, g_2\mu, y_2),\end{aligned}$$

where the last equality is obtained by observing that (1) implies that

$$(x_1^{-1}, g_1\mu, y_1, x_2^{-1}, g_2\mu, y_2)\in\mathbf{D}^*.$$

Now

$$\Pi^*(x_1^{-1}, g_1\mu, y_1, x_2^{-1}, g_2\mu, y_2)=\Pi^*(x_1^{-1}(g_1\mu)y_1, x_2^{-1}(g_2\mu)y_2)=\Pi^*(w^*).$$

The case $w=(C_1, ..., C_n)$ with $n>2$ differs in no essential way from the case $n=2$, so the above argument establishes that γ is a homomorphism.

The next step will be to show the following fact:

(2) For each $P, Q\in\Delta^+$, the mapping $\gamma_{P,Q}: N_{\mathcal{L}^+}(P, Q)\to N_{\mathcal{L}^*}(P, Q)$ is surjective.

Of course, we may assume that $N_{\mathcal{L}^*}(P, Q)$ is non-empty. If $P\in\Delta$ then $Q\in\Delta$ and $\gamma_{P,Q}=\beta_{P,Q}$ is bijective. So assume that $P\notin\Delta$. Then P contains a unique $\mathcal{L}$-conjugate U of T, and hence $U\trianglelefteq P$. Let $f^*\in N_{\mathcal{L}^*}(P, Q)$ and set $V=U^{f^*}$. By Lemma 5.4 (b), there exist elements $x, y\in\mathcal{L}$ with $T^x=U$ and $T^y=V$, and such that $N_{S_x}(T)^x=N_S(U)$ and $N_{S_y}(T)^y=N_S(V)$. Then $(x, f^*, y^{-1})\in\mathbf{D}^*$ via T, and we set $h^*=\Pi^*(x, f^*, y^{-1})$. Then

$(x^{-1}, x, f^*, y^{-1}, y) \in \mathbf{D}^*$ via U, so $f^* = \Pi^*(x^{-1}, h^*, y)$ by Definition 2.1 (3). Moreover, we have $h^* \in N_{\mathcal{L}^*}(T)$. Now $[x^{-1}, h^*\mu^{-1}, y]$ is mapped to f^* by γ, and so (2) holds.

Next, we show

(3) $\mathrm{Ker}(\gamma) = \mathbf{1}$.

As $\gamma|_{\mathcal{L}} = \mathrm{id}$, the set of non-identity elements of $\mathrm{Ker}(\gamma)$ is contained in $\mathcal{L}_0^+$. Let $C = [x^{-1}, g, y] \in \mathrm{Ker}(\gamma)$, and let U and V be the conjugates of T associated with C by Lemma 5.6. Then $\mathbf{1}^* = C\gamma = \Pi^*(x^{-1}, g\mu, y)$, and (1) implies that $U = V = T$. Hence

$$C = [\mathbf{1}, x^{-1}\lambda \cdot g \cdot y\lambda, \mathbf{1}] \in N_{\mathcal{L}^+}(T).$$

Since μ is injective, we conclude that $C = \mathbf{1}^+$, and thus (3) holds.

Since $S_h \in \Delta$ for all $h \in \mathcal{L}'$, it is immediate from (2) that γ is surjective, and from (1) that $S_f = S_{f\gamma}$ for all $f \in \mathcal{L}$. Then (3) and Lemma 3.6 imply that γ is an isomorphism, and hence a rigid isomorphism by construction. Thus, it remains only to establish the uniqueness of γ, subject to the conditions that $\gamma|_{\mathcal{L}} = \mathrm{id}_{\mathcal{L}}$ and that $\gamma|_{N_{\mathcal{L}}(T)} = \lambda^+ \circ \mu$. Let $\gamma' \colon \mathcal{L}^+ \to \mathcal{L}^*$ be another such isomorphism. Then, for any $C = [x^{-1}, g, y] \in \mathcal{L}_0^+$, we get

$$C\lambda' = ([x^{-1}, 1_M, \mathbf{1}][\mathbf{1}, g, \mathbf{1}][\mathbf{1}, 1_M, y])\lambda' = \Pi^*(x^{-1}, g\mu, y) = C\lambda,$$

and this completes the proof. □

Proof of Theorem 5.15 (c). Assuming that there exists a rigid isomorphism

$$N_{\mathcal{L}}(T) \longrightarrow \mathcal{L}_{\Delta_T}(M),$$

Theorem 5.14 yields the existence of a locality $(\mathcal{L}^*, \Delta^+, S)$ with the required properties, and Lemma 5.5 (b) implies that $\mathcal{L}^*$ is rigidly isomorphic to some $\mathcal{L}^+(\lambda)$. Assuming further that all rigid automorphisms of $\mathcal{L}_{\Delta_T}(M)$ are M-equivalent, Theorem 5.15 (b) yields the uniqueness of $\mathcal{L}^*$ up to rigid isomorphism. That is, Theorem 5.15 (c) holds, and the proof is complete. □

This completes the proof of Theorem 5.15.

It will be convenient, for the applications in the next two sections, to state a corollary concerning a special case of Theorem 5.15.

COROLLARY 5.16. *Assume Hypothesis* 5.3, *and let* $(\mathcal{L}^*, \Delta^+, S)$ *be a locality such that the restriction of* $\mathcal{L}^*$ *to* Δ *is equal to* $\mathcal{L}$, *and with* $N_{\mathcal{L}^*}(T) = M$. *Let* β *be a rigid automorphism of* $\mathcal{L}$ *and let* $\lambda := \beta_T$ *be the automorphism of* $N_{\mathcal{L}}(T)$ *given by restricting* β.

(a) *There exists a unique rigid isomorphism* $\alpha \colon \mathcal{L}^+(\lambda) \to \mathcal{L}^*$ *such that* β *is the restriction of* α *to* $\mathcal{L}$, *and such that* λ^+ *is the restriction of* α *to* $N_{\mathcal{L}^+(\lambda)}(T)$.

(b) β *extends to an automorphism of* $\mathcal{L}^*$ *if and only if* λ *extends to an automorphism of* M.

Proof. The locality $\mathcal{L}^*$ is rigidly isomorphic to a locality of the form $\mathcal{L}^+(\lambda)$, by Theorem 5.15 (a) (and with μ being the identity automorphism of M). Then also Theorem 5.15 (a) yields an isomorphism $\alpha: \mathcal{L}^+(\lambda) \to \mathcal{L}^*$ with the properties required in point (a). Point (b) is then given by Theorem 5.15 (b). □

Definition 5.17. Let $(\mathcal{L}, \Delta, S)$ be a locality with fusion system $\mathcal{F} = \mathcal{F}_S(\mathcal{L})$. Let $\{\Delta_i\}_{i=0}^N$ be a sequence of subsets of Δ, and let $\{R_i\}_{i=0}^N$ be a sequence of subgroups of S, such that each R_i is fully normalized in $\mathcal{F}$. Then $(\Delta_i, R_i)_{i=0}^N$ is an *$\mathcal{F}$-filtration of* Δ if the following conditions hold:

(1) R_0 is weakly closed in $\mathcal{F}$, and Δ_0 is the set of overgroups of R_0 in S;

(2) For $i > 0$, Δ_i is the union of Δ_{i-1} and the set $\mathcal{R}_i$ of subgroups of S which contain an $\mathcal{F}$-conjugate of R_i;

(3) For any $U, V \in \mathcal{R}_i$, $\langle U, V \rangle \in \mathcal{R}_{i-1}$ if and only if $U \neq V$;

(4) $\Delta = \Delta_N$.

Lemma 5.18. *Let $(\mathcal{L}, \Delta, S)$ be a locality, let $\mathcal{F} = \mathcal{F}_S(\mathcal{L})$ be its fusion system, and let $R \in \Delta$ be weakly closed in $\mathcal{F}$. Then there exists an $\mathcal{F}$-filtration $\mathbf{F} = (\Delta_i, R_i)_{i=0}^N$ for $\mathcal{L}$, such that $R_0 = R$, and such that, for all $i > 0$, R_i is of maximal order among subgroups of S in $\Delta_i \setminus \Delta_{i-1}$.*

Proof. Take $R_0 = R$ and let Δ_0 be the set of all overgroups of R in S. Then Δ_0 is $\mathcal{F}$-invariant as R is weakly closed in $\mathcal{F}$. Now let m be an index with $1 \leqslant m \leqslant N$. Suppose that Δ_{m-1} has been given, and that Δ_{m-1} is $\mathcal{F}$-invariant and overgroup closed in S, and that $\Delta_{m-1} \neq \Delta$. Choose $Q \in \Delta \setminus \Delta_{m-1}$ so that $|Q|$ is as large as possible, and let R_m be a fully normalized $\mathcal{F}$-conjugate of Q. Define Δ_m to be the union of Δ_{m-1} and the set of subgroups of S which contain an $\mathcal{F}$-conjugate of R_m. Then Δ_m is $\mathcal{F}$-invariant and overgroup closed, so the process may be iterated until arriving at an index N with $\Delta_N = \Delta$. The points (1), (2), and (4) of Definition 5.17 are given at once by this construction, while point (3) is immediate from the maximality in the choice of R_m. □

6. The reduction to FF-pairs

Our aim in this section is to establish the main theorem, modulo a result (Proposition 6.10) on localities in finite groups. In order to state that result (to be proved in the following section), we begin by reviewing some notions from finite group theory.

For any finite p-group P, the set of elements $z \in Z(P)$ such that $z^p = 1$ is a characteristic subgroup of P, often denoted $\Omega_1(Z(P))$, but which we shall write as Z_P. A p-group V is *elementary abelian* if $V = Z_V$. Equivalently, V is elementary abelian if V is the underlying group of a finite-dimensional vector space $\widetilde{V}$ over the field $\mathbb{F}_p$ of p elements.

Let A be a group, and let D be a group on which A acts (from the right). Then $[D,A]$ is by definition the subgroup of D generated by the set of commutators $[g,a]=g^{-1}a^{-1}ga$, with $g\in D$ and $a\in A$. If $[D,A]\leqslant C_D(A)$, one says that A *acts quadratically* on D, and one also expresses this condition by writing $[D,A,A]=1$.

We begin with two elementary (and well-known) results.

LEMMA 6.1. *Let D be an abelian p-group admitting action* (*from the right*) *by a group X, and set $V=Z_D$. Let $\mathcal{A}$ be the set of elements $a\in X$ of order p such that $[D,a,a]=1$, and suppose that $G=\langle\mathcal{A}\rangle$. Then $[D,G]\leqslant V$.*

Proof. Let $g\in D$, let $a\in\mathcal{A}$, and set $h=[g,a]$. Then $h\in C_D(a)$, and so $h=h^{a^k}$ for all integers k. Thus,

$$h=g^{-1}g^a=(g^a)^{-1}g^{a^2}=...=(g^{a^{p-1}})^{-1}g^{a^p}.$$

One then observes that $h^p=g^{-1}g^{a^p}$, and then $h^p=1$ since $a^p=1$.

As D is abelian, we have $[g,a]^{-1}=[g^{-1},a]$, and $[g_1g_2,a]=[g_1,a][g_2,a]$ for all $g_1,g_2\in D$. Thus

$$V\geqslant\{[g,a]\mid g\in D \text{ and } a\in\mathcal{A}\}=[D,G]. \qquad \square$$

LEMMA 6.2. *Let X be a finite group, let S be a Sylow p-subgroup of X, and let D be an abelian p-group. Assume that there is given an action $X\to\operatorname{Aut}(D)$ of X on D. Set $V=Z_D$, and set*

$$W=[V,O^p(X)]C_V(X)/C_V(X).$$

Then the following hold:

(a) $C_D(S)\leqslant[D,O^p(X)]C_D(X)$.

(b) *Suppose that $X=KS$, where $K\trianglelefteq X$ is generated by elements that act quadratically on D. Then $C_D(S)=C_V(S)C_D(X)$.*

(c) *Suppose that $[C_W(S),X]=1$. Then $[C_D(S),X]=1$.*

Proof. As $X=O^p(X)S$, there exists a right transversal $\{x_1,...,x_r\}$ for S in X such that each x_i is in $O^p(X)$. Thus $\Omega=\{Sx_1,...,Sx_r\}$ is the set of right cosets of S in X, and each $x\in X$ defines a permutation of Ω, by right multiplication. That is, $(Sx_i)x=Sx_j$ for some j. Let $g\in C_D(S)$, and set

$$h=g^{x_1}...g^{x_r}.$$

Then $h^x=h$ for all $x\in X$, while also

$$h=g^r[g,x_1]...[g,x_r]=g^rd,$$

where $d\in[D,O^p(X)]$. As $(p,r)=1$ and D is a p-group, we get $h^n=gd^n$ for some n, and thus $g=d^{-n}h^n\in[D,O^p(X)]C_D(X)$. That is, point (a) holds.

We continue the preceding setup in order to prove the following:

$$\text{if } [C_V(S), X] = 1 \text{ then } [C_D(S), X] = 1. \qquad (*)$$

Suppose by way of contradiction that $[C_V(S), X]=1$ but that $[C_D(S), X]\neq 1$. The element g in the proof of (a) may then be chosen so that $g\notin C_D(X)$ and with $g^p \in C_D(X)$. Then $g^p\neq 1$. As $(g^x)^p=(g^x)^p=g^p$ for all $x\in X$, we get $h^p=g^{pr}$ (where h is defined as in the proof of (a)), and then $h^p\neq 1$ as r is relatively prime to p. Set $D_0=\langle d^p | d\in D\rangle$. We may assume that $D=\langle g^X\rangle$, so $D_0=\langle g^p\rangle=\langle h^p\rangle$ is cyclic. Then $D=V\langle h\rangle$, and

$$C_D(S) = C_V(S)\langle h\rangle = C_D(X).$$

This contradiction completes the proof of $(*)$.

Now suppose that $[C_W(S), X]=1$. Applying (a) with V in the role of D, we obtain $C_V(S)\leqslant [V, X]C_V(X)$. Then $[C_V(S), X]\leqslant C_V(X)$ by the definition of W, and so $[C_V(S), O^p(X)]=1$. As $X=O^p(X)S$, (c) is proved.

Finally, assume the hypothesis of (b). Then $O^p(X)\leqslant K$, and $[D, O^p(X)]\leqslant V$ by Lemma 6.1. Thus (a) yields $C_D(S)\leqslant C_D(X)V$, and so

$$C_D(S) = C_D(S)\cap C_D(X)V = C_D(X)(C_D(S)\cap V) = C_D(X)C_V(S),$$

and (b) holds. □

Definition 6.3. Let M be a finite group, let S be a Sylow p-subgroup of M, and set $Y=O_p(M)$. Then M is *p-reduced*, and (M, S, Y) is a *reduced setup*, if

$$C_M(Y)\leqslant Y, \quad C_S(Z(Y)) = Y \quad \text{and} \quad O_p(M/C_M(Z(Y))) = 1.$$

LEMMA 6.4. *Let (M, S, Y) be a reduced setup, and set $D=Z(Y)$, $V=Z_Y$, and*

$$G = M/C_M(Z(Y)).$$

Let A be an abelian p-subgroup of G. Then $V=Z_D$, and the following facts hold:

(a) $C_M(D)=C_M(V)$.

(b) *If A acts quadratically on V, then A is elementary abelian. In particular, if A acts quadratically on D, then A is elementary abelian.*

Proof. Evidently $V=Z_D$, and $C_M(D)\leqslant C_M(V)\trianglelefteq M$. But also $O^p(C_M(V))\leqslant C_M(D)$, by [11, Theorem 5.3.10]. Thus, the image of $C_M(V)$ in $M/C_M(D)$ is a normal p-subgroup of $M/C_M(V)$, and then, since M is p-reduced, we obtain point (a). Point (b) is given by [13, statement 9.1.1 (c)]. □

The following result shows how to isolate a reduced setup from any finite group M such that $C_M(O_p(M))\leqslant O_p(M)$.

LEMMA 6.5. *Let M be a finite group with $C_M(O_p(M))\leqslant O_p(M)$, and let S be a Sylow p-subgroup of M. Then there exists a unique largest (with respect to inclusion) subgroup D of $Z(O_p(M))$ such that $Z(S)\leqslant D\trianglelefteq M$ and such that $O_p(M/C_M(D))=1$. Moreover, the following hold for $Y:=C_S(D)$, $H:=N_M(Y)$, and $\mathcal{F}:=\mathcal{F}_S(M)$:*

(a) *(H,S,Y) is a reduced setup;*

(b) *Y is strongly closed in $\mathcal{F}$.*

Proof. We aim first of all to define subgroups Y_k and D_k of S, for all $k\geqslant 0$, with the following properties:

(1) Y_k is strongly closed in $\mathcal{F}$, and $C_M(Y_k)\leqslant Y_k$;

(2) $D_k=Z(Y_k)\trianglelefteq M$.

The conditions (1) and (2) are satisfied by $Y_0:=O_p(M)$ and $D_0=Z(Y_0)$. We shall define Y_k and D_k for $k\geqslant 1$, in the following recursive way. Take Y_k to be the preimage in S of $O_p(M/C_M(D_{k-1}))$ and take $D_k=Z(Y_k)$. We now check that conditions (1) and (2) hold for Y_k and D_k under the asssumption that they hold for Y_{k-1} and D_{k-1}.

As $D_{k-1}\trianglelefteq M$, also $C_M(D_{k-1})\trianglelefteq M$, and then $C_M(D_{k-1})P\trianglelefteq M$ where P is defined to be the preimage in S of $O_p(M/C_M(D_{k-1}))$. As P is a Sylow p-subgroup of the normal subgroup $C_M(D_{k-1})P$ of M, P is strongly closed in $\mathcal{F}$. Again as P is Sylow in $C_M(D_{k-1})P$, we have $C_S(D_{k-1})\leqslant P$, and so $Y_{k-1}\leqslant P$. As $C_M(Y_{k-1})\leqslant Y_{k-1}$, we obtain $C_M(P)\leqslant P$, and $Z(P)\leqslant Z(Y_{k-1})=D_{k-1}$. Here $M=C_M(D_{k-1})N_M(P)$ by the Frattini lemma, so $Z(P)\trianglelefteq M$. Thus (1) and (2) hold, where $Y_k=P$ and $D_k=Z(P)$.

Since M is finite, there exists n minimal subject to $D_n=D_{n+1}$. Set $D=D_n$ and $Y=Y_{n+1}$. Then $C_S(D)\leqslant Y$ and $D=Z(Y)$, so $Y=C_S(D)$ is a Sylow p-subgroup of $C_M(D)$. The Frattini lemma yields $M=C_M(D)H$, where $H=N_M(Y)$. Also,

$$YC_M(D)/C_M(D)=O_p(M/C_M(D)),$$

so as $Y\leqslant C_M(D)$ we have $O_p(M/C_M(D))=1$. Since $M=C_M(D)H$, we get $M/C_M(D)\cong H/C_H(D)$, so also $O_p(H/C_H(D))=1$. As $O_p(H)C_H(D)/C_H(D)\leqslant O_p(H/C_H(D))=1$, it follows that $O_p(H)\leqslant C_S(D)=Y$, so $O_p(H)=Y$. Since $C_H(Y)\leqslant Y$ by (1), (H,S,Y) is then a reduced setup.

We now establish the uniqueness and maximality of D. Thus, let U be a subgroup of $Z(Y_0)$ such that $Z(S)\leqslant U\trianglelefteq M$ and such that $O_p(M/C_M(U))=1$. It will suffice to show that $U\leqslant D$. Assuming otherwise, we have $[U,Y]\neq 1$. Since $O_p(M/C_M(U))=1$, we have $U\leqslant Z(O_p(M))$. That is, $U\leqslant D_0$, and so there is a largest index n such that $U\leqslant D_n$. Then $C_M(U)Y_{n+1}/C_M(U)$ is a non-identity normal p-subgroup of $M/C_M(U)$, contrary to $O_p(M/C_M(U))=1$. We conclude that $U\leqslant D$, as required. □

In what follows, the group $H{=}N_M(Y)$ in the preceding lemma will be called the *reduced core of M with respect to S.*

We next review the definition of a certain characteristic subgroup of an arbitrary finite p-group S, and of some terms related to it. All of the themes, and much of the terminology and notation that will be introduced here, have their origin in early work of John Thompson, and they have been of fundamental importance to the p-local viewpoint in finite group theory ever since.

Let $d(S)$ be the maximum, taken over all abelian subgroups A of S, of the numbers $|A|$. As in [20], we take $\mathcal{A}(S)$ to be the set of all abelian subgroups A of S such $|A|{=}d(S)$, and we set

$$J(S) := \langle \mathcal{A}(S) \rangle. \tag{$*$}$$

Notice that $J(S)$ is the unique subgroup of S which is isomorphic to $J(S)$. Because of this, the operator J has the following inheritance property: If R is a subgroup of S and $J(S){\leqslant}R$, then $J(S){=}J(R)$. In particular, $J(S)$ is weakly closed in any fusion system on S. One observes that $J(S)$ has the further property that it is centric in any fusion system on S.

Remark. The tendency in the last thirty years or more has been to define $\mathcal{A}(S)$ to be the set of *elementary abelian* subgroups of S of maximal order, and then to define $J(S)$ by the formula $(*)$. But the formulation that we have chosen is the one which is needed for the task at hand.

Let G be a finite group, let D be a finite abelian p-group, and suppose that there is given a faithful group action (from the right, as always) of G on D. An abelian p-subgroup A of G is an *offender* (on D, in G) if $|A|\,|C_D(A)|{\geqslant}|D|$. An offender A is *non-trivial* if $A{\neq}1$, and A is a *best offender* if $|A|\,|C_D(A)|{\geqslant}|B|\,|C_D(B)|$ for every subgroup B of A. A *quadratic offender* is an offender A such that $[D, A, A]{=}1$. Write $\mathcal{A}_D(G)$ for the set of best offenders in G on D, and set $J_D(G){=}\langle \mathcal{A}_D(G) \rangle$.

We shall most often be interested in the situation where D is a normal abelian p-subgroup of a finite group M, and where $G{=}M/C_M(D)$. In this case, we say that (G, D) is an FF-*pair* if $O_p(G){=}1$ and $J_D(G){=}G$. The structure of FF-pairs in the case that D is elementary abelian is analyzed in [15], using the CFSG (the classification of the finite simple groups), and the preceding terminology is adapted from [15].

Lemma 6.6. *Let D be a finite abelian p-group and let G be a finite group acting faithfully on D.*

(a) *Every non-trivial offender in G (on D) contains a non-trivial best offender.*

(b) *Every non-trivial best offender in G contains a non-trivial quadratic best offender.*

(c) *If A is a best offender in G, and U is an A-invariant subgroup of D, then $A/C_A(U)$ is a best offender in $N_G(U)/C_G(U)$ on U.*

Proof. Statement (a) is a triviality: If $A \neq 1$ is an offender on D, one has only to choose a subgroup $B \neq 1$ of A so as to maximize $|B|\,|C_D(B)|$, in order to obtain a best offender on D.

Point (b) is essentially given by the Timmesfeld replacement theorem [21]; but here it must be noted that in the hypothesis of Timmesfeld's theorem the groups D and A are assumed to be elementary abelian. A one-page proof of this result (with the extra, but in fact extraneous, hypothesis concerning D and A) is given as [9, Theorem 2]. At no point in the proof is the extra hypothesis used.

The argument for (c) is again a case of asking the reader to check a very short proof (about half a page) in which there is an extraneous hypothesis as mentioned in the proof of (b). In this case, the relevant result is [14, Lemma 2.5 (c)]. □

LEMMA 6.7. *Let M be a finite group, let $S \in \mathrm{Syl}_p(M)$, let D be a normal abelian p-subgroup of M, and set $G = M/C_M(D)$. Let $A \in \mathcal{A}(S)$, and set $Y = C_S(D)$. Then the following facts hold*:

(a) *the image of A in G is a best offender on D*;

(b) $J(S) \nleq Y \Leftrightarrow J(S) \neq J(Y) \Rightarrow J_D(G) \neq 1$.

Proof. We provide the standard argument for the convenience of the reader. Let $\bar{A}$ be the image of A in G, and suppose that $\bar{A}$ is not a best offender on D. Thus, there exists a subgroup $\overline{B} \leqslant \bar{A}$ such that

(1) $|\overline{B}|\,|C_D(\overline{B})| > |\bar{A}|\,|C_D(\bar{A})|$.

Let B be the preimage of $\overline{B}$ in A, and set $B^* = C_D(B)B$. Then,

(2) $C_B(D) = C_A(D)$.

As $A \in \mathcal{A}(S)$ we have $C_D(A) = D \cap A$, and then also $D \cap A = C_D(B) \cap A$. Thus,

(3) $C_D(B) \cap A = D \cap A = C_D(A)$.

We now obtain

$$\begin{aligned}
|B^*| &= |C_D(B)|\,|B|/|C_D(B) \cap B| \\
&= |C_D(\overline{B})|\,|\overline{B}|\,|C_B(D)|/|C_D(B) \cap B| \\
&\geqslant |C_D(\overline{B})|\,|\overline{B}|\,|C_B(D)|/|C_D(B) \cap A| \\
&> |C_D(\bar{A})|\,|\bar{A}|\,|C_A(D)|/|C_D(B) \cap A| \quad \text{by (1) and (2)} \\
&= |C_D(\bar{A})|\,|\bar{A}|\,|C_A(D)|/|C_D(A)| \quad \text{by (3)} \\
&= |C_D(A)|\,|A|/|C_D(A)| \\
&= |A|.
\end{aligned}$$

This contradicts the maximality of $|A|$ among the abelian subgroups of S, and completes the proof of (a). Point (b) follows from (a) and from the inheritance property of the J-operator, mentioned above. □

This completes the background material for this section. The next result lists some criteria for extending rigid automorphisms of localities within finite groups to automorphisms of the groups themselves.

LEMMA 6.8. *Let M be a finite group, let $S\in\mathrm{Syl}_p(M)$, set $X=O_p(M)$, and assume that $C_M(X)\leqslant X$. Set $\mathcal{F}=\mathcal{F}_S(M)$, and let Γ be a non-empty, $\mathcal{F}$-invariant, overgroup-closed set of subgroups of S, such that $X\leqslant Q$ for all $Q\in\Gamma$. Set $\mathcal{L}=\mathcal{L}_\Gamma(M)$.*

(a) *Suppose that $M=C_M(Z(X))S$. Then $\mathcal{L}$ is the unique $\mathcal{F}$-natural Γ-linking system, up to a unique rigid isomorphism. In particular, the identity automorphism is the unique rigid automorphism of $\mathcal{L}$.*

(b) *Let H be the reduced core of M with respect to S, set $Y=O_p(H)$, and let Γ_Y be the set of all $Q\in\Gamma$ such that $Y\leqslant Q$. Let γ be a rigid automorphism of $\mathcal{L}$, let γ_H be the restriction of γ to $\mathcal{L}_{\Gamma_Y}(H)$ (see Lemma 3.9), and suppose that γ_H extends to an automorphism of H. Then γ extends to an automorphism of M.*

(c) *Set $D=Z(X)$ and assume that $C_M(D)/X$ is a p'-group. Assume also that there exists $Q\in\Gamma$ such that $M=C_M(D)N_M(Q)$, and an automorphism γ of $\mathcal{L}$ such that γ restricts to the identity automorphism of $N_M(Q)$. Then γ is the identity automorphism of $\mathcal{L}$.*

Proof. (a) Let M be a minimal counterexample, and choose an $\mathcal{F}$-filtration

$$(\Gamma_k, T_k)_{k=0}^N$$

for Γ, with $\Gamma_0=\{S\}$ (see Definition 5.17). Let $\mathcal{L}_k$ be the restriction of $\mathcal{L}$ to Γ_k. Then $\mathcal{L}_0=N_M(S)$. Since $Z(S)\leqslant Z(X)$, we have $Z(S)\leqslant Z(M)$, and then Proposition 1.10 (b) shows that for any $\mathcal{F}$-natural Δ_0-linking system $\mathcal{K}_0$, there is a unique rigid isomorphism $\mathcal{K}_0\to\mathcal{L}_0$.

Let n be the largest index such that $\mathcal{L}_n$ is unique up to a unique rigid isomorphism, in the preceding sense. Then, without loss of generality, we may assume that $n=N-1$. Let $\mathcal{K}$ be an $\mathcal{F}$-natural Γ-linking system, and let $\mathcal{K}_n$ be the restriction of $\mathcal{K}$ to Γ_n. There is then no loss of generality in taking $\mathcal{K}_n=\mathcal{L}_n$. Set $T=T_N$, and set $\mathcal{K}_T=N_{\mathcal{K}}(T)$ and $\mathcal{L}_T=N_{\mathcal{L}}(T)$. Observe that

$$O^p(N_M(T))\leqslant O^p(M)\leqslant C_M(Z(X))\leqslant C_M(Z(T)).$$

Thus, the hypothesis of Lemma 6.8 (a) holds with $N_M(T)$ in place of M, and with $\Gamma_T:=\{Q\in\Gamma\,|\,T\trianglelefteq Q\}$ in place of Γ. As M is a minimal counterexample, we conclude that the identity automorphism of $\mathcal{L}_T$ is the unique rigid automorphism of $\mathcal{L}_T$. Now Theorem 5.15 (b) yields a rigid isomorphism $\gamma: \mathcal{K} \to \mathcal{L}$. Since γ restricts to the identity automorphism on $\mathcal{L}_n$ and on $N_M(T)$, γ is uniquely determined, by Corollary 5.16 (a).

(b) Set $E = Z(Y)$ and note that $M = C_M(E)H$ by Lemma 6.5. Let β_H be an extension of γ_H to an automorphism of H. Then $\beta_H = c_z$ for some $z \in Z(S)$, by Proposition 1.10 (c). Now let β be the automorphism c_z of M. In order to complete the proof of (b) it suffices to show that β restricts to γ on $\mathcal{L}$. Both c_z and γ restrict to the identity map on $C_M(E)S$, by (a). Set $\mathcal{K} = C_{\mathcal{L}}(D)$. Then $\mathcal{K}$ is a partial normal subgroup of $\mathcal{L}$ by Lemma 3.9, and the Frattini lemma for localities (Corollary 4.8) yields the result that each $f \in \mathcal{L}$ is a product $f = gh$, with $g \in \mathcal{K}$ and $h \in H$. Thus

$$f\gamma = (gh)\gamma = (g\gamma)(h\gamma) = g(h\beta) = gh^z = (gh)^z = f^z = f\beta,$$

as required.

(c) Let $\mathbf{A}_0$ be the set of $\mathcal{L}$-essential subgroups of S (see Definition 2.14), and set $\mathbf{A} = \mathbf{A}_0 \cup \{S\}$. By hypothesis, each member of Γ contains X, so $C_M(P) \leqslant P$ for all $P \in \Gamma$. By Proposition 2.17, each $f \in \mathcal{L}$ is $\mathbf{A}$-decomposable, so in order to prove (c) it suffices to show that γ restricts to the identity map on $O^{p'}(N_M(P))$ for each $P \in \mathbf{A}_0$, and to the identity map on $N_M(S)$.

Fix $P \in \mathbf{A}$, and set $N = O^{p'}(N_M(P))$ (or $N = N_M(S)$ if $P = S$). Suppose first that $Q \leqslant P$ and let $g \in N$. By hypothesis, $g = g_1 g_2$ (where the product is taken in M), and where $g_1 \in N_M(Q)$ and $g_2 \in C_M(D)$. Then $Q^{g_2} = Q^g \leqslant P$ (and thus $(g_1, g_2) \in \mathbf{D}(\mathcal{L})$ via P). As $C_M(D)/X$ is a p'-group, and $X \leqslant Q$, it follows that $Q^{g_2} = Q$, and thus $N \leqslant N_M(Q)$. Since γ restricts to the identity map on $N_M(Q)$, there is no more to prove in this case. In particular, we have shown that γ restricts to the identity map on $N_M(S)$.

We are now reduced to the case where $P \in \mathbf{A}_0$, and where $Q \nleqslant P$. Then $N_Q(P) \nleqslant P$. On the other hand, $N_Q(P)C_M(D) \trianglelefteq C_M(D)N$, and thus $N_Q(P)C_N(D)P/C_N(D)$ is a non-identity normal p-subgroup of $N/C_N(D)$, properly containing $C_N(D)P/C_N(D)$. Since N/P has a strongly p-embedded subgroup, it follows that $N/C_N(D)$ is a p-group. Thus $N = C_N(D)N_S(P)$, and point (a) implies that γ restricts to the identity map on N. □

LEMMA 6.9. *Let $\mathcal{F}$ be a constrained fusion system on S, let M be a model for $\mathcal{F}$, let H be the reduced core of M with respect to S, and set $Y = O_p(H)$. Suppose that there is given an $\mathcal{F}$-natural Γ-linking system $(\mathcal{L}, \Gamma, S)$ such that $Y \in \Gamma$, and such that $O_p(M) \leqslant P$ for all $P \in \Gamma$. Suppose also that there is given an isomorphism $\beta: N_{\mathcal{L}}(Y) \to H$ of groups, such that β restricts to the identity map on S. Then β extends to an isomorphism $\mathcal{L} \to \mathcal{L}_\Gamma(M)$.*

Proof. As Y is weakly closed in $\mathcal{F}$, by Lemma 6.5 (b), Lemma 5.18 implies that there is an $\mathcal{F}$-filtration $\mathbf{F} = (\Delta_i, T_i)_{i=0}^N$ of Γ, in which Δ_0 is the set of overgroups of Y in S. For each k, $1\leqslant k\leqslant N$, let $\mathcal{L}_k$ be the restriction of $\mathcal{L}$ to Γ_k, and set $\mathcal{M}_k=\mathcal{L}_{\Gamma_k}(M)$. As $Y\in\Gamma$, β is an isomorphism $\mathcal{L}_0\to\mathcal{M}_0=H$. Let n be the largest index such that β extends to an isomorphism $\beta_n\colon\mathcal{L}_n\to\mathcal{M}_n$. Thus $n\geqslant 0$, and we may assume that $n<N$ as otherwise there is nothing to prove. There is then no loss of generality in assuming further that $n=N-1$.

Set $\mathcal{M}=\mathcal{L}_\Gamma(M)$. Then $C_M(O_p(M))\leqslant O_p(M)$ as M is a model for the constrained fusion system $\mathcal{F}=\mathcal{F}_S(M)$. Since each $P\in\Gamma$ contains $O_p(M)$, by assumption, $\mathcal{M}$ is then a Γ-linking system. Set $T=T_N$, $K=N_M(T)$, $\Sigma=\{P\in\Gamma_n\,|\,T\trianglelefteq P\}$ and $\mathcal{K}=\mathcal{L}_\Sigma(K)$. As $T\in\Gamma$, both $N_{\mathcal{L}}(T)$ and K are models for $\mathcal{F}_{N_S(T)}(K)$, so Proposition 1.10 (b) yields an isomorphism $\gamma\colon N_{\mathcal{L}}(T)\to K$ which restricts to the identity map on $N_S(T)$. We may therefore apply Theorem 5.15 (a) with K in the role of M, and find that there are rigid isomorphisms λ and λ' from $N_{\mathcal{L}_n}(T)$ to $\mathcal{K}$ such that $\mathcal{L}$ and $\mathcal{M}$ are rigidly isomorphic to $\mathcal{L}^+(\lambda)$ and $\mathcal{L}^+(\lambda')$, respectively. Moreover, λ is given explicitly as $\beta_T\circ\gamma^{-1}$, where β_T is the restriction of β_n to $N_{\mathcal{L}_n}(T)$, while λ' is the composition $\beta_T\circ\iota$, where ι is the identity map on K.

Let α be the rigid automorphism $\lambda^{-1}\circ\lambda'$ of $\mathcal{K}$. By Theorem 5.15 (b), it suffices to show that α extends to an automorphism of K in order to conclude that β extends to an isomorphism $\mathcal{L}\to\mathcal{M}$.

Since $T\notin\Gamma_0$, we have $Y\nleqslant T$, so $T<N_Y(T)T$, and then $N_Y(T)T\in\Gamma_n$ by the construction of $\mathbf{F}$. Thus α restricts to an automorphism α_0 of $N_K(N_Y(T)T)$ which centralizes $N_S(T)$, and then Proposition 1.10 (c) yields $\alpha_0=c_z$ for some $z\in Z(N_S(T))$. We now observe that $N_Y(T)=N_S(T)\cap C_S(D)$ is a Sylow subgroup of $C_K(D)$, and hence

$$K=C_K(D)N_K(N_Y(T))=C_K(D)N_K(N_Y(T)T)$$

by the Frattini lemma. By Lemma 3.9, $C_{\mathcal{K}}(D)$ is a partial normal subgroup of $\mathcal{K}$, and evidently $N_S(T)\cap C_{\mathcal{K}}(D)=N_Y(T)$. The Frattini lemma for localities (Corollary 4.8) then yields

$$\mathcal{K}=C_{\mathcal{K}}(D)N_{\mathcal{K}}(N_Y(T))=C_{\mathcal{K}}(D)N_K(N_Y(T)T),$$

since $N_{\mathcal{K}}(N_Y(T))=N_{\mathcal{K}}(N_Y(T)T)$, and since $N_Y(T)T\in\Sigma$.

Let $\mathcal{C}$ be the locality $\mathcal{L}_\Sigma(C_K(D)N_S(Y))$. Then the identity automorphism of $\mathcal{C}$ is the unique rigid automorphism of $\mathcal{C}$ by Lemma 6.8 (a). In particular, the restriction α_1 of α to $\mathcal{C}$ is the identity automorphism, and so c_z induces α_1 on $\mathcal{C}$. By Corollary 4.8, each $f\in\mathcal{K}$ is a product gh taken in $\mathcal{K}$, where $g\in C_{\mathcal{K}}(D)$ and $h\in N_K(N_Y(T))$. Since $Y\leqslant T$ we have $z\in C_M(Y)=D$, and so

$$f\alpha=(gh)\alpha=(g\alpha_1)(h\alpha_0)=gh^z=(gh)^z=f^z,$$

and thus α extends to the automorphism c_z of K. As remarked earlier, this suffices to complete the proof. □

Let (M,S,Y) be a reduced setup, and set $D{=}Z(Y)$ and $V{=}Z_Y$. Recall from Lemma 6.4 (a) that $C_M(V){=}C_M(D)$. Set $G{=}M/C_M(V)$, and recall that, for any subgroup K of G, $J_D(K)$ is defined to be the subgroup of K generated by the best offenders in K on D, and similarly for $J_V(K)$. For the remainder of this paper, whenever such a setup is given, and whenever H is a subgroup of M, we write $J(H,D)$ for the preimage in H of $J_D(H/C_H(D))$. We define $J(H,V)$ analogously, relative to $J_V(H/C_H(V))$.

The proof of the following proposition will be postponed to the next (concluding) section.

Proposition 6.10. *Let (M,S,Y) be a reduced setup and set $D{=}Z(Y)$. Let Γ be the set of all overgroups Q of Y in S such that $J(Q,D){\neq}Y$, and assume that $S{\in}\Gamma$. Set $\mathcal{L}{=}\mathcal{L}_\Gamma(M)$ and let γ be a rigid automorphism of $\mathcal{L}$. Then γ extends to an automorphism of M.*

Proposition 6.11. *Let M be a finite group, and assume that Proposition 6.10 holds for all reduced setups (M',S',Y') with $|M'|{<}|M|$. Let S be a Sylow p-subgroup of M, and let X be a normal p-subgroup of M with $C_M(X){\leqslant}X$. Set $Y{=}O_p(M)$, $\mathcal{F}{=}\mathcal{F}_S(M)$, $D{=}Z(Y)$, and let Γ be an $\mathcal{F}$-invariant, overgroup-closed collection of overgroups of X in S such that*

$$Q\in\Gamma \quad\Longrightarrow\quad J(Q,D)\in\Gamma.$$

Assume that $J(S,D){\in}\Gamma$. Then every rigid automorphism of $\mathcal{L}_\Gamma(M)$ extends to an automorphism of M.

Among all pairs (M,Γ) for which Proposition 6.11 fails, choose one so that first $|M|$ is as small as possible, and then so that $|X|$ is as large as possible. Set $\mathcal{L}{=}\mathcal{L}_\Gamma(M)$, and fix a rigid automorphism γ of $\mathcal{L}$ such that γ has no extension to an automorphism of M. Set $V{=}Z_D$.

Fact 6.12. *$X{=}Y$, and $Y{\notin}\Gamma$.*

Proof. If $Y{\in}\Gamma$ then $\mathcal{L}_\Gamma(M){=}M$, and the conclusion of Proposition 6.11 holds trivially. Thus $Y{\notin}\Gamma$. Now suppose that X is a proper subgroup of Y, and let Γ_Y be the set of all $Q{\in}\Gamma$ such that $Y{\leqslant}Q$. The maximality of $|X|$ in the choice of (M,Γ) then implies that every rigid automorphism of $\mathcal{L}_{\Gamma_Y}(M)$ extends to an automorphism of M. Let γ_Y be the restriction of γ to $\mathcal{L}_{\Gamma_Y}(M)$ and let β be an extension of γ_Y to an automorphism of M.

Let $Q{\in}\Gamma$. Then $QY{\in}\Gamma_Y$ and $N_M(Q){\leqslant}N_M(QY)$, so γ and β agree on $N_M(Q)$ for all $Q{\in}\Gamma$. By Lemma 3.8, β restricts to an automorphism β_0 of $\mathcal{L}_\Gamma(M)$, and now Lemma 3.10 shows that $\beta_0{\circ}\gamma$ is the identity automorphism of $\mathcal{L}_\Gamma(M)$. Thus, β is an extension of γ to an automorphism of M. □

Fact 6.13. *(M, S, Y) is a reduced setup. In particular, $C_M(D){=}C_M(V)$.*

Proof. Suppose this is false, let H be the reduced core of M with respect to S, set $R{=}O_p(H)$, and set $\mathcal{L}_H{=}\mathcal{L}_\Gamma(H)$. If $R{\in}\Gamma$ then $\mathcal{L}_H{=}N_M(R){=}H$, and then γ extends to an automorphism of M by Lemma 6.8 (b). We conclude that $R{\notin}\Gamma$, and hence no member of Γ is contained in R.

Let Γ_R be the set of all products RP with $P{\in}\Gamma$. Thus, Γ_R also has the usual meaning. Namely, Γ_R is the set of members Q of Γ such that $R{\trianglelefteq}Q$. We note that since $R{=}O_p(H)$ and H is reduced, we have $R{=}C_S(Z(R))$. For $Q{\in}\Gamma_R$ write $J(Q, Z(R))$ for the preimage in Q of $J_{Z(R)}(Q/R)$. Since $Y{\leqslant}R$, and since $J(Q, D){\in}\Gamma$ for all $Q{\in}\Gamma$ by hypothesis, Lemma 6.6 (c) implies that $J(Q, Z(R)){\in}\Gamma_R$ for each $Q{\in}\Gamma_R$. Since $R{\notin}\Gamma_R$, by the preceding paragraph, the hypothesis of Proposition 6.11 is fulfilled with (H, Γ_R) in place of (M, Γ). Here H is a proper subgroup of M as (M, S, Y) is not a reduced setup, so we conclude that the restriction γ_H of γ to $\mathcal{L}_{\Gamma_R}(H)$ extends to an automorphism of H. We appeal again to Lemma 6.8 (b) and conclude that γ extends to an automorphism of M, contrary to our choice of (M, γ). Thus, (M, S, Y) is reduced, and then $C_M(D){=}C_M(V)$ by Lemma 6.4 (a). □

Fact 6.14. *$M{=}J(M, D)$.*

Proof. Set $K{=}J(M, D)$. So K is the preimage in M of the subgroup of $M/C_M(D)$ generated by best offenders on D, and thus $K{\trianglelefteq}M$. Set $S_0{=}S{\cap}K$ and let Φ be the set of all $Q{\in}\Gamma$ with $Q{\leqslant}S_0$. Then Φ is $\mathcal{F}$-invariant. Since $Q{\in}\Gamma$ implies $J(Q, D){\in}\Gamma$, by the hypothesis in Proposition 6.11, we get $Q{\cap}S_0{\in}\Phi$ for all $Q{\in}\Gamma$. In particular, $S_0{\in}\Phi$ since (by hypothesis) $S{\in}\Gamma$. Then Φ is a non-empty, overgroup-closed collection of subgroups of S_0.

Set $\mathcal{K}{=}\mathcal{L}_\Phi(K)$. Then β restricts to a rigid automorphism $\varkappa$ of $\mathcal{K}$, by Lemma 3.8. Assume now that $K{\neq}M$, and set $X_0{=}X{\cap}K$. The hypothesis of Proposition 6.11 is satisfied with $(K, S_0, X_0, \mathcal{F}_{S_0}(K), \Phi)$ in place of $(M, S, X, \mathcal{F}, \Gamma)$, and hence, by the minimality of $|M|$ as a counterexample to Proposition 6.11, $\varkappa$ extends to an automorphism λ of K. But then γ extends to an automorphism of M by Lemma 4.9, and contrary to the choice of (M, Γ). Thus $M{=}K$. □

Let Φ now denote the set of all subgroups P of S such that $Y{\leqslant}P$ and such that $J(P, D){\neq}Y$. If $\Phi{=}\Gamma$ then Facts 6.12–6.14 yield the hypothesis of Proposition 6.10, and then Proposition 6.10 yields an extension of γ to an automorphism of M. Thus Γ is a proper subset of Φ. Choose $T{\in}\Phi{\setminus}\Gamma$ so that

(1) T is fully normalized in $\mathcal{F}$,
(2) $|J(T, D)|$ is as large as possible subject to (1), and
(3) $|T|$ is as small as possible subject to (2).

Fact 6.15. *Y is a proper subgroup of T, and $N_M(T)$ is a proper subgroup of M.*

Proof. By assumption, $Y \leqslant T$, and since $\Phi \neq \Gamma$ we know that $Y \neq T$. Then $T \nleq Y$, and so T is not normal in M. □

Fact 6.16. *The following statements hold*:
(a) $T = J(T, D)$;
(b) *if P and P' are distinct $\mathcal{F}$-conjugates of T, then $\langle P, P' \rangle \in \Gamma$.*

Proof. Since $T \in \Phi$ we have $J(T, D) \neq Y$, and therefore $J(T, D) \in \Phi$ by definition. As $J(T, D)$ has a fully normalized $\mathcal{F}$-conjugate, point (a) then follows from the minimality condition (3) in the choice of T. Then also $P = J(P, D)$ for any $\mathcal{F}$-conjugate P of T. For $\mathcal{F}$-conjugates P and P' of T, the definition of the "$J(\cdot, D)$"-operator then yields $\langle P, P' \rangle = J(\langle P, P' \rangle, D)$. For $P \neq P'$ we get $|J(\langle P, P' \rangle, D)| > |J(T, D)|$, and then (b) follows from condition (2) in the choice of T. □

Set $M_T = N_M(T)$, set $R = N_S(T)$, and let Γ^+ be the union of Γ and the set of subgroups of S which contain an $\mathcal{F}$-conjugate of T. Let Γ_T be the set of all $Q \in \Gamma$ with $T \leqslant Q \leqslant R$, and set $\mathcal{L}_T = N_{\mathcal{L}}(T)$. Thus,

$$\mathcal{L}_T = N_{\mathcal{L}}(T) = \mathcal{L}_{\Gamma_T}(M_T).$$

Fact 6.17. *Let γ_T be the restriction of γ to $\mathcal{L}_T$. Then γ_T does not extend to an automorphism of M_T.*

Proof. Suppose γ_T extends to an automorphism of M_T. Then Corollary 5.16 (b), with M_T in the role of M and with γ in the role of β, yields an extension of γ to an automorphism of M. Thus, as (M, γ) is a counterexample to Proposition 6.11, no such extension of γ_T exists. □

Let H be the reduced core of M_T with respect to R, set $X = O_p(H)$, and set $U = Z(X)$. Thus $H = N_{M_T}(X)$, and the Frattini lemma yields

$$M_T = C_{M_T}(U) H. \tag{$*$}$$

Set $\mathcal{H} = N_{\mathcal{L}_T}(X)$. Then $C_{\mathcal{L}_T}(U)$ is a partial normal subgroup of $\mathcal{L}_T$ by Lemma 3.9, and then Corollary 4.8 yields

$$\mathcal{L}_T = C_{\mathcal{L}_T}(U) \mathcal{H}. \tag{$**$}$$

Fact 6.18. *Let β be the restriction of γ_T to $\mathcal{H}$. Then β does not extend to an automorphism of H. In particular, $X \notin \Gamma$.*

Proof. If β extends to an automorphism of $\mathcal{H}$, then γ_T extends to an automorphism of M_T, by Lemma 6.8 (b), and contrary to Fact 6.17. Thus no such extension of β exists. If $X \in \Gamma$ then $\mathcal{H}=N_{\mathcal{H}}(X)=N_H(X)=H$, so we conclude that $X \notin \Gamma$. □

Proof of Proposition 6.10. Since H is a proper subgroup of M by Fact 6.15, the minimality of M in the choice of a counterexample to Proposition 6.11 implies that the conclusion of Proposition 6.11 holds with (H, Γ_X) in place of (M, Γ). This contradicts Fact 6.18, and so the proof is complete. □

LEMMA 6.19. *Let M, S, $Y=X$, D, $\mathcal{F}$, and Γ be as in Lemma* 3.10, *and assume that Proposition* 6.10 *holds for all reduced setups (M', S', Y') with $|M'|<|M|$. Then every $\mathcal{F}$-natural Γ-linking system is rigidly isomorphic to $\mathcal{L}_\Gamma(M)$.*

Proof. Set $\mathcal{L}=\mathcal{L}_\Gamma(M)$ and let $\mathcal{L}'$ be any other $\mathcal{F}$-natural Γ-linking system. Set $T_0=J(S, D)$ and set $\mathcal{L}_0=N_{\mathcal{L}}(T_0)$ and $\mathcal{L}'_0=N_{\mathcal{L}'}(T_0)$. Then $\mathcal{L}_0$ and $\mathcal{L}'_0$ are isomorphic groups, via an isomorphism which restricts to the identity map on S, by Proposition 1.10 (b).

Let Γ_0 be the set of overgroups of T_0 in S. Among all groups $T_1 \in \Gamma \setminus \Gamma_0$ such that T_1 is fully normalized in $\mathcal{F}$, choose T_1 so as first to maximize $|J(T_1, D)|$ and then so as to minimize $|T_1|$. Then $T_1=J(T_1, D)$ and, as in the proof of Fact 6.16 (b), we find that any two distinct $\mathcal{F}$-conjugates P and P' of T_1 generate a member of Γ_0. Let Γ_1 be the union of Γ_0 and the set of subgroups X of S such that X contains an $\mathcal{F}$-conjugate of T_1, and then iterate this procedure, so as to obtain an $\mathcal{F}$-filtration $\mathbf{F}=(\Gamma_i, T_i)_{i=0}^N$ of Γ. Let $\mathcal{L}_i$ and $\mathcal{L}'_i$ be the restrictions of $\mathcal{L}$ and $\mathcal{L}'$ to Γ_i, and let n be an index such that there exists a rigid isomorphism $\mathcal{L}_n \to \mathcal{L}'_n$. Then $n<N$, or else there is nothing to prove. Set $T=T_{n+1}$, and set $\Delta=\Gamma_n$.

Set $\mathcal{K}=\mathcal{L}_n$ and $\mathcal{K}_T=N_{\mathcal{K}}(T)$, and similarly define $\mathcal{K}'$ and $\mathcal{K}'_T$. Then

$$\mathcal{K}_T=\mathcal{L}_{\Delta_T}(N_M(T)),$$

and $\mathcal{L}_{n+1}$ is rigidly isomorphic to $\mathcal{K}^+(\iota)$, where ι is the identity automorphism of $\mathcal{K}_T$, by Theorem 5.15 (a). But also, $\mathcal{L}'_{n+1}$ is rigidly isomorphic to $(\mathcal{K}')^+(\lambda)$ for some rigid isomorphism $\lambda: \mathcal{K}_T \to \mathcal{L}_{\Delta_T}(N_M(T))$, again by Theorem 5.15 (a). Now, by Theorem 5.15 (b) (i), it suffices to show that all rigid automorphisms of $\mathcal{L}_{\Delta_T}(N_M(T))$ extend to automorphisms of $N_M(T)$, in order to complete the proof.

Set $M_T=N_M(T)$, $Y_T=O_p(M_T)$, $D_T=Z(Y_T)$, and set $R=N_S(T)$. Then $R \in \Delta$ by Lemma 5.4 (a). Thus $J(R, D) \neq T$, by construction of the filtration $\mathbf{F}$. Then, again by construction of $\mathbf{F}$, we get $J(R, D) \in \Delta$. If $J(R, D) \leqslant Y_T$ then we have $Y_T \in \Delta$ and $\mathcal{L}_{\Delta_T}(M_T)=M_T$. Since there is nothing to prove in that case, we may assume that $J(R, D) \nleq Y_T$. Then also $J(R, D_T) \nleq Y_T$, by Lemma 6.6 (c). In fact, the preceding argument shows that $J(Q, D_T) \nleq Y_T$ for any $Q \in \Delta$. We may then apply Proposition 6.11, with (M_T, R, Δ) in place of (M, S, Γ), in order to conclude that all rigid automorphisms of $\mathcal{L}_{\Delta_T}(M_T)$ extend to automorphisms of M_T, and to thereby complete the proof. $\square$

Recall from Definition 2.9 that a locality $(\mathcal{L}, \Delta, S)$ is a centric linking system if it is a Δ-linking system, where Δ is the set of all $\mathcal{F}_S(\mathcal{L})$-centric subgroups of P. Recall also from Proposition 2.18 (a) that if $\mathcal{L}$ is a centric linking system then $\mathcal{F}_S(\mathcal{L})$ is saturated.

On the other hand, let $\mathcal{F}$ be a given saturated fusion system on S. By an *$\mathcal{F}$-centric linking system* we mean a centric linking system $(\mathcal{L}, \Delta, S)$ such that $\mathcal{F}=\mathcal{F}_S(\mathcal{L})$. Thus, Δ is the set of $\mathcal{F}$-centric subgroups of S if $\mathcal{L}$ is an $\mathcal{F}$-centric linking system.

Assume now that the main theorem is false. We express this as a hypothesis, as follows.

Hypothesis 6.20. Proposition 6.10 holds, and $\mathcal{F}$ is a saturated fusion system on the p-group S such that one of the following conditions hold:

(i) There exists no $\mathcal{F}$-centric linking system on S;

(ii) There exist $\mathcal{F}$-centric linking systems on S which are not rigidly isomorphic.

Set $X_0=J(S)$, and define Δ_0 to be the set of all overgroups of X_0 in S. Then Δ_0 is closed under $\mathcal{F}$-conjugation since $J(S)$ is weakly closed in $\mathcal{F}$, and since $J(S)=J(Q)$ for all $Q\in\Delta_0$. As remarked earlier, it is immediate from the definition of $J(S)$ that $J(S)\in\mathcal{F}^c$, and hence $\Delta_0\subseteq\mathcal{F}^c$.

Now suppose that $\Delta_0\neq\mathcal{F}^c$, and let $\mathbf{X}=\mathbf{X}_1$ be the set of all $X\in\mathcal{F}^c\setminus\Delta_0$ such that X is fully normalized in $\mathcal{F}$. Among all $X\in\mathbf{X}_1$, choose X so that

(1) $d(X)$ is as large as possible,

(2) $|J(X)|$ is as large as possible (subject to (1)),

(3) $J(X)\in\mathcal{F}^c$, if possible, (subject to (1) and (2)), and

(4) subject to conditions (1)–(3), $|X|$ is as small as possible if $J(X)\in\mathcal{F}^c$, and otherwise $|X|$ is as large as possible.

Set $X_1=X$, and define Δ_1 to be the union of Δ_0 and the set of subgroups of S which contain an $\mathcal{F}$-conjugate of X_1. If $\Delta_1\neq\mathcal{F}^c$ we then repeat the above procedure, taking $\mathbf{X}_2$ to be the set of all $X\in\mathcal{F}^c\setminus\Delta_1$ such that X is fully normalized in $\mathcal{F}$, and choosing $X_2\in\mathbf{X}_2$ according to the rules (1)–(4). By iteration, we arrive at a sequence of pairs

$$\mathbf{F}=(\Delta_i, X_i)_{i=0}^N,$$

where $\Delta_N=\mathcal{F}^c$. Recall now the notion of $\mathcal{F}$-filtration from Definition 5.17.

LEMMA 6.21. *$\mathbf{F}$ is an $\mathcal{F}$-filtration of $\mathcal{F}^c$. Moreover, each X_i may be chosen so that $J(X_i)$ is fully normalized in $\mathcal{F}$.*

Proof. As X_0 is weakly closed in $\mathcal{F}$, point (1) of Definition 5.17 holds, while points (2) and (4) (with $\mathcal{F}^c$ in the role of Δ) hold by construction. Assuming now that $\mathbf{F}$ is not an $\mathcal{F}$-filtration of $\mathcal{F}^c$, we conclude that point (3) of Definition 5.17 fails to hold. There is then a smallest index n such that there exist $\mathcal{F}$-conjugates P and P' of $X:=X_n$ such that $P\neq P'$ and $\langle P,P'\rangle\notin\Delta_{n-1}$.

Set $Q=\langle P,P'\rangle$. Then $d(Q)=d(X)$, and $|J(Q)|=|J(X)|$, and thus $J(P)=J(Q)=J(P')$. If $J(P)\in\mathcal{F}^c$ then $X=J(X)$ by the minimality condition in (4). But in that case we also obtain $P=J(P)$ and $P'=J(P')$, and so $P=P'$, contrary to the hypothesis. Thus $J(X)\notin\mathcal{F}^c$. But then $P=Q=P'$ by the maximality condition in (4), and again contrary to the hypothesis. Thus, $\mathbf{F}$ is an $\mathcal{F}$-filtration of $\mathcal{F}^c$. The second part of the lemma follows from Lemma 1.7. □

LEMMA 6.22. *Let n be an index with $1\leqslant n\leqslant N$. Suppose that $J(X_n)$ is $\mathcal{F}$-centric and let $Q\in\Delta_{n-1}$ with $X_n\leqslant Q$. Then $J(Q)\in\Delta_{n-1}$.*

Proof. Suppose this is false, and let n be the smallest index for which the lemma fails. Set $X=X_n$ and set $\Delta=\Delta_{n-1}$. As $J(X)$ is $\mathcal{F}$-centric, by assumption, condition (4) in the choice of X implies that $X=J(X)$. If $d(Q)>d(X)$, or if $d(Q)=d(X)$ but $J(Q)>J(X)$, then $J(Q)\in\Delta$ by the maximality conditions (1) and (2) in the choice of X. So, we conclude that $J(Q)=J(X)$, and then $J(Q)=X$.

As $Q\in\Delta$, there exists an index m with $0\leqslant m<n$ such that Q contains an $\mathcal{F}$-conjugate U of X_m. Then the construction of $\mathbf{F}$ yields $d(U)\geqslant d(X)$. But $d(U)\leqslant d(Q)=d(X)$, so in fact $d(U)=d(X)$. Similarly, we obtain $|J(U)|=|J(X)|$, and hence $J(U)\cong J(X)$. Here $J(U)$ is $\mathcal{F}$-centric by condition (3) in the construction of Δ_m, so $U=J(U)=J(X)=X$. This is contrary to $m<n$, and completes the proof. □

By Proposition 1.10 (a), there exists a model M_0 for the fusion system $N_{\mathcal{F}}(X_0)$, and M_0 may then be viewed as an $\mathcal{F}$-natural Δ_0-linking system. Any two such linking systems are rigidly isomorphic by Proposition 1.10 (b), so there is a largest index n such that there exists an $\mathcal{F}$-natural Δ_n linking system, and such that all such linking systems are rigidly isomorphic. By Hypothesis 6.20, we have $n<N$. Set $\Delta=\Delta_n$, and let $\mathcal{L}$ be the unique (up to rigid isomorphism) $\mathcal{F}$-natural Δ-linking system.

Set $X=X_{n+1}$, set $R=N_S(X)$, let M_X be a model for $N_{\mathcal{F}}(X)$, and let H be the reduced core of M_X with respect to R. Set $Y=O_p(H)$, and set $D=Z(Y)$. In view of Lemma 6.21, we may assume that $J(X)$ is fully normalized in $\mathcal{F}$.

LEMMA 6.23. *Suppose that $Y\notin\Delta$, and let Δ_X be the set of all $P\in\Delta$ with $X\trianglelefteq P$. Then the following hold*:

(a) *Δ_X is the set of all subgroups Q of R, properly containing X, and such that $J(Q)\neq X$*;

(b) $X = J(X) = J(Y)$;

(c) $R = N_S(Y) \in \Delta$;

(d) *Y is fully normalized in $\mathcal{F}$, and H is a model for $N_{\mathcal{F}}(Y)$.*

Proof. Let Q be a subgroup of R containing X, and suppose that $Q \notin \Delta$. The condition (1) in the choice of X then yields $d(X) = d(Q)$, and thus $\mathcal{A}(X) \subseteq \mathcal{A}(Q)$ and $J(X) \leqslant J(Q)$. Now Lemma 6.22 will complete the proof of (a), once it is shown that $J(X)$ is $\mathcal{F}$-centric.

Set $B = C_S(J(X))$. Then B is X-invariant, and

$$N_B(X) = C_R(J(X)) \leqslant C_R(Z(X)) \leqslant C_R(D) = Y,$$

and thus $N_B(X) = C_Y(J(X))$. But $J(X) = J(Y)$ as $Y \notin \Delta$, and so

$$N_B(X) = C_Y(J(Y)) \leqslant J(Y) \leqslant X.$$

Thus $B \leqslant X$, and since $J(X)$ is fully normalized in $\mathcal{F}$, we conclude that $J(X)$ is $\mathcal{F}$-centric. Then $X = J(X)$ by condition (4) in the choice of X. This completes the proof of (a) and of (b).

Suppose that $R \notin \Delta$. Then $J(R) = J(X) = X$, by (a) and (b). Then

$$N_S(R) \leqslant N_S(J(R)) = N_S(X) = R,$$

and therefore $R = S$. Hence $X = J(S)$, and $Y \in \Delta_0$, contrary to $Y \notin \Delta$. Thus, $R \in \Delta$. We have $Y \trianglelefteq R$ by Lemma 6.5. Since $N_S(Y) \leqslant N_S(J(Y))$ and $J(Y) = X$, we conclude that $R = N_S(Y)$, completing the proof of (c).

Let $\phi \in \mathrm{Hom}_{\mathcal{F}}(R, S)$ be chosen so that $Y' := Y\phi$ is fully normalized in $\mathcal{F}$, and set $X' = X\phi$. Then $N_S(Y') \leqslant N_S(X')$ by (b), and since X is fully normalized it follows that $|N_S(Y')| \leqslant |R|$. Thus $N_S(Y') = R\phi$, and so Y is fully normalized in $\mathcal{F}$. Point (d) then follows from Lemma 1.11. □

LEMMA 6.24. *$Y \in \Delta$.*

Proof. Suppose that $Y \notin \Delta$. We check that the hypothesis of Proposition 6.11 (and hence also of Lemma 6.19) holds, with the role of $(M, S, Y, \mathcal{F}_S(M), \Gamma)$ being taken by $(M_X, R, X, N_{\mathcal{F}}(X), \Delta_X)$. First, $C_{M_X}(X) \leqslant X$ as M_X is a model of the constrained fusion system $N_{\mathcal{F}}(X)$. Next, by Lemma 6.23 (a), $Q \in \Delta_X$ implies that $J(Q) \nleq X$, while Lemma 6.23 (b) implies that $X = J(X)$ (and therefore that $J(X)$ is $\mathcal{F}$-centric). Then, Lemma 6.22 yields $J(Q) \in \Delta_X$, and so $J(Q, Z(X)) \in \Delta_X$ by Lemma 6.7 (b). In particular, $J(R, Z(X)) \in \Delta_X$, and so the claim has been verified. Since we are assuming Proposition 6.10, we are free to apply Lemma 6.19 to the setup with M_X; and so $\mathcal{L}_{\Delta_X}(M_X)$ is the unique $\mathcal{F}_{\Delta_X}(M_X)$-natural linking system, up to rigid isomorphism. Then all rigid isomorphisms $N_{\mathcal{L}}(X)\to\mathcal{L}_{\Delta_X}(M_X)$ are M_X-equivalent, by Theorem 5.15 (b). Now Theorem 5.15 (c) applies with M_X in the role of M, and we conclude that there exists an $\mathcal{F}$-natural Δ_{n+1}-linking system, and that any two such are rigidly isomorphic. This contradicts the maximality of n. □

LEMMA 6.25. *$Y\notin\Delta$.*

Proof. Suppose that $Y\in\Delta$. Then $Y\in\Delta_X$ as $X\leqslant Y$. Set $\mathcal{H}=\mathcal{L}_{\Delta_X}(H)$. Then

$$\mathcal{H}=N_H(Y)=H,$$

and therefore every rigid automorphism of $\mathcal{H}$ is in fact an automorphism of H. Hence Lemma 6.8 (b) applies, with M_X in the role of M, and so every rigid automorphism of $\mathcal{L}_{\mathcal{D}_X}(M_X)$ extends to an automorphism of M_X. That is, all rigid automorphisms of $\mathcal{L}_{\Delta_X}(M_X)$ are M_X-equivalent. As in the proof of Lemma 6.25, we conclude via (b) and (c) in Theorem 5.15 that there exists an $\mathcal{F}$-natural Δ_{n+1}-linking system, that any two such are rigidly isomorphic, and thereby contradict the maximality of n. □

With Lemmas 6.24 and 6.25 we now have a contradiction to Hypothesis 6.20. This contradiction provides a proof of the main theorem modulo Proposition 6.10. Thus, in order to complete the proof of the main theorem, it remains to prove Proposition 6.10.

7. The main theorem

Our aim in this section is to give a proof of Proposition 6.10, using the classification of the finite simple groups (CFSG). As was pointed out at the end of the preceding section, this will complete the proof of the main theorem.

We continue using the terminology and notation relating to FF-pairs. In particular, it is important to recall that our definition of $J(P)$, for a p-group P, is given in terms of abelian (and not elementary abelian) subgroups of P of maximal order.

For ease of reference, we restate Proposition 6.10, as follows.

PROPOSITION 7.1. *Let (M,S,Y) be a reduced setup, and set $D=Z(Y)$. Let Γ be the set of all overgroups Q of Y in S such that $J(Q,D)\neq Y$, and assume that $S\in\Gamma$. Set $\mathcal{L}=\mathcal{L}_\Gamma(M)$, and let γ be a rigid automorphism of $\mathcal{L}$. Then γ extends to an automorphism of M.*

Among all pairs (M,γ) satisfying the hypothesis of Proposition 7.1, and such that γ does not extend to an automorphism of M, fix (M,γ) so that $|M|$ is as small as possible.

We note that proposition Proposition 6.11 may then be applied to groups of order less than $|M|$.

Set $\mathcal{F}{=}\mathcal{F}_S(\mathcal{L})$, and recall the definition of $\mathcal{F}$-essential subgroup of S, from Definition 2.14.

LEMMA 7.2. *Let* $\mathbf{A}$ *be the union of* $\{S\}$ *and the set of* $\mathcal{F}_S(\mathcal{L})$*-essential subgroups of* S, *and set* $M_0{=}\langle N_M(Q)\,|\,Q{\in}\mathbf{A}\rangle$. *Then* $M_0{=}M$. *In particular, there exists no proper subgroup of* M *which contains the partial subgroup* $\mathcal{L}$ *of* M.

Proof. Suppose that M_0 is a proper subgroup of M. We first show

(1) $\mathcal{L}{\subseteq}M_0$.

Indeed, in order to establish (1) it suffices, by Proposition 2.17, to show that $C_M(P){\leqslant}P$ for all $P{\in}\Gamma$. But $C_M(P){\leqslant}C_M(Y){=}D{=}Z(Y)$ as (M,S,Y) is a reduced setup. Thus (1) holds.

Since $\mathcal{L}{=}\mathcal{L}_\Gamma(M)$ we have also $\mathcal{L}{=}\mathcal{L}_\Gamma(M_0)$. We now claim that the hypothesis of Proposition 6.11 is fulfilled with (M_0,Y,Γ) in the role of (M,X,Δ). Set $Y_0{=}O_p(M_0)$, $D_0{=}Z(Y_0)$, and $D{=}Z(Y)$. Then $Y{\leqslant}Y_0$, and so $D_0{\leqslant}D$. We must show that $J(Q,D_0){\in}\Gamma$ for each $Q{\in}\Gamma$. Set $\widetilde{M}{=}M/C_M(D)$. Since $Q{\in}\Gamma$, we have $J(Q,D){\neq}Y$, and so there exists A with $Y{\leqslant}A{\leqslant}Q$ such that the image $\tilde{A}$ of A in $\widetilde{M}$ is a best offender on D. By Lemma 6.6 (c), $\mathrm{Aut}_A(D_0)$ is a best offender on D_0, so $J(Q,D)C_Q(D_0){\leqslant}J(Q,D_0)$. Thus $J(Q,D_0){\in}\Gamma$, as required.

Now Proposition 6.11 yields an extension of γ to an automorphism β of M_0. Then $\beta{=}c_z$ for some $z{\in}Z(S)$ by Proposition 1.10 (c), and c_z is then also an extension of γ to an automorphism of M. □

In what follows, set $V{=}Z_Y$. That is, V is the subgroup (to be regarded as a vector space over the field $\mathbb{F}_p$ of p elements) of D consisting of those elements $x{\in}D$ such that $x^p{=}1$. Set $G{=}M/C_M(D)$, and recall from Lemma 6.4 (a) that also $G{=}M/C_M(V)$.

LEMMA 7.3. *G is generated by quadratic best offenders on D, and any quadratic best offender on D is also a quadratic best offender on V.*

Proof. Let G_0 be the subgroup of G generated by the set of all subgroups A of G such that A is a quadratic best offender on D. Let M_0 be the preimage of G_0 in M, and set $S_0{=}S{\cap}M_0$. Then $Y{\leqslant}O_p(M_0)$. The reverse inclusion holds since $M_0{\trianglelefteq}M$, so $Y{=}O_p(M_0)$.

Let $Q{\in}\Gamma$, and set $P{=}Q{\cap}S_0$. Here $Y{=}C_S(D){\leqslant}S_0$, and the image $\widetilde{Q}$ of Q in G contains a best offender on D, so by Lemma 6.6 (b) there is a non-trivial quadratic best offender $\widetilde{B}{\leqslant}\widetilde{Q}$. The preimage B of $\widetilde{B}$ in S is contained in P, by the definition of M_0, so

$\widetilde{P}\neq 1$. Now let Φ be the set of all subgroups $Q\cap S_0$ of S_0 with $Q\in\Gamma$. Thus,

$$\Phi\subseteq\Gamma. \tag{$*$}$$

Let $\mathcal{K}$ be the partial normal subgroup $M_0\cap\mathcal{L}$ of $\mathcal{L}$, as given by Lemma 3.9. Then Lemma 3.10 gives $\mathcal{K}$ the structure of a locality $\mathcal{K}=\mathcal{L}_\Phi(M_0)$. Further, the hypothesis of Lemma 6.19 holds with (M_0,S_0,Γ_0) in the role of (M,S,Γ). By Lemma 4.9 (a), γ restricts to an automorphism γ_0 of $\mathcal{K}$, and if $M_0\neq M$ then γ_0 extends to an automorphism β_0 of M_0 by Lemma 6.19. Then γ extends to an automorphism of M by Lemma 4.9 (b), contrary to the hypothesis, and completing the proof. □

Any group that acts faithfully and quadratically on an elementary abelian p-group is itself elementary abelian, by [13, statement 9.1.1 (c)]. For that reason the preceding lemma effects the transition from "abelian offenders on abelian p-groups" to "elementary abelian offenders on vector spaces over $\mathbb{F}_p$" that is needed in order to apply the results of Meierfrankenfeld and Stellmacher [15, Theorems 1 and 2] on FF-pairs.

The set of all non-identity subgroups X of G such that $X=[X,G]$ is a poset, with respect to inclusion, consisting of normal subgroups of G. Define $\mathbf{X}$ to be the set of minimal elements of this poset. The elements of $\mathbf{X}$ are the *J-components* of G. The product of the set $\mathbf{X}$ of J-components is then a normal subgroup of G, henceforth to be denoted by G_0. Set $V=Z_Y$ and set

$$W=[V,G_0]C_V(G_0)/C_V(G_0).$$

For any $X\in\mathbf{X}$ set $W_X=[W,X]$ and $V_X=[V,X]$.

It will turn out that for each $X\in\mathbf{X}$, either X is quasisimple, or $p\in\{2,3\}$ and X is isomorphic to the commutator subgroup of $\mathrm{SL}_2(p)$ (a group of order 3 if $p=2$, and a quaternion group of order 8 if $p=3$). It will be convenient to set up some further notation, in order to accomodate such solvable J-components. Thus, let $\mathbf{X}_{\mathrm{sol}}$ be the set of all subgroups X of G such that X is a direct factor of G, $X\cong\mathrm{SL}_2(p)$ (with $p=2$ or $p=3$), $[X,X]\in\mathbf{X}$, and $|V_X|=p^2$. Let $\mathbf{X}^*$ be the union of $\mathbf{X}_{\mathrm{sol}}$ and the set of non-solvable J-components of G. The elements of $\mathbf{X}^*$ will be referred to as the *J^*-components* of G. Set $G_0^*=\langle\mathbf{X}^*\rangle$.

Theorem 7.4. *The following statements hold*:

(a) *Each J^*-component of G is normal in G.*

(b) *G_0^* is the direct product of the J-components of G.*

(c) *Let $A\leqslant G$ be a best offender on V. Then A is a best offender on every A-invariant subspace of V and on every A-invariant subspace of W.*

(d) *If* $A \leqslant G$ *is a best offender on* V, *and* $X \in \mathbf{X}$, *then* $C_A(X)=C_A(V_X)=C_A(W_X)$, *and either* $[X,A]=1$ *or* $[X,A]=X$.

(e) W *is the direct sum of its subspaces* W_X *for* $X \in \mathbf{X}^*$, *and* $[V,X_1,X_2]=0$ *whenever* X_1 *and* X_2 *are distinct members of* $\mathbf{X}^*$.

(f) $G=G_0R$ *where* R *is the image of* S *in* G.

Proof. See [15, Theorem 1]. We remark that points (a)–(e) are "elementary" in that they are proved without appealing to the CFSG. □

In Theorem 7.5 and Proposition 7.6 we sum up the remaining parts of the general FF-module theorem in the form that will be needed here, and eliminate some special cases. As in [15] we write $U^{(r)}$ for the direct sum of r copies of a module U.

THEOREM 7.5. *Suppose that* G *has only one* J^**-component. Then one of the following holds* (*where* q *is a power of* p):

(1) G *is a linear group* $\mathrm{SL}_n(q)$, *and* W *is a direct sum* $U^{(r)} \oplus (U^*)^{(s)}$, *where* U *is a natural module for* G *and* U^* *is the dual of* U. *Moreover*, *if both* r *and* s *are non-zero then* $n \geqslant 4$.

(2) G *is a classical group* (*unitary*, *symplectic*, *or orthogonal*) *in characteristic* p, *and* W *is a direct sum* $U^{(r)}$ *of natural modules for* G. *Specifically*,

(i) $G \cong \mathrm{Sp}_{2n}(q)$ $(n \geqslant 2)$;

(ii) $G \cong \mathrm{SU}_n(q)$ $(n \geqslant 4)$;

(iii) p *is odd*, $G \cong \Omega_{2n+1}(q)$ $(n \geqslant 2)$;

(iv) $G \cong \Omega^{\varepsilon}_{2n}(q)$ $(n \geqslant 3)$; *or*

(v) $p=2$, $G \cong O^{\varepsilon}_{2n}(q)$ $(n \geqslant 3)$.

(3) $p=2$, G *is a symmetric group of degree* $n \geqslant 5$, *and* W *is a natural module for* G.

Proof. Let R be the image of S in G. In the list of groups and their modules given by [15], all but the three types listed above are eliminated by Lemma 7.2. In detail: by [15, Theorem 2], G_0 is either a group of Lie type in characteristic p or an alternating group. If G_0 is of Lie type in characteristic p then, in the cases other than the above three, [15] states that either R contains a unique quadratic best offender A or that $G \cong \mathrm{Spin}_7(q)$ and that W is a spin module of order q^8. In the case of a unique quadratic offender A, it follows from Lemma 6.6 (b) that the image in G of the normalizer in M of any object in Γ is contained in $N_G(A)$. Since every element of $\mathcal{L}$ is a product of elements of normalizers of objects, by Proposition 2.17, it follows that $\mathcal{L}$ is contained in the proper subgroup $C_M(D)N_M(B)$, where B is the preimage of A in S; and we thereby obtain a contradiction to Lemma 7.2. In the case where $G \cong \mathrm{Spin}_7(q)$, it is pointed out in [15, Theorem 2] that every quadratic best offender A has the same commutator space on W. Then $\mathcal{L}$ is contained in the M-stabilizer of that subspace, and so in either case we contradict Lemma 7.2.

Finally, if $G_0=G$ is an alternating group, then [15] (see Proposition 7.6 immediately below) says that R contains a unique quadratic best offender, leading again to a contradiction to Lemma 7.2. □

PROPOSITION 7.6. *Suppose that G is a symmetric group* $\mathrm{Sym}(n)$ $(n\geqslant 5)$, *and that W is a natural module for G. Let R be the image of S in G. Then every offender in G on W is a best offender, and one of the following holds*:

(1) *n is odd, and each offender A is generated by transpositions*;

(2) *n is even, and for any quadratic offender $A\leqslant R$, there exists a set $\{t_1,...,t_k\}$ of pairwise commuting transpositions in R such that one of the following holds*:

(a) $A=\langle t_1,...,t_k\rangle$;

(b) *$n=2k$ and $A=\langle t_1t_2,...,t_{l-1}t_l\rangle\times\langle t_{l+1},...,t_k\rangle$ for some l with $1<l\leqslant k$*;

(c) *$n=2k$ and $A=\langle t_1t_2, s_1s_2\rangle\times\langle t_3,...,t_k\rangle$, where s_1 and s_2 are commuting transpositions distinct from t_1 and t_2, and where* $\mathrm{Supp}(s_1s_2)=\mathrm{Supp}(t_1t_2)$;

(d) *$n=8=|A|=|W/C_W(A)|$, and A acts regularly on the standard G-set.*

Moreover, if A is a quadratic offender and $|A|>|W/C_W(A)|$, then n is even, and A is generated by the set of all transpositions in R.

Proof. See [15, Theorem 2]. □

LEMMA 7.7. *G has a unique J^*-component.*

Proof. Suppose this is false, let $\overline{K}:=\overline{K}_1$ be a J^*-component of G, and let $\overline{K}_2$ be the product of the J^*-components other than $\overline{K}$. Let K_i be the preimage of $\overline{K}_i$ in M. Then K_i is S-invariant by Theorem 7.4 (a). Set $W_i=C_W(K_i)$. Then W_i is S-invariant, as is $C_M(W_i)$. Set $M_i=C_M(W_i)S$. Then $M_i=K_iS$ by Theorem 7.4 (e).

Set $\mathcal{L}_i=\mathcal{L}_\Gamma(M_i)$, and set $\mathcal{L}=\mathcal{L}_\Gamma(M)$. Further, define $\mathcal{C}_i$ to be the set of all $g\in\mathcal{L}$ such that $[W_i,g]=1$. Then $\mathcal{C}_i$ is a partial subgroup of $\mathcal{L}$, and in fact a partial normal subgroup since each W_i is M-invariant. Since Γ is S-invariant, an element g of M is in $\mathcal{L}_1$ if and only if $g=hs$ for some $h\in C_M(W_2)$ such that $S_g\in\Gamma$, and thus $\mathcal{L}_i=\mathcal{C}_iS$. Also, for any $h\in\mathcal{C}_i$ we have $h\gamma\in\mathcal{C}_i$, since γ centralizes V by rigidity. Thus $\mathcal{L}_i$ is a γ-invariant locality contained in $\mathcal{L}$. Let γ_i be the restriction of γ to $\mathcal{L}_i$.

We have $M_i=K_iS$ by Theorem 7.4 (e), so M_i is a proper subgroup of M. We may then apply Proposition 6.11 with (M_i,Y) in place of (M,X), and thereby conclude that γ_i extends to an automorphism β_i of M_i. Then β_i centralizes S, and since $C_M(Y)\leqslant Y$ it follows from Proposition 1.10 (c) that β_i is conjugation by z_i for some $z_i\in Z(S)$.

Set $D_i=[D,K_i]$. Then $Z(S)\leqslant C_D(K_i)D_i$ by Lemma 6.2 (a), and we may therefore take $z_i\in D_i$. We now claim that z_1 centralizes K_2 (and by symmetry of argument, that z_2

centralizes K_1). To prove the claim, we recall that $[W, K_1]$ centralizes K_2. Observe also that D_1 is K_2-invariant, since $K_1 \trianglelefteq M$. Set $V_1 = V \cap D_1$ and set $U_1 = [V_1, K_1]/C_{[V_1,K_1]}(K_1)$. Then U_1 is M-isomorphic to W_1, and so $[U_1, K_2] = 1$. We now apply Lemma 6.2 (c) with D_1, V_1, and K_2 in place of D, V, and X, and conclude that $[C_{D_1}(S), K_2] = 1$. Thus, $[z_1, K_2] = 1$ as claimed, and similarly $[z_2, K_1] = 1$.

Since $M = M_1 M_2 = K_1 K_2 S$ it now follows that g extends to the automorphism $c_{z_1 z_2}$ of M. This contradicts the choice of (M, γ), and completes the proof. □

LEMMA 7.8. *Let T be a subgroup of S, such that T is weakly closed in $\mathcal{F}$ and properly contains Y. Let Γ^+ be the union of Γ and the set of subgroups P of S such that $T \leqslant P$. Then γ extends to an automorphism of $\mathcal{L}_{\Gamma^+}(M)$.*

Proof. Set $\Phi = \Gamma^+$, $\mathcal{K} = \mathcal{L}_\Phi(M)$, $\mathcal{L}_T = N_{\mathcal{L}}(T)$, and let Γ_T be the set of all $Q \in \Gamma$ such that $T \leqslant Q$. Then $\mathcal{L}_T = \mathcal{L}_{\Gamma_T}(N_M(T))$ since $T \leqslant S_w$ for all $w \in \mathbf{D}(\mathcal{L}) \cap \mathbf{W}(N_M(T))$.

Let H be the reduced core of $N_M(T)$ with respect to S, set $Y_T = O_p(H)$, and set $D_T = Z(Y_T)$. Also, set $\mathcal{H} = N_{\mathcal{L}}(Y_T)$, let γ_T be the restriction of γ to $\mathcal{L}_T$, and let η be the restriction of γ to $\mathcal{H}$. Recall that Y_T is weakly closed in $N_{\mathcal{F}}(T)$ by Lemma 6.5. Then, since T is weakly closed in $\mathcal{F}$, it follows that also Y_T is weakly closed in $\mathcal{F}$. If $Y_T \in \Gamma$ then $\mathcal{H} = H$ by Lemma 2.3 (c), and η is an automorphism of H. On the other hand, suppose that $Y_T \notin \Gamma$, and let $\mathcal{R}$ be the set of all $R \in \Gamma$ containing Y_T. Since $J(Q, V) \in \Gamma$ for all $Q \in \Gamma$, by Lemma 6.6 (c), it follows that $J(R, V) \nleq Y_T$ for any $R \in \mathcal{R}$. Setting $V_T = \Omega_1(D_T)$, Lemma 6.6 (c) yields also $J(R, V_T) \neq Y_T$ for $R \in \mathcal{R}$. Similarly, $J(S, D) \nleq Y_T$, and then $J(S, D_T) \neq Y_T$. Thus, the hypothesis of Proposition 6.11 is satisfied with $(H, \mathcal{R})$ in place of (M, Γ), and we conclude that η extends to an automorphism of H. Thus, in any case, η either is, or extends to, an automorphism of H, and then Lemma 6.8 (b) implies that γ_T extends to an automorphism β_T of $N_M(T)$.

Since T is weakly closed in $\mathcal{F}$, it is vacuously true that any pair of distinct $\mathcal{F}$-conjugates of T generates a member of Γ. Then Theorem 5.15 (a) implies that γ extends to an automorphism β of $\mathcal{K}$, such that β restricts to γ on $\mathcal{L}$ and to β_T on $N_M(T)$. This yields the lemma. □

Definition 7.9. Let $\overline{S}$ be the image of S in G, and let $\mathcal{Q}$ be the set of all non-identity subgroups U of $\overline{S}$ such that $N_{\overline{S}}(U) \in \mathrm{Syl}_p(N_G(U))$ and such that $N_G(U)/U$ has a strongly p-embedded subgroup. We say that G has an *essential splitting* if there exists a subgroup H of G having the following properties:

(1) $N_G(\overline{S}) \leqslant H$;

(2) $O_p(H) \neq 1$ and $O_p(H)$ is weakly closed in $\mathcal{F}_{\overline{S}}(G)$;

(3) For each $U \in \mathcal{Q}$, either $O^{p'}(N_G(U)) \leqslant H$ or $O^{p'}(N_G(U))$ centralizes $C_W(U)$.

PROPOSITION 7.10. *G has no essential splitting.*

Proof. Suppose this is false, and let H be a subgroup of G satisfying conditions (1)–(3) in Definition 7.9. Set $\overline{B}=O_p(H)$, and let B be the preimage of $\overline{B}$ in S. Then we have that $B=S\cap C_M(D)B$ is a Sylow p-subgroup of $C_M(D)B$, and it follows that B is weakly closed in $\mathcal{F}:=\mathcal{F}_S(M)$.

Let Σ be the union of Γ and the set of overgroups of B in S. By Definition 5.17 there is an $\mathcal{F}$-filtration $\mathbf{F}=(\Sigma_i,R_i)_{i=0}^N$ of Σ with $R_0=B$, and with the property that R_i is fully normalized in $\mathcal{F}$, and of maximal order subject to $R_i\notin\Sigma_{i-1}$ $(1\leqslant i\leqslant N)$. Thus, Σ_0 is the set of overgroups of B in S, and $\Sigma_N=\Sigma$.

Set $\mathcal{K}=\mathcal{L}_\Sigma(M)$ and let $\mathcal{K}_i$ be the restriction of $\mathcal{K}$ to Σ_i, as in Example/Lemma 2.10. Thus $\mathcal{K}_0=N_M(B)$. Let σ be an extension of γ to an automorphism of $\mathcal{K}$, as given by Lemma 7.8, and let σ_i be the restriction of σ to $\mathcal{K}_i$. Then $\sigma_N=\sigma$, while σ_0 is a rigid automorphism of the group $N_M(B)$. Since H contains the image of S in G, we have $S\leqslant N_M(B)$, and then Proposition 1.10 (c) yields $\sigma_0=c_z$ (conjugation by z) for some $z\in Z(S)$. If also σ is given on all of $\mathcal{K}$ by z-conjugation, then γ is given by z-conjugation on $\mathcal{L}$, and γ extends to the automorphism c_z of M. Since (M,γ) is a counterexample to Proposition 7.1, we conclude that the largest index m such that $\sigma_m=c_z$ is smaller than N.

Set $R=R_{m+1}$, $L=N_M(R)$, and let X be the subgroup of L generated by $C_L(Z(S))$ together with the set of all $N_L(P)$ as P varies over the set of proper overgroups of R in $N_S(R)$. Each such P is in Σ_m by the construction of $\mathbf{F}$. The restriction of σ_L to L (see Lemma 3.8) acts on X, and $\sigma_L\circ c_{z^{-1}}$ centralizes a set of generators for X, so σ_L acts as c_z on X. If $X=L$ then Theorem 5.15 (a) implies that σ_{m+1} is given by c_z on all of $\mathcal{K}_{m+1}$, contrary to the choice of m. Thus $X\neq L$, and hence X/R is strongly p-embedded in L/R. Moreover, we now have $[Z(S),O^{p'}(L)]\neq 1$.

For any subgroup E of L, let $\overline{E}$ be the image of E in G. Then $\bar{L}\neq\overline{X}$ as $C_L(D)\leqslant X$. Then $\overline{X}/\bar{R}$ is strongly p-embedded in $\bar{L}/\bar{R}$, and then condition (3) in Definition 7.9 says that either $O^{p'}(\bar{L})\leqslant H$ or $[C_W(S),O^{p'}(\bar{L})]=1$. Suppose $O^{p'}(\bar{L})\leqslant H$. As $B\leqslant R_i$ for all i, where B is weakly closed in $\mathcal{F}$, it follows that $L\leqslant N_M(B)$, and hence $\sigma_L=c_z$. Again, Theorem 5.15 (a) implies that $\sigma_{m+1}=c_z$ on $\mathcal{K}_{m+1}$, contradicting the choice of m. Thus, $O^{p'}(\bar{L})$ centralizes $C_W(S)$, and then $O^{p'}(\bar{L})$ centralizes $Z(S)$ by Lemma 6.2 (c). Then also $O^{p'}(L)$ centralizes $Z(S)$.

Set $\mathcal{C}=\mathcal{L}_\Sigma(C_M(Z(S)))$. Then $\mathcal{C}$ is the set of all $f\in\mathcal{K}$ such that $[Z(S),f]=1$, and thus $\mathcal{C}$ is σ-invariant. Since $|\mathcal{C}|<|\mathcal{L}|$, Lemma 6.19 applies and yields an extension of σ to an automorphism σ^* of $C_M(Z(S))$. Then σ^* is the identity automorphism, by Proposition 1.10 (c), and thus σ_L is the identity automorphism of L. Hence $\sigma_L=c_z$, and we again have a contradiction via Theorem 5.15 (a). □

In the remaining arguments, whenever X is a subgroup of M we write $\overline{X}$ for the image $C_M(D)X/C_M(D)$ of X in G.

LEMMA 7.11. *Assume that G is one of the classical groups that appear in Theorem 7.5. Then either G has an essential splitting in the sense of Definition 7.9, or $G{=}\mathrm{SL}_2(q)$ (q being a power of p).*

Proof. In Theorem 7.5 it is given that W is a direct sum of copies of the natural G-module U, or that $G{=}\mathrm{SL}_n(q)$ ($n{\geqslant}4$) and W is a direct sum of copies of U and its dual U^*. Then $C_G(C_W(S))$ is equal to either $C_G(C_U(S))$ or, in the exceptional case, to $C_G(C_U(S)){\cap}C_G(C_{U^*}(S))$. Since the definition of essential splitting depends only on G, and on the subgroup $C_G(C_W(S))$ of G, we may therefore assume, for the sake of simplicity, that $W{=}U$ or, exceptionally, that $W{=}U{\oplus}U^*$. In either case, write W_0 for the subspace U of W.

Set $G_0{=}[G,G]$ if $p{=}2$ and $G{=}O^{\varepsilon}_{2n}(q)$, and otherwise set $G_0{=}G$. Let T be the Sylow p-subgroup $\bar{S}$ of G, set $T_0{=}T{\cap}G_0$, let $\mathcal{P}$ be the set of minimal parabolic subgroups L of G_0 over T_0 such that $C_W(T_0)$ is not L-invariant, and set $H{=}\langle\mathcal{P}\rangle T$. Let $\mathcal{Q}$ be the set of all subgroups Q of T such that $N_T(Q){\in}\mathrm{Syl}_p(N_G(Q))$ and such that $N_G(Q)/Q$ has a strongly p-embedded subgroup. Set $K{=}C_G(C_W(T))$. It will then suffice to show

(a) $O_p(H)$ is weakly closed in $\mathcal{F}_T(G)$, and if $G{\neq}\mathrm{SL}_2(q)$ then $O_p(H){\neq}1$;

(b) there exists a parabolic subgroup K^* of G_0 such that $O^{p'}(K^*)T{\leqslant}K{\leqslant}K^*T$;

(c) for each $Q{\in}\mathcal{Q}$, either $N_G(Q){\leqslant}H$ or $O^{p'}(N_G(Q))$ centralizes $C_U(T)$.

Indeed, only (a) and (c) are needed, but (b) will play a role in obtaining these points.

We note at the outset that $|G/G_0|{\leqslant}2$ and that $p{=}2$ if $|G/G_0|{\neq}1$.

Set $K^*{=}N_{G_0}(C_W(T_0))$. If W is irreducible then $C_W(T_0)$ is a 1-dimensional subspace of W, and K^* is a maximal parabolic subgroup of G_0 over T_0. On the other hand, if W is reducible, so that $G{=}SL_n(q)$ with $n{\geqslant}4$, then K^* is the parabolic subgroup of Lie corank 2 obtained as the intersection of the two maximal parabolic subgroups L_1 and L_2 such that $L_i/O_p(L_i){\cong}\mathrm{GL}_{n-1}(q)$. Then $O^{p'}(K^*)$ centralizes $C_U(T_0)$ and we obtain (b). Further,

(1) either there is a unique minimal parabolic subgroup X of G_0 over T_0 not contained in K^*, or there are two such (to be denoted X_1 and X_2). In the latter case, X_1X_2 is a group, and $O^{p'}(X_1X_2)/O_p(X_1X_2){\cong}\mathrm{SL}_2(q){\times}\mathrm{SL}_2(q)$.

Set $H_0{=}H{\cap}G_0$. Then (1) shows that either $G{=}G_0$ and H is a parabolic subgroup of G; or else $G{\neq}G_0$, $H{=}H_0T$, and H_0 is a minimal parabolic subgroup of G_0. Set $P{=}O_p(H)$ and set $P_0{=}P{\cap}G_0$. By Lemma 1.12, P_0 is weakly closed in $\mathcal{F}_{T_0}(G_0)$. As $H{=}N_G(P_0){=}N_G(P)$, P is weakly closed in $\mathcal{F}{:=}\mathcal{F}_T(G)$. If $H{=}G$ then G_0 is itself a minimal parabolic subgroup of G_0, since the exceptional case where H_0 is not a minimal parabolic subgroup occurs only when the Lie rank of G_0 is greater than that of H_0. In the list of groups under consideration from Theorem 7.5, only $\mathrm{SL}_2(q)$ has Lie rank equal to 1, so (a) holds.

Let $Q\in\mathcal{Q}$, set $N=N_G(Q)$ and $N_0=N\cap G_0$. Also, set $E=O^{p'}(N)$, $E_0=E\cap G_0$, and $Q_0=Q\cap G_0$. In proving (c) we may assume that $[C_W(T),E]\neq 0$.

As N/Q has a strongly p-embedded subgroup we have $Q=O_p(N)$. Suppose that $N\leqslant G_0$. Since $N_T(Q)\in\mathrm{Syl}_p(N_G(Q)$ by the definition of $\mathcal{Q}$, a theorem of Borel and Tits [12, Theorem 3.1.3] implies that $N_{G_0}(Q)$ is a parabolic subgroup of G_0 over T_0. Thus N/Q is a group of Lie type, as is E/Q. The only groups of Lie type having a strongly p-embedded subgroup are those of Lie rank 1, so N is a minimal parabolic subgroup of G_0. Since $[C_U(T),E]\neq 0$, we have $[C_U(T_0),E]\neq 0$, and thus N is either the unique minimal parabolic subgroup over T_0 which does not normalize $C_U(T_0)$, or N is one of the two such minimal parabolic subgroups given by (1). Thus $N\leqslant H$, and we have (c) in this case. We have thus reduced the proof of (c) to the case where $G\neq G_0$, so $p=2$ and $G=O^{\varepsilon}_{2n}(q)$ for some sign ε.

Suppose next that $Q\leqslant G_0$. Then $O_2(N_0)\leqslant O_2(N)=Q$, so the Borel–Tits theorem implies that N_0 is a parabolic subgroup of G_0 over T_0. Let K be an overgroup of Q in N such that K/Q is strongly embedded in N/Q, and set $K_0=K\cap G_0$. Then $|K_0/Q|$ is divisible by 2 since $N_{T_0}(Q)\leqslant K$, and hence K_0/Q is strongly embedded in N_0/Q. As in the preceding paragraph, it follows that N_0 is a minimal parabolic subgroup over T_0, and N_0 is then the unique minimal parabolic subgroup over T_0 which does not normalize $C_W(T_0)$. Then $N_0\leqslant H$, and since $N=N_0T$ where $T\leqslant H$, (c) holds in this case. We may therefore assume that $Q\nleqslant G_0$.

Set $R=O_p(N_{G_0}(Q_0))$. Then R is E-invariant, as is $N_R(Q_0)$. Hence $N_{QR}(Q)\leqslant Q$ as $Q=O_p(N)$, and thus $R=Q_0$. By the Borel–Tits theorem, $N_{G_0}(Q_0)$ is a parabolic subgroup of G_0 over T_0.

Suppose $Q_0=1$. Then $|Q|=2$ and $O_2(C_{G_0}(Q))=1$. Hence [4, statement 8.7] implies that $E_0\cong Sp_{2n-2}(q)$. But $E_0\cong E/Q$ in this case, so E_0 has a strongly embedded subgroup. This yields $n=2$, whereas Theorem 7.5 (2) excludes $O^{\varepsilon}_4(q)$. Thus $\overline{Q}_0\neq 1$, and $N_{G_0}(Q_0)$ is a proper parabolic subgroup of G_0.

Set $L=O^{2'}(N_{G_0}(Q_0))$, and set $\tilde{L}=L/Q_0$. The Levi decomposition for $N_G(Q_0)$ yields a direct product decomposition

$$\tilde{L}=\tilde{L}_1\times\ldots\times\tilde{L}_k,$$

where each $\tilde{L}_i$ is a (possibly disconnected) group of Lie type, and where each $\tilde{L}_i$ is $\widetilde{Q}$-invariant. We choose such a decomposition so as to maximize k. Then

$$\widetilde{E}_0=C_{\widetilde{E}_1}(\widetilde{Q})\times\ldots\times C_{\widetilde{E}_k}(\widetilde{Q}),$$

and each $C_{\widetilde{E}_i}(\widetilde{Q})$ has even order. But $\widetilde{E}_0 \cong E/Q$, so $\widetilde{E}_0$ has a strongly embedded subgroup. We conclude that $k=1$.

Suppose that $N_{G_0}(Q_0)$ is disconnected. Then $\tilde{L}$ is a direct product of two factors (each of them a group of Lie type) which are interchanged by $\widetilde{Q}$, and $\widetilde{E}$ is isomorphic to each of the factors. Thus $\widetilde{E} \cong \mathrm{SL}_2(q)$ as $\widetilde{E}$ has a strongly embedded subgroup, and $N_{G_0}(Q_0)$ is a product of two minimal parabolic subgroups, with root system of type $A_1 \times A_1$ acted on non-trivially by T/T_0. But there is a unique minimal parabolic subgroup X over T_0 which does not normalize $C_U(T_0)$, and hence $N_{G_0}(Q_0)$ normalizes $C_U(T_0)$. Hence $[C_U(T_0), E_0]=1$ and $[C_U(T), E]=1$, contrary to the choice of Q. We conclude that $N_{G_0}(Q_0)$ is a connected parabolic subgroup, and moreover, $X=N_{G_0}(Q_0)$.

From the discussion in [12, §2.7] on the relation between classical groups and groups defined as groups of Lie type, it now follows that $N_{G_0}(Q_0)$ is contained in the stabilizer of a singular j-space U for some $j>1$, and that L/Q_0 acts as $\mathrm{SL}_j(q)$ on U. Let t be an $\mathbb{F}_q$-transvection in T on U. Then t centralizes every singular T_0-invariant subspace of U, so $[L,t] \leqslant O_2(L)=Q_0$. Since $E_0 \leqslant L$, $Q_0\langle t\rangle$ is then an E-invariant subgroup of T. But Q_0 is the unique largest E_0-invariant subgroup of T_0, so Q is the unique largest E_0-invariant subgroup of T, and thus $Q=Q_0\langle t\rangle$. Then Q is L-invariant, and so $L=E_0$. Again, as E_0/Q_0 has a strongly embedded subgroup, we conclude that L/Q_0 has Lie rank 1, and thus $E_0=O^{2'}(X)$. As $H=XT$ we obtain $E=E_0T \leqslant H$. Further, we now have $H=N_G(Q_0)$, and so $N \leqslant H$. This completes the proof of (c). □

Remark. The group Sym(8) is well known to be isomorphic to $O_6^+(2)$, by means of a quadratic form on the natural irreducible Sym(8)-module preserved by Sym(8). Thus, the preceding lemma applies to the case where $G=\mathrm{Sym}(8)$ and W is the natural irreducible module.

LEMMA 7.12. *Suppose that $p=2$, and suppose that $G=\mathrm{Sym}(n)$ is a symmetric group, with $n \geqslant 5$, and that W is isomorphic to the natural irreducible G-module over $\mathbb{F}_2$. Then $n=8$, and W is a natural module for G.*

Proof. We assume throughout that $n \neq 8$. Let $\Omega=\{1,2,...,n\}$ be the standard G-set, and let $G_0=\mathrm{Alt}(n)$ be the subgroup of index 2 in G. Let $\widetilde{V}$ be the natural permutation module for G, identified with the set 2^Ω of subsets of Ω (where addition is given by symmetric difference of subsets). Let $\widetilde{W}$ be the submodule of $\widetilde{V}$ consisting of subsets of Ω of even order, and let $\widetilde{Z}$ be the 1-dimensional submodule $\{\varnothing, \Omega\}$ of $\widetilde{V}$. By definition, $W \cong \widetilde{W}$ if n is odd, and $W \cong \widetilde{W}/Z$ if n is even.

Recall the notation $V=\Omega_1(D)$. We now write $V=V_0 \times V_1$, where $V_1 \leqslant C_V(G)$, and where V_1 is a G-submodule of V chosen to be as small as possible subject to $V=V_0C_V(G)$. Thus, V_0 is indecomposable for G. We claim that V_0 is isomorphic to a G-submodule of $\widetilde{V}$ or of $\widetilde{V}/Z$. Indeed, by an elementary calculation [1, p. 74, Exercise 3] we have $|H^1(G_0,W)|=1$ if n is odd, $|H^1(G_0,W)|=2$ if n is even, and $\widetilde{V}$ indecomposable for G_0 if n is even. The claim follows in a straightforward way from this exercise, and from the observations made in the preceding paragraph. In particular, transpositions in G are transvections on V_0, and hence also on V.

Set $K=C_M(C_W(S))$ and let $\overline{K}$ be the image of K in G. Then $K\leqslant C_M(C_D(S))$ by Lemma 6.2 (c). By Lemma 3.8 there is an automorphism γ_K of $\mathcal{L}_\Gamma(K)$ given by restricting γ. Since $|K|<|M|$, and since best offenders on D are also best offenders on $C_D(O_2(K))$ by Lemma 6.6 (c), we may apply Lemma 6.19 with $(K,S,O_2(K),\Gamma)$ in the role of (M,S,Y,Γ). Thus γ_K extends to an automorphism of K. As $[Z(S),K]=1$, Proposition 1.10 (c) yields

(1) The identity map on K is the unique extension of γ_K to an automorphism of K.

Let $\mathcal{T}$ be the set of subgroups T of S such that $Y\leqslant T$, and such that $\overline{T}$ is generated by a transposition. For any subgroup P of S set

$$P_{\mathcal{T}}=\langle T\in\mathcal{T}\mid T\leqslant P\rangle Y.$$

Set $R=S_{\mathcal{T}}$, set $R_0=R\cap M_0$, where M_0 is the preimage of $\mathrm{Alt}(n)$ in M, and set $H=N_M(R)$. Then $H=N_M(R_0)$, and $\overline{H}\cong 2^m\rtimes\mathrm{Sym}(m)$, where m is the greatest integer $\leqslant\frac{1}{2}n$.

If the set $\{\bar{R},\bar{R}_0\}$ contains all of the quadratic best offenders on D contained in $\bar{S}$, then every member of Γ contains R_0, and then $\mathcal{L}=\mathcal{L}_\Gamma(M)=H$, contrary to Lemma 7.2. Thus there exists a quadratic best offender $\bar{A}\leqslant\bar{S}$ with $\bar{A}\notin\{\bar{R},\bar{R}_0\}$. As $n\neq 8$ by assumption, Proposition 7.6 implies that $|W/C_W(\bar{A})|=|\bar{A}|$ and that $\bar{A}$ contains a transposition t. Notice that

$$|W/C_W(\bar{A})|\leqslant|V/C_V(\bar{A})|\leqslant|D/C_D(\bar{A})|. \qquad (*)$$

Since $\bar{A}$ is a best offender on D, and hence also on V by Lemma 6.6 (c), we conclude that the inequalities in $(*)$ are equalities, and hence that $D=C_D(\bar{A})V$. Then $C_D(\bar{A})C_V(t)$ has index 2 in D, and thus $|D/C_D(t)|=2$. Since G is generated by $n-1$ transpositions, we conclude that $|D/C_D(G)|\leqslant 2^{n-1}$, and hence

(2) $D=C_D(G)V$.

Further, since $|D/C_D(t)|=2$, the preimage in S of any subgroup of $\langle\mathcal{T}\rangle$ generated by transpositions is in Γ. Thus

(3) $\mathcal{T}\subseteq\Gamma$, and $Q_{\mathcal{T}}\in\Gamma$ for any $Q\in\Gamma$ such that $N_M(Q)\nleq H$.

Notice that (3) yields $O_2(H)\in\Gamma$, so $H=N_{\mathcal{L}}(O_2(H))$, and then γ restricts to an automorphism of H. By Proposition 1.10 (c), $\gamma|_H=c_z$ for some $z\in Z(S)$, and where of course c_z is also an automorphism of M. Replacing γ with $\gamma\circ c_z^{-1}$, we may assume

(4) γ restricts to the identity automorphism on H.

Suppose that $n{=}2m$ is even, let $Q{\in}\Gamma$ with $N_M(Q){\nleqslant}H$, and let Q_0 be a fully normalized $\mathcal{F}$-conjugate of $Q_{\mathcal{T}}$. As H is m-transitive on $\mathcal{T}$, Q_0 is in fact an H-conjugate of $Q_{\mathcal{T}}$. Set $X{=}N_M(Q_0)$. Then $\overline{X}{\cong}(2^k{\rtimes}\mathrm{Sym}(k)){\times}\mathrm{Sym}(n{-}2k)$, where k is the number of transpositions in $\overline{Q}_0$. Since n is even it follows from [8, Lemma 2.8] that $\overline{X}$ is generated by its subgroups $\overline{X}{\cap}\overline{H}$ and $\overline{X}{\cap}\overline{K}$. Since $C_X(D){\leqslant}K$, we then have $X{=}\langle X{\cap}H, X{\cap}K\rangle$. Let γ_X be the restriction of γ to X. Then (1) and (5) imply that γ_X induces the identity map on X. Here $N_M(Q_{\mathcal{T}}){=}X^h$ for some $h{\in}H$. We have $(h^{-1},x,h){\in}\mathbf{D}(\mathcal{L}_\Gamma(M))$ via $Q_{\mathcal{T}}$ for each $x{\in}X$, so γ restricts to the identity on $N_M(Q_{\mathcal{T}})$ by (5). Since $N_M(Q){\leqslant}N_M(Q_{\mathcal{T}})$ we conclude that γ restricts to the identity map on $N_M(Q)$. Thus γ is the identity automorphism of $\mathcal{L}$ in the case where n is even, and we may therefore assume that $n{=}2m{+}1$ is odd.

Let G_1 be a subgroup of G such that $\overline{S}{\leqslant}G_1$, and with $G_1{\cong}\mathrm{Sym}(2m)$. Let M_1 be the preimage of G_1 in M and set $\mathcal{L}_1{=}\mathcal{L}_\Gamma(M_1)$. Then $\mathcal{L}_1$ is γ-invariant by Lemma 3.8. Let γ_1 be the restriction of γ to $\mathcal{L}_1$. Then $M_1{=}\langle H,K\rangle$ (again by [8, Lemma 2.8]). The hypothesis of Proposition 7.1 holds with M_1 in place of M, so the minimality of $|M|$ implies that γ_1 extends to an automorphism β_1 of M_1. Then (1) and (5) imply that β_1 is the identity automorphism.

Now let G_2 be a subgroup of G such that $G_2{\cong}\mathrm{Sym}(m){\times}\mathrm{Sym}(m{+}1)$, and such that $\overline{S}{\cap}G_2{\in}\mathrm{Syl}_2(G_2)$. Let M_2 be the preimage of G_2 in M, set $S_2{=}S{\cap}M_2$, and let Γ_2 be the set of all subgroups Q of S_2 with $Q{\in}\Gamma$. One observes that $R{\leqslant}M_2$, so Γ_2 is non-empty. Set $\mathcal{L}_2{=}\mathcal{L}_{\Gamma_2}(M_2)$. Then $\mathcal{L}_2$ is γ-invariant by Lemma 3.8. Let γ_2 be the restriction of γ to $\mathcal{L}_2$. As $|M_2|{<}|M|$, Lemma 6.19 applies with (M_2,S_2,Γ_2) in the role of (M,S,Γ), yielding an extension of γ_2 to an automorphism β_2 of M_2. Then $\beta_2{=}c_u$ for some $u{\in}Z(S_2)$. As n is odd, and as remarked above, we have $H^1(G,W){=}0$. Thus V may be identified with $W{\times}C_V(G)$. As G is generated by $n{-}1$ transpositions, it follows that $|D/C_D(G)|{=}2^{n-1}$ and that $D{=}W{\times}C_D(G)$. Thus, we may take $u{\in}W$.

Recall that Ω denotes the standard G-set. We may then take G_1 to be the stabilizer in G of n, and we may take G_2 to be the stabilizer in G of the partition (Δ_1,Δ_2) of Ω, where $\Delta_1{=}\{1,...,m\}$. Identify W with the set of even-order subsets of Ω. As β_1 is the identity map on M_1, c_u centralizes $M_1{\cap}M_2$, and it follows that c_u is either the identity map on M_2 or that c_u is given on M_2 by taking $u{=}\Omega{\setminus}\{n\}$. In either case, the automorphism c_u of M induces β_i on M_i $(i{=}1,2)$. Replacing γ with $\gamma{\circ}c_u^{-1}$, we may assume that both β_1 and β_2 are identity maps.

We now argue as we did in the case where n is even, taking an arbitrary $Q{\in}\Gamma$, taking Q_0 to be a fully normalized $\mathcal{F}$-conjugate of $Q_{\mathcal{T}}$, and setting $X{=}N_M(Q_0)$. As before, we have $\overline{X}{\cong}(2^k{\rtimes}\mathrm{Sym}(k)){\times}\mathrm{Sym}(n{-}2k)$, and now $\overline{X}$ is generated by its subgroups $\overline{X}{\cap}G_1$ and $\overline{X}{\cap}G_2$. Thus $X{=}\langle X{\cap}M_1, X{\cap}M_2\rangle$. As each β_i is an identity map, the restriction

γ_X of γ to X is the identity map on X. Since $N_M(Q) \leqslant N_M(\Gamma_{\mathcal{T}})$, and $N_M(Q_{\mathcal{T}})$ is an H-conjugate of X, it follows as in the case where n is even that γ restricts to the identity map on $N_M(Q)$. Thus, γ is the identity automorphism of $\mathcal{L}$, and we have obtained a contradiction to the assumed non-existence of an extension of γ to an automorphism of M. □

Proof of Proposition 7.1. By Lemma 7.7, G has a unique J-component, and Theorem 7.5 then yields the possibilities for the structure of G and for the action of G on W. Recall that (M, γ) is a counterexample to Proposition 7.1. By Lemma 7.12, if G is a symmetric group $\mathrm{Sym}(n)$, and W is its natural irreducible module, then $n=8$. Since $\mathrm{Sym}(8) \cong O_6^+(2)$ via a quadratic form on the natural irreducible module for $\mathrm{Sym}(8)$ (see the remark following Proposition 7.10), Proposition 7.10 and Lemma 7.11 imply that G is a symmetric group $\mathrm{Sym}(n)$ with $n \neq 8$ and with W being the natural module—or else that $G = \mathrm{SL}_2(q)$. Thus, we have only the case where $G = \mathrm{SL}_2(q)$ and (by Theorem 7.5) W is the natural $\mathrm{SL}_2(q)$-module left to consider. But in this last case, G has a strongly p-embedded subgroup $N_G(\overline{S})$, and hence $C_M(D)N_M(S)$ is a proper subgroup of M containing $\mathcal{L}_\Gamma(M)$. This contradicts Lemma 7.2, and thus the proof of Proposition 7.1 is complete. □

Proof of the main theorem. As pointed out at the beginning of this section, Proposition 7.1 provides the remaining step required for the proof of Proposition 6.10. Then Proposition 6.11, and Lemmas 6.21–6.25—which were proved under the assumption that Proposition 6.10 holds—yield a contradiction to the presumed non-existence or non-uniqueness of a centric linking system $\mathcal{L}$ whose fusion system $\mathcal{F}_S(\mathcal{L})$ is a given saturated fusion system. □

Appendix A.

In [16], Bob Oliver and Joana Ventura introduced a category $\mathbf{T}$ of "transporter systems" and isomorphisms of transporter systems. Part of the structure of any given transporter system consists of a functor $\varrho: \mathcal{T} \to \mathcal{F}$, where $\mathcal{T}$ is a category and where $\mathcal{F}$ is a fusion system on a finite p-group; and one says that the given transporter system is a "transporter system on (or over) $\mathcal{F}$". The category $\mathbf{T}$ has a full subcategory $\mathbf{T}^c$ of "centric linking systems" whose definition is far different from the one given in Definition 2.9 here. The aim of this section is to show that transporter systems are the "same" as localities, that the two definitions of centric linking systems are essentially equivalent, and to obtain the following result.

Theorem A. *Let $\mathcal{F}$ be a saturated fusion system on a finite p-group S. Then there exists a centric linking system $\mathcal{T}$ over $\mathcal{F}$ (in the sense of* [16] *or* [7]*), and $\mathcal{T}$ is unique up to isomorphism of transporter systems.*

In this way we will establish that our main theorem yields existence and uniqueness of "centric linking systems" in either sense of this term. In order to do this, we first review the definitions in [16].

Let S be a finite p-group and let $\mathbf{X}$ be a collection of subgroups of S with $S\in\mathbf{X}$. There is then a category $\mathcal{T}_{\mathbf{X}}(S)$ whose set of objects is $\mathbf{X}$, and whose morphism-sets are given by

$$\mathrm{Mor}_{\mathcal{T}_{\mathbf{X}}(S)}(P,Q)=N_S(P,Q)$$

for $P,Q\in\mathbf{X}$. Composition is given by multiplication in S.

Here is the definition of transporter system from [16], but with the notions of left and right composition reversed from their original meanings, in order to maintain consistency with our policy of taking all categories in the right-handed sense.

Definition A.1. Let $\mathcal{F}$ be a fusion system over a finite p-group S. A *transporter system* associated with $\mathcal{F}$ is a non-empty finite category $\mathcal{T}$, together with a pair of functors

$$\mathcal{T}_{\mathrm{Ob}(\mathcal{T})}(S)\xrightarrow{\ \varepsilon\ }\mathcal{T}\xrightarrow{\ \varrho\ }\mathcal{F}$$

satisfying the following conditions:

(A1) $\mathrm{Ob}(\mathcal{T})\subseteq\mathrm{Ob}(\mathcal{F})$, and $\mathrm{Ob}(\mathcal{T})$ is closed under $\mathcal{F}$-conjugacy and overgroups. Also, ε is the identity on objects and ϱ is the inclusion on objects.

(A2) For each $P,Q\in\mathrm{Ob}(\mathcal{T})$, the kernel

$$E(P)\overset{\mathrm{def}}{=}\mathrm{Ker}[\varrho_P:\mathrm{Aut}_{\mathcal{T}}(P)\to\mathrm{Aut}_{\mathcal{F}}(P)]$$

acts freely on $\mathrm{Mor}_{\mathcal{T}}(P,Q)$ by left composition, and $\varrho_{P,Q}$ is the orbit map for this action. Also, $E(Q)$ acts freely on $\mathrm{Mor}_{\mathcal{T}}(P,Q)$ by right composition.

(B) For each $P,Q\in\mathrm{Ob}(\mathcal{T})$, $\varepsilon_{P,Q}\colon N_S(P,Q)\to\mathrm{Mor}_{\mathcal{T}}(P,Q)$ is injective, and the composition $\varrho_{P,Q}\circ\varepsilon_{P,Q}$ sends $g\in N_S(P,Q)$ to $c_g\in\mathrm{Hom}_{\mathcal{F}}(P,Q)$.

(C) For all $\phi\in\mathrm{Mor}_{\mathcal{T}}(P,Q)$ and all $g\in P$, the diagram

$$\begin{array}{ccc} P & \xrightarrow{\ \phi\ } & Q \\ {\scriptstyle\varepsilon_P(g)}\downarrow & & \downarrow{\scriptstyle\varepsilon_Q(g')} \\ P & \xrightarrow{\ \phi\ } & Q \end{array}$$

commutes in $\mathcal{T}$, where g' is the image of g under $\varrho(\phi)$.

(I) $\varepsilon_S(S)\in\mathrm{Syl}_p(\mathrm{Aut}_{\mathcal{T}}(S))$.

(II) Let $\phi\in\mathrm{Iso}_{\mathcal{T}}(P,Q)$, and let $P\trianglelefteq\overline{P}\leqslant S$ and $Q\trianglelefteq\overline{Q}\leqslant S$ be such that

$$\phi^{-1}\circ\varepsilon_P(\overline{P})\circ\phi\leqslant\varepsilon_Q(\overline{Q}).$$

Then there exists $\bar{\phi}\in\mathrm{Mor}_{\mathcal{T}}(\overline{P},\overline{Q})$ such that $\varepsilon_{P,\overline{P}}(1)\circ\bar{\phi}=\phi\circ\varepsilon_{Q,\overline{Q}}(1)$.

If moreover $\mathcal{F}$ is saturated, $\mathrm{Ob}(\mathcal{T})=\mathcal{F}^c$, and $E(P)=Z(P)$ for all objects P, then $\mathcal{T}$ is a *centric linking system*.

Definition A.2. Let $\mathcal{T}=(\mathcal{T},\varepsilon,\varrho)$ and $\mathcal{T}'=(\mathcal{T}',\varepsilon',\varrho')$ be transporter systems over a fusion system $\mathcal{F}$ on S, with $\mathrm{Ob}(\mathcal{T})=\mathrm{Ob}(\mathcal{T}')$. An *isomorphism* $\mathcal{T}\to\mathcal{T}'$ (of transporter systems) consists of an invertible functor $\alpha:\mathcal{T}\to\mathcal{T}'$ (of categories) such that, in right-hand notation, $\varepsilon\circ\alpha=\varepsilon'$ and $\alpha\circ\varrho'=\varrho$.

Let $\mathcal{L}=(\mathcal{L},\Delta,S)$ be a locality over S. Set

$$\mathcal{T}=\mathrm{Cat}(\mathcal{L},\Delta)$$

(as defined in Remark 2.8 (1)) and let $\mathcal{F}$ be the fusion system $\mathcal{F}_S(\mathcal{L})$ on S, generated by the conjugation maps between objects. There is a functor

$$\varepsilon:\mathcal{T}_\Delta(S)\longrightarrow\mathcal{T}$$

for which $\varepsilon_{\mathrm{Ob}}:\Delta\to\mathrm{Ob}(\mathcal{T})$ is the identity map, and where each

$$\varepsilon_{P,Q}:N_S(P,Q)\longrightarrow\mathrm{Mor}_{\mathcal{T}}(P,Q)$$

is an inclusion map. There is also a functor

$$\varrho:\mathcal{T}\longrightarrow\mathcal{F}$$

such that $\varrho_{\mathrm{Ob}}:\Delta\to\mathrm{Ob}(\mathcal{F})$ is the inclusion map of Δ into the set of all subgroups of S, and such that $\varrho_{P,Q}(\phi)$ is the conjugation map $c_g:P\to Q$, where g is the unique element of $\mathcal{L}$ such that $\phi=(g,P,Q)$. (See the discussion in Remark 2.8 (1)). We note that the functoriality of ϱ depends on condition (O2) in Definition 2.6 of “objective partial groups”.

Proposition A.3. (a) *Let $\mathcal{L}=(\mathcal{L},\Delta,S)$ be a locality. Then the diagram*

$$\mathcal{T}_\Delta(S)\xrightarrow{\ \varepsilon\ }\mathcal{T}\xrightarrow{\ \varrho\ }\mathcal{F}\qquad(*)$$

of categories and functors is a transporter system, and if $\mathcal{L}$ is a centric linking system in the sense of Definition 2.9, then $\mathcal{T}$ is a centric linking system in the sense of [7] or [16].

(b) *Let $\mathcal{L}=(\mathcal{L},\Delta,S)$ and $\mathcal{L}'=(\mathcal{L}',\Delta,S)$ be localities having the same set Δ of objects, and let $\beta:\mathcal{L}\to\mathcal{L}'$ be a rigid isomorphism. Define $(\mathcal{T},\varepsilon,\varrho)$ as above, and define $(\mathcal{T}',\varepsilon',\varrho')$ in the analogous way. There is then an isomorphism $\mathcal{T}\to\mathcal{T}'$ of localities, given on objects by $P\mapsto P$ and on morphisms by $(f,P,Q)\mapsto(f\beta,P,Q)$.*

Proof. (a) By the definition of $\mathcal{T}$, $\Delta=\mathrm{Ob}(\mathcal{T})$, and then also $\Delta\subseteq\mathrm{Ob}(\mathcal{F})$ as Δ is a set of subgroups of S. Since $\mathcal{F}$ is generated by the conjugation maps $c_f: P\to Q$ with $f\in N_{\mathcal{L}}(P,Q)$, where $P,Q\in\Delta$, condition (O2) in Lemma 2.5 implies that Δ is closed under $\mathcal{F}$-conjugacy. Since Δ is overgroup closed by Definition 2.9, we then have (A1).

Let $P,Q\in\Delta$ and define $E(P)$ and $E(Q)$ as in (A2). Since left and right cancellation holds in $\mathcal{L}$, by Lemma 2.2, $E(P)$ acts freely on $\mathrm{Mor}_{\mathcal{T}}(P,Q)$ by left composition, and $E(Q)$ acts freely by right composition. Let (f,P,Q) and $(g,P,Q)\in\mathrm{Mor}_{\mathcal{T}}(P,Q)$ lie in the same fiber of the map $\varrho_{P,Q}: \mathrm{Mor}_{\mathcal{T}}(P,Q)\to\mathrm{Hom}_{\mathcal{F}}(P,Q)$. Then the conjugation maps c_f and c_g from P to Q are equal. Set $P'=P^f\,(=P^g)$ and regard $c_{g^{-1}}$ as a map from P' to P. Then $c_{fg^{-1}}=c_f\circ c_{g^{-1}}$ is the identity map on P, so that $fg^{-1}\in E(P)$. This shows that each fiber of $\varrho_{P,Q}$ is contained in an orbit of $E(P)$. The reverse inclusion holds since

$$\varrho_{P,Q}(hf)=\varrho(hf)=\varrho(h)\varrho(f)=\varrho(f)$$

for any $h\in E(P)$. Thus (A2) holds.

Condition (B) follows immediately from the definitions of the functors ε and ϱ. The commutativity of the diagram in (C) is no more than the observation that if $(f,P,Q)\in\mathrm{Mor}_{\mathcal{T}}(P,Q)$ and $g\in P$, then g^f is defined (via P^f), and $g^f=g(\varrho(f))\in Q$. Condition (I) is given by the hypothesis, in Definition 2.9, that $S\in\Delta$, so it only remains to establish (II). Here $N_{\mathcal{L}}(P)$ is isomorphic to $\mathrm{Aut}_{\mathcal{T}}(P)$ via the map $g\mapsto(g,P,P)$ for $P\in\Delta$. Let $P,\overline{P},Q,\overline{Q}\in\Delta$, with $P\trianglelefteq\overline{P}$ and $Q\trianglelefteq\overline{Q}$, and let $f\in\mathcal{L}$ with $P^f=Q$. Then c_f induces an isomorphism $N_{\mathcal{T}}(P)\to N_{\mathcal{T}}(Q)$. If $(\overline{P})c_f=\overline{Q}$ then the $\mathcal{T}$-isomorphism (f,P,Q) extends to the $\mathcal{T}$-isomorphism $(f,\overline{P},\overline{Q})$. Thus (II) holds, and $\mathcal{T}$ is a transporter system.

Now suppose that $\mathcal{L}$ is a centric linking system in the sense of Definition 2.9. That is, assume that Δ is the set of all $\mathcal{F}$-centric subgroups, and that $C_{\mathcal{L}}(P)\leqslant P$ for all $P\in\Delta$. Then $\mathcal{F}$ is saturated, by Proposition 2.18 (a), and $E(P)=Z(P)$ for all $P\in\Delta$, so $\mathcal{T}$ is a centric linking system in the sense of [16]. Thus (a) holds.

(b) Let $\mathcal{T}$, $\mathcal{T}'$, and $\beta: \mathcal{L}\to\mathcal{L}'$ be as given. Let $\alpha: \mathcal{T}\to\mathcal{T}'$ be the pair of maps, given on $\mathrm{Ob}(\mathcal{T})=\Delta$ by $P\mapsto P$, and on morphisms by $(f,P,Q)\mapsto(f\beta,P,Q)$. That α is then a functor is immediate from the fact that β is a homomorphism which sends each subgroup P of S to P. The invertibility of α is immediate from the invertibility of β, and it is trivially verified that $\varepsilon\circ\alpha=\varepsilon'$ (in right-hand notation). In order that $\alpha\circ\varrho'$ be equal to ϱ it is necessary and sufficient that each conjugation map $c_f: P\to Q$ with $P,Q\in\Delta$ be equal to the conjugation map $c_{f\beta}$. Thus, let $x\in P$. Then $x^f\in Q$, so $x^f\in S$, and then

$$x^f=(x^f)\beta=(x\beta)^{f\beta}=x^{f\beta}.$$

Hence $c_f=c_{f\beta}$ as required, and (b) holds. □

Corollary A.4. *Let $\mathcal{F}$ be a saturated fusion system on a finite p-group S. Then there exists a centric linking system $(\mathcal{T},\varepsilon,\varrho)$ whose fusion system is $\mathcal{F}$.*

Proof. The main theorem provides a centric linking system $(\mathcal{L},\Delta,S)$ in the sense of Definition 2.9, with $\mathcal{F}=\mathcal{F}_S(\mathcal{L})$, and then Proposition A.3 (a) provides the required centric linking system in the sense of [16]. □

For the remainder of this appendix, let

$$\mathcal{T}_\Delta(S) \xrightarrow{\ \varepsilon\ } \mathcal{T} \xrightarrow{\ \varrho\ } \mathcal{F}$$

be a transporter system. Set $\iota_{P,Q}=\varepsilon_{P,Q}(1)$, write ι_P for $\varepsilon_P(1)$, and observe that ι_P is the identity element of $\mathrm{Aut}_{\mathcal{T}}(P)$ by Definition A.1 (C). The morphisms $\iota_{P,Q}$ are called *inclusion morphisms*, and condition (B) implies that ϱ sends inclusion morphisms in $\mathcal{T}$ to inclusion maps in $\mathcal{F}$. Whenever $P\leqslant P'\leqslant S$ and $Q\leqslant Q'\leqslant S$ are in $\mathrm{Ob}(\mathcal{T})$, and whenever

$$\begin{array}{ccc} P & \xrightarrow{\ \phi\ } & Q \\ {\scriptstyle \iota_{P,P'}}\downarrow & & \downarrow{\scriptstyle \iota_{Q,Q'}} \\ P' & \xrightarrow{\ \phi'\ } & Q' \end{array}$$

is a commutative square in $\mathcal{T}$, we say that ϕ is a *restriction* of ϕ' (and sometimes write $\phi'|_{P,Q}=\phi$); or we may say that ϕ' is an *extension* of ϕ. Some of the results that follow can be found in [18, §24]. Most notably, point (a) of Lemma A.8 (and which is the key point of this appendix) appears to be prefigured in [18, Remark 24.12].

The following lemma collects what are for our purposes the key properties of $\mathcal{T}$, established in [16].

Lemma A.5. *The following statements hold*:

(a) *All morphisms of $\mathcal{T}$ are both monomorphisms and epimorphisms in the categorical sense. That is, we have left and right cancellation for morphisms in $\mathcal{T}$.*

(b) *For every $\phi\in\mathrm{Mor}_{\mathcal{T}}(P,Q)$, and every $P_0,Q_0\in\mathrm{Ob}(\mathcal{T})$ such that $P_0\leqslant P$, $Q_0\leqslant Q$, and $\varrho(\phi)$ maps P_0 into Q_0, there is a unique $\phi_0\in\mathrm{Mor}_{\mathcal{T}}(P_0,Q_0)$ such that $\phi_0=\phi|_{P_0,Q_0}$. In particular, every morphism in $\mathcal{T}$ is the composition of an isomorphism followed by an inclusion morphism.*

(c) *Let ϕ and ϕ' be $\mathcal{T}$-homomorphisms $P\to Q$, and let P_0 and Q_0 be objects of $\mathcal{T}$ with $P_0\leqslant P$ and $Q_0\leqslant Q$. Suppose that $\varrho(\phi)$ and $\varrho(\phi')$ map P_0 into Q_0, and that $\phi|_{P_0,Q_0}=\phi'|_{P_0,Q_0}$. Then $\phi=\phi'$.*

(d) *Let P, $\overline{P}$, Q and $\overline{Q}$ be objects of $\mathcal{T}$, with $P\trianglelefteq\overline{P}$ and $Q\trianglelefteq\overline{Q}$. If $\bar\phi\in\mathrm{Mor}_{\mathcal{T}}(\overline{P},\overline{Q})$*

is an extension of $\phi \in \mathrm{Iso}_{\mathcal{T}}(P,Q)$, *then the square*

$$\begin{array}{ccc} P & \xrightarrow{\ \phi\ } & Q \\ {\scriptstyle \varepsilon_P(x)}\Big\downarrow & & \Big\downarrow{\scriptstyle \varepsilon_Q(x(\varrho(\phi)))} \\ P & \xrightarrow{\ \phi\ } & Q \end{array}$$

commutes for all $x \in \overline{P}$.

Proof. Three of these points are given by the following results in [16]: (a) by Lemma 3.2 (b) and Lemma 3.8, (b) by Lemma 3.2 (c), and (d) by Lemma 3.3.

For the proof of (c): write ψ for $\phi|_{P_0,Q_0}$, and hence also for $\phi'|_{P_0,Q_0}$. Then

$$\iota_{P_0,P} \circ \phi = \psi \circ \iota_{Q_0,Q} = \iota_{P_0,P} \circ \phi',$$

and (c) follows from left cancellation. □

LEMMA A.6. *Let* $\phi_0: P_0 \to Q_0$, $\phi: P \to Q$, *and* $\phi': P' \to Q'$ *be* $\mathcal{T}$*-isomorphisms, and suppose that both* ϕ *and* ϕ' *are extensions of* ϕ_0. *Then the following statements hold*:

(a) $P = P'$ *if and only if* $Q = Q'$;

(b) *There is a unique extension of* ϕ_0 *to a* $\mathcal{T}$*-isomorphism* $\phi_1: P \cap P' \to Q \cap Q'$, *and each of* ϕ *and* ϕ' *is an extension of* ϕ_1.

Proof. (a) Suppose that $P = P'$. Let $x \in N_P(P_0)$, and let y and y' be the images of x under $\varrho(\phi)$ and $\varrho(\phi')$, respectively. Then Lemma A.5 (d), with ϕ_0 in the role of ϕ, yields

$$\phi_0^{-1} \circ \varepsilon_{P_0}(x) \circ \phi = \varepsilon_{Q_0}(y) = \varepsilon_{Q_0}(y').$$

As ε_{Q_0} is injective, by condition (B), we get $y = y'$, and thus $\varrho(\phi)$ and $\varrho(\phi')$ agree on $P_1 := N_P(P_0)$. Let Q_1 be the image of P_1 under $\varrho(\phi)$. By Lemma A.5 (b), there is a restriction $\phi_1: P_1 \to Q_1$ of ϕ and a restriction $\phi_1': P_1 \to Q_1$ of ϕ', and Definition A.2 (c) implies that $\phi_1 = \phi_1'$. Replacing ϕ_0 by ϕ_1 in (a), and applying induction on the index of P_0 in P, we obtain $Q = Q'$ as desired. On the other hand, if $Q = Q'$ then we obtain $P = P'$ by working with ϕ_0^{-1}, ϕ^{-1}, and $(\phi')^{-1}$.

(b) Set $P_1 = P \cap P'$ and $Q_1 = Q \cap Q'$. Then ϕ and ϕ' have restrictions ϕ_1 and ϕ_1' to P_1 which, in turn, restrict to ϕ_0. Then (a) implies that ϕ_1 and ϕ_1' are $\mathcal{T}$-isomorphisms $P_1 \to Q_1$, and Lemma A.5 (c) yields $\phi_1 = \phi_1'$. □

Define a relation $\uparrow$ on the set $\mathrm{Mor}(\mathcal{T})$ of morphisms of $\mathcal{T}$ by $\phi \uparrow \phi'$ if ϕ' is an extension of ϕ. That is, $\phi \uparrow \phi'$ if $\phi: P \to Q$ and $\phi': P' \to Q'$ with $P \leqslant P'$, $Q \leqslant Q'$, and with

$$\iota_{P,P'} \circ \phi' = \phi \circ \iota_{Q,Q'}.$$

We may write also $\phi' \downarrow \phi$ for $\phi \uparrow \phi'$.

Lemma A.7. *The following statements hold*:

(a) *The relation* $\uparrow$ *induces a partial order on* $\mathrm{Iso}(\mathcal{T})$.

(b) *The relation* $\uparrow$ *respects composition of morphisms. That is, if* $\phi\uparrow\phi'$ *and* $\psi\uparrow\psi'$, *and the compositions* $\phi\circ\psi$ *and* $\phi'\circ\psi'$ *are defined, then* $(\phi\circ\psi)\uparrow(\phi'\circ\psi')$.

Proof. The transitivity of the relation $\uparrow$ is easily verified. Suppose that both $\phi\uparrow\phi'$ and $\phi\downarrow\phi'$, where $\phi\in\mathrm{Iso}_{\mathcal{T}}(P,Q)$ and $\phi'\in\mathrm{Iso}_{\mathcal{T}}(P',Q')$. Then $P=P'$, $Q=Q'$, $\iota_{P,P'}=\iota_P$, and $\iota_{Q,Q'}=\iota_Q$. Further, $\iota_P\phi'=\phi\circ\iota_Q$ and then $\phi'=\phi$ since ι_P and ι_Q are identity morphisms in $\mathcal{T}$. Thus (a) holds.

Suppose that we are given $\phi\uparrow\phi'$ and $\psi\uparrow\psi'$, with $\phi\circ\psi$ and $\phi'\circ\psi'$ defined on objects P and P' respectively. Set $Q=P\phi$ and $R=Q\psi$, and set $Q'=P'\phi'$ and $R'=Q'\psi'$. The following diagram, in which the vertical arrows are inclusion morphisms, adequately demonstrates that $\phi\circ\psi\uparrow\phi'\circ\psi'$:

$$\begin{array}{ccccc} P' & \xrightarrow{\phi'} & Q' & \xrightarrow{\psi'} & R' \\ \uparrow & & \uparrow & & \uparrow \\ P & \xrightarrow{\phi} & Q & \xrightarrow{\psi} & R. \end{array}$$

This yields (b). □

Let $\equiv$ be the equivalence relation on $\mathrm{Iso}(\mathcal{T})$ generated by the restriction of $\uparrow$ to isomorphisms. Let $\mathcal{L}$ be the set $\mathrm{Iso}(\mathcal{T})/\equiv$ of equivalence classes. For $\phi\in\mathrm{Iso}(\mathcal{T})$ we write $[\phi]$ for the equivalence class containing ϕ.

Lemma A.8. *Let* $f\in\mathcal{L}$.

(a) *There is a unique maximal* $\phi\in f$ *with respect to* $\uparrow$, *and* ϕ^{-1} *is then maximal in* $[\phi^{-1}]$.

(b) $f\cap\mathrm{Iso}_{\mathcal{T}}(P,Q)$ *has cardinality at most* 1 *for any* $P,Q\in\mathrm{Ob}(\mathcal{T})$.

Proof. Let $\phi\colon P\to Q$ be maximal in f with respect to $\uparrow$. Suppose that there exists $\phi'\colon P'\to Q'$ in f such that ϕ is not an extension of ϕ'. Then ϕ' may be chosen so that there exists $\phi_0\colon P_0\to Q_0$ in f with $\phi_0\uparrow\phi$ and $\phi_0\uparrow\phi'$. Among all such pairs (ϕ',ϕ_0), choose one so that $|P_0|$ is as large as possible. Then Lemma A.6 (b) implies that $P_0=P\cap P'$ and $Q_0=Q\cap Q'$. It follows that $N_{P'}(P_0)\nleq P$, and so we may replace ϕ' by the restriction of ϕ' to $N_{P'}(P_0)\to N_{Q'}(Q_0)$. That is, we may assume that $P_0\trianglelefteq P'$ and $Q_0\trianglelefteq Q'$.

Let $\lambda\colon\mathrm{Aut}_{\mathcal{T}}(P_0)\to\mathrm{Aut}_{\mathcal{T}}(Q_0)$ be the isomorphism induced by conjugation by ϕ_0. Set $P_1=N_P(P_0)$ and $Q_1=N_Q(Q_0)$. Also, set $P''=\langle P_1,P'\rangle$ and $Q''=\langle Q_1,Q'\rangle$. Then Lemma A.5 (d) implies that λ maps $\varepsilon_{P_0}(P'')$ onto $\varepsilon_{P_0}(Q'')$. By condition (II) in Definition A.1, there is an extension of ϕ_0 to a $\mathcal{T}$-isomorphism $P''\to Q''$, and the maximality of

P_0 then yields $P''\leqslant P$. Thus $P'\leqslant P$, and we have a contradiction. Hence f has a unique maximal element ϕ.

Set $\psi=\phi^{-1}$ and let $\psi\uparrow\bar{\psi}$. Then $\bar{\psi}: \overline{Q}\to\overline{P}$ for some $\overline{Q}$ containing Q and some $\overline{P}$ containing P. Then $\phi\uparrow\bar{\psi}^{-1}$, so $\phi=\bar{\psi}^{-1}$ and $\psi=\bar{\psi}$. Thus ϕ^{-1} is maximal in its $\equiv$-class, and (a) holds.

In order to prove (b), let $\psi,\psi'\in f\cap \mathrm{Iso}_{\mathcal{T}}(P,Q)$. Then both ψ and ψ' are restrictions of a single $\phi\in f$, by (a). Now Lemma A.5 (b) implies that $\psi=\psi'$. □

Define $\mathbf{D}$ to be the set of words $w=(f_1,...,f_n)\in\mathbf{W}(\mathcal{L})$ such that there exists a sequence $(\phi_1,...,\phi_n)$ of $\mathcal{T}$-isomorphisms with $\phi_i\in f_i$, and a sequence $(P_0,...,P_n)$ of objects of $\mathcal{T}$ with $\phi_i: P_{i-1}\to P_i$ for all i. We also say that $w\in\mathbf{D}$ via $(P_0,...,P_n)$, or via P_0. Define

$$\Pi: \mathbf{D}\longrightarrow\mathcal{L}$$

by setting $\Pi(w)=f$, where f is the unique maximal element of $[\phi_1\circ...\circ\phi_n]$ given by Lemma A.8 (a). That Π is well defined follows from Lemma A.7 (b) and an obvious induction on the length of w. Set $\mathbf{1}=[\iota_S]$, and for any $f\in\mathcal{L}$ let f^{-1} be the equivalence class of ϕ^{-1}, where ϕ is the unique maximal member of f.

Proposition A.9. *$\mathcal{L}$, with the above structures, is a partial group. Moreover, the following statements hold*:

(a) *For any $g\in S$, $[\varepsilon_S(g)]$ is the set of all $\varepsilon_{P,Q}(g)$ such that $P^g=Q$, and $\varepsilon_S(g)$ is the maximal member of its class*;

(b) *$[\iota_S]$ is the set of all ι_P, $P\in\mathrm{Ob}(\mathcal{T})$, and ι_S is the maximal member of its class*;

(c) *For any $\phi\in\mathrm{Iso}(\mathcal{T})$, $[\phi^{-1}]$ is the set of inverses of the members of $[\phi]$.*

Proof. We first check that $\mathcal{L}$ is a partial group. Of course $\mathcal{L}$ is non-empty since $\mathcal{T}$ is non-empty. For any $f\in\mathcal{L}$ and any representative ϕ of f, f is a $\mathcal{T}$-isomorphism between objects of $\mathcal{T}$, so the word (f) of length 1 is in $\mathbf{D}$. Now let $w=(f_1,...,f_n)$ be in $\mathbf{D}$. Clearly, any prefix $u=(f_1,...,f_k)$ and any suffix $v=(f_{k+1},...,f_n)$ of w is in $\mathbf{D}$, so Definition 2.1 (1) holds for $\mathcal{L}$. By definition $\Pi(f)=f$ for $f\in\mathcal{L}$, so Definition 2.1 (2) holds. Definition 2.1 (3) is a straightforward consequence of associativity of composition of isomorphisms in $\mathcal{T}$, and of the definition of Π.

Lemma A.8 (a) implies that the inversion map $f\mapsto f^{-1}$ is an involutory bijection. Now let $u=(f_1,...,f_n)\in\mathbf{D}$ via $(P_0,...,P_n)$, and set $u^{-1}=(f_n^{-1},...,f_1^{-1})$. Then $u^{-1}\in\mathbf{D}$ via $(P_n,...,P_0)$, so $u^{-1}\circ u\in\mathbf{D}$. One obtains a representative in the class $\Pi(u^{-1}\circ u)$ via a sequence of cancellations $\phi_k^{-1}\circ\phi$ of representatives $\phi_k\in f_i$, so $\Pi(u^{-1}\circ u)$ is the equivalence class containing ι_{P_0}. Since $\iota_{P_0}\uparrow\iota_S$, and since $\mathbf{1}=[\iota_S]$ by definition, we get $\Pi(u^{-1}\circ u)=\mathbf{1}$. Thus Definition 2.1 (4) holds in $\mathcal{L}$, and $\mathcal{L}$ is a partial group.

We now prove (a). Let $P \leqslant P'$ and $Q \leqslant Q'$ in $\mathrm{Ob}(\mathcal{T})$, and let g be an element of S such that $P^g = Q$ and $(P')^g = Q'$. The functoriality of ε yields

$$\varepsilon_{P,P'}(1) \circ \varepsilon_{P',Q'}(g) = \varepsilon_{P,Q'}(g) = \varepsilon_{P,Q}(g) \circ \varepsilon_{Q,Q'}(1),$$

which means that $\varepsilon_{P,Q}(g) \uparrow \varepsilon_{P',Q'}(g)$. In particular, we get $\varepsilon_{P,Q}(g) \uparrow \varepsilon_S(g)$. In order to complete the proof of (a), it now suffices to show that for any $\phi \in \mathrm{Iso}_{\mathcal{T}}(P,Q)$ with $\varepsilon_S(g) \equiv \phi$, we have $\phi = \varepsilon_{P,Q}(g)$.

Suppose this is false, and let $\sigma = (\phi_1, ..., \phi_n)$ be a sequence of $\mathcal{T}$-isomorphisms with $\phi = \phi_1$, $\varepsilon_S(g) = \phi_n$, and with either $\phi_i \uparrow \phi_{i+1}$ or $\phi_i \downarrow \phi_{i+1}$ for all i with $1 \leqslant i < n$. Among all (ϕ, P, Q) with $\phi \neq \varepsilon_{P,Q}(g)$ and $\varepsilon_S(g) \equiv \phi$, choose (ϕ, P, Q) so that the length of such a chain σ is as small as possible. Set $\psi = \phi_2$. Then $\psi = \varepsilon_{X,Y}(g)$, where X and Y are objects of $\mathcal{T}$ with $X^g = Y$. Suppose $\phi \uparrow \psi$. Applying the functor ϱ to the commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{\varepsilon_{X,Y}(g)} & Y \\ {\scriptstyle \iota_{P,X}}\big\downarrow & & \big\downarrow{\scriptstyle \iota_{Q,Y}} \\ P & \xrightarrow{\phi} & Q, \end{array}$$

and applying condition (B) in Definition A.1 to $\varrho(\varepsilon_{X,Y}(g))$, we conclude that $\varrho(\phi)$ is the restriction of c_g to the homomorphism $\varrho(\phi): P \to Q$. In particular, we get $P^g = Q$, so that also $\varepsilon_{P,Q}(g)$ is a restriction of $\varepsilon_{X,Y}(g)$. Then Lemma A.5 (a) yields $\phi = \varepsilon_{P,Q}(g)$, contrary to assumption. On the other hand, if $\phi \downarrow \psi$, then $\phi = \varepsilon_{P,Q}(g)$ by Lemma A.6, again contrary to assumption. This completes the proof of (a), and then (b) is the special case of (a) given by $g = 1$.

Let $f = [\phi]$ be an equivalence class, with ϕ maximal in f. One checks (by reversing pairs of arrows in the appropriate diagrams) that if ψ is a $\mathcal{T}$-isomorphism, and ψ is a restriction of ϕ, then the $\mathcal{T}$-isomorphism ψ^{-1} is a restriction of ϕ^{-1}. Point (c) follows from this observation. □

In view of Proposition A.9 (a), there is no harm in writing g to denote the equivalence class $[\varepsilon_S(g)]$, for $g \in S$.

Lemma A.10. *Let $\phi: Z \to W$ be a $\mathcal{T}$-isomorphism which is maximal in its $\equiv$-class. Let X and Y be objects of $\mathcal{T}$ contained in Z, and let U and V be the images of X and Y, respectively, under $\varrho(\phi)$. Suppose that there exist elements g and g' in S such that the following diagram commutes*:

$$\begin{array}{ccc} X & \xrightarrow{\phi|_{X,U}} & U \\ {\scriptstyle \varepsilon_{X,Y}(g)}\big\downarrow & & \big\downarrow{\scriptstyle \varepsilon_{U,V}(g')} \\ Y & \xrightarrow{\phi|_{Y,V}} & V. \end{array} \qquad (*)$$

Then $g\in Z$, *and* g' *is the image of* g *under* $\varrho(\phi)$.

Proof. Let ϕ' be the composition (in right-hand notation)

$$\phi' = \varepsilon_{Z^g,Z}(g^{-1})\circ\phi\circ\varepsilon_{W,W^{g'}}(g').$$

Thus, $\phi'\in\mathrm{Iso}_{\mathcal{T}}(Z^g, W^{g'})$, and the commutativity of $(*)$ yields $\phi|_Y=\phi'|_Y$. Hence, $\phi\equiv\phi'$, and the maximality of $\phi\colon Z\to W$ implies that $Z^g\leqslant Z$ and $W^{g'}\leqslant W$. That is, $g\in N_S(Z)$ and $g'\in N_S(W)$. There is then a commutative diagram as follows:

$$\begin{array}{ccc} Z & \xrightarrow{\phi} & W \\ {\scriptstyle\varepsilon_Z(g)}\downarrow & & \downarrow{\scriptstyle\varepsilon_W(g')} \\ Z & \xrightarrow{\phi} & W. \end{array}$$

Condition (I) in Definition A.1 implies that there is an extension of ϕ to a $\mathcal{T}$-isomorphism $\langle Z,g\rangle\to\langle W,g'\rangle$, and the maximality of ϕ then yields $g\in Z$ and $g'\in W$. Condition (C) in Definition A.1 implies that g' is the image under $\varrho(\phi)$ of g. □

Set $\Delta=\mathrm{Ob}(\mathcal{T})$.

Corollary A.11. *Let* $f\in\mathcal{L}$ *and let* $P\in\Delta$ *with the property that, for all* $x\in P$, $(f^{-1},x,f)\in\mathbf{D}$ *and* $\Pi(f^{-1},x,f)\in S$. *Let* Q *be the set of all such products* $\Pi(f^{-1},x,f)$. *Then* $Q\in\Delta$ *and there exists* $\psi\in f$ *such that* $\psi\in\mathrm{Iso}_{\mathcal{T}}(P,Q)$.

Proof. As $(f^{-1},x,f)\in\mathbf{D}$, there exist $U,X,Y,V\in\Delta$ and representatives ψ and $\bar\psi$ of f such that

$$U \xrightarrow{\bar\psi^{-1}} X \xrightarrow{\varepsilon_{X,Y}(x)} Y \xrightarrow{\psi} V$$

is a chain of $\mathcal{T}$-isomorphisms, and where the middle arrow in the diagram is indeed $\varepsilon_{X,Y}(x)$ by Proposition A.9 (a). Since $\Pi(f^{-1},x,f)\in S$, there exists $x'\in S$ such that $\bar\psi^{-1}\circ\varepsilon_{X,Y}(x)\circ\psi=\varepsilon_{U,V}(x')$. Let $\phi\colon Z\to W$ be the maximal element of f. Then Lemma A.10 implies that $x\in Z$, and x' is the image of x under $\varrho(\phi)$. In particular, we have $P\leqslant Z$ and $Q\leqslant W$, and we may therefore take $X=Y=P$ and $U=V=Q$, obtaining $\psi\in\mathrm{Iso}_{\mathcal{T}}(P,Q)$. □

Lemma A.12. *Let* $\psi\colon P\to Q$ *be a* $\mathcal{T}$*-isomorphism, and let* $f=[\psi]$ *be the equivalence class of* ψ. *Then* $P\leqslant\mathbf{D}(f)$, *and* $P^f=Q$ *in the partial group* $\mathcal{L}$.

Proof. For any $g\in P$, we have the composable sequence

$$Q \xrightarrow{\phi^{-1}} P \xrightarrow{\varepsilon_P(g)} P \xrightarrow{\phi} P$$

of $\mathcal{T}$-isomorphisms, so (f^{-1},g,f) is in $\mathbf{D}$, and $P\subseteq\mathbf{D}(f)$. By Definition A.1 (C),

$$\psi^{-1}\circ\varepsilon_P(g)\circ\psi=\varepsilon_Q(g'),$$

where $g'\in Q$. The class $[\varepsilon_Q(g')]$ is the same as $[\varepsilon_S(g')]$ by Lemma A.8 (a); and we recall that we have introduced the convention to denote this class simply as g'. Thus $g^f=g'$, and so $P^f\subseteq Q$. The conjugation map $g\mapsto g^f$ is injective by Example 2.4 (c), so $P^f=Q$, as required. □

Proposition A.13. *$(\mathcal{L},\Delta,S)$ is a locality, and if $(\mathcal{T},\varepsilon,\varrho)$ is a centric linking system (in the sense of [7] and [16]) then $\mathcal{L}$ is a centric linking system in the sense of Definition 2.9.*

Proof. First, $\mathcal{L}$ is a partial group, by Proposition A.9. In order to show that $(\mathcal{L},\Delta)$ is objective, let $w=(f_1,...,f_n)\in\mathbf{D}$. By definition, there exist representatives ψ_i of the classes f_i, and a sequence $(P_0,...,P_n)$ of objects of $\mathcal{T}$, such that each ψ_i is a $\mathcal{T}$-isomorphism $P_{i-1}\to P_i$. Then $P_{i-1}^{f_i}=P_i$ for all i, by Lemma A.12. Conversely, given $w=(f_1,...,f_n)\in\mathbf{W}$, and given $(P_0,...,P_n)\in\mathbf{W}(\Delta)$ with $P_{i-1}^{f_i}=P_i$ for all i, it follows from Corollary A.11 that $w\in\mathbf{D}$. Thus, $(\mathcal{L},\Delta)$ satisfies condition (O1) of Definition 2.6. Condition (O2) is given by Corollary A.11, so $(\mathcal{L},\Delta)$ is objective. That is, condition (L1) for a locality holds. Also, since $\mathcal{T}$ is finite by Definition A.1, $\mathcal{L}$ is finite.

The mapping $\mathrm{Aut}_{\mathcal{T}}(S)\to N_{\mathcal{L}}(S)$ given by $\psi\mapsto[\psi]$ is a homomorphism, as follows from Lemma A.7. It is surjective by the definition of $\mathcal{L}$, and injective by Lemma A.8 (b). As $\varepsilon_S(S)\in\mathrm{Syl}_p(\mathrm{Aut}_{\mathcal{T}}(S))$, by Definition A.1 (I), we conclude that $S\in\mathrm{Syl}_p(N_{\mathcal{L}}(S))$, and hence S is maximal in the poset of p-subgroups of $\mathcal{L}$. That is, (L2) holds for $\mathcal{L}$, and thus $\mathcal{L}$ is a locality.

Now suppose that $(\mathcal{T},\varepsilon,\varrho)$ is a centric linking system. That is, suppose that $\Delta=\mathrm{Ob}(\mathcal{T})$ is the set $\mathcal{F}^c$ of $\mathcal{F}$-centric subgroups of S, and suppose for each object P that $Z(P)=\mathrm{Ker}(\varrho_P)$. Let $\mu:\mathrm{Aut}_{\mathcal{T}}(P)\to N_{\mathcal{L}}(P)$ be the mapping $\phi\mapsto[\phi]$. Then μ is a homomorphism by Lemma A.7 (b). Let $\phi\in\mathrm{Ker}(\mu)$. Then $[\phi]=[\iota_S]$, so $\phi\uparrow\iota_S$, and then $\phi=\iota_P$ by Proposition A.9 (b). That is, ϕ is the identity element of $\mathrm{Aut}_{\mathcal{T}}(P)$, and thus $\mathrm{Ker}(\mu)=1$. Now let $f\in N_{\mathcal{L}}$ and let $\psi\in f$ be the maximal element. Then ψ restricts to a $\mathcal{T}$-automorphism ϕ of P by Corollary A.11, so μ is surjective, and hence an isomorphism. Since $Z(P)=\mathrm{Ker}(\varrho_P)=C_{\mathrm{Aut}_{\mathcal{T}}}(P)$, we conclude that $C_{\mathcal{L}}(P)=Z(P)$, and hence $\mathcal{L}$ is a centric linking system in the sense of Definition 2.9. □

Let $\phi:P\to Q$ be a morphism in $\mathcal{T}$ (and not necessarily a $\mathcal{T}$-isomorphism). Let Q_0 be the image of P under the homomorphism $\varrho(\phi)$. Then, by Lemma A.5 (b), there is a well-defined restriction $\phi_0=\phi|_{P,Q_0}$ of ϕ to a $\mathcal{T}$-isomorphism $P\to Q_0$.

Lemma A.14. *There is a functor $\eta:\mathcal{T}\to\mathrm{Cat}(\mathcal{L},\Delta)$ such that η is the identity map on the set of objects, and such that*

$$\eta_{P,Q}:\mathrm{Mor}_{\mathcal{T}}(P,Q)\longrightarrow\mathrm{Mor}_{\mathrm{Cat}(\mathcal{L},\Delta)}(P,Q)$$

is the mapping $\phi\mapsto(f,P,Q)$, *where* f *is the* $\equiv$*-class of the* $\mathcal{T}$*-isomorphism* $\phi_0\colon P\to Q_0$, *and where* Q_0 *is the image of* P *under* $\varrho(\phi)$.

Proof. Let $\phi\colon P\to Q$ and $\psi\colon Q\to R$ be composable morphisms in $\mathcal{T}$, let Q_0 be the image of P under $\varrho(\phi)$, and let R_1 be the image of Q_0 under $\varrho(\psi)$. The restrictions $\phi_0\phi|_{P_0,Q_0}$ and $\psi_1=\psi|_{Q_0,R_1}$ are then composable $\mathcal{T}$-isomorphisms. Set $\theta=\phi_0\circ\psi_1$. Then the product $[\phi_0][\psi_1]$ of $\equiv$-classes is defined in $\mathcal{L}$, and is equal to $[\theta]$, by Lemma A.7 (b). Set $f=[\phi_0]$ and $g=[\psi_1]$. Thus $[\theta]=fg$, so $(f,P,Q)\circ(g,Q,R)=(fg,Q,R)$ in $\mathrm{Cat}(\mathcal{L},\Delta)$. This shows that η is a functor. □

LEMMA A.15. *There is a functor* $\xi\colon\mathrm{Cat}(\mathcal{L},\Delta)\to\mathcal{T}$ *such that* ξ *is the identity map on the set of objects, and such that*

$$\xi_{P,Q}\colon \mathrm{Mor}_{\mathrm{Cat}(\mathcal{L},\Delta)}(P,Q)\longrightarrow \mathrm{Mor}_{\mathcal{T}}(P,Q)$$

is the mapping $(f,P,Q)\mapsto\phi|_{P,P^f}\circ\iota_{P^f,Q}$, *where* ϕ *is the maximal element in the* $\equiv$*-class* f. *Moreover,* ξ *is invertible, and its inverse is* η.

Proof. Let (f,P,Q) and (g,Q,R) be composable morphisms in $\mathrm{Cat}(\mathcal{L},\Delta)$, and let $\phi\in f$ and $\psi\in g$ be maximal. Then the composition $\xi(f,P,Q)\circ\xi(g,Q,R)$ is defined in $\mathcal{T}$, as the following calculation shows:

$$\begin{aligned}\xi(f,P,Q)\circ\xi(g,Q,R)&=(\phi|_{P,P^f}\circ\iota_{P^f,Q})\circ(\psi|_{Q,Q^g}\circ\iota_{Q^g,R})\\&=\phi|_{P,P^f}\circ\psi|_{P^f,P^{fg}}\circ\iota_{P^{fg},Q^g}\circ\iota_{Q^g,R}\\&=\phi|_{P,P^f}\circ\psi|_{P^f,P^{fg}}\circ\iota_{P^{fg},R}.\end{aligned}\qquad(*)$$

Set $\theta_0=\phi|_{P,P^f}\circ\psi|_{P^f,P^{fg}}$. Then $fg=[\theta_0]$, by the definition of the product in $\mathcal{L}$. Let θ be the maximal element of fg. Then $\theta_0=\theta|_{P,P^{fg}}$, and $(*)$ then yields

$$\xi(f,P,Q)\circ\xi(g,Q,R)=\theta|_{P,P^{fg}}\circ\iota_{P^{fg},R}=\xi(fg,P,R).$$

Thus, ξ is a functor.

Set $P'=P^f$. By Corollary A.11, there exists $\gamma\in f$ such that $\gamma=\phi|_{P,P'}$. The functor $\varrho\colon\mathcal{T}\to\mathcal{F}$ sends $\mathrm{Mor}_{\mathcal{T}}(P,P')$ to $\mathrm{Hom}_{\mathcal{F}}(P,P')$, so P^f is the image of P under $\varrho(\gamma)$. Then also P^f is the image of P under $\varrho(\phi)$, since $\varrho(\gamma)$ is a restriction of the homomorphism $\varrho(\phi)$ by Definition A.1 (C). We now note that

$$\eta(\xi(f,P,Q))=\eta(\phi|_{P,P^f}\circ\iota_{P^f,Q}).$$

By the definition of η, $\eta(\xi(f,P,Q))$ is then (f',P,Q), where f' is the $\equiv$-class of the $\mathcal{T}$-isomorphism $\phi|_{P,P^f}$. That is, $f'=f$, and the composition ξ followed by η is the identity functor on $\mathrm{Cat}(\mathcal{L},\Delta)$.

In the other order: consider $\xi(\eta(\theta))$, where $\theta\colon A\to B$ is an arbitrary $\mathcal{T}$-morphism. Let B_0 be the image of A under $\varrho(\theta)$. Then $\eta(\theta)=(h,A,B)$, where $h=[\theta_0]$ and where $\theta_0\colon A\to B_0$ is the restriction $\theta|_{A,B_0}$. Applying ξ to (h,A,B) yields the $\mathcal{T}$-morphism θ', where

$$\theta'=\theta^*|_{A,B_0}\circ\iota_{B_0,B},$$

and where θ^* is the maximal element in the $\equiv$-class h. Maximality of θ^* yields $\theta_0\uparrow\theta^*$, and then $\theta^*|_{A,B_0}=\theta_0$. Now Lemma A.5 (b) yields $\theta'=\theta$, and thus η followed by ξ is the identity morphism on $\mathcal{T}$, completing the proof. □

We are now able to prove Theorem A, and to thereby translate the main theorem into the language of [7] and [16].

Theorem A. *Let $\mathcal{F}$ be a saturated fusion system on the finite p-group S. Then, up to isomorphism of transporter systems, there exists a unique centric linking system $(\mathcal{T},\varepsilon,\varrho)$ over $\mathcal{F}$.*

Proof. Existence is given by Corollary A.4. Now let $(\mathcal{T},\varepsilon,\varrho)$ and $(\mathcal{T}',\varepsilon',\varrho')$ be centric linking systems over $\mathcal{F}$ in the sense of [16]. Set $(\mathcal{L},\Delta,S)=(\mathrm{Iso}(\mathcal{T})/{\equiv},\mathrm{Ob}(\mathcal{T}),S)$, and similarly define $(\mathcal{L}',\Delta',S)$. Then $\Delta=\Delta'$ is the set of $\mathcal{F}$-centric subgroups of S. By Proposition A.13 both $\mathcal{L}$ and $\mathcal{L}'$ are $\mathcal{F}$-centric linking systems in the sense of Definition 2.9, so the main theorem yields a rigid isomorphism $\beta\colon\mathcal{L}\to\mathcal{L}'$. Then Proposition A.3 yields an isomorphism $\beta^*\colon\mathrm{Cat}(\mathcal{L},\Delta)\to\mathrm{Cat}(\mathcal{L}',\Delta)$ of categories. We now apply Lemmas A.14 and A.15 to obtain a sequence

$$\mathcal{T}\xrightarrow{\ \eta\ }\mathrm{Cat}(\mathcal{L},\Delta)\xrightarrow{\ \beta^*\ }\mathrm{Cat}(\mathcal{L}',\Delta)\xrightarrow{\ \xi'\ }\mathcal{T}'$$

of isomorphisms. Let $\alpha\colon\mathcal{T}\to\mathcal{T}'$ be the composition. It now remains to show that (in right-hand notation) $\varepsilon\circ\alpha=\varepsilon'$ and $\alpha\circ\varrho'=\varrho$, in order to conclude that α fulfills the requirements of Definition A.2 for an isomorphism of transporter systems.

Let $\delta\colon\mathcal{T}_\Delta(S)\to\mathrm{Cat}(\mathcal{L},\Delta)$ be the functor which is the identity map on the set Δ of objects, and which sends $x\in N_S(P,Q)$ to (x,P,Q). Let $\sigma\colon\mathrm{Cat}(\mathcal{L},\Delta)\to\mathcal{F}$ be the functor which is the inclusion map $\Delta\to\mathrm{Ob}(\mathcal{F})$ on objects and which sends (f,P,Q) to $c_f\colon P\to Q$. Define δ' and σ' with respect to $\mathrm{Cat}(\mathcal{L}',\Delta)$ in the analogous way. We now check that the following diagram of categories and functors (in which "=" indicates the identity functor) commutes:

$$\begin{array}{ccccccc}
\mathcal{T}_\Delta(S) & = & \mathcal{T}_\Delta(S) & = & \mathcal{T}_\Delta(S) & = & \mathcal{T}_\Delta(S) \\
\downarrow{\scriptstyle \varepsilon} & & \downarrow{\scriptstyle \delta} & & \downarrow{\scriptstyle \delta'} & & \downarrow{\scriptstyle \varepsilon'} \\
\mathcal{T} & \xrightarrow{\eta} & \mathrm{Cat}(\mathcal{L},\Delta) & \xrightarrow{\beta^*} & \mathrm{Cat}(\mathcal{L}',\Delta) & \xrightarrow{\xi'} & \mathcal{T}' \\
\downarrow{\scriptstyle \varrho} & & \downarrow{\scriptstyle \sigma} & & \downarrow{\scriptstyle \sigma'} & & \downarrow{\scriptstyle \varrho'} \\
\mathcal{F} & = & \mathcal{F} & = & \mathcal{F} & = & \mathcal{F}.
\end{array}$$

Note that, by Lemma A.15, ξ' is the inverse of a corresponding η', so by symmetry it will suffice to check the two left-hand squares and the middle squares in this diagram for commutativity. Note also that all the arrows in the diagram act trivially on objects, so the problem is to check for commutivity when the arrows are applied to morphisms. We shall write all mappings to the right in the following calculations.

We first consider the two middle squares. Given $x \in N_S(P,Q)$, one obtains

$$(x\delta_{P,Q})\beta^* = (x,P,Q)\beta^* = (x\beta,P,Q) = (x,P,Q) = x\delta'_{P,Q},$$

since β is the identity on S. Also, given a morphism (f,P,Q) in $\mathrm{Cat}(\mathcal{L},\Delta)$, one obtains

$$((f,P,Q)\beta^*)\sigma' = (f\beta,P,Q)\sigma' = [c_{f\beta}: P \to Q] = [c_f: P \to Q] = (f,P,Q)\sigma,$$

since $c_{f\beta} = c_f$ on any subgroup of S_f, again by the rigidity of β. Thus $\delta \circ \beta^* = \delta'$ and $\beta^* \circ \sigma' = \sigma$.

Next, in order to show that $\varepsilon \circ \eta = \delta$, we need to verify that $(x\varepsilon_{P,Q})\eta = (x,P,Q)$ for $x \in N_S(P,Q)$. By definition, η maps $x\varepsilon_{P,Q}$ to $([\phi_0],P,Q)$, where ϕ_0 is the restriction $x\varepsilon_{P,P^x}$ of $x\varepsilon_{P,Q}$. Since $x \in S$, the maximal element of $[\phi_0]$ is $x\varepsilon_S$, and $[x\varepsilon_S]$ is (by the convention established earlier) the element x of $\mathcal{L}$. Thus $\varepsilon \circ \eta = \delta$.

Finally, let $\phi: P \to Q$ be a $\mathcal{T}$-morphism and set $\phi_0 = \phi_{P,P'}$, where P' is the image of P under $(\phi)\varrho$. Applying $\eta \circ \sigma$ to ϕ we obtain $c_f: P \to Q$, where $f = [\phi_0]$. Then $P' = P^f$ by Lemma A.12. For any $x \in P$, Definition A.1 (C) yields $\phi_0^{-1} \circ x\varepsilon_P \circ \phi_0 = x'\varepsilon_{P'}$, where x' is the image of x under $(\phi_0)\varrho$. Thus, conjugation $c_f: P \to P'$ is, by the definition of the product Π in $\mathcal{L}$, given by

$$x = [x\varepsilon_P] \longmapsto [\phi_0^{-1} \circ x\varepsilon_P \circ \phi_0] = [x'\varepsilon_{P'}] = x',$$

and this shows that $((\phi_0)\eta)\sigma = (\phi_0)\varrho$. But $\phi = \phi_0 \circ \iota_{P',Q}$, where $(\iota_{P',Q})\eta = (\mathbf{1},P',Q)$, and where $(\mathbf{1},P',Q)\sigma$ is the inclusion map $P' \subseteq Q$. Functoriality of η and σ then yields that $(\phi)(\eta \circ \sigma)$ is just $(\phi_0)\varrho$ followed by inclusion. Since also ϱ sends inclusion morphisms to inclusion maps, the result is that $(\phi)(\eta \circ \sigma) = (\phi)\varrho$. This completes the proof that the big diagram commutes, and hence that $\alpha: \mathcal{T} \to \mathcal{T}'$ is an isomorphism of transporter systems. □

References

[1] Aschbacher, M., *Finite Group Theory*. Cambridge Studies in Advanced Mathematics, 10. Cambridge Univ. Press, Cambridge, 1986.

[2] — Normal subsystems of fusion systems. *Proc. Lond. Math. Soc.*, 97 (2008), 239–271.

[3] Aschbacher, M., Kessar, R. & Oliver, B., *Fusion Systems in Algebra and Topology*. London Mathematical Society Lecture Note Series, 391. Cambridge Univ. Press, Cambridge, 2011.

[4] Aschbacher, M. & Seitz, G. M., Involutions in Chevalley groups over fields of even order. *Nagoya Math. J.*, 63 (1976), 1–91.

[5] Benson, D. J., Cohomology of sporadic groups, finite loop spaces, and the Dickson invariants, in *Geometry and Cohomology in Group Theory* (Durham, 1994), London Mathematical Society Lecture Note Series, 252, pp. 10–23. Cambridge Univ. Press, Cambridge, 1998.

[6] Broto, C., Castellana, N., Grodal, J., Levi, R. & Oliver, B., Subgroup families controlling p-local finite groups. *Proc. Lond. Math. Soc.*, 91 (2005), 325–354.

[7] Broto, C., Levi, R. & Oliver, B., The homotopy theory of fusion systems. *J. Amer. Math. Soc.*, 16 (2003), 779–856.

[8] Bundy, D., Hebbinghaus, N. & Stellmacher, B., The local $C(G,T)$ theorem. *J. Algebra*, 300 (2006), 741–789.

[9] Chermak, A., Quadratic action and the $\mathcal{P}(G,V)$-theorem in arbitrary characteristic. *J. Group Theory*, 2 (1999), 1–13.

[10] Goldschmidt, D. M., A conjugation family for finite groups. *J. Algebra*, 16 (1970), 138–142.

[11] Gorenstein, D., *Finite Groups*. Harper & Row, New York, 1968.

[12] Gorenstein, D., Lyons, R. & Solomon, R., *The Classification of the Finite Simple Groups*. Mathematical Surveys and Monographs, 40. Amer. Math. Soc., Providence, RI, 1994.

[13] Kurzweil, H. & Stellmacher, B., *The Theory of Finite Groups*. Universitext. Springer, New York, 2004.

[14] Meierfrankenfeld, U. & Stellmacher, B., The other $\mathcal{P}(G,V)$-theorem. *Rend. Semin. Mat. Univ. Padova*, 115 (2006), 41–50.

[15] — The general FF-module theorem. *J. Algebra*, 351 (2012), 1–63.

[16] Oliver, B. & Ventura, J., Extensions of linking systems with p-group kernel. *Math. Ann.*, 338 (2007), 983–1043.

[17] Puig, L., Frobenius categories. *J. Algebra*, 303 (2006), 309–357.

[18] — *Frobenius Categories versus Brauer Blocks*. Progress in Mathematics, 274. Birkhäuser, Basel, 2009.

[19] Stancu, R., Equivalent definitions of fusion systems. Preprint, 2004.

[20] Thompson, J. G., A replacement theorem for p-groups and a conjecture. *J. Algebra*, 13 (1969), 149–151.

[21] Timmesfeld, F., A remark on Thompson's replacement theorem and a consequence. *Arch. Math.* (*Basel*), 38 (1982), 491–495.

Andrew Chermak
Kansas State University
Mathematics Department
138 Cardwell Hall
Manhattan, KS 66506
U.S.A.
chermak@math.ksu.edu

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