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Some estimates related to fractal measures and Laplacians on manifolds

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Abstract. Let Δ be the Laplace–Beltrami operator on an n-dimensional complete C^∞ manifold M. In this paper, we establish an estimate of $e^{t\Delta}(d\mu)$ valid for all $t>0$, where $d\mu$ is a locally uniformly α-dimensional measure on M, $0\le\alpha\le n$. The result is used to study the mapping properties of $(I-t\Delta)^{-\beta}$ considered as an operator from $L^p(M,d\mu)$ to $L^p(M,dx)$, where dx is the Riemannian measure on M, $\beta>(n-\alpha)/2p'$, $1/p+1/p'=1$, $1\le p\le\infty$.

1. Introduction

Let M be an n-dimensional complete C^∞ Riemannian manifold. We will denote by Δ the Laplace–Beltrami operator on M and by $h_t(x,y)$ the heat kernel for the heat semi-group $e^{t\Delta}$. A Borel measure μ on M is said to be locally uniformly α-dimensional if there exists a constant C such that

$$\sup_{\substack{1\ge r>0\\ x\in M}} r^{n-\alpha}\frac{\mu(B_r(x))}{V(B_r(x))}\le C,$$

where $B_r(x)$ is the geodesic ball with center x and radius r, and $V(B_r(x))$ is the volume of $B_r(x)$ with respect to the Riemannian measure dx on M. If M is of positive injective radius and the sectional curvature $K(M)$ of M satisfies $\tilde{k}\ge K(M)\ge -k$ for some positive constants k and $\tilde{k}$, then M is said to be of bounded geometry. Assuming that M is of bounded geometry, R. S. Strichartz in [5] established several results about the asymptotic behavior of $e^{t\Delta}(f\,d\mu)$ as $t\to 0$, where $f\in L^p(M,d\mu)$, $1\le p\le\infty$, and

$$e^{t\Delta}(f\,d\mu)(x)=\int_M h_t(x,y)f(y)\,d\mu(y).$$

Later, A. G. Setti [3] obtained similar results about L^p-weakly α-dimensional measure with respect to a weighted Laplacian. The results in [3] and [5] are based on estimates of the heat kernel.

From $e^{t\Delta}$ one can construct a family of operators $\{(I-t\Delta)^{-\beta}; t>0\}$ (with $\beta>0$ fixed) by setting

$$(I-t\Delta)^{-\beta} = \frac{1}{\Gamma(\beta)} \int_0^\infty s^{\beta-1} e^{-s} e^{st\Delta}\, ds.$$

The kernel $J_t(x,y)$ corresponding to $(I-t\Delta)^{-\beta}$ is given by

$$J_t^\beta(x,y) = \frac{1}{\Gamma(\beta)} \int_0^\infty s^{\beta-1} e^{-s} h_{st}(x,y)\, ds.$$

It is interesting to know the asymptotic behavior of $(I-t\Delta)^{-\beta}(f\,d\mu)$ as $t\to 0$. The aim of this paper is to show that a key to this problem is to get certain estimates of $e^{t\Delta}(\mu)$ valid for all $t>0$ rather than to estimate the kernel $J_t^\beta(x,y)$ directly. Actually, by our reproach, one may find that most of the results in [5] valid for $e^{t\Delta}(f\,d\mu)$ are essentially valid for $(I-t\Delta)^{-\beta}(f\,d\mu)$. We do not intend to give every detail here, but present the main steps. These are contained in our proof for the L^p-boundedness of $(I-t\Delta)^{-\beta}$ and the generalization of Wiener's theorem. Moreover, under the suggestion of R. S. Strichartz, we here also present an estimation for $\|(I-t\Delta)^{-\beta}(\nu)\|_{L^p(dx)}$ where ν is an L^p-weakly α-dimensional measure (see the next section for its definition).

I am especially grateful to Professor R. S. Strichartz. He sent me several of his papers on fractal measures, which brought my attention to this subject, and later gave me many helpful comments and suggestions on my original manuscript, which led to the results in this paper. I also would like to thank Professor Bo Berndtsson who supplied me a copy of A. G. Setti's paper.

2. Statement of the results

We make the convention that all the constants, if not specified, are denoted by C which may be different in different occurences. A simply connected n-dimensional manifold with $K(M)=-k$ is denoted by $H^{n,k}$ and called a hyperbolic space. As usual, p' denotes the adjoint number of p (≥ 1), i.e., $1/p+1/p'=1$. Following A. G. Setti [3], a locally finite (complex) measure ν on M is said to be L^p-weakly α-dimensional, if

$$\sup_{0<r\leq 1} r^{(n-\alpha)/p'} \cdot \left\| \frac{\nu(B_r(x))}{V(B_r(x))} \right\|_{L^p(M,dx)} < C.$$

The main results of this paper are as follows.

Theorem 1. *Let $h_t(x,y)$ be the heat kernel on a complete n-dimensional C^∞ Riemannian manifold M. Suppose that M is of positive injectivity radius, $\inf_{x\in M} V(B_1(x))>0$, and the Ricci curvature $\mathrm{Ric}(M)\geq -(n-1)k$. If μ is a locally uniformly α-dimensional measure on M, $(0\leq\alpha\leq n)$, then there exist constants $C_1=C_1(M,\mu)$ and $\tau=\tau(M)$ such that*

$$\int_M h_\varrho(x,y)\,d\mu(y)\leq C_1(\varrho^{-(n-\alpha)/2}+e^{\tau\varrho})$$

for all $x\in M$ and $\varrho>0$. In particular, if $M=H^{n,k}$, one can choose $\tau=\tau(H^{n,k})=0$.

Theorem 2. *Let M and μ be given as in Theorem 1. Suppose that $f\in L^p(M,d\mu)$ and $\beta>(n-\alpha)/2p'$, $1\leq p\leq\infty$. Then there exists a constant*

$$C_2=C_2(M,\mu,\beta,p)$$

such that

$$\|(I-t\Delta)^{-\beta}(f\,d\mu)\|_{L^p(dx)}\leq C_2(t^{-(n-\alpha)/2p'}+1)\|f\|_{L^p(d\mu)} \tag{1}$$

for all $0<t<1/2\tau$. In particular, if $M=H^{n,k}$ then (1) holds for all $t>0$.

Theorem 3. *Let M be given as in Theorem 1. Suppose that ν is an L^p-weakly α-dimensional measure on M. Then, for $\beta>(n-\alpha)/2p'$, $1\leq p\leq\infty$, there exists a constant $C_3=C_3(M,\nu,\beta,p)$ such that*

$$\|(I-t\Delta)^{-\beta}(\nu)\|_{L^p(dx)}\leq C_3(t^{-(n-\alpha)/2p'}+1) \tag{2}$$

for $0<t<1/2\tau$. Moreover, if $M=H^{n,k}$, then (2) holds for all $t>0$.

Theorem 4. *Let M be a complete n-dimensional C^∞ manifold with bounded geometry and μ a locally uniformly 0-dimensional measure on M. Suppose that $\mu=\mu_c+\sum c_j\mu_{x_j}$ is the decomposition of μ into its continuous and discrete parts. Then, for $f\in L^p(M,d\mu)$ and $\beta>n/2p'$ with $1<p<\infty$,*

$$\lim_{t\to 0} t^{n/2p'}\|(I-t\Delta)^{-\beta}(f\,d\mu)\|_{L^p(dx)}=C(n,\beta,p)\Big(\sum f(x_j)^p c_j^p\Big)^{1/p},$$

where

$$C(n,\beta,p)=\frac{2^{n/2+1-\beta}}{(4\pi)^{n/2}\Gamma(\beta)}\|\,|x|^{\beta-n/2}K_{\beta-n/2}(|x|)\|_{L^p(R^n)},$$

and K_1 is the l-th K-Bessel function (cf. [6]).

Theorem 4 can be viewed as a generalization of Wiener's theorem (cf. [5]); its proof will be given in the last section. In the next section we will prove Theorem 1 and Theorem 2 and deduce some interesting consequences.

3. Proof of Theorem 1 and Theorem 2

Keep the notations previously used. The following two facts crucial to our discussion are valid under the assumption that $\mathrm{Ric}(M) \geq -(n-1)k$ (cf. [2], [3]):

(F_1) $\sup_{x \in M} V(B_r(x)) \leq V(n,k,r)$, $V(n,k,r) = \mathbf{O}\big(e^{\sqrt{k}\,nr}\big)$ as $r \to \infty$, where $V(n,k,r)$ is the volume of a ball of radius r contained in $H^{n,k}$.

(F_2) For any $0<\varepsilon<1$, there exists a constant $C_4 = C_4(n,k,\varepsilon)$ such that

$$h_t(x,y) \leq C_4 \big[V(B_{\sqrt{t}}(x)) V(B_{\sqrt{t}}(y))\big]^{-1/2} e^{(\varepsilon - \lambda_M)t} \exp(-d(x,y)^2/4(1+\varepsilon)t)$$

for all $x, y \in M$ and $t>0$, where $0 \leq \lambda_M \leq (n-1)^2 k/4$.

Remark 1. It is easy to check that the Riemannian measure dx is a locally uniformly α-dimensional measure for all $0 \leq \alpha \leq n$.

Remark 2. Given a locally uniformly α-dimensional measure μ, a paving argument (cf. [5]) simply yields that $\mu(B_r(x)) \leq C \cdot V(B_r(x))$ for all $x \in M$ and $r>1$. Consequently, one gets from (F_1) that $\mu(B_r(x)) \leq C \cdot e^{\sqrt{k}\,nr}$. This inequality is obviously valid for all $x \in M$ and $r>0$.

Proof of Theorem 1. For $0<\varrho \leq 1$, the assertion has been proved by Setti in [3, p. 1073]. For $\varrho>1$ we know from (F_2) (choose $\varepsilon = \frac{1}{2}$) and the assumption $\inf_{x \in M} V(B_1(x))>0$ that

$$h_\varrho(x,y) \leq C \cdot e^{(1/2-\lambda_M)\varrho} \cdot e^{-d(x,y)^2/6\varrho}, \quad x, y \in M, \ \varrho>1.$$

Applying the above inequality and noting Remark 2, we have

$$\begin{aligned}
\int_M h_\varrho(x,y)\, d\mu(y) &\leq C \cdot e^{(1/2-\lambda_M)\varrho} \int_M e^{-d(x,y)^2/6\varrho}\, d\mu(y) \\
&= C \cdot e^{(1/2-\lambda_M)\varrho} \int_0^\infty -(e^{-r^2/6\varrho})' \mu(B_r(x))\, dr \\
&\leq C \cdot e^{(1/2-\lambda_M)\varrho} \int_0^\infty e^{-r^2/6\varrho} \cdot \frac{r}{3\varrho} \cdot e^{\sqrt{k}\,nr}\, dr \\
&\leq C \cdot e^{(1/2-\lambda_M)\varrho} \int_0^\infty r e^{-r^2} e^{\sqrt{6k\varrho}\,nr}\, dr.
\end{aligned}$$

But, $\int_0^\infty r e^{-r^2} e^{br}\, dr \leq e^{b^2/2}$ for $b>0$. Hence, $\int_M h_\varrho(x,y)\, d\mu(y) \leq C \cdot e^{\tau\varrho}$ for all $x \in M$ and $\varrho>1$, where $\tau = \frac{1}{2} - \lambda_M + 3kn^2$.

Now, consider the special case $M = H^{n,k}$. In this case, the heat kernel can be written as $h_t(r)$ with $r = d(x,y)$. And $h_t(r)$ has the property (cf. [1]): for any $t>0$ and $r>0$,

$$h_t(r) \leq (4\pi t)^{-n/2} e^{-r^2/4t} \quad \text{and} \quad h_t'(r) \leq 0,$$

where the derivative is taken about the variable r. This fact together with Remark 2 yields

$$\begin{aligned}\int_M h_\varrho(x,y)\,d\mu(y) &= \int_0^\infty h_\varrho(r)\,d\mu(B_r(x)) = \int_0^\infty -h'(r)\,\mu(B_r(x))\,dr \\ &\le C\left(\varrho^{-n/2}\int_0^1 e^{-r^2/4\varrho}\cdot\frac{r}{2\varrho}\cdot r^\alpha dr + \int_1^\infty -h'_\varrho(r)V(B_r(x))\,dr\right) \\ &\le C\left(\varrho^{-(n-\alpha)/2}\int_0^1 r^{1+\alpha}e^{-r^2}dr + \int_0^\infty -h'_\varrho(r)V(B_r(x))\,dr\right) \\ &\le C(\varrho^{-(n-\alpha)/2}+1), \quad x\in M,\ \varrho>0.\end{aligned}$$

This completes the proof.

Proof of Theorem 2. First note that in the proof of Theorem 1 we have used the identity $\int_M h_\varrho(x,y)\,dx\equiv 1$. This identity and the conclusion of Theorem 1 together with an interpolation argument shows

$$\|e^{\varrho\Delta}(f\,d\mu)\|_{L^p(dx)} \le C(\varrho^{-(n-\alpha)/2}+e^{\tau\varrho})^{1/p'}\|f\|_{L^p(d\mu)}$$

for $1\le p\le\infty$ and $\varrho>0$. Then, an application of Minkowski's inequality yields that, for $\beta>(n-\alpha)/2p'$ and $1\le p\le\infty$,

$$\begin{aligned}\|(I-t\Delta)^{-\beta}f\,d\mu)\|_{L^p(dx)} &\le C\int_0^\infty s^{\beta-1}e^{-s}\|e^{(st)\Delta}(f\,d\mu)\|_{L^p(dx)}\,ds \\ &\le C(t^{-(n-\alpha)/2p'}+1)\|f\|_{L^p(d\mu)}, \quad \text{if } 0<t<\frac{p'}{2\tau}.\end{aligned} \tag{3}$$

If $M=H^{n,k}$ then $\tau=0$, and hence (3) holds for all $t>0$. The proof is completed.

The operator $(I-\Delta)^{-\beta}$, usually called a Bessel potential, is of particular interest. A consequence of Theorem 2 is the following

Corollary 1. *Let M and μ be given as in Theorem* 1. *Assume that $p\ge 1$ and $\beta>(n-\alpha)/2p'$. If $p'>n^2k-\lambda_M$ then the Bessel potential $(I-\Delta)^{-\beta}$ is bounded from $L^p(M,d\mu)$ to $L^p(M,dx)$. In particular, if* $\mathrm{Ric}(M)\ge 0$ *or $K(M)=-k$ then $(I-\Delta)^{-\beta}$ is bounded from $L^p(M,d\mu)$ to $L^p(M,dx)$ for all $p\ge 1$.*

Proof. In fact, one can deduce from (F_1) and (F_2) that, for any $0<\varepsilon<1$,

$$h_\varrho(x,y)\le C\cdot e^{(\varepsilon-\lambda_M)\varrho}\cdot e^{-d(x,y)^2/4(1+\varepsilon)\varrho}, \quad \text{if } \varrho>1.$$

And a simple calculation shows that for any $\theta>0$ there exists a constant $C(\theta)$ such that

$$\int_o^\infty re^{-r^2}\cdot e^{br}\,dr \le C(\theta)e^{(1+\theta)b^2/4}, \quad b>0.$$

Hence, it is easy to see that (3) (in its proof, we have choosen $\varepsilon=\frac{1}{2}$ and $\theta=1$) is still valid for all $0<t\le t_0<p'/\widetilde{\tau}$, where $\widetilde{\tau}=(\varepsilon-\lambda_M)+(1+\varepsilon)(1+\theta)kn^2$. Hence, if $p'>kn^2-\lambda_M$ then one can choose $\varepsilon>0$ and $\theta>0$ small enough so that $p'>\widetilde{\tau}$. Thus, the first part of the conclusion follows. If $K(M)=-k$ the $\tau=0$ and the conclusion is already contained in Theorem 2. If $\mathrm{Ric}(M)\ge 0$ then one can choose $k>0$ arbitrarily small. But $p'\ge 1$, hence $p'>kn^2-\lambda_M$. This completes the proof.

Corollary 2. *Let μ be a locally uniformly α-dimensional measure on the hyperbolic space $H^{n,k}$, $0\le\alpha\le n$. If $\mathrm{Re}(z)<(\alpha-n)/2$ then the Riesz potential $(-\Delta)^z$ is bounded from $L^\varrho(H^{n,k},d\mu)$ to $L^\varrho(H^{n,k},dx)$ for $1<p<\infty$.*

Proof. Let w and z be complex numbers. Bear in mind that the manifold presently discussed is the hyperbolic space. It follows from Theorem 2 that, for $w>0$ and $z<(\alpha-n)/2$ (the same proof works for $\mathrm{Re}(w)>0$ and $\mathrm{Re}(z)<(\alpha-n)/2$), the operator $(wI-\Delta)^z$ is bounded from $L^\infty(d\mu)$ to $L^\infty(dx)$ and from $L^1(d\mu)$ to $L^1(dx)$ as well. On the other hand, it is also a consequence of Theorem 2 that

$$\|E_\lambda(f\,d\mu)\|_{L^2(dx)}\le C(\lambda^{(n-\alpha)/2}+1)\|f\|_{L^2(d\mu)},\quad \lambda>0,$$

where E_λ is the spectral projection onto the portion of the spectrum of Δ in the interval $[-\lambda^2,0]$. This fact implies that $(wI-\Delta)^z$ is bounded from $L^2(d\mu)$ to $L^2(dx)$ provided that $\mathrm{Re}(w)>-((n-1)/2)^2k$ and $\mathrm{Re}(z)<(\alpha-n)/4$. A complex interpolating argument (cf. [4, p. 69]) yields the conclusion.

4. Proof of Theorem 3 and Theorem 4

Proof of Theorem 3. It has been shown in [3, p. 1075] that

$$\sup_{0<\varrho\le 1}\varrho^{(n-\alpha)/2p'}\|e^{\varrho\Delta}(\nu)\|_{L^p(dx)}<C. \tag{4}$$

Moreover, fixing a paving $\{M_j\}$ of size $\frac{1}{4}$, one can define a measure $\mu_{1/4}$ by

$$\mu_{1/4}(A)=\frac{|\nu|(A)}{|\nu|(M_j)}V(M_j),\quad \text{if } A\subseteq M_j$$

so that $\nu=f_{1/4}\mu_{1/4}$ with $\|f_{1/4}\|_{L^p(d\mu_{1/4})}=C\left(\frac{1}{4}\right)^{(\alpha-n)/p'}$.

For any $x\in M$ and $r>1$, let $J=\{j;M_j\cap B_r(x)\ne\emptyset\}$. Then, one must have

$$\bigcup_{j\in J}M_j\subset B_{2r}(x).$$

Hence,

$$\mu_{1/4}(B_r(x)) \le \sum_{j\in J} \mu_{1/4}(M_j) = \sum_{j\in J} V(M_j) \le V(B_{2r}(x)).$$

Arguing as in the proof of Theorem 1, one gets $\int_M h_\varrho(x,y)\, d\mu_{1/4}(y) \le Ce^{\tau\varrho}$, and hence

$$\|e^{\varrho\Delta}(\nu)\|_{L^p(dx)} \le Ce^{\tau\varrho/p'} \quad \text{for } \varrho > 1. \tag{5}$$

The conclusion follows from (4) and (5) immediately.

To prove Theorem 4, one can treat all the cases ($1<p<\infty$) using the same approach. But we will use a different method in the case that $p=2$, which is of particular interest.

Proof of Theorem 4. *The special case* ($p=2$). For $t>0$ and $\beta>0$, $(I-t\Delta)^{-\beta}$ is selfadjoint and has the semigroup property with respect to β. This fact together with Fubini's theorem yields

$$\begin{aligned}
&\|(I-t\Delta)^{-\beta}(f\,d\mu)\|^2_{L^2(dx)} \\
&\quad = \int_M J_t^{2\beta}(x,y) f(x)\overline{f(y)}\, d\mu(x)\, d\mu(y) \\
&\quad = \frac{1}{\Gamma(2\beta)} \int_0^\infty s^{2\beta-1} e^{-s} \left\{ \int_M h_{st}(x,y) f(x)\overline{f(y)}\, d\mu(x)\, d\mu(y) \right\} ds \\
&\quad = \frac{1}{\Gamma(2\beta)} \int_0^\infty s^{2\beta-1} e^{-s} \|e^{(st/2)\Delta}(f\,d\mu)\|^2_{L^2(dx)}\, ds.
\end{aligned}$$

The last equality follows from the semigroup property of $e^{t\Delta}$. Hence, we have

$$\begin{aligned}
&t^{n/2}\|(I-t\Delta)^{-\beta}(f\,d\mu)\|^2_{L^2(dx)} \\
&\quad = \frac{2^{n/2}}{\Gamma(2\beta)} \int_0^\infty s^{2\beta-n/2-1} e^{-s} \left\{ \left(\tfrac{1}{2}st\right)^{n/2} \|e^{(st/2)\Delta}(f\,d\mu)\|^2_{L^2(dx)} \right\} ds.
\end{aligned} \tag{6}$$

By Theorem 1 (see the proof of Theorem 2), the integrand in (6) is dominated by $C(s^{2\beta-n/2-1}e^{-s} + s^{2\beta-1}e^{-s/2})$ for $0<t<\min(1, 1/2\tau)$. Thus, the conclusion follows from Theorem 3.2 in [5] and the Lebesgue dominated convergence theorem. In particular, we obtain

$$C(n,\beta,2) = \left(\frac{\Gamma(2\beta-\frac{1}{2}n)}{(4\pi)^{n/2}\Gamma(2\beta)} \right)^{1/2} \quad \text{for } \beta > \frac{n}{4}.$$

The general case $(1<p<\infty)$. According to the argument in [5, p. 197], it suffices to verify the following two facts:

(I) For any fixed $\delta>0$ and $y\in M$,

$$\lim_{t\to 0} t^{n/2p'}\left(\int_{d(x,y)\leq\delta} J_t^{\beta}(x,y)^p\,dx\right)^{1/p}=0.$$

(II)

$$t^{n/2p'}\left(\int_{d(x,y)>\delta} J_t^{\beta}(x,y)^p\,dx\right)^{1/p}$$

approaches to the constant $C(n,\beta,p)$ independent of δ and y.

By an argument similar to the one used in the proof of Theorem 1, one can show that, for a locally uniformly α-dimensional measure μ,

$$\left(\int_M h_\varrho(x,y)^p\,d\mu(y)\right)^{1/p}\leq C(\varrho^{-n/2+\alpha/2p}+e^{\tau\varrho}). \tag{7}$$

Recall that dx is a locally uniformly n-dimensional measure. Then (I) follows from the known result about the heat kernel $h_t(x,y)$ and the Minkowski inequality together with (7) combining with a dominated convergence argument. Furthermore, a similar argument shows that, for any fixed $\delta>0$ and $\eta>0$,

$$\lim_{t\to 0} t^{n/2p'}\left(\int_{d(x,y)\leq\delta}\left|\int_{\eta/t}^{\infty} s^{\beta-1}e^{-s}h_{st}(x,y)\,ds\right|^p dx\right)^{1/p}=0.$$

Thus, to show (II) we need only to deal with

$$t^{n/2p'}\left(\int_{d(x,y)\leq\delta}\left|\int_{0}^{\eta/t} s^{\beta-1}e^{-s}h_{st}(x,y)\,ds\right|^p ds\right)^{1/p}. \tag{8}$$

Since $\delta>0$ and $\eta>0$ can be arbitrarily small, one can replace $h_{st}(x,y)$ by $\tilde{h}_{st}(x,y)=(4\pi st)^{-n/2}\exp(-|x-y|^2/4st)$ and compute (8) in the Euclidean sense. Finally, we get

$$\begin{aligned}
C(n,\beta,p)&=\lim_{t\to 0}\frac{t^{n/2p'}}{\Gamma(\beta)}\left(\int_{R^n}\left|\int_0^{\infty} s^{\beta-1}e^{-s}h_{st}(x,0)\,ds\right|^p dx\right)^{1/p}\\
&=\frac{1}{(4\pi)^{n/2}\Gamma(\beta)}\left(\int_{R^n}\left|\int_0^{\infty} s^{\beta-n/2-1}e^{-s}e^{-|x|^2/4s}\,ds\right|^p dx\right)^{1/p}\\
&=\frac{2^{n/2+1-\beta}}{(4\pi)^{n/2}\Gamma(\beta)}\|\,|x|^{\beta-n/2}K_{\beta-n/2}(|x|)\|_{L^p(R^n)},
\end{aligned}$$

where dx denotes the Lebesgue measure on R^n. The proof is completed.

In general, it is not simple to compute $C(n,\beta,p)$ since the integral involves a Bessel function. But for $\beta=\frac{1}{2}(n+1)$, it is easy to check that

$$C\big(n,\tfrac{1}{2}(n+1),p\big)=p^{-n/p}\left(\frac{\sqrt{\pi}}{(4\pi)^{n/2}\Gamma\big(\frac{1}{2}(n+1)\big)}\right)^{1/p'}.$$

Moreover, looking back to the proof of Theorem 4, one may easily see from Theorem 6.2 in [5] that

$$C(n,\beta,p)\leq(4\pi)^{-n/2p'}p^{-n/2p}.$$

An interesting by-product to Theorem 4 is the following equality

$$\|\,|x|^{\beta-n/2}K_{\beta-n/2}(|x|)\|_{L^2(R^n)}=2^{\beta-n/2-1}(4\pi)^{n/4}\Gamma(\beta)\left(\frac{\Gamma\big(2\beta-\frac{1}{2}n\big)}{\Gamma(2\beta)}\right)^{1/2}$$

for $\beta>\frac{1}{4}n$, $n=2,3,\dots$.

Remark 3. The assumption that M is of positive injectivity radius is needed to carry out the paving procedure on M globally.

Remark 4. In [2], S. T. Yau gave an example of M with $\mathrm{Ric}(M)\geq 0$ but $\inf_{x\in M}V(B_1(x))=0$. In our approach, we need a uniform control on $V(B_r(x))^{-1}$ for all $x\in M$ and $r>1$. Hence, we posed the condition that $\inf_{x\in M}V(B_1(x))>0$, which is valid if $K(M)\leq\tilde{k}$.

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