REGULAR AND SEMI-REGULAR POSITIVE TERNARY QUADRATIC FORMS.

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1. *Introduction.* For any ternary quadratic form $f(x, y, z)$ with integral coefficients there are usually congruences $f \equiv h \pmod{m}$ which are not solvable, whence no number $mn + h$ is represented by f, where n is an integer. For instance, $f = x^2 + y^2 + z^2 \equiv 3 \pmod 4$ implies that x, y and z are odd, whence $f \equiv 3 \pmod 8$. It follows that f represents no number $8n + 7$ where n is an integer. Similarly f may be shown to represent no number $4^k(8n + 7)$. In this case, these are the only numbers congruentially excluded. For any form the numbers so excluded consist of certain arithmetic progressions of the forms $2^r(8n + a)$, $p^s(pn + b)$, where r and s range over some or all non-negative integers, a is odd, p is an odd prime factor of the determinant of f, and b ranges over the quadratic residues or non-residues of p or both. H. J. S. Smith's definition of *genus*[1] in terms of the characters $(f \mid p)$ etc., of the form and its reciprocal, is equivalent[2] to the following: two forms of the same determinant are in the same genus if the progressions associated, as above, with the forms are the same. Two forms are of the same genus, as proved by H. J. S. Smith, if and only if one can be carried into the other by a linear transformation of determinant 1 and whose coefficients are rational numbers whose denominators are prime to twice the determinant of the forms. It is therefore natural in this article that the solution of problems in genera of several classes[3] is found by use of such

[1] H. J. S. SMITH, *Collected Papers*, vol. 1, pp. 455—509; *Philosophical Transactions*, vol. 157, pp. 255—298.

[2] B. W. JONES, *Trans. Amer. Math. Soc.*, vol. 33 (1931), pp. 92—110; also ARNOLD ROSS *Proc. Nat. Acad. Sc.*, vol. 18 (1932), pp. 600—608.

[3] Two forms are of the same *class* if one may be taken into the other by a linear trans formation with *integral* coefficients and of determinant 1; i. e. by a unimodular transformation.

rational transformations. A second important property is that, given a genus and its associated progressions, every number not in one of the progressions is represented by at least one form of the genus.[1] If it happens that one form represents all the numbers not in the progressions, that form is called regular.[2] It follows that whenever there is but one class in the genus, that class (and hence every form in the class) is regular.

Though, subject to certain restrictions on the invariants, there is in each genus[3] of *indefinite* ternaries only one class, this is not so generally the case for *positive* forms. Hence problems concerning the numbers represented by positive forms are generally more difficult than is the case for indefinite forms. We consider in this paper only positive forms.

A few positive regular forms were studied previous to their designation as such. The first complete proof of the fact that $x^2 + y^2 + z^2$ represents exclusively all positive integers $\neq 4^k(8n + 7)$ was given by Legendre and was followed by simpler proofs by Gauss and Dirichlet.[4] Similar results for $x^2 + y^2 + az^2$ where $a = 2$, 3 or 5 were obtained by Lebesgue, Dirichlet and Liouville. A limited number of allied forms had also been dealt with. Since all these forms are in genera of one class, their regularity now follows from the second property of genera mentioned above. In 1916 Ramanujan[5] employed a number of such results, empirically obtained, in making his list of positive forms $ax^2 + by^2 + cz^2 + dt^2$ which represent all positive integers. It was this and his remark that the odd integers not represented by $x^2 + y^2 + 10z^2$ seemed to follow no definite law, that led to Dickson's definition of regularity and the systematic investigation which followed.

Using Dickson's methods and extensions of them it was found[6] that every primitive form (a, b, c) not in table I (p. 190) was irregular.[7] Ninety-six of

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[1] B. W. Jones, *Trans. Amer. Math. Soc.*, vol. 33 (1931), pp. 111—124.

[2] L. E. Dickson, *Annals of Math.*, (2), vol. 28 (1927), pp. 333—341.

[3] A. Meyer gave a partial proof in *Journal für Mathematik*, vol. 108 (1891), pp. 125—139. For a complete proof with further references see L. E. Dickson, *Studies in the Theory of Numbers*, chap. 4.

[4] For references see Dickson, *History of the Theory of Numbers*, vol. 2.

[5] S. Ramanujan, *Proc. Cambridge Phil. Soc.*, vol. 19 (1916), pp. 11—21; also *Collected Papers*, pp. 169—178.

[6] B. W. Jones, »The Representation of Integers by Positive Ternary Quadratic Forms», a University of Chicago thesis (1928), unpublished.

[7] In the thesis the form (1, 5, 200) was erroneously reported to be regular. It fails to represent 44 and hence is irregular. The rest of the table has been checked and found to be correct.

these forms were proved regular in the thesis or previous to it — some by laborious methods. In this paper we prove certain theorems, which, starting with certain basic forms, may be used to show quickly that eighty-two of these forms are each in genera of one class and hence are regular. These eighty-two forms are the only primitive positive ternary quadratic forms without cross-products which are in genera of one class. We also sketch the methods used in the thesis to prove the regularity of several forms in genera of more than one class. By one or other of these methods it may be established that ninety-three of the 102 forms of table I are regular. The form (1, 1, 16)[1] was proved regular by using theta function expansions[2] and later (1, 2, 32) yielded to the same method. The regularity of (1, 4, 16), (1, 16, 16) and (1, 8, 32) follow directly from these results. However, the regularity of the remaining forms

(A) (1, 8, 64), (1, 3, 36)

and two derivable from the latter has hitherto remained unproved. It may be noted that the forms (1, 1, 16), (1, 16, 16), (1, 3, 36), (1, 12, 36), (1, 4, 16) and (1, 8, 64) are the only regular forms of the table which are in genera of more than one class and whose reciprocal forms are also regular.

In this paper we prove by means of the rational automorphs of $x^2+y^2+\lambda z^2$ ($\lambda = 1, 2, 3$), in the convenient guise of quaternions, that the forms (A), (1, 1, 16), (1, 2, 32) and a few others of special interest are regular. We have succeeded in proving regular all forms which we have been able to discover as apparently regular. With the exception of (1, 48, 144) which belongs to a genus of four classes, all regular forms (a, b, c) belong to genera of one or two classes. The companion class we find, in many cases, is regular except that either it fails to represent a finite number of integers represented by forms of the genus, or it fails to represent an infinite number specified by a finite number of formulas involving square factors: for example, all odd squares whose every prime factor is in some cases $\equiv 1 \pmod 4$ and in other cases $\equiv 1 \pmod 3$. These almost regular forms are new and are one of the most significant products of the method of proof. We may call attention to the form $g = (8, 12, 21\ -6, 0, 0)$,

[1] that is $x^2+y^2+16z^2$. Similarly $ax^2+by^2+cz^2+2ryz+2sxz+2txy$ is denoted by (a, b, c, r, s, t).

[2] Nazimoff, *Applications of the Theory of Elliptic Functions to the Theory of Numbers* (Russian) translated by Arnold Chaimovitch. The proof for this form was indicated by Nazimoff and carried out by Chaimovitch.

the companion of the regular form $f = (5, 5, 72, 0, 0, -2)$; g is regular with the *single* exception of the number 5. $(4, 8, 9, 0, -2, 0)$ has a similar property. In table II we list all regular primitive forms (a, b, c) with more than one class in a genus, and their companion forms; in addition, two examples with cross products.

Ramanujan's form $(1, 1, 10)$ was observed by him to be regular for even numbers and he found that the following odds were not represented: 3, 7, 21, 31, 33, 43, 67, 79, 87, 133, 217, 219, 223, 253, 307, 391. If he had gone farther he would have found only one more odd number less than 2000 not represented, viz. 679. Although we have no complete proof, this form seems to be regular with these seventeen exceptions.

In this connection, some results of Tartakowsky[1] with regard to forms of s variables for $s \geqq 4$ are of interest. He claims to prove that if $s \geqq 5$, all forms in a genus represent the same sufficiently large numbers and a similar result with a restriction if $s = 4$. Our results as listed in table II would indicate that his theorem would be true for $s = 3$ in some cases, e. g. for the genus of $(1, 2, 32)$ and false in some other cases, e. g. for the genus of $(1, 1, 16)$.

The regularity of the forms (A), $(1, 1, 16)$ and $(1, 2, 32)$ is connected with special cases proved in Theorem 5 of a phenomenon in the representation of quadratic residues (mod $8d$) by ternary quadratic forms of determinant d. Other examples are easily obtained empirically, and perhaps can be proved by methods like those in section 4. Several examples connected with $(1, 1, 1)$ have been given as consequences of elliptic identities by Jacobi and Glaisher[2] and these were recently generalized.[3] One of the most interesting examples is the following: if $24n + 1 = s^2$ $(s > 0)$, then all proper solutions of $24n+1 = x^2 + 2y^2 - 2yz + 2z^2$ satisfy $x \equiv \pm 1 \pmod{12}$ if $s \equiv 1$ or 5 but $x \equiv \pm 5 \pmod{12}$ if $s \equiv 7$ or 11 (mod 12); but if $24n + 1 \neq s^2$, there are equally many solutions of each type. This has recently been verified by E. Rosenthall.

2. Though, to prove a form regular, it is sufficient, from the above discus-

[1] W. A. Tartakowsky, *Comptes Rendus de l'Académie des Sciences,* vol. 186 (1928), pp. 1337—1340, 1401—1403, 1684—1687. Errata in the second paper are corrected in vol. 187, p. 155. Complete paper in Bull. Ak. Sc. U. R. S. S. (7) (1929), pp. 111—22, 165—96.

[2] For references see Dickson, *History of the Theory of Numbers,* vol. 2, pp. 261—3 and p. 268 respectively. For example Glaisher states the following in *Messenger of Mathematics,* new series vol. 6, (1877), p. 104: The excess of the number of representations of $8n+1$ in the form $x^2+4y^2+4z^2$ with y and z even over the number of representations with y and z odd is zero if $8n+1$ is not a square and $2(-1)^{(s-1)/2}s$ if $8n+1 = s^2$.

[3] Gordon Pall, *Amer. Journ. of Math.* (1937), vol. 59, pp. 895—913.

sion, to prove that it is in a genus of one class, such a proof is usually very tedious especially if the form in question lies outside the range of the table of reduced forms.[1] We hence prove in this section a new theorem which, with its modifications not only proves with considerable celerity that most of the forms in table I are in genera of one class but determines the number of classes in the genera of the remaining forms. We shall use the following

Lemma: Given two primitive ternary quadratic forms f and g of the same genus, then for every ∇ whose every prime factor is a factor of their determinant, there exists a form φ equivalent to f whose coefficients are congruent to the corresponding coefficients of g (mod ∇).

This may be proved as follows. By a theorem quoted above, there is a transformation (t_{ij}/r) taking f into g where t_{ij} are integers and r is an integer prime to twice the determinant of f. Then for any ∇ of the lemma we find an s such that $rs \equiv 1 \pmod{\nabla}$. The transformation $(s\,t_{ij})$ will take f into a form $\equiv g \pmod{\nabla}$ and the determinant of the transformation is $\equiv 1 \pmod{\nabla}$. Then by a theorem of Smith[2] we can find a transformation (u_{ij}) of determinant 1 such that $u_{ij} \equiv s\,t_{ij} \pmod{\nabla}$ for every i and j. This transformation will take f into $\varphi \equiv g \pmod{\nabla}$.

Theorem 1. *If $g = \alpha_1 x^2 + \beta_1 y^2 + \gamma_1 z^2$ is primitive, if $\delta = \varrho^2 \sigma$ is a common factor of β_1 and γ_1 where σ is without a square factor and if $f = \sigma \alpha_1 x^2 + (\beta_1/\delta) y^2 + (\gamma_1/\delta) z^2$ is in a genus of one class, g is in a genus of one class provided*

$$\text{(B)} \qquad f \equiv \sigma \alpha_1 \pmod{\sigma \Omega^4} \text{ implies } y \equiv z \equiv 0 \pmod{\varrho \sigma}$$

where Ω is the g. c. d. of $\alpha_1 \beta_1$, $\alpha_1 \gamma_1$, $\beta_1 \gamma_1$.

To prove this consider a form h in the same genus as g. Then, by the lemma, we may assume that $h \equiv g \pmod{\Omega^4}$. Now

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[1] Eisenstein, *Journal für Mathematik*, vol. 41 (1851), pp. 141—190 gives a table for determinants from 1 to 100.

Arnold Ross, in *Studies in the Theory of Numbers* by L. E. Dickson, pp. 181—185 has a table for determinants from 1 to 50.

E. Borissow, *Reduction of Positive Ternary Quadratic Forms by Selling's Method, with a Table of Reduced Forms for all Determinants from 1 to 200.* St. Petersburg (1890), 1—108; tables 1—116 (Russian).

B. W. Jones, *A Table of Eisenstein-reduced Positive Ternary Quadratic Forms of Determinant $\leqq 200$* (1935), Bulletin No. 97 of the National Research Council.

[2] H. J. S. Smith, *Collected Papers*, vol. 2, p. 635; also *Mémoires présentés par divers Savants à l'Académie des Sciences de l'Institut de France* (2), vol. 29 (1887), No. 1, 72 pp.

$$U = \begin{pmatrix} \varrho\sigma & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

takes g into δf and will take h into a form $\delta\varphi$ of the same genus as δf, since[1] f represents a number N if and only if δN is represented by g while φ represents N if and only if δN is represented by h, that is, the progressions associated with f and with φ are the same. Then there is a unimodular transformation R taking f into φ. Hence $K = U R U^{-1}$ takes g into h and if $R = (r_{ij})$ we have

$$K = \begin{pmatrix} r_{11} & \varrho\sigma r_{12} & \varrho\sigma r_{13} \\ r_{21}/\varrho\sigma & r_{22} & r_{23} \\ r_{31}/\varrho\sigma & r_{32} & r_{33} \end{pmatrix}.$$

Hence g and h will be equivalent if $r_{21} \equiv r_{31} \equiv 0 \pmod{\varrho\sigma}$. But the coefficient of x^2 in h is then $\alpha_1 r_{11}^2 + (\beta_1 r_{21}^2 + \gamma_1 r_{31}^2)/\delta\sigma$ which must be an integer $\equiv \alpha_1 \pmod{\nabla}$. Thus, if (B) holds, g is equivalent to h.

Modification 1. If f has an automorph T, then replacing f by $T' f T$ above has the effect of replacing R by TR. T will have the same effect on r_{11}, r_{21}, r_{31} as it will on x, y, z and hence if, for every r_{11}, r_{21}, r_{31} there exists an automorph T taking r_{11}, r_{21}, r_{31} into $r_{11}^1, r_{21}^1, r_{31}^1$ such that $r_{21}^1 \equiv r_{31}^1 \equiv 0 \pmod{\varrho\sigma}$ we may conclude that g and h are equivalent and hence that g is in a genus of one class.

Corollary 1. If $\sigma\alpha_1 = \gamma_1/\delta$ and $f \equiv \sigma\alpha_1 \pmod{\sigma\Omega^4}$ implies $y \equiv 0 \pmod{\varrho\sigma}$ and either x or $z \equiv 0 \pmod{\varrho\sigma}$, g is in a genus of one class. This follows from the modification above since $r_{31} \not\equiv 0 \pmod{\varrho\sigma}$ would imply $r_{11} \equiv 0 \pmod{\varrho\sigma}$ and the transformation $x = -z^1$, $y = -y^1$, $z = -x^1$ is an automorph and would interchange r_{11} and r_{31}. Similar results follow if $\beta_1/\delta = \sigma\alpha_1$ or $\beta_1/\delta = \gamma_1/\delta = \sigma\alpha_1$.

Corollary 2. If $\varrho\sigma = 2$, $\beta_1/\delta = 1$, $\gamma_1/\delta = 3$ and if $f \equiv \sigma\alpha_1 \pmod{\sigma\Omega^4}$ implies $y \equiv z \pmod 2$, the theorem still holds, for

$$T_1 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1/2 & -3/2 \\ 0 & -1/2 & 1/2 \end{pmatrix} \text{ and } T_2 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1/2 & 3/2 \\ 0 & 1/2 & 1/2 \end{pmatrix}$$

are automorphs of f and take r_{21} and r_{31} into $-(r_{21} + 3r_{31})/2$, $(-r_{21} + r_{31})/2$ or

[1] B. W. Jones, *A New Definition of Genus* . . . see earlier reference.

$(-r_{21}+3r_{31})/2$, $(r_{31}+r_{21})/2$. If r_{21} and r_{31} are odd, one of these pairs consists of even integers.

Modification 2. If (B) holds, or the modification above, and f is in a genus of more than one class, the number of classes in the genus of g is not more than the number of classes in the genus of f. For, suppose the number of classes in the genus of f is s. Then, if (B) holds for one form of the genus of g it will hold for a representative of each class of forms. The transformation U will lead to not more than s non-equivalent forms. And any two forms of the genus of g which lead to equivalent forms are themselves equivalent.

Modification 3. If $\sigma = 1$ and $f \equiv \alpha_1 \pmod{\Omega^4}$ in the theorem implies that (B) holds or one of the conditions, $C_2, C_3, \ldots, C_r$ on the variables holds and if for every C_i there is a transformation T_i^{-1} of determinant ϱ^2 taking f into a form of the genus of g; if further all the coefficients in the second and third columns of $\varrho T_i R U^{-1}$ are integers and under condition C_i the coefficients of the first column are also; then the number of classes in the genus of g does not exceed rs where s is the number of classes in the genus of f.

This may be seen as follows: if φ is equivalent to f and if T_i^{-1} takes f into g_i we have $T_i' g_i T_i = f$. Hence $\varrho T_i R U^{-1}$ takes g_i into h and if the coefficients of the first column of R satisfy C_i, g_i is equivalent to h. Hence h will be equivalent to one of the g_i. If, on the other hand, φ is not equivalent to f, the reasoning of modification 2 applies.

Corollary 3. If $\varrho = 2$, $r = 2$, $\sigma = 1$ and C_2 is one of the following, the number of classes in the genus of g is $\leqq 2s$: y even and $x \equiv z \pmod 2$; $x \equiv y \equiv z \equiv 1 \pmod 2$; $x \equiv y \equiv 0 \pmod 2$. In the first case take as T_2^{-1}: $x = 2x^1 + z^1$, $y = 2y^1$, $z = z^1$ and see that the first column of $2T_2 R U^{-1}$ is $(r_{11} - r_{31})/2$, $r_{21}/2$, r_{31} while all the other elements are integers. C_2 implies that all are integers. In the second case T_2^{-1} is $x = 2x^1 + z^1$, $y = 2y^1 + z^1$, $z = z^1$ and in the third case $x = 2x^1$, $y = 2y^1$, $z = z^1$.

The theorem and the first two modifications suffice to prove that all forms in table I which are not marked are in genera of one class if one first ascertains from a table of reduced forms that the following are in genera of one class: (1, 1, 1), (1, 1, 2), (1, 1, 3), (1, 1, 5), (1, 1, 6), (1, 1, 21), (1, 2, 3), (1, 2, 5), (1, 3, 10). We show this for a few simple cases.

a) If $g = (1, r, r^2)$ where $r = 2, 3, 5$; then f of the theorem is $rx^2 + y^2 + rz^2$,

$f \equiv r \pmod{r^2}$ implies $y = r y_1$ and $x^2 + r y_1^2 + z^2 \equiv 1 \pmod{r}$ which implies that x or $z \equiv 0 \pmod{r}$. Corollary 1 applies with condition (B) to prove our result since $(1, 1, r)$ is in a genus of one class and it is the reciprocal form of f.

b) if $g = (1, 1, r)$ where $r = 4, 9, 12$ or 24, g will be in a genus of one class if and only if its reciprocal $(1, r, r)$ is. Replace g by its reciprocal and $f = \sigma x^2 + y^2 + z^2 \equiv \sigma \pmod{r^2}$ implies $y \equiv z \equiv 0 \pmod{\varrho\sigma}$ or corollary 1 applies. If $r = 8$ we take f to be $(1, 2, 2)$.

c) If $g = (4, 3, 12)$, take $\delta = 3$ and have $f = 12x^2 + y^2 + 4z^2 \equiv 0 \pmod{3}$ implies that $y \equiv z \equiv 0 \pmod{3}$. Hence g is in a genus of one class if f is. Then repeat the process using corollaries 1 and 2 on $(1, 4, 12)$.

Corollary 3 may be used to prove that all the forms of table II are in genera of two classes except that $(1, 48, 144)$ is in a genus of four classes. Again we prove this for a few typical cases.

a) $g = (1, 2, 32)$ has a reciprocal $g = (1, 16, 32)$ which we consider in its place. Then if we take $\delta = 4$, $f = x^2 + 4y^2 + 8z^2 \equiv 1 \pmod{8}$ implies $y \equiv z \equiv 0 \pmod{2}$ or $x \equiv z \equiv 1 \pmod{2}$ with y even and since f is in a genus of one class from table I, the corollary applies to prove that g is in a genus of 1 or 2 classes. Table II exhibits another reduced form of the same genus as g.

b) $g = (a, 4b, 12b)$ where a is odd, $b \equiv 2, 4$ or $6 \pmod{8}$ and $(a, b, 3b)$ is in a genus of one class. Taking $\delta = 4$ we have $f = ax^2 + by^2 + 3bz^2 \equiv a \pmod{8}$ implies $y \equiv z \pmod{2}$ and x odd and hence the corollary applies.

c) $g = (1, 48, 144)$. Take $\delta = 4$ and $f = (1, 12, 36)$ and the corollary shows that the number of classes in the genus of g is $\leqq 2s$. If now we take $g = (1, 12, 36)$ and take $\delta = 4$ we have $f = (1, 3, 9)$ and the application of the corollary shows that since $(1, 3, 9)$ is in a genus of one class, $(1, 12, 36)$ is in a genus of not more than two classes and $(1, 48, 144)$ in a genus of not more than four classes. The table exhibits three other reduced forms of the genus.

d) $g = (5, 5, 72, 0, 0, -1)$. The replacement of x by $x - 3y$ takes g into $g^1 = 5x^2 + 56y^2 + 72z^2 - 32xy \equiv 5x^2 + 56y^2 + 72z^2 \pmod{32}$. Hence taking $\delta = 4$, $f = 5x^2 + 14y^2 + 18z^2 - 16xy \equiv 5x^2 + 14y^2 + 18z^2 \pmod{16}$. Hence $f \equiv 5 \pmod{16}$ implies $y \equiv z \pmod{2}$, x odd and the corollary shows that the number of classes in the genus of g is not more than two if f is in a genus of one class. That this is the case follows from the fact that $x = x_1 - 2y_1$, $y = x_1 - y_1$, $z = z_1$ takes f into $3x_1^2 + 2y_1^2 + 18z_1^2$ which, from table I, is in a genus of one class.

e) That the form $(1, 1, 3, 0, -1/2, 0)$ is in a genus of two classes may be verified from the table.

We prove: $g = (a, 4b, 12b)$ where a is odd and $b \equiv 2, 4$ or $6 \pmod 8$ is regular if and only if $f = (a, b, 3b)$ is. $f \equiv a \pmod 8$ implies $y \equiv z \pmod 2$. If $f \equiv a \pmod 8$ with y and z odd we may choose the sign of z so that $y \equiv z \pmod 4$ and both of $y^1 = (y + 3z)/2$ and $z^1 = (y - z)/2$ will be even. This transformation, however, is an automorph of f. Hence, if f represents an odd number with y and z odd, it represents that same number with y and z even. Hence g represents the same odds that f does. The multiples of 4 represented by g are 4 times the integers represented by f. This, together with the above theory, suffices to prove the regularity of all forms of table II except $(1, 4, 36)$ and those dealt with later in this paper.

That $f = (1, 4, 36)$ is regular as to multiples of 3 or 4 is easily shown. Using the form $(1, 1, 1)$ it is not hard to prove that f represents all $12n + 1$. To prove that it represents all $12n + 5$ replace y by $y + 3z$ and have the form $g = x^2 + 2y^2 + 2(6z + y)^2$ equivalent to f. Since $h = x^2 + 2y^2 + 2Z^2$ represents all $12n + 5$ we need merely to show that there is a representation with $Z \equiv y \pmod 6$. We may choose Z prime to 3. We can show that $x^2 + 2y^2 = 12n + 5 - 2Z^2$ implies the existence of an r and s prime to 3 for which $x^2 + 2y^2 = r^2 + 2s^2$. For if $a^2 + 2b^2 = k$ with a or b prime to 3, $(a + 4b)^2 + 2(2a - b)^2 = 9k$ where, after an interchange of b and $-b$ if necessary, $a + 4b$ and $2a - b$ are both prime to 3. Repetition of this argument shows that if $x = 3^p x_1$, $y = 3^p y_1$ with x_1 or y_1 prime to 3 and $x_1^2 + 2y_1^2 = (12n + 5 - 2Z^2)/9^p$ then an r and s of the desired type exists, i. e. h represents $12n + 5$ with x, y and Z prime to 3. Then $y \equiv \pm Z \pmod 6$ and replacing Z by $-Z$ if necessary makes our proof complete.

3. $x^2 + y^2 - yz + 3z^2$ is regular. The forms of determinant $11/4$,

$$f = x^2 + y^2 - yz + 3z^2 \quad \text{and} \quad g = x^2 + y^2 + 4z^2 + xy + yz + zx,$$

represent a genus of two classes which represent between them all positive integers $n \neq A$, where $A = 11^{2k+1}(11k + 2, 6, 7, 8 \text{ or } 10)$. Similarly, every $n \neq A$ is represented in either f_1 or g_1 where

$$f_1 = x^2 + y^2 + 11z^2, \quad g_1 = x^2 + 3y^2 - 2yz + 4z^2$$

represent the two classes of a genus of determinant 11. Now

$$n = f_1 \text{ yields } 4n = 4x^2 + (2y)^2 + 11(2z)^2,$$

$$n = g_1 \text{ yields } 4n = 4x^2 + (4z - y)^2 + 11y^2.$$

Hence for every $n \neq A$, $4n$ is represented in $(4, 1, 11)$, that is

$$4n = 4x^2 + (2y - z)^2 + 11z^2, \quad n = x^2 + y^2 - yz + 3z^2,$$

whence f is regular.

It is interesting to note that g and g_1 represent the same numbers, but that f represents numbers that f_1 does not, e.g. 3.

The reduced form for f is $x^2 + y^2 + 3z^2 - xz$.

4. The letters $a, b, c, t, \ldots, z$ will denote in this section integral quaternions of the type

(1) $$t = t_0 + i t_1 + j t_2 + k t_3, \quad t_0, \ldots, t_3 \text{ rational integers,}$$

where

(2) $$i^2 = -1, \quad j^2 = -\lambda, \quad k^2 = -\lambda,$$
$$ij = -ji = k, \quad ki = -ik = j, \quad jk = -kj = \lambda i,$$

λ denoting a fixed positive integer. For this section we assume that

(3) $$\lambda = 1, 2 \text{ or } 3.$$

Conjugates are defined as usual (with i replaced by $-i$, j by $-j$, k by $-k$). Then the norm of t is given by

(4) $$Nt = t\bar{t} = \bar{t}t = t_0^2 + t_1^2 + \lambda t_2^2 + \lambda t_3^2.$$

The unit-quaternions, of norm 1, to be denoted by θ, are respectively

(5) $$\pm 1, \ \pm i, \ \pm j \text{ and } \pm k, \text{ if } \lambda = 1,$$

(6) $$\pm 1 \text{ and } \pm i, \text{ if } \lambda = 2 \text{ or } 3.$$

With any quaternion t we link the class of its left-associates θt, θ ranging over the σ values (5) or (6), (and similarly for right-associates). Here

(7) $$\sigma = 8 \text{ if } \lambda = 1, \quad \sigma = 4 \text{ if } \lambda = 2 \text{ or } 3.$$

A quaternion is called *proper* if its coordinates are coprime.

We require the following fundamental result:

Theorem 2. *A proper quaternion x of norm divisible by a positive odd integer m, has exactly σ right-divisors (left-divisors) of norm m, these forming a class of left-associates (right-associates).*

This was first proved in the case $\lambda = 1$ and m prime by Lipschitz. For the cases $\lambda = 1$ and 3, but m prime, it follows immediately from Hilfssätze 8 and 10 of L. E. Dickson's Algebren und ihren Zahlentheorie, pp. 167 and 170 (it being necessary to transform Dickson's integral quaternions into ours by suitable unit factors).

Let us note first that if the theorem is true for x of odd norm it follows for Nx even. For if $x = ut$ where $Nt = m$, then $x + m = (u + \bar{t})t$; whence x and $m + x$ have the same right-divisors of norm m.

Second we extend the theorem from m prime to m composite.

I. Existence. Assume the truth of the theorem for products m of $r - 1$ primes. Write $m = np$, n being a product of $r - 1$ primes, p a prime. Then $x = ut$, $Nt = p$; and since $n \mid Nu$, $u = vw$, where $Nw = n$; hence $x = vwt$, and $N(wt) = m$.

II. Uniqueness up to a left-unit factor. With the same hypothesis assume if possible that $x = uv = u'v'$, where $Nv = Nv' = m = np$. We can set $v = wt$ and $v' = w't'$, $Nt = p = Nt'$. Since t and t' are right-divisors of norm p of x, $t' = \theta t$ for a unit θ, and $uw = u'w'\theta$ follows on cancelling the right-factor t. Here the divisor uw of x is proper and has both w and $w'\theta$ as right-divisors of norm n. By the induction-hypothesis w and $w'\theta$ are left-associates and the same follows for v and v'.

Third we extend the theorem from $\lambda = 1$ to $\lambda = 2$. There is a $(1, 1)$ correspondence between the quaternions

$$x = x_0 + i x_1 + j x_2 + k x_3, \quad x_2 \equiv x_3 \pmod 2, \quad x_0, \ldots, x_3 \text{ integers} \tag{8}$$

$$\text{in which } \lambda = 1 \text{ (i. e. } j^2 = k^2 = -1,\ jk = i, \text{ etc.),}$$

and the quaternions

$$y = y_0 + I y_1 + J y_2 + K y_3, \quad y_0, \ldots, y_3 \text{ integers,} \tag{9}$$

$$\text{in which } \lambda = 2 \text{ (i. e. } I^2 = -1,\ J^2 = K^2 = -2,\ JK = 2I, \text{ etc.).}$$

This correspondence is set up under the transformations

$$I = i, \quad J = j - k, \quad K = j + k, \tag{10}$$
$$x_0 = y_0, \quad x_1 = y_1, \quad x_2 = y_3 + y_2, \quad x_3 = y_3 - y_2.$$

The norm is preserved under these transformations:

$$x_0^2 + x_1^2 + x_2^2 + x_3^2 = y_0^2 + y_1^2 + 2\,y_2^2 + 2\,y_3^2.$$

Every relation in quaternions (8) is immediately interpretable in quaternions (9) and conversely. If $x = u\,t$, x and t being of type (8), the same is true of u if $N t$ is odd; for a product of quaternions of type (8) must be of the same type, in view of the correspondence with (9), and $u = x\,\bar{t}/N t$.

Finally, consider a quaternion (9) of odd norm divisible by m. The corresponding quaternion (8) has eight right-divisors $\theta\, t$ of norm m. Exactly four of these have their last two coordinates congruent (mod 2). The corresponding quaternions of type (9) are the right-divisors sought in Theorem 2 for $\lambda = 2$.

Theorem 3. *Let* $x = i\,x_1 + j\,x_2 + k\,x_3$ *be a proper pure quaternion of norm*

$$\lambda\, m^2 = x_1^2 + \lambda\, x_2^2 + \lambda\, x_3^2, \tag{11}$$

where m *is odd and positive, and* (3) *holds. Then* x *is of the form*

$$x = \bar{t}\, a\, t, \tag{12}$$

where t *is of norm* m, *and* a *is a pure quaternion of norm* λ.

For by Theorem 2 we can write $x = v\,t$, $N\,t = m$. Further since $\bar{x} = \bar{t}\bar{v} = -x$, $\bar{t}$ and its right-associates are the only left-divisors of x of norm m. But $m \mid N v$. Hence $v = \bar{t}\, a$, where a has integer coordinates, and (as is seen on taking norms) $N a = \lambda$. Thus $x = \bar{t}\, a\, t$. Evidently a is pure along with $\bar{t}\, a\, t$.

By (11), $\lambda \mid x_1$. Replacing x_1 by $\lambda\, y_1$ we obtain

$$m^2 = \lambda\, y_1^2 + x_2^2 + x_3^2. \tag{13}$$

Using merely the fact that $m^2 \equiv 1 \pmod 8$ if $\lambda = 1$ or 2, $m^2 \equiv 1 \pmod{24}$ if $\lambda = 3$, we obtain for (13) the following mutually exclusive and exhaustive possibilities A and B:

	A	B
if $\lambda = 1$,	x_2 or $x_3 \equiv 0 \pmod 4$	x_2 or $x_3 \equiv 2 \pmod 4$
if $\lambda = 2$,	x_2 or $x_3 \equiv 0 \pmod 8$	x_2 or $x_3 \equiv 4 \pmod 8$
if $\lambda = 3$, $3 \nmid m$,	x_2 or $x_3 \equiv 0 \pmod 6$	x_2 or $x_3 \equiv 3 \pmod 6$.

Theorem 4. *Let m be positive and prime to 2λ. All proper solutions of* (13) *satisfy A if $(-\lambda|m) = 1$, but B if $(-\lambda|m) = -1$.*

By Theorem 3 it suffices to show that if a is a pure quaternion of norm λ and $Nt = m$, then $x = \bar{t}at = ix_1 + jx_2 + kx_3$ satisfies A if $(-\lambda|m) = 1$, and B if $(-\lambda|m) = -1$. Now if $a = ia_1 + ja_2 + ka_3$, x is given by

$$
\begin{aligned}
&x_1 = (t_0^2 + t_1^2 - \lambda t_2^2 - \lambda t_3^2)a_1 + 2\lambda(t_0 t_3 + t_1 t_2)a_2 + 2\lambda(-t_0 t_2 + t_1 t_3)a_3, \\
(14) \qquad &x_2 = 2(-t_0 t_3 + t_1 t_2)a_1 + (t_0^2 + \lambda t_2^2 - t_1^2 - \lambda t_3^2)a_2 + 2(t_0 t_1 + \lambda t_2 t_3)a_3, \\
&x_3 = 2(t_0 t_2 + t_1 t_3)a_1 + 2(-t_0 t_1 + \lambda t_2 t_3)a_2 + (t_0^2 + \lambda t_3^2 - t_1^2 - \lambda t_2^2)a_3.
\end{aligned}
$$

First consider $\lambda = 1$. It will be seen that x_1, x_2, and x_3 are obtained from each other by permuting subscripts 1, 2, 3 cyclically. Hence by symmetry we can take $a = i$, that is $a_1 = 1$, $a_2 = a_3 = 0$. If $m = t_0^2 + t_1^2 + t_2^2 + t_3^2 \equiv 1 \pmod 4$, three of the t_f are even, one odd, and a glance at (14) shows that x_2 or $x_3 \equiv 0 \pmod 4$. If $m \equiv 3 \pmod 4$ three t_f are odd, one even, and (14) shows that x_2 or $x_3 \equiv 2 \pmod 4$.

Second let $\lambda = 2$. We take $a_2 = 1$, $a_1 = a_3 = 0$ as a typical case. Now $m = t_0^2 + t_1^2 + 2t_2^2 + 2t_3^2$. If $m \equiv 1$ or $3 \pmod 8$, then if t_2 and t_3 are odd, one of t_0 and t_1 is odd, the other double of an odd, $x_3 \equiv 2(-2 + 2) \equiv 0 \pmod 8$; if t_2 or t_3 is even, then t_0 or t_1 is odd, the other divisible by 4, whence $x_3 \equiv 2(0 + 0) \equiv 0 \pmod 8$. If $m \equiv 5$ or $7 \pmod 8$, and t_2 and t_3 are odd, then t_0 or t_1 is odd, the other divisible by 4, $x_3 \equiv 2(0 + 2) \equiv 4 \pmod 8$; but if t_2 or t_3 is even, then t_0 or t_1 is odd, the other double an odd, $x_3 \equiv 2(2+0) \equiv 4 \pmod 8$.

Third let $\lambda = 3$. We take $a_2 = 1$, $a_1 = a_3 = 0$ as typical. Now $m = t_0^2 + t_1^2 + 3t_2^2 + 3t_3^2$. If $m \equiv 1 \pmod 6$, t_0 or t_1 is divisible by 3, the other prime to 3, whence $x_3 = 2(-t_0 t_1 + 3t_2 t_3) \equiv 0 \pmod 3$. Evidently x_3 is also even. If $m \equiv 5 \pmod 6$ t_0 and t_1 are both prime to 3, and $x_2 = t_0^2 + 3t_2^2 - t_1^2 - 3t_3^2$ is divisible by 3; it is also odd.

These results become more interesting in the light of

Theorem 5. *If $\lambda = 1$ or 2 and n is of the form $8f + 1$, or if $\lambda = 3$ and $n = 24f + 1$, but n is not a square, then*

$$n = \lambda y_1^2 + x_2^2 + x_3^2 \tag{15}$$

possesses equally many solutions satisfying A or B.

We observed before Theorem 4 that all solutions of (15) for the given forms of n satisfy A or B but not both. We shall set up a $(\frac{1}{2}\sigma, \frac{1}{2}\sigma)$ correspondence between the solutions of the two types.

To do this we fix upon a prime p satisfying both of

$$(p|n) = -1, \quad (-\lambda|p) = -1. \tag{16}$$

This is possible since n is not a square in virtue of Dirichlet's theorem on the existence of primes in an arithmetical progression. Since $n \equiv 1 \pmod 4$, (16) implies $(-\lambda n|p) = 1$; hence we can choose an integer x_0 such that

$$\lambda x_0^2 + n \equiv 0 \pmod p. \tag{17}$$

Since the property A or B is unaffected by the removal of a common odd factor from y_1, x_2, and x_3, and since in all solutions x_2 or x_3 is prime to λ, we can restrict attention to proper solutions. Let $\xi = \lambda i y_1 + j x_2 + k x_3$ represent a proper solution of (15). Then

$$N\xi = \lambda n, \text{ and } N(\lambda x_0 + \xi) = \lambda(\lambda x_0^2 + n) \equiv 0 \pmod p. \tag{18}$$

By Theorem 1, $\lambda x_0 + \xi$ possesses a right-divisor t of norm p:

$$\lambda x_0 + \xi = u t, \quad N t = p. \tag{19}$$

From (19) we obtain at once

$$t u - \lambda x_0 = t \xi \bar{t}/p. \tag{20}$$

Thus $(t \xi \bar{t})/p$ has integral coordinates, is pure (along with ξ), has its coefficient of i divisible by λ (as will be evident from (22)), and is of norm $(N t \cdot N \xi \cdot N \bar{t})/p^2 = N\xi = \lambda n$, and hence represents another proper integral solution of (15), proper since any common divisor of the coordinates of

$$\eta = (t \xi \bar{t})/p = \lambda i w_1 + j v_2 + k v_3 \tag{21}$$

divides the coordinates of $\xi = (\bar{t} \eta t)/p$. Set $t \xi \bar{t} = i z_1 + j z_2 + k z_3$; then

$$\begin{aligned} z_1 &= (t_0^2 + t_1^2 - \lambda t_2^2 - \lambda t_3^2)\lambda y_1 + 2\lambda(-t_0 t_3 + t_1 t_2) x_2 + 2\lambda(t_0 t_2 + t_1 t_3) x_3, \\ z_2 &= 2(t_0 t_3 + t_1 t_2)\lambda y_1 + (t_0^2 + \lambda t_2^2 - t_1^2 - \lambda t_3^2) x_2 + 2(-t_0 t_1 + \lambda t_2 t_3) x_3, \\ z_3 &= 2(-t_0 t_2 + t_1 t_3)\lambda y_1 + 2(t_0 t_1 + \lambda t_2 t_3) x_2 + (t_0^2 + \lambda t_3^2 - t_1^2 - \lambda t_2^2) x_3. \end{aligned} \tag{22}$$

If t is replaced in (19) by a left-associate θt, then η in (21) is replaced by $\theta \eta \bar{\theta}$ which (as is easily verified) is obtained from η by

$$\text{(23)}\qquad \begin{array}{l} \text{merely changing the signs of } v_2 \text{ and } v_3, \text{ if } \lambda = 2 \text{ or } 3, \\ \text{merely changing signs of two of } w_1,\ v_2,\ v_3, \text{ if } \lambda = 1. \end{array}$$

If the same sequence of operations be applied to η instead of ξ, with the same p, but with $-x_0$ in place of x_0, we obtain $\bar{t}$ for a right-divisor and are led back to ξ; for by (20), $-\lambda x_0 + \eta = (-\bar{u})\bar{t}$. Also the $\frac{1}{2}\sigma$ quaternions $\bar{\theta}\xi\theta$ lead in (19) to $t\theta$, and hence again to

$$\eta = (t\xi\bar{t})/p = (t\theta\cdot\bar{\theta}\xi\theta\cdot\bar{\theta}\bar{t})/p.$$

Let us anticipate the proof below that if ξ is of type A then η is of type B and vice-versa. Then to each set of $\frac{1}{2}\sigma$ representations of type A we correspond the set of type B obtained by means of p and x_0; but for sets of type B we use p with $-x_0$. Two sets of type A cannot correspond in this way to the same set of type B: for by the above argument the latter set must lead back to both of the former, contrary to the statement about (23).

Finally we prove that if ξ is of type A, η is of type B. The converse will follow by parity. Since p is prime to 2λ, it suffices to show that $t\xi\bar{t}$ is of type B.

Let $\lambda = 1$. Then $p \equiv 3 \pmod 4$, three t_f are odd, one even. We can suppose by symmetry that $x_2 \equiv x_3 \equiv 0 \pmod 4$. Then by (22) obviously $z_2 \equiv z_3 \equiv 2$.

Let $\lambda = 2$. Then $p \equiv 5$ or $7 \pmod 8$. By symmetry we can take $x_3 \equiv 0 \pmod 8$, x_2 odd. By residues (mod 8) in (15) y_1 must be even. Since

$$t_0^2 + t_1^2 + 2t_2^2 + 2t_3^2 \equiv 5 \text{ or } 7 \pmod 8,$$

one of t_0 and t_1 is odd, the other $\equiv 2$ or $0 \pmod 4$ according as $t_2 t_3$ is even or odd; hence $z_3 \equiv 2(t_0 t_1 + 2t_2 t_3) \equiv 4 \pmod 8$.

Let $\lambda = 3$. Then $p \equiv 5 \pmod 6$, $p = t_0^2 + t_1^2 + 3t_2^2 + 3t_3^2$, t_0 and t_1 prime to 3. Suppose $x_3 \equiv 0 \pmod 6$. By (15), x_2 is odd and prime to 3, y_1 is even, $y_1 \equiv x_3 \equiv 0$ or $2 \pmod 4$. Hence $z_2 \equiv (t_0^2 + 3t_2^2 - t_1^2 - 3t_3^2)x_2 \equiv 0 \pmod 3$ and also $\equiv t_0 + t_2 + t_1 + t_3 \equiv 1 \pmod 2$; that is $z_2 \equiv 3 \pmod 6$.

Apart from similar cases this completes the proof of Theorem 5. To take an example, $73 = 3x_1^2 + x_2^2 + x_3^2$ has the solutions (4; 5, 0) and (2; 5, 6) of type A; and (4; 4, 3) and (0; 8, 3) of type B.

Theorem 6. *Every positive integer of the form* $8n + 1$ *is represented in*

$$(1,\ 1,\ 16),\ (1,\ 4,\ 16),\ (1,\ 16,\ 16),\ (1,\ 2,\ 32),\ (1,\ 8,\ 32),\ \textit{and}\ (1,\ 8,\ 64);$$

and every positive integer of the form $24n+1$ *is represented in*

$$(1, 3, 36),\ (1, 12, 36),\ \textit{and}\ (1, 48, 144).$$

All the results of Theorem 6 are trivial for the case of a square. For a non-square the required representation follows at once from Theorem 5 in the case of

$$(1, 1, 16),\ (2, 1, 64),\ (3, 1, 36).$$

From $8n+1=(1, 1, 16)$ follows x_1 or $x_2 \equiv 0 \pmod 4$; which takes care of $(1, 4, 16)$ and $(1, 16, 16)$. From $8n+1=(2, 1, 64)$ follows x_1 even, $8n+1= =(8, 1, 64)$. From $8k+1=(3, 1, 36)$ follows x_1 even, $8k+1=(12, 1, 36)$. From $8k+1=(1, 12, 36)$ follows $x_2 \equiv x_3 \pmod 2$, whence $8k+1=(1, 48, 144)$ unless x_2 and x_3 are odd; then $8k+1=x_1^2+48y_2^2+144y_3^2$ with

$$y_2=\tfrac{1}{4}(x_2 \pm 3x_3), \quad y_3=\tfrac{1}{4}(x_2 \mp x_3).$$

By Theorem 5 with $\lambda=1$, $8n+1=x_1^2+4y_2^2+4y_3^2$ with y_2 and y_3 odd, if $8n+1$ is not a square, this being a representation of type B. Hence

$$8n+1=x_1^2+2(y_2+y_3)^2+2(y_2-y_3)^2,$$

where by choice of signs, $y_2+y_3 \equiv 2 \pmod 4$, $y_2-y_3 \equiv 0 \pmod 4$. The results stated for $(1, 2, 32)$ and $(1, 8, 32)$ follow.

That all other integers of the genera of these forms can be represented thereby can easily be proved and was proved in B. W. Jones' Chicago Dissertation. For example to represent $8n+3$ in $(1, 2, 32)$ we start with a representation

$$8n+3=y_1^2+y_2^2+y_3^2,$$

wherein the y_i are necessarily odd and we can choose their signs and order to secure $y_2 \equiv y_3 \pmod 8$; then $8n+3=y_1^2+2(\tfrac{1}{2}y_2+\tfrac{1}{2}y_3)^2+32((y_2-y_3)/8)^2$.

Corollary. *All the forms listed in Theorem 6 are regular.*

5. There are also interesting properties of the companion forms in the genera of each of the forms listed in Theorem 6. Consider for example

$$f=x^2+y^2+16z^2 \text{ and } g=2x^2+2y^2+5z^2-2yz-2zx=(x+y-z)^2+(x-y)^2+4z^2,$$

which are the reduced forms of a genus of determinant 16 (cf. Table II). To every representation of an $8n+1$ in f corresponds a solution of

(E) $$8n+1=y_1^2+4y_2^2+4y_3^2$$

with y_2 and y_3 even; and to every representation in g corresponds a solution of (E) with y_2 and y_3 odd. Hence Theorem 5 together with the fact that every $8n+1$ is a sum of three squares shows that every $8n+1$ not a square is represented equally often in both f and g. On the other hand f obviously represents every m^2. But g represents properly no m^2 for which $m \equiv 1 \pmod 4$ (m positive), and hence cannot represent (properly or improperly) any m^2 all of whose prime factors are $\equiv 1 \pmod 4$. However g does represent properly any m^2 for which $m \equiv 3 \pmod 4$ (Th. 4), and hence g represents every m^2 for which m has some prime factor $\equiv 3$. This proves the result stated in the first line of Table II. The proofs of the other results of the table, in which m^2 or w^2 appears, are similar.

In the case of the form $f=(1, 2, 32)$ the situation is somewhat different. The companion form $g=(2, 4, 9, -2, 0, 0)$ *seems* to represent every $8n+3$ except 3, 43, and 163, but we have not been able to prove this. However we can prove as follows that g represents every $8n+1$ with the single exception 1. For if m is odd, $m=g$ if and only if $m=x^2+2y^2+8z^2=x^2+(y+2z)^2+(y-2z)^2$ with z odd, that is

(F) $$m=x_1^2+x_2^2+x_3^2, \quad x_2 \equiv x_3+4 \pmod 8.$$

We have seen above that unless $8n+1$ is a square containing no prime factor $4k+3$, (E) is solvable with y_2 and y_3 odd: then $2y_2 \equiv \pm 2y_3+4 \pmod 8$ by choice of sign. It remains only to prove the solvability of (F) when $m=p^2$ with p a prime $4k+1$. Setting $p=t_0^2+t_1^2+t_2^2+t_3^2$ we have

$$p^2=x_1^2+x_2^2+x_3^2, \quad x_1=t_0^2+t_1^2-t_2^2-t_3^2, \quad x_2=2(t_0t_3+t_1t_2),$$
$$x_3=2(-t_0t_2+t_1t_3).$$

If $p \equiv 5 \pmod 8$ we can take $p=t_0^2+t_2^2$, $t_1=t_3=0$, whence $x_2=0$ and $x_3=-2t_0t_2 \equiv 4 \pmod 8$. If $p \equiv 1 \pmod 8$ it has by the above a representation $t_0^2+t_1^2+t_2^2$ with $t_3=0$ and $t_1 \equiv t_2 \equiv 2 \pmod 4$, t_0 odd; hence $x_2=2t_1t_2 \equiv 0 \pmod 8$, $x_3=-2t_0t_2 \equiv 4 \pmod 8$. The result for $(4, 8, 9, 0, -2, 0)$ follows.

6 a. The classes of forms represented by

(1) $$f=5x^2-2xy+5y^2+72z^2 \text{ and } g=8x^2+12y^2-12yz+21z^2$$

constitute a genus and are rather noteworthy in that

Theorem 7. *f is regular, and g represents exactly the same numbers as f except that g does not represent the number 5.*

Both forms are derived from $x^2 + y^2 + 3z^2$:

$$f = (x + y + 6z)^2 + (x + y - 6z)^2 + 3(x - y)^2, \tag{2}$$

$$g = (2x + 3z)^2 + (2x - 3z)^2 + 3(2y - z)^2. \tag{3}$$

Either of $2n = f$ or $2n = g$ leads to $\frac{1}{2}n = 2X^2 + 3Y^2 + 18Z^2$; either of $3n = f$ or $3n = g$ yields $n = 4X^2 - 4XY + 7Y^2 + 24Z^2$. Hence f and g represent the same numbers $2n$ and $3n$; since a genus is always regular, f and g are each regular for multiples of 2 and 3.

The only remaining numbers possibly representable in f or g are those of the form $24n + 5$.

To represent $24n + 5$ in f it suffices (by (2)) to solve

$$24n + 5 = x^2 + y^2 + 3z^2 \tag{4}$$

in integers x, y, z for which the equations

$$X + Y + 6Z = x, \quad X + Y - 6Z = y, \quad X - Y = z \tag{5}$$

yield integer solutions X, Y, Z. The condition for this is

$$x \equiv y \pmod{12}, \quad y \equiv z \pmod{2}, \tag{6}$$

which in the particular case of (4), may be replaced by

$$x \equiv y \pmod{12}, \quad xyz \text{ odd}. \tag{7}$$

Similarly in view of (3), to represent $24n + 5$ in g it suffices to solve (4) in integers x, y, z satisfying

$$x \equiv y + 6 \pmod{12}, \quad xyz \text{ odd}. \tag{8}$$

6 b. Thus Theorem 7 will follow if we prove

Theorem 8. *Every $24n + 5$ is represented in $x^2 + y^2 + 3z^2$ with x, y, z odd and $x \equiv y$ (mod 12); and every $24n + 5$ except 5 is represented therein with x, y, z odd and $x \equiv y + 6$ (mod 12).*

That (4) is solvable in integers x, y, z is well-known. Either x, y, z are all odd; or one of x or y is odd, the others even. In the latter case the even ones are incongruent (mod 4), and it is evident that the application of one of the following automorphic transformations will produce a representation having x, y, z odd:

$$\begin{aligned} T\colon\ (x, y; z) &\to (x, \tfrac{1}{2}(y-3z); \tfrac{1}{2}(y+z)), \\ T'\colon\ & \quad\ (x, \tfrac{1}{2}(y+3z); \tfrac{1}{2}(y-z)), \\ U\colon\ & \quad\ (y, \tfrac{1}{2}(x-3z); \tfrac{1}{2}(x+z)), \\ U'\colon\ & \quad\ (y, \tfrac{1}{2}(x+3z); \tfrac{1}{2}(x-z)), \end{aligned}$$

The proof of Theorem 8 will involve a finite sequence, of arbitrary length, of alternate applications of these automorphs, (which, we may note, correspond to $t = 1 \pm j$ or $1 \pm k$ with $\lambda = 3$, in § 3).

6 c. If x, y, z are odd, then in (4) either

$$x \equiv \pm y + 6 \text{ or } x \equiv \pm y \pmod{12}. \tag{9}$$

Starting with a solution of either type (9_1) or (9_2) we shall try to derive one of the other type.

If x, y, z are determined to modulus 24, the result of applying $T, \ldots, U'$ is determined to modulus 12. Thus, under T, $(1, 5; 1) \pmod{24} \to (1, 1; 3) \pmod{12}$, wherein $x \equiv y \pmod{12}$; and evidently this resolves the step from (9_1) to (9_2) also for $(\pm 1, \pm 5; \pm 1)$ and $(\pm 5, \pm 1; \pm 1) \pmod{24}$, that is the residues may be taken as least absolute residues (mod 24) and the x and y interchanged. In a similar way, applying T, the reader can immediately complete the step from (9_1) to (9_2) in the following cases (mod 24): $(5, 1; -3)$, $(5, 1; 5)$, $(1, 5; -7)$, $(1, 5; 9)$, $(5, 1; -11)$, $(1, 7; 3)$, $(1, 7; -5)$, $(1, 7; 11)$, $(5, 11; -1)$, $(5, 11; 7)$, $(5, 11; -9)$, $(7, 11; -1)$, $(11, 7; 3)$, $(11, 7; -5)$, $(7, 11; 7)$, $(7, 11; -9)$, $(11, 7; 11)$. There remain to be treated only the six cases:

$$(1, 7; 1),\ (1, 7; 7),\ (1, 7; 9),\ (5, 11; 3),\ (5, 11; 5),\ (5, 11; 11),\ \pmod{24}. \tag{10}$$

Similarly, starting with (9_2) the transit to (9_1) is obtained by one application of T in the cases $(1, 1; -3)$, $(1, 1; 5)$, $(1, 1; -11)$, $(1, 11; -1)$, $(11, 1; -3)$, $(11, 1; 5)$, $(1, 11; 7)$, $(1, 11; -9)$, $(11, 1; -11)$, $(11, 11; 1)$, $(11, 11; 7)$, $(11, 11; -9)$, $(5, 5; 1)$, $(5, 5; -7)$, $(5, 5; 9)$, $(7, 5; 1)$, $(5, 7; 3)$, $(5, 7; -5)$, $(7, 5; -7)$, $(7, 5; 9)$, $(5, 7; 11)$, $(7, 7; 3)$, $(7, 7; -5)$, $(7, 7; 11)$. There remain here twelve cases (mod 24):

$$(11) \quad \begin{gathered}(1,\ 1;\ 1),\ (1,\ 1;\ 7),\ (1,\ 1;\ 9),\ (7,\ 7;\ 1),\ (7,\ 7;\ 7),\ (7,\ 7;\ 9),\\ (5,\ 5;\ 3),\ (5,\ 5;\ 5),\ (5,\ 5;\ 11),\ (11,\ 11;\ 3),\ (11,\ 11;\ 5),\ (11,\ 11;\ 11).\end{gathered}$$

All cases (10) can be reduced to $(1,\ 7;\ 1)$. For example, if

$$24n + 5 = x^2 + y^2 + 3z^2, \quad x \equiv 5, \quad y \equiv 11, \quad z \equiv 5 \pmod{24},$$

then

$$24(25n + 5) + 5 = (5x)^2 + (5y)^2 + 3(5z)^2, \quad 5x \equiv 1, \quad 5y \equiv 7, \quad 5z \equiv 1;$$

and it is obvious that application of automorphs $T, \ldots, U'$ (which are the only transformations to be used) cannot eliminate the divisor 5 of $(5x,\ 5y;\ 5z)$. Similarly $(1,\ 7;\ 7)$ reduces to $(1,\ 7;\ 1)$ through a factor 7; and $(5,\ 11;\ 11)$ to $(11,\ 5;\ 5)$. Next, $(1,\ 7;\ 9) \pmod{24} \to (1,\ 17;\ -1)$, where $\pm x \equiv 1$ is still determined to modulus 24, the 17 and -1 to modulus 12; this separates (mod 24) into $(1,\ 7;\ 1)$ and the three trivial cases $(1,\ 7;\ 11)$, $(1,\ 5;\ 1)$, $(1,\ 5;\ 11)$. Similarly for $(5,\ 11;\ 3)$.

6 d. We require the fact that if $n > 0$, (4) is solvable with

$$(12) \qquad x^2,\ y^2, \text{ and } z^2 \text{ odd, but not all equal.}$$

The only case of doubt is $24n + 5 = 5m^2$, m positive and prime to 6. There seems to exist a simple formula for the number of solutions of

$$(13) \qquad 5m^2 = 3x_1^2 + x_2^2 + x_3^2, \quad x_1,\ x_2,\ x_3 \text{ odd},$$

from which we might see that if $m > 1$ there are solutions besides $x_1^2 = x_2^2 = x_3^2 = m^2$. However we shall be content with a brief proof, based on the solvability of

$$(14) \qquad t_0^2 + t_1^2 + 3t_2^2 + 3t_3^2 = m,$$

that if $m > 1$, m prime to 6, (13) cannot have all its solutions divisible by m. We assume that m is a prime > 3; the stated result will then follow for any m on multiplying (13) by a factor s^2. We set

$$3ix_1 + jx_2 + kx_3 = (t_0 - it_1 - jt_2 - kt_3)(3i + j + k)(t_0 + it_1 + jt_2 + kt_3),$$

the quaternions being of the type with $j^2 = k^2 = -3$. We have

$$x_1 = (t_0^2 + t_1^2 - 3t_2^2 - 3t_3^2) + 2(t_0t_3 + t_1t_2) + 2(-t_0t_2 + t_1t_3),$$

$$x_2 = 6(-t_0t_3 + t_1t_2) + (t_0^2 + 3t_2^2 - t_1^2 - 3t_3^2) + 2(t_0t_1 + 3t_2t_3), \quad x_3 = \cdots,$$

which are odd; and, on taking norms, obtain (13). If x_1, x_2, x_3 could be divisible by m for all solutions of (14) they will remain divisible by m if t_0 and t_1, or t_2 and t_3, are interchanged or changed in sign. Changing the signs of t_0, t_1 in x_1 and adding, yields

$$m \mid 2(t_0^2 + t_1^2 - 3t_2^2 - 3t_3^2), \quad m \mid t_0^2 + t_1^2 \text{ and } t_2^2 + t_3^2,$$

the latter using (14). Since $t_2^2 + t_3^2 < m$, $t_2 = t_3 = 0$. Interchanging t_0, t_1 in x_2 now gives $m \mid t_0^2 - t_1^2$, whence $m \mid t_0^2$ and t_1^2, a contradiction.

6 e. Assuming $n > 0$ and (12), we can reduce all cases (11) to

(15) $\qquad (1,\ 1;\ 1)$ with x, y, z not all equal, or to $(1,\ 1;\ 7)$, (mod 24).

For example if $x \equiv y \equiv z \equiv 5$ we multiply through by 5^2 and use $(5x,\ 5y;\ 5z)$; similarly for $(7,\ 7;\ 7)$ and $(11,\ 11;\ 11)$. In the same way $(7,\ 7;\ 1)$, $(5,\ 5;\ 11)$, and $(11,\ 11;\ 5)$ reduce to $(1,\ 1;\ 7)$; and $(7,\ 7;\ 9)$, $(5,\ 5;\ 3)$, $(11,\ 11;\ 3)$ reduce to $(1,\ 1;\ 9)$. Finally $(1,\ 1;\ 9)$ transforms under T into one of $(1,\ 11;\ 5)$, $(1,\ 1;\ 5)$, $(1,\ 11;\ 7)$, and $(1,\ 1;\ 7)$ (mod 24); the three first were treated as trivial in § 6 c.

6 f. There remain to be treated solutions of (4) of the three types:

$$E = (a + 24h,\ b + 24k;\ c + 24l), \tag{16}$$

where $(a,\ b;\ c) = (1,\ -7;\ 1)$, $(1,\ 1;\ 1)$, or $(1,\ 1;\ -7)$. (Cases 1, 2, 3).

We shall form virtually all sets of odd integers obtained by applying to (16) the automorphs $T, \ldots, U'$. To begin with we have

$$\begin{aligned} ET &= (a + 24h,\ a' + 12k - 36l;\ a'' + 12k + 12l), \\ EU &= (b + 24k,\ b' + 12h - 36l;\ b'' + 12h + 12l), \end{aligned} \tag{17}$$

where

$$(a,\ a',\ a'') = (1,\ -5,\ -3),\ (1,\ -1,\ 1),\ (1,\ 11,\ -3), \text{ resp.},$$

$$(b,\ b',\ b'') = (-7,\ -1,\ 1),\ (1,\ -1,\ 1),\ (1,\ 11,\ -3), \text{ resp.}$$

Let A stand for either ET or EU, and write

$$\begin{aligned} &(UU')^r = (UU')(UU') \ldots \text{ to } r \text{ factor-pairs}, \\ &A(UU')^r = (u_r,\ v_r;\ w_r), \quad A(UU')^r U = (x_r,\ y_r;\ z_r), \quad (r = 0,\ 1,\ \ldots). \end{aligned} \tag{18}$$

We shall prove, for every $r \geqq 0$, that

Lemma 1. *If, in the respective three cases,*

$$h \equiv l \equiv k + \tfrac{1}{3}(4^r - 1)$$

(19) $$h \equiv k \equiv l,$$

$$h \equiv k \equiv l + \tfrac{1}{3}(4^r - 1), \quad (\text{mod } 4^r),$$

then both of the sets of solutions of (4) *expressed by*

(20) $$E\,T(U\,U')^r\,U \text{ and } E\,U(U\,U')^r\,U$$

are integral, and one of them satisfies

(21) $x \equiv \pm y \pmod{12}$ *in case* 1, $x \equiv \pm y + 6 \pmod{12}$ *in cases* 2 *and* 3,

unless (respectively)

$$h \equiv l \equiv k + \tfrac{1}{3}(4^{r+1} - 1),$$

(22) $$h \equiv k \equiv l,$$

$$h \equiv k \equiv l + \tfrac{1}{3}(4^{r+1} - 1), \; (\text{mod } 2.4^r).$$

And if (22) *holds, then both of the sets of solutions*

(23) $$E\,T(U\,U')^{r+1} \text{ and } E\,U(U\,U')^{r+1}$$

are integral, and one of them satisfies (21) *unless* (19) *holds with* $r+1$ *in place of* r.

It should be observed that $\tfrac{1}{3}(4^r - 1) + 4^r = \tfrac{1}{3}(4^{r+1} - 1)$.

Conditions (19) being vacuous if $r = 0$, Theorem 8 will follow. For no set of values can satisfy (19) and (22) for all values of r, except in case 2 with $h = k = l$. The latter case can be excluded as in § 6 d unless $n = 0$.

From $(u_r,\, v_r;\, w_r)\,U\,U' = (u_{r+1},\, v_{r+1};\, w_{r+1})$ follow

$$u_{r+1} = \tfrac{1}{2}u_r \qquad - \tfrac{3}{2}w_r,$$

$$v_{r+1} = \tfrac{3}{4}u_r + \tfrac{1}{2}v_r + \tfrac{3}{4}w_r,$$

$$w_{r+1} = -\tfrac{1}{4}u_r + \tfrac{1}{2}v_r - \tfrac{1}{4}w_r,$$

and hence

(24) $$\begin{bmatrix} u_r \\ v_r \\ w_r \end{bmatrix} = K^r \begin{bmatrix} u_0 \\ v_0 \\ w_0 \end{bmatrix}, \text{ where } K = \begin{bmatrix} \tfrac{1}{2} & 0 & -\tfrac{3}{2} \\ \tfrac{3}{4} & \tfrac{1}{2} & \tfrac{3}{4} \\ -\tfrac{1}{4} & \tfrac{1}{2} & -\tfrac{1}{4} \end{bmatrix}.$$

To evaluate K^r we bring K to a diagonal form by a collineatory transformation, employing

$$K = M D M^{-1}, \tag{25}$$

$$M = \begin{bmatrix} \frac{1}{2} & \omega - 1 & \bar{\omega} - 1 \\ -\frac{1}{2} & \frac{1}{2}\omega & \frac{1}{2}\bar{\omega} \\ \frac{1}{2} & (2-\omega)/6 & (2-\bar{\omega})/6 \end{bmatrix}, \quad D = \begin{bmatrix} -1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \bar{\omega} \end{bmatrix},$$

where ω and $\bar{\omega}$ denote the roots of the equation

$$4\omega^2 - 7\omega + 4 = 0. \tag{26}$$

Hence $K^r = M D^r M^{-1} =$

$$\frac{1}{5}\begin{bmatrix} (-1)^r + 2e(r) & -(-1)^r - 3e(r) + 4e(r+1) & 3(-1)^r - 5e(r) + 4e(r+1) \\ -(-1)^r + 4e(r) - 4e(r+1) & (-1)^r + 2e(r) & -3(-1)^r - 2e(r) + 4e(r+1) \\ (-1)^r + \frac{2}{3}e(r) - \frac{4}{3}e(r+1) & -(-1)^r + \frac{5}{3}e(r) - \frac{4}{3}e(r+1) & 3(-1)^r + e(r) \end{bmatrix} \tag{27}$$

where $e(r) = \omega^r + \bar{\omega}^r$.

Since $\omega + \bar{\omega} = 7/4$ and $\omega\bar{\omega} = 1$ it is evident that $f(r) = 4^r e(r)$ is an integer. It is easy to verify for every $r \geqq 0$, that

$$f(r+2) - 7f(r+1) + 16f(r) = 0, \tag{28}$$

and since $f(0) = 2$, $f(1) = 7$, and $f(2) = 17$, that $f(r+1)$ is odd, and

$$f(r+1) + f(r) \equiv 0 \pmod 3, \quad f(r) \equiv 2 \pmod 5. \tag{29}$$

We shall adopt temporarily the following abbreviations:

$$\begin{aligned} &[a,\ b,\ c] = a(-1)^r + b\,e(r) + c\,e(r+1), \quad \alpha = 24(h + k + 3l), \\ &\beta = 48(h - 2k + l), \qquad \gamma = 96(k - l), \quad \delta = 16(h + 2k - 3l), \\ &\varepsilon = 24(2h + 3k - 5l), \quad \zeta = 32(h - k), \quad \eta = 8(5k - 2h - 3l), \end{aligned} \tag{30}$$

and indicate by a prime the act of interchanging h and k; e. g. $\beta' = 48(k - 2h + l)$. In these notations, using (17), (24), and (27) we obtain formulas for $u_r, \ldots, z_r$. (The mode of formation of $u_r, \ldots, z_r$ by applying $T, \ldots, U'$ shows that they either are integers or have powers of 2 for denominators.)

Case 1. $E\,T(U\,U')^r = (u_r,\ v_r;\ w_r)$, where

$$\begin{aligned} 5u_r &= [\alpha - 3,\ \beta + 32,\ \gamma - 32], \quad 5v_r = [3 - \alpha,\ \gamma',\ \beta' - 16], \\ 5w_r &= [\alpha - 3,\ \delta - 32/3,\ 16/3 - \delta']; \end{aligned} \tag{31}$$

$E\,U(U\,U')^r = (u_r,\ v_r;\ w_r)$, where $5\,u_r$, $5\,v_r$, $5\,w_r$ are given by

(32) $[\alpha - 3,\ \beta' - 16,\ \gamma'],\quad [3 - \alpha,\ \gamma - 32,\ \beta + 32],\quad [\alpha - 3,\ \delta' - 16/3,\ 32/3 - \delta]$.

For $E\,T(U\,U')^r\,U$ and $E\,U(U\,U')^r\,U$ we therefore have respectively

(33) $x_r = v_r,\ 5\,y_r = [3-\alpha,\ 32-\gamma,\ \varepsilon-24],\quad 5\,z_r = [\alpha-3,\ 32/3+\zeta,\ \eta-40/3]$;

(34) $x_r = v_r,\ 5\,y_r = [3-\alpha,\ \ -\gamma',\ \varepsilon'-16],\quad 5\,z_r = [\alpha-3,\ \zeta'-32/3,\ \eta'+16/3]$.

By (29_1) and since $3 \mid \alpha,\ \beta,\ \gamma$, and ε, (31)—(34) satisfy

(35) $$u_r \equiv v_r,\quad x_r \equiv -\,y_r \pmod 3),$$

It is therefore sufficient to show that if (19_1) holds, i. e. if

(36) $$k = h + 4^r\varkappa - \tfrac{1}{3}(4^r - 1),\quad l = h + 4^r\lambda,$$

where $\varkappa$ and λ are integers, then x_r, y_r, z_r in both (33) and (34) are integral, and that for one of them,

(37) $$x_r \equiv -\,y_r \pmod 4)$$

unless (22_1) holds; and that if (22_1) holds with $r - 1$ in place of r, i. e.

(38) $$k = h + 2.4^{r-1}\varkappa - \tfrac{1}{3}(4^r - 1),\quad l = h + 2.4^{r-1}\lambda,$$

then unless (19_1) holds, the $(u_r,\ v_r;\ w_r)$ in both (31) and (32) are integral, and for one of them,

(39) $$u_r \equiv v_r \pmod 4).$$

Substituting from (36) into (33) we obtain, to modulus 4,

(40) $$x_r \equiv -(-1)^r,\quad y_r \equiv -(-1)^r + 2\varkappa + 2\lambda,\quad z_r \text{ odd}.$$

The details for y_r are typical: $5\,y_r = (3 - \alpha)(-1)^r +$

$$[32 - 96\{4^r\varkappa - \tfrac{1}{3}(4^r - 1) - 4^r\lambda\}]\,e(r) + 24\,[3.4^r\varkappa - (4^r - 1) - 5.4^r\lambda - 1]\,e(r+1).$$

Here we replace $4^r e(r)$ by the integer $f(r)$ and obtain (mod 4)

$$-(-1)^r - (96\varkappa - 96\lambda + 32)f(r) + 6(3\varkappa - 1 - 5\lambda)f(r+1) \equiv -(-1)^r - 0 + 2\varkappa + 2 + 2\lambda,$$

since $f(r + 1)$ is odd. Similarly in (34),

(41) $$x_r \equiv -(-1)^r,\quad y_r \equiv -(-1)^r + 2\lambda,\quad z_r \text{ is odd}.$$

Hence $x_r \equiv -y_r$ in (41) unless λ is even, and then $x_r \equiv -y_r$ in (40) unless $\varkappa$ is odd; that is unless (22_1) holds.

Substituting from (38) in (31) and (32) we obtain (mod 4)

$$(42)\qquad u_r \equiv (-1)^r, \quad v_r \equiv -(-1)^r + 2\varkappa + 2\lambda, \quad w_r \text{ odd};$$

$$(43)\qquad u_r \equiv (-1)^r, \quad v_r \equiv -(-1)^r + 2\lambda, \qquad w_r \text{ odd}.$$

Hence $u_r \equiv v_r$ in (42) or (43) unless λ and $\varkappa$ are even, i. e. (19_1) holds.

Cases 2 and 3. The results for $EU\ldots$ are deduced from those for $ET\ldots$ by interchanging h and k. For $ET\ldots$ we obtain

$$5u_r = [\alpha + 5,\ \beta,\ \gamma], \quad 5v_r = [-5-\alpha,\ \gamma',\ \beta'], \quad 5w_r = [5+\alpha,\ \delta,\ -\delta'],$$

$$x_r = v_r, \quad 5y_r = [-5-\alpha,\ -\gamma,\ \varepsilon], \quad 5z_r = [\alpha + 5,\ \zeta,\ \eta],$$

in case 2; and respectively for the same quantities in case 3,

$$[\alpha - 19,\ \beta - 16,\ \gamma + 32], \quad [19 - \alpha,\ \gamma' + 32,\ \beta' - 16], \quad [\alpha - 19,\ \delta + 16,\ -\delta' - 16],$$

$$x_r = v_r, \quad [19 - \alpha,\ -\gamma - 32,\ \varepsilon + 40], \quad [\alpha - 19,\ \zeta,\ \eta + 8].$$

In both cases, $u_r \equiv -v_r$ and $x_r \equiv y_r$ (mod 3). Hence (21_2) will be attained if, first, given

$$(44)\qquad k = h + 4^r\varkappa \text{ and } l = h + 4^r\lambda, \text{ in case 2},$$

$$(45)\qquad k = h + 4^r\varkappa \text{ and } l = h + 4^r\lambda - \tfrac{1}{3}(4^r - 1), \text{ in case 3},$$

then $(x_r, y_r; z_r)$ are integral for both ET and EU and satisfy (37) for one of them, unless $\varkappa$ and λ are even in (44), or $\varkappa$ is even and λ is odd in (45); and, second, a like result holds in regard to (39) unless $\varkappa$ and λ are even, being given

$$(46)\qquad k = h + 2.4^{r-1}\varkappa \text{ and } l = h + 2.4^{r-1}\lambda, \text{ in case 2},$$

$$(47)\qquad k = h + 2.4^{r-1}\varkappa \text{ and } l = h + 2.4^{r-1}\lambda - \tfrac{1}{3}(4^r - 1), \text{ in case 3}.$$

That all this goes through as stated is easily verified. With a little patience we find, by virtue of (44)—(47) in their proper places,

z_r and w_r are odd integers, $x_r \equiv -(-1)^r \equiv -u_r$ (mod 4), in all cases;

$v_r \equiv -(-1)^r + 2\lambda$ for EU, $\quad v_r \equiv -(-1)^r + 2\varkappa + 2\lambda$ for ET;

in (44), $\quad y_r \equiv -(-1)^r + 2\lambda$ for EU, $\qquad y_r \equiv -(-1)^r + 2\varkappa + 2\lambda$ for ET;

in (45), $\quad y_r \equiv -(-1)^r + 2\lambda + 2$ for EU, $\quad y_r \equiv -(-1)^r + 2\lambda + 2\varkappa + 2$ for ET.

Table I.

All primitive regular forms $ax^2 + by^2 + cz^2$; $a \leqq b \leqq c$.

a) Self-reciprocal forms: $(1, r, r^2)$ where $r = 1, 2, 3, 4$[1], $5, 8$[1].

b) Forms whose reciprocals are regular:

$(1, 1, r)$ and $(1, r, r)$ where $r = 2, 3, 4, 5, 6, 8, 9, 12, 16$[1], $21, 24$.

$(1, 2, r)$ and $(2, r, 2r)$ where $r = 3$ or 5.

$(1, 2, r)$ and $(1, r/2, r)$ where $r = 6, 8, 10, 16$.

$(1, 3, r)$ and $(3, r, 3r)$ where $r = 4$ or 10.

$(1, 3, r)$ and $(1, r/3, r)$ where $r = 12, 18, 30, 36$[1]

$(1, 4, 6)$ and $(2, 3, 12)$; $(1, 4, 24)$ and $(1, 6, 24)$.

$(1, 5, 8)$ and $(5, 8, 40)$; $(1, 5, 40)$ and $(1, 8, 40)$.

$(1, 6, 9)$ and $(2, 3, 18)$; $(1, 6, 16)$ and $(3, 8, 48)$.

$(1, 9, r)$ and $(3, r/3, 3r)$ where $r = 12, 21, 24$.

$(1, 16, 24)$ and $(2, 3, 48)$.

$(2, 2, 3)$ and $(2, 3, 3)$.

$(2, 3, 8)$ and $(3, 8, 12)$; $(2, 3, 9)$ and $(2, 6, 9)$.

$(2, 5, 6)$ and $(5, 6, 15)$; $(2, 5, 15)$ and $(2, 6, 15)$.

$(3, 3, r)$ and $(3, r, r)$ where $r = 4, 7, 8$.

c)[1] Forms whose reciprocals are not regular:

$(1, 2, 32)$; $(1, 4, 36)$; $(1, 8, 24)$; $(1, 8, 32)$; $(1, 16, 48)$;
$(1, 24, 72)$; $(1, 40, 120)$; $(1, 48, 144)$; $(3, 8, 24)$; $(3, 16, 48)$;
$(3, 40, 120)$; $(5, 8, 24)$; $(8, 9, 24)$; $(8, 15, 24)$.

Table II.

The primitive regular forms $ax^2 + by^2 + cz^2$ in genera of more than one class and two regular forms with cross products.

In this table D is the determinant of the form, f the regular form, g (or in the case of $D = 6912$: g_1, g_2, g_3) is the other reduced form in its genus. m represents an odd whose every prime factor is $\equiv 1 \pmod 4$, w an odd whose every prime factor is $\equiv 1 \pmod 3$. $g \neq m^2$, for instance, means that g is regular except that it represents no m^2. When no notation occurs after a form g, the results are not known.

[1] Forms so marked are in genera of more than one class.

D	f	g
16	(1, 1, 16)	$(2, 2, 5, -1, -1, 0) \neq m^2$
64	(1, 2, 32)	$(2, 4, 9, -2, 0, 0) \neq 1$ and certain $8n+3$
64	(1, 4, 16)	$(4, 4, 5, 0, -2, 0) \neq m^2$
108	(1, 3, 36)	$(3, 4, 9) \neq w^2$
144	(1, 4, 36)	(4, 4, 9)
192	(1, 8, 24)	$(4, 8, 9, -4, -2, 0)$
256	(1, 8, 32)	$(4, 8, 9, 0, -2, 0) \neq 1$
256	(1, 16, 16)	$(4, 9, 9, 1, 2, 2) \neq m^2$
432	(1, 12, 36)	$(4, 9, 12) \neq w^2$
512	(1, 8, 64)	$(4, 8, 17, 0, -2, 0) \neq m^2$
576	(3, 8, 24)	(8, 11, 11, 5, 4, 4)
768	(1, 16, 48)	$(4, 16, 17, -8, -2, 0)$
960	(5, 8, 24)	(8, 13, 13, 3, 4, 4)
1728	(1, 24, 72)	$(4, 24, 25, -12, -2, 0)$
1728	(8, 9, 24)	(8, 17, 17, 5, 4, 4)
2304	(3, 16, 48)	$(12, 16, 19, -8, -6, 0)$
2880	(8, 15, 24)	(8, 23, 23, 11, 4, 4)
4800	(1, 40, 120)	$(4, 40, 41, -20, -2, 0)$
6912	(1, 48, 144)	$g_1 = (9, 16, 48) \neq w^2, 4w^2$
		$g_2 = (4, 48, 49, -24, -2, 0)$
		$g_3 = (16, 25, 25, 7, 8, 8)$
14, 400	(3, 40, 120)	$(12, 40, 43, -20, -6, 0)$
1728	$(5, 5, 72, 0, 0, -1)$	$(8, 12, 21, -6, 0, 0) \neq 5$
11/4	$(1, 1, 3, 0, -1/2, 0)$	$(1, 1, 4, 1/2, 1/2, 1/2)$