COMPLEX GEOMETRY AND OPERATOR THEORY(1)

BY

M. J. COWEN(2) and R. G. DOUGLAS

State University of New York, Stony Brook, N.Y., U.S.A.

Dedicated to M. G. Krein

Most of the progress which has been made in the study of bounded linear operators on Hilbert space is restricted to operators with thin spectrum or operators which can be written as a direct sum of such operators. Our aim in this paper is to begin a systematic study of a class of operators at the opposite extreme, namely those which possess an open set of eigenvalues. We relate the study of such operators to certain problems in complex Hermitian geometry by showing that the eigenspaces form a Hermitian holomorphic bundle over an open set of the complex plane **C**. Along the way we generalize a rigidity theorem of Calabi for embeddings in Grassmannians, obtain a characterization generalizing that of Griffiths of which Hermitian bundles can be induced by such an embedding, and obtain rather refined results on the equivalence problem for Hermitian holomorphic bundles over an open set of **C**.

The paper is organized as follows. In § 1 we discuss operator theory including definitions, examples, and our main results. Geometric terminology is introduced in § 2 and related to the operator theory. Also the rigidity theorem is proved. In § 3 the equivalence problem is formulated and solved. We conclude in § 4 with various related results, examples, and open questions both in operator theory and complex geometry. Partial announcement of this work has been made in [5] and [6].

Since this paper is directed to readers in more than one area of mathematics, we have included more detail in many instances than would be usual.

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§ 1. The class $\mathcal{B}_n(\Omega)$

1.1. Let $\mathcal{H}$ be a complex separable Hilbert space and $\mathcal{L}(\mathcal{H})$ denote the collection of bounded linear operators on $\mathcal{H}$. A basic problem in operator theory is to determine when two operators S and T in $\mathcal{L}(\mathcal{H})$ are *unitarily equivalent*, that is, when there exists a unitary operator U on $\mathcal{H}$ satisfying $S = U^*TU$. In a real sense the problem has no general solution but one restricts attention to special classes of operators. An important approach to this problem is via spectral theory in which one attempts to synthesize operators from elementary "local operators", where "local" refers to the spectrum. For example, a normal operator on a finite dimensional space can be obtained as the orthogonal direct sum of scalar operators on eigenspaces, where the scalars are just the eigenvalues which together with their multiplicity determine the operator up to unitary equivalence. On infinite dimensional spaces direct sum must be replaced by a continuous direct sum or direct integral but the result is essentially the same. For an arbitrary operator on a finite dimensional space the direct sum is no longer orthogonal, consists of generalized eigenspaces, and the local operators are scalars plus nilpotents.

Conventional spectral theory attempts to extend such representations to as large a class of operators as possible. However, there exist operators which can not be synthesized in this sense from local operators. One example is the backward shift U_+^* on l^2 defined by

$$U_+^*(\alpha_0, \alpha_1, \alpha_2, \dots) = (\alpha_1, \alpha_2, \alpha_3, \dots)$$

for $(\alpha_0, \alpha_1, \alpha_2, \dots)$ in l^2. Since

$$U_+^*(1, \lambda, \lambda^2, \dots) = \lambda(1, \lambda, \lambda^2, \dots)$$

and $(1, \lambda, \lambda^2, \dots)$ is in l^2 for $|\lambda| < 1$, we see that the open unit disc $\mathbf{D}$ consists of eigenvalues for U_+^*. Such behavior is quite different from that for the finite dimensional case. Moreover, it can easily be shown that one can't express $l^2 = \mathcal{M} + \mathcal{N}$, where $\mathcal{M}$ and $\mathcal{N}$ are invariant for U_+^*. Thus one can not study U_+^* using conventional spectral theory.

However, we probably know as much about this operator and its adjoint, as we do about any single operator. In the functional representation, the adjoint U_+ acts as multiplication by z on the Hardy space H^2 and an enormous literature exists on it (cf. [8], [15]). This theory does not apply, however, to operators as closely related to the shift as multiplication by z on the Bergman space. What we hope to supply in this paper is the beginnings of a systematic approach which will apply to a whole class of operators. We define this class after introducing some notation.

For T in $\mathcal{L}(\mathcal{H})$ let $\operatorname{ran} T = \{Tx : x \in \mathcal{H}\}$ and $\ker T = \{x \in \mathcal{H} : Tx = 0\}$.

Definition 1.2. For Ω a connected open subset of $\mathbf{C}$ and n a positive integer, let $\mathcal{B}_n(\Omega)$ denote the operators T in $\mathcal{L}(\mathcal{H})$ which satisfy:

(a) $\Omega \subset \sigma(T) = \{\omega \in \mathbf{C}: T - \omega \text{ not invertible}\}$;

(b) $\operatorname{ran}(T-\omega) = \mathcal{H}$ for ω in Ω;

(c) $\bigvee \ker_{\omega\in\Omega}(T-\omega) = \mathcal{H}$; and

(d) $\dim\ker(T-\omega) = n$ for ω in Ω.

The collection $\mathcal{B}_n(\Omega)$ is void unless $\mathcal{H}$ is infinite dimensional. Conditions (a) and (b) insure that Ω is contained in the point spectrum of T and that $T-\omega$ is right invertible for ω in Ω. And since we intend to study T by investigating its eigenspaces, it's not unreasonable to assume that they span $\mathcal{H}$ which is (c). Lastly, since (a) and (b) imply that $\dim\ker(T-\omega)$ is constant, condition (d) imposes only that it is finite.

To see that the dimension is constant we recall a few facts about semi-Fredholm operators. An operator T in $\mathcal{L}(\mathcal{H})$ is said to be *semi-Fredholm* if ran T is closed and at least one of ker T and ker T^* is finite dimensional. The *index* is defined for a semi-Fredholm operator T by $\operatorname{ind}(T) = \dim\ker T - \dim\ker T^*$, is continuous, and satisfies $\operatorname{ind}(ST) = \operatorname{ind}(S) + \operatorname{ind}(T)$ for semi-Fredholm operators S and T as long as $\operatorname{ind}(S)+\operatorname{ind}(T)$ is defined. Now since $T-\omega$ is right invertible we see that $\operatorname{ind}(T-\omega) = \dim\ker(T-\omega)$ is constant.

1.3. If Ω_0 is an open subset of Ω, then $\mathcal{B}_n(\Omega) \subset \mathcal{B}_n(\Omega_0)$ because $\bigvee \ker_{\omega\in\Omega_0}(T-\omega) = \bigvee \ker_{\omega\in\Omega}(T-\omega)$. We shall prove this later in this section. Thus T can be associated with any open subset of Ω. There would seem to be some advantage in choosing Ω as large as possible. One kind of hypothesis implying that is the assumption that the closure of Ω is a *K-spectral set* for T. This means that $\|r(T)\| \leq K\|r\|_\infty$ for each rational function with poles outside $\bar{\Omega}$, where $\|r\|_\infty$ denotes the supremum norm on $\bar{\Omega}$. In general, for T in $\mathcal{B}_n(\Omega)$ there is no open set Λ which fulfills this hypothesis. A further possibility is to replace the supremum norm $\|r\|_\infty$ by a norm involving derivatives of r. We shall not explore this further in this paper.

1.4. Now $\mathcal{B}_n(\Omega)$ is an especially rich class of operators containing the adjoint of many subnormal, hyponormal, and weighted unilateral shift operators. For example if μ is a finite measure supported on $\partial\mathbf{D}$, m is normalized Lebesgue measure on $\partial\mathbf{D}$, and $H^2(\mu+m)$ is the closure of the analytic polynomials in $L^2(\mu+m)$, then the adjoint of multiplication by z belongs to $\mathcal{B}_1(\mathbf{D})$ [4]. If $l^2_{\mathbf{C}^n}$ denotes the Hilbert space of square-summable $\mathbf{C}^n$-valued sequences and $\{P_k\}$ is a sequence of positive operators on $\mathbf{C}^n$ satisfying $0 < cI \leq P_k \leq CI$, then the backward shift operator S on $l^2_{\mathbf{C}^n}$ defined by $(Sf)(k) = P_k f(k+1)$ belongs to $\mathcal{B}_n(\mathbf{D})$.

Moreover, operators in these classes have been the subject of intense investigation during recent years [15], [20].

1.5. The local operators associated with T in $\mathcal{B}_n(\Omega)$ are defined as follows. Since $T-\omega$ is a semi-Fredholm operator for ω in Ω and ran $(T-\omega)^k=\mathcal{H}$ for each positive integer k, it follows that

$$\text{(1.5.1)} \qquad \dim\ker(T-\omega)^k = \operatorname{ind}(T-\omega)^k = k\operatorname{ind}(T-\omega) = nk \quad \text{for } \omega \text{ in } \Omega.$$

Now the generalized eigenspace $\ker(T-\omega)^k$ is invariant for T and hence we can define an operator $N_\omega^{(k)}=(T-\omega)|\ker(T-\omega)^{k+1}$ with $N_\omega=N_\omega^{(n)}$. The *local operator* associated to T at ω is N_ω. Since $(N_\omega)^k=(T-\omega)^k|\ker(T-\omega)^{n+1}$ we see that N_ω is nilpotent of order $n+1$ and thus our spectral picture for an operator in $\mathcal{B}_n(\Omega)$ is very reminiscent of the finite dimensional situation. However, whereas there one must assume information about the relative angles between the different generalized eigenspaces, in our situation the geometry makes that unnecessary and we obtain as our main result:

THEOREM 1.6. *Operators T and $\tilde{T}$ in $\mathcal{B}_n(\Omega)$ are unitarily equivalent if and only if N_ω is unitarily equivalent to $\tilde{N}_\omega$ for each ω in Ω.*

We prove this as a consequence of our equivalence results on Hermitian holomorphic bundles in § 4.2. Actually, as we shall see in § 4 for "most" operators T in $\mathcal{B}_n(\Omega)$ it is enough to know $N_\omega^{(3)}$ for ω in Ω.

1.7. Before continuing we make several remarks. First, we emphasize that there is no requirement on the behaviour of the unitary which implements the equivalence of N_ω and $\tilde{N}_\omega$ as a function of ω. Secondly, if one knows that $N_{\omega_0}^{(k)}$ and $\tilde{N}_{\omega_0}^{(k)}$ are unitarily equivalent for a fixed ω_0 in Ω but for *all* k, then it is easy to show that T and $\tilde{T}$ are unitarily equivalent. One chooses a weak limit of a sequence of partial unitaries which effect the unitary equivalence of the local operators. Next one shows that it is unitary and intertwines the two operators T and $\tilde{T}$. For this one needs the fact that (c) is equivalent to

$$\text{(1.7.1)} \qquad \bigvee_{k=1}^{\infty} \ker(T-\omega_0)^k = \mathcal{H}.$$

Thus the depth of the theorem lies in concluding unitary equivalence on the behavior of T on the generalized eigenspaces of order $n+1$.

Further, it is possible to obtain a complete set of unitary invariants for T in $\mathcal{B}_n(\Omega)$ by recalling the result of Specht [21] and Pearcy [18] that a complete set of unitary invariants for an operator F on an N-dimensional Hilbert space is provided by the traces of a finite collection of words in F and F^*. Using this we obtain a finite collection of real analytic functions on Ω, the number depending on n, which form a complete set of unitary invariants for T in $\mathcal{B}_n(\Omega)$. We treat in detail only the case $n=1$.

Suppose N is a non-zero nilpotent operator of order two defined on a two-dimensional space. If e_1 is a unit vector in ker N, then choose a unit vector e_2 orthogonal to e_1 such that $(Ne_2, e_1)=h>0$. Relative to this orthonormal basis the matrix for N has the form

$$\begin{pmatrix} 0 & h \\ 0 & 0 \end{pmatrix} \tag{1.7.2}$$

and h is a complete unitary invariant for N. In connection with our earlier comments observe that $h^2=\text{trace}\,(N^*N)$. Applying this to an operator T in $\mathcal{B}_1(\Omega)$ and observing that $N_\omega^* N_\omega = P_\omega T^* T P_\omega$, where P_ω is the orthogonal projection onto ker $(T-\omega)^2$, we obtain

COROLLARY 1.8. *Operators T and $\tilde{T}$ in $\mathcal{B}_1(\Omega)$ are unitarily equivalent if and only if* trace $(N_\omega^* N_\omega)=$ trace $(\tilde{N}_\omega^* \tilde{N}_\omega)$ *for ω in Ω or equivalently if and only if* trace $(P_\omega T^* T P_\omega)=$ trace $(\tilde{P}_\omega \tilde{T}^* \tilde{T} \tilde{P}_\omega)$ *for ω in Ω.*

Thus unitary equivalence for operators in $\mathcal{B}_1(\Omega)$ is reduced to the equality of two functions on Ω. We see later in this section that trace $(N_\omega^* N_\omega)$ is real-analytic and we calculate some examples. The problem of characterizing which positive real-analytic functions on Ω can occur as trace $(N_\omega^* N_\omega)$ for some T in $\mathcal{B}_1(\Omega)$ and reconstructing T from it is solved in § 4.

As we mentioned earlier certain weighted backward shift operators belong to $\mathcal{B}_1(\mathbf{D})$. The preceding result can be used to characterize those operators in $\mathcal{B}_1(\mathbf{D})$ which are.

COROLLARY 1.9. *An operator T in $\mathcal{B}_1(\mathbf{D})$ is a weighted unilateral shift operator if and only if the function* trace $(N_\omega^* N_\omega)$ *depends only on* $|\omega|$.

Proof. Since T and $e^{i\theta}T$ are unitarily equivalent for T a weighted unilateral shift operator [20], one direction is clear. Conversely, by Corollary 1.8 if trace $(N_{re^{i\theta}}^* N_{re^{i\theta}})$ depends only on r, then T and $e^{i\theta}T$ are unitarily equivalent. Thus $T\,|\,\ker T^k$ and $e^{i\theta}T\,|\,\ker T^k$ are unitarily equivalent for each $e^{i\theta}$ and $k=1, 2, 3, \ldots$. We can choose an orthonormal basis for ker T^{k+1} such that the matrix for $T\,|\,\ker T^{k+1}$ is

$$\begin{pmatrix} 0 & \alpha_1 & \beta_{12} \ldots \beta_{1k} \\ 0 & 0 & \alpha_2 \;\ldots \beta_{2k} \\ \vdots & \vdots & \vdots \qquad \vdots \\ 0 & 0 & 0 \;\;\ldots \alpha_k \\ 0 & 0 & 0 \qquad 0 \end{pmatrix}$$

where $\alpha_j > 0$, $j = 1, 2, \ldots, k$. Moreover, such a matrix is unique. Now the corresponding matrix for $e^{i\theta} T | \ker T^{k+1}$ is

$$\begin{pmatrix} 0 & \alpha_1 & e^{i\theta}\beta_{12} \ldots e^{i(k-1)\theta}\beta_{1k} \\ 0 & 0 & \alpha_2 \quad \ldots e^{i(k-1)\theta}\beta_{2k} \\ 0 & 0 & 0 \quad \ldots \quad \alpha_k \\ 0 & 0 & 0 \quad \ldots \quad 0 \end{pmatrix}$$

and hence if T is unitarily equivalent to $e^{i\theta} T$ for θ/π irrational, then $\beta_{jl} = 0$, $1 \leq j < l \leq k$. Therefore T is a weighted shift operator.

Although we have stated this result only for $\Omega = \mathbf{D}$, on the basis of Corollary 1.8 one can see that for Ω containing the origin 0 in $\mathbf{C}$, that an operator T in $\mathcal{B}_1(\Omega)$ is a weighted backward shift if trace $(N^*_{re^{i\theta}} N_{re^{i\theta}})$ depends only on r on some small disk about 0. The obvious generalization of the corollary to T in $\mathcal{B}_n(\mathbf{D})$ is not correct.

1.10. To proceed further we must examine more carefully the nature of the subspace valued mapping $\omega \to \ker (T - \omega)$. To do this we need to introduce the notion of a Hermitian holomorphic vector bundle. Let Λ be a manifold with a complex structure and n be a positive integer. A *rank n holomorphic vector bundle* over Λ consists of a manifold E with a complex structure together with a holomorphic map π from E onto Λ such that each fibre $E_\lambda = \pi^{-1}(\lambda)$ is isomorphic to $\mathbf{C}^n$ and such that for each λ_0 in Λ there exists a neighborhood Δ of λ_0 and holomorphic functions $\gamma_1(\lambda), \ldots, \gamma_n(\lambda)$ from Δ to E whose values form a basis for E_λ at each λ in Δ. The functions $\gamma_1, \ldots, \gamma_n$ are said to be a *frame* for E on Δ. The bundle is said to be *trivial* if Δ can be taken to be all of Λ. A *cross-section* of E is a map γ from Λ to E such that $\pi(\gamma(\lambda)) = \lambda$ for λ in Λ. A *bundle map* is a map φ between two bundles E and $\tilde{E}$ over Λ which is holomorphic, and defines a linear transformation from E_λ to $\tilde{E}_\lambda$ for λ in Λ.

The trivial bundle over Ω of rank n is obtained by taking $E^n = \Omega \times \mathbf{C}^n$ and defining $\pi(\omega, x) = \omega$. A holomorphic cross-section of this bundle is just a $\mathbf{C}^n$-valued holomorphic function on Ω. A function f from an open set Δ in $\mathbf{C}$ to a Banach space $\mathfrak{X}$ is said to be *holomorphic* if it can be defined locally by a power series with vector coefficients which converges in norm. A bundle map for E^n is just an $M_n(\mathbf{C})$-valued holomorphic function on Ω.

The mapping $\omega \to \ker (T - \omega)$ will be shown to define a rank n holomorphic vector bundle E_T over Ω for T in $\mathcal{B}_n(\Omega)$. Since all holomorphic bundles over Ω are trivial as holomorphic bundles by Grauert's theorem [12] and the fact that all such bundles over Ω are topologically trivial, we shall be interested in additional structure which E_T possesses.

A *Hermitian holomorphic vector bundle* E over Λ is a holomorphic vector bundle such that each fibre E_λ is an inner product space. The bundle is said to have a real-analytic $[C^\infty]$ metric if $\lambda \to \|\gamma(\lambda)\|^2$ is real-analytic $[C^\infty]$ for each holomorphic cross-section of E. Two Hermitian holomorphic vector bundles E and $\tilde{E}$ over Λ will be said to be *equivalent* if there exists an isometric holomorphic bundle map from E onto $\tilde{E}$. In what follows we shall sometimes use the terminology "complex bundle" to refer to a Hermitian holomorphic vector bundle.

For T an operator in $\mathcal{B}_n(\Omega)$ let (E_T, π) denote the sub-bundle of the trivial bundle $\Omega \times \mathcal{H}$ defined by

$$E_T = \{(\omega, x) \in \Omega \times \mathcal{H}: x \in \ker (T - \omega)\} \quad \text{and } \pi(\omega, x) = \omega.$$

That E_T is a complex bundle over Ω is due to Šubin [22]. We offer a slight generalization of his result which will be useful in future work. First we need to recall a few more facts about Fredholm operators. If $T - \omega_0$ is Fredholm, then $T - \omega$ is Fredholm for ω in some neighborhood of ω_0 but whereas ind $(T - \omega)$ is locally constant, dim ker $(T - \omega)$ need not be. It is, however, except at isolated points. Thus we call ω_0 a *point of stability* for T if $T - \omega_0$ is Fredholm and dim ker $(T - \omega)$ is constant on some neighborhood of ω_0.

PROPOSITION 1.11. *If ω_0 is a point of stability for T in $\mathcal{L}(\mathcal{H})$, then there exist holomorphic $\mathcal{H}$-valued functions $\{e_i(\omega)\}_{i=1}^n$ defined on some neighborhood Δ of ω_0 such that $\{e_1(\omega), e_2(\omega), \ldots, e_n(\omega)\}$ forms a basis for* ker $(T - \omega)$ *for ω in Δ.*

Proof. We assume that $\omega_0 = 0$. There exists S in $\mathcal{L}(\mathcal{H})$ such that $ST = I - P$, where P is the projection onto ker T. If we define $S(\omega) = (I - \omega S)^{-1} S$ and $P(\omega) = (I - \omega S)^{-1} P$ for ω in $\Delta = \{\omega \in \mathbf{C}: |\omega| < 1/\|S\|\}$, then $P(\omega)$ is rank n and

$$\ker (T - \omega) \subset \ker (S(\omega)(T - \omega)) = \ker (I - P(\omega)) \subset \operatorname{ran} P(\omega).$$

Since dim ker $(T - \omega) = n = \dim \operatorname{ran} P(\omega)$, it follows that ker $(T - \omega) = \operatorname{ran} P(\omega)$. Hence, if $e_1, e_2, \ldots, e_n$ is a basis for ker T, then the functions $e_i(\omega) = P(\omega) e_i$, $i = 1, 2, \ldots, n$ defined for ω in Δ have the required properties.

COROLLARY 1.12. *For T in $\mathcal{B}_n(\Omega)$ the mapping $\omega \to$* ker $(T - \omega)$ *defines a complex bundle E_T over Ω.*

Before continuing we use the proposition to take care of some unfinished business.

COROLLARY 1.13. *If $\Omega_0 \subset \Omega$ are bounded connected open subsets of $\mathbf{C}$, then $\mathcal{B}_n(\Omega) \subset \mathcal{B}_n(\Omega_0)$.*

Proof. It is enough to prove that $\bigvee_{\omega\in\Omega_0}\ker(T-\omega)=\bigvee_{\omega\in\Omega}\ker(T-\omega)$. Suppose x in $\mathcal{H}$ is orthogonal to $\ker(T-\omega)$ for ω in Ω_0. If ω_0 is a boundary point of $\Delta =$ interior of $\{\omega\in\Omega\colon x\perp\ker(T-\omega)\}$ in Ω, then by the proposition there exists an open set Δ_0 of Ω about ω_0 and holomorphic functions $e_1(\omega), e_2(\omega), \ldots, e_n(\omega)$ defined on Δ_0 which form a basis for $\ker(T-\omega)$ for each ω in Δ_0. Since the holomorphic functions $(e_i(\omega), x)$ for $i=1, 2, \ldots, n$ vanish on Δ, they vanish identically and hence Δ_0 is contained in Δ. Thus $\Delta=\Omega$ which completes the proof.

THEOREM 1.14. *Operators T and $\tilde{T}$ in $\mathcal{B}_n(\Omega)$ are unitarily equivalent if and only if the complex bundles E_T and $E_{\tilde{T}}$ are equivalent as Hermitian holomorphic vector bundles.*

This is a consequence (see § 4.1) of our generalization of Calabi's rigidity theorem which is stated and proved in the next chapter.

1.15. There is an older concept in operator theory which is related to our approach. Let T be an operator in $\mathcal{B}_1(\Omega)$ and γ be a non-zero holomorphic cross-section of E_T (which exists by Grauert's theorem [12]). Then corresponding to γ there is a natural representation Γ of $\mathcal{H}$ as a space of holomorphic functions on $\Omega^*=\{\bar{\omega}\colon \omega\in\Omega\}$ defined by $(\Gamma x)(\omega)=\langle x, \gamma(\bar{\omega})\rangle$ for x in $\mathcal{H}$ and ω in Ω^*.(¹) Moreover, since $(\Gamma T^* x)(\omega)=(x, T\gamma(\bar{\omega}))=(x, \bar{\omega}\gamma(\bar{\omega}))=\omega(\Gamma(x))(\omega)$ for ω in Ω^*, we see that T is the adjoint of multiplication by ω. If we set $k(\lambda, \omega)=(\gamma(\bar{\omega}), \gamma(\bar{\lambda}))$, then k is a *reproducing kernel* for this space of holomorphic functions or $\Gamma\mathcal{H}$ is a kernel Hilbert space [1], [15]. Now different cross-sections yield different representations. However, if γ_1 and γ_2 are both non-zero holomorphic cross-sections of E_T, then there exists a non-zero holomorphic function f defined on Ω such that $\gamma_2(\omega)=f(\omega)\gamma_1(\omega)$ and thus $\Gamma_1\mathcal{H}$ and $\Gamma_2\mathcal{H}$ differ by a holomorphic multiplier.

In general, there is no canonical representation of $\mathcal{H}$ as a space of holomorphic functions on Ω, since there is no canonical cross-section of E_T. However, in some instances there is a preferred or natural choice. For example the Szegő kernel $k(z, \omega)=(1-\bar{\omega}z)^{-1}$ corresponds to the preferred cross-section $\gamma(\omega)=(1, \omega, \omega^2, \ldots)$ for the shift operator U_+^*. Moreover, if $B^2(\mathbf{D})$ denotes the closure of the analytic polynomials in $L^2(m^2)$, where m^2 is normalized Lebesgue measure on $\mathbf{D}$, and B_+ is the operator on $B^2(\mathbf{D})$ defined by $(B_+ f)(\omega)=\omega f(\omega)$ for f in $B^2(\mathbf{D})$, then B_+^* belongs to $\mathcal{B}_1(\mathbf{D})$ and a preferred cross-section corresponds to the Bergman kernel $k(z, \omega)=\pi^{-1}(1-\bar{\omega}z)^{-2}$.

1.16. Now although cross-sections are not themselves invariants for an operator in $\mathcal{B}_1(\Omega)$, they can be used to calculate the invariant described in Corollary 1.8 as follows.

(¹) If Ω is symmetric with respect to the real axis then it is possible to take γ anti-holomorphic and represent $\mathcal{H}$ as holomorphic functions on Ω.

Let T be an operator in $\mathcal{B}_1(\Omega)$ and γ be a non-zero holomorphic cross-section of E_T. Thus $(T-\omega)\gamma(\omega)=0$ and differentiating we obtain

$$0=\{(T-\omega)\gamma(\omega)\}'=-\gamma(\omega)+(T-\omega)\gamma'(\omega)$$

or $(T-\omega)\gamma'(\omega)=\gamma(\omega)$. Therefore, ker $(T-\omega)^2$ is spanned by $\gamma(\omega)$ and $\gamma'(\omega)$. An orthonormal basis for ker $(T-\omega)^2$ is

$$e_1(\omega)=\frac{\gamma(\omega)}{\|\gamma(\omega)\|}$$

$$e_2(\omega)=\frac{(\gamma(\omega),\gamma(\omega))\gamma'(\omega)-(\gamma'(\omega),\gamma(\omega))\gamma(\omega)}{\{\|\gamma(\omega)\|^4\|\gamma'(\omega)\|^2-|(\gamma'(\omega),\gamma(\omega))|^2\|\gamma(\omega)\|^2\}^{1/2}}$$

and we have by (1.7.2)

$$h(\omega)=((T-\omega)\,e_2(\omega),\,e_1(\omega))=\frac{\|\gamma(\omega)\|^2}{\{\|\gamma(\omega)\|^2\|\gamma'(\omega)\|^2-|(\gamma'(\omega),\gamma(\omega))|^2\}^{1/2}}.$$

Thus we obtain

THEOREM 1.17. *A complete unitary invariant for T in $\mathcal{B}_1(\Omega)$ is the real-analytic function*

$$\mathcal{K}_T(\omega)=\frac{|(\gamma'(\omega),\gamma(\omega))|^2-\|\gamma(\omega\|^2\|\gamma'(\omega)\|^2}{\|\gamma(\omega)\|^4}=-\frac{\partial^2\log\|\gamma(\omega)\|^2}{\partial\omega\partial\bar\omega},\tag{1.17.1}$$

where γ is any non-zero holomorphic cross-section of E_T.

The reason for choosing this particular function is that, as we shall see in § 2, $\mathcal{K}(\omega)$ is the curvature of the bundle E_T.

As a consequence of this calculation, we see that trace $(N_\omega^* N_\omega)=-\mathcal{K}_T(\omega)^{-1}$ is real-analytic as promised. Also using the Szegö and Bergman kernels we calculate that

$$\text{trace}\,(U_{+\omega}U_{+\omega}^*)=(1-|\omega|^2)^2$$

and

$$\text{trace}\,(B_{+\omega}B_{+\omega}^*)=\tfrac{1}{2}(1-|\omega|^2)^2.$$

Thus we see that U_+ and B_+ are not unitarily equivalent. Although there are easier ways to prove this (using for example the fact that U_+ is an isometry), it's not always easy to decide when two operators in $\mathcal{B}_1(\Omega)$ are unitarily equivalent.

One direct connection between T and E_T is contained in

PROPOSITION 1.18. *An operator T in $\mathcal{B}_n(\Omega)$ is reducible if and only if the complex bundle E_T is reducible as a Hermitian holomorphic vector bundle.*

Proof. Suppose $\mathcal{H}=\mathcal{M}\oplus\mathcal{N}$, where $\mathcal{M}$ and $\mathcal{N}$ reduce T. If x in $\ker(T-\omega)$ is written $x=x_1\oplus x_2$ for x_1 in $\mathcal{M}$ and x_2 in $\mathcal{N}$, then $Tx_1\oplus Tx_2=Tx=\omega x=\omega x_1\oplus\omega x_2$ implying that both x_1 and x_2 are in $\ker(T-\omega)$. Thus $\ker(T-\omega)$ decomposes into $\{\ker(T-\omega)\cap\mathcal{M}\}\oplus\{\ker(T-\omega)\cap\mathcal{N}\}$ and hence $T_1=T|\mathcal{M}$ is in $\mathcal{B}_{n_1}(\Omega)$ and $T_2=T|\mathcal{N}$ is in $\mathcal{B}_{n_2}(\Omega)$, where $n=n_1+n_2$. Therefore $E_T=E_{T_1}\oplus E_{T_2}$ is reducible.

Now suppose $E_T=E_1\oplus E_2$ is a reduction of E_T. Fix ω_0 in Ω and let Δ be a neighborhood of ω_0 in Ω on which there exists a trivialization $e_1(\omega), \ldots, e_n(\omega)$ of E_T, where $e_1(\omega), \ldots, e_{n_1}(\omega)$ span E_1 and $e_{n_1+1}(\omega), \ldots, e_n(\omega)$ span E_2. To show that T is reducible, it is enough to show that $\bigvee_{\omega\in\Delta}\{e_1(\omega), \ldots, e_{n_1}(\omega)\}$ and $\bigvee_{\omega\in\Delta}\{e_{n_1+1}(\omega), \ldots, e_n(\omega)\}$ are orthogonal, since as in the proof of Corollary 1.13 together they span $\mathcal{H}$. But $(e_i(\omega), e_j(\omega))=0$ for $1\leqslant i\leqslant n_1<j\leqslant n$ and differentiating with respect to ω (viewing the functions as functions of the variables ω and $\bar{\omega}$) we obtain

$$0=\frac{\partial}{\partial\omega}(e_i(\omega), e_j(\omega))=(e_i'(\omega), e_j(\omega)).$$

The second term in the product rule vanishes because the right hand side of the inner product is anti-holomorphic. Similarly we have $(e_i^{(k)}(\omega), e_j(\omega))=0$ for $k=0, 1, 2, \ldots$ and therefore

$$(e_i(\omega), e_j(\omega_0))=\left(\sum_{k=0}^{\infty}\frac{e_i^{(k)}(\omega_0)}{k!}(\omega-\omega_0)^k, e_j(\omega_0)\right)=0$$

for $|\omega-\omega_0|$ sufficiently small. Again as in the proof of Corollary 1.13 we are finished.

COROLLARY 1.19. *An operator T in $\mathcal{B}_1(\Omega)$ is irreducible.*

1.20. Important in the study of an operator T is an explicit characterization of its commutant $(T)'$, that is, the weakly closed algebra of operators which commute with T. For each of the operators U_+^* and B_+^*, the commutant can be identified with the algebra $H^\infty(\mathbf{D})$ of bounded holomorphic function on $\mathbf{D}$. We show that the commutant of an operator in $\mathcal{B}_1(\Omega)$ can always be identified with a subalgebra of $H^\infty(\Omega)$ and for T in $\mathcal{B}_n(\Omega)$ we identify $(T)'$ as a subalgebra of the bounded bundle endomorphisms on E_T. Recall that a bundle map Φ from E to E is a holomorphic map such that $\Phi(\omega)=\Phi|E_\omega$ is a linear endomorphism on the fibre E_ω over ω in Ω and we shall say that it is bounded if $\sup_{\omega\in\Omega}\|\Phi(\omega)\|<\infty$. We denote the collection of bounded bundle endomorphisms on E by $H^\infty_{\mathcal{L}(E)}(\Omega)$. For E the trivial bundle over Ω, $H^\infty_{\mathcal{L}(E)}\Omega)$ is just the bounded holomorphic matrix-valued functions on Ω.

PROPOSITION 1.21. *For T in $\mathcal{B}_n(\Omega)$ there is a contractive monomorphism Γ_T from the commutant $(T)'$ into $H^\infty_{\mathcal{L}(E_T)}(\Omega)$. In general, Γ_T is not onto.*

Proof. If $XT = TX$, then $X \ker(T-\omega) \subset \ker(T-\omega)$ for ω in Ω. Moreover, if $e(\omega)$ is a local holomorphic cross-section of E_T, then so is $Xe(\omega)$. Therefore X defines a holomorphic bundle map $\Gamma_T X$ on E_T. Since $\|(\Gamma_T X)(\omega)\| = \|X \,|\ker(T-\omega)\| \leqslant \|X\|$ we see that $\Gamma_T X$ lies in $H^\infty_{\mathcal{L}(E_T)}(\Omega)$ and Γ_T is contractive. That Γ_T is a homomorphism is clear and it is one-to-one since $\bigvee_{\omega\in\Omega} \ker(T-\omega) = \mathcal{H}$. Since Γ_T is not onto for T the Dirichlet operator (cf. [23]), it is not onto in general. (1)

Which bounded bundle maps are in the range of Γ_T? Before answering that we need the following result on a basis for generalized eigenspaces.

LEMMA 1.22. *If $\gamma_1, \ldots, \gamma_n$ are holomorphic functions from Ω to $\mathcal{H}$ such that $\gamma_1(\omega), \gamma_2(\omega), \ldots,$ $\gamma_n(\omega)$ forms a basis for* $\ker(T-\omega)$ *for each ω in Ω, then*

(i) $(T-\omega)\gamma_i^{(k)}(\omega) = k\gamma_i^{(k-1)}(\omega)$ *for all k and $i = 1, 2, \ldots, n$;*

and

(ii) $\gamma_1(\omega), \ldots, \gamma_n(\omega), \ldots, \gamma_1^{(k-1)}(\omega), \ldots, \gamma_n^{(k-1)}(\omega)$ *form a basis for* $\ker(T-\omega)^k$ *for $k \geqslant 1$ and ω in Ω.*

Proof. We first prove (i) by induction on k. Differentiating the equation $T(\gamma_i(\omega)) = \omega\gamma_i(\omega)$ we obtain

$$T(\gamma_i'(\omega)) = \gamma_i(\omega) + \omega\gamma_i'(\omega$$

which proves (i) for $k=1$. Assuming (i) holds for k and differentiating we obtain

$$-\gamma_i^{(k)}(\omega) + (T-\omega)\gamma_i^{(k+1)}(\omega) = k\gamma_i^{(k)}(\omega)$$

which is (i) for $k+1$. Moreover we have

$$(T-\omega)^k \gamma_i^{(k)}(\omega) = k!\,\gamma_i(\omega) \tag{1.22.1}$$

for all k and ω in Ω.

To prove (ii) we need only show that $\ker(T-\omega)^k/\ker(T-\omega)^{k-1}$ has basis $[\gamma_1^{(k-1)}(\omega)], \ldots,$ $[\gamma_n^{(k-1)}(\omega)]$, where $[\ ]$ denotes residue class. Now if $\sum_{i=1}^n a_i[\gamma_i^{(k-1)}(\omega)] = 0$, then

$$(T-\omega)^{k-1}\left(\sum_{i=1}^n a_i\gamma_i^{(k-1)}(\omega)\right) = 0$$

and by (1.22.1) we have

$$(T-\omega)^{k-1}\left(\sum_{i=1}^n a_i\gamma_i^{(k-1)}(\omega)\right) = \sum_{i=1}^n a_i(k-1)!\,\gamma_i(\omega).$$

(1) We will define the Dirichlet operator later in this section.

Therefore $a_i=0$ for $i=1, 2, \ldots, n$ so that $[\gamma_1^{(k-1)}(\omega)], \ldots, [\gamma_n^{(k-1)}(\omega)]$ are independent which completes the proof since the dimension of $\ker(T-\omega)^k/\ker(T-\omega)^{k-1}$ is n by (1.5.1).

1.23. Now suppose Φ is an element of $H^\infty_{\mathcal{L}(E_T)}(\Omega)$ for which there exists a bounded operator X in $(T)'$ such that $\Gamma_T X=\Phi$. If $\gamma_1(\omega), \ldots, \gamma_n(\omega)$ is a basis of local holomorphic cross-sections for E_T, then by differentiating we obtain

$$\begin{aligned}
X\gamma_i'(\omega) &= (X\gamma_i(\omega))' = \Phi(\omega)\gamma_i'(\omega)+\Phi'(\omega)\gamma_i(\omega)\\
X\gamma_i''(\omega) &= (X\gamma_i(\omega))'' = \Phi(\omega)\gamma_i''(\omega)+2\Phi'(\omega)\gamma_i'(\omega)+\Phi''(\omega)\gamma_i(\omega)\\
&\vdots \qquad\qquad \vdots \qquad \vdots\\
X\gamma_i^{(N)}(\omega) &= (X\gamma_i(\omega))^{(N)} = \Phi(\omega)\gamma_i^{(N)}(\omega)+N\Phi'(\omega)\gamma_i^{(N-1)}(\omega)+\ldots+\Phi^{(N)}(\omega)\gamma_i(\omega).
\end{aligned}$$

In other words the block matrix for $X|\ker(T-\omega)^{N+1}$ relative to the basis $\{\gamma_i^{(j)}(\omega)\}_{i=1}^{n}{}_{j=1}^{N}$ is

$$\begin{bmatrix}
\Phi(\omega) & \Phi'(\omega) & \Phi''(\omega) & \ldots & \Phi^{(N)}(\omega)\\
0 & \Phi(\omega) & 2\Phi'(\omega) & \ldots & N\Phi^{(N-1)}(\omega)\\
0 & 0 & \Phi(\omega) & \ldots & \frac{N(N-1)}{2}\Phi^{(N-2)}(\omega)\\
\vdots & \vdots & \vdots & & \vdots\\
0 & 0 & 0 & \ldots & \Phi(\omega)
\end{bmatrix} \tag{1.23.1}$$

Let $\Phi_N(\omega)$ denote the operator defined on $\ker(T-\omega)^{N+1}$ by this matrix. Note that to calculate $\|\Phi_N(\omega)\|$ we need only know the value of Φ and its derivatives at ω together with the Gram-Schmidt orthogonalization of the basis $\{\gamma_i^{(j)}(\omega)\}_{i=1}^{n}{}_{j=0}^{N}$. Our result characterizing the range of Γ_T is:

PROPOSITION 1.24. *For Φ in $H^\infty_{\mathcal{L}(E_T)}(\Omega)$ the following are equivalent:*

(1) $\Phi=\Gamma_T X$ *for some X in $(T)'$;*

(2) $\sup\{\|\Phi_N(\omega)\|: \omega\in\Omega, N=0, 1, 2, \ldots\}=C_1<\infty$, *and*

(3) $\sup\{\|\Phi_N(\omega_0)\|: N=0, 1, 2, 3, \ldots\}=C_2<\infty$ *for some ω_0 in Ω.*

Moreover, if these conditions hold, then $\|X\|=C_1=C_2$.

Proof. That (1) implies (2) implies (3) is obvious as is the inequality $\|X\|\geq C_1\geq C_2$. To show (3) implies (1) recall that $\bigvee_{k=1}^\infty \ker(T-\omega_0)^k=\mathcal{H}$ and take a weak limit of the uniformly bounded sequence of operators $\Phi_N(\omega_0)$ defined to be 0 on $[\ker(T-\omega_0)^{N+1}]^\perp$. If we call X the limit, then $\|X\|\leq C_2$, X commutes with T, and $\Gamma_T X=\Phi$ since $(\Gamma_T X)^{(k)}(\omega_0)=\Phi^{(k)}(\omega_0)$ for $k=0, 1, 2, \ldots$.

This procedure becomes much simpler in case $n=1$ when $\Phi=\varphi$ is a function in $H^\infty(\Omega)$. Let us calculate the matrix for $\Phi_1(\omega)$ relative to the orthonormal basis obtained in § 1.16. We obtain

$$\Phi_1(\omega)e_1(\omega)=\varphi(\omega)e_1(\omega)$$

and

$$\Phi_1(\omega)e_2(\omega)=\varphi(\omega)e_2(\omega)+\varphi'(\omega)h(\omega)e_1(\omega),$$

and hence the desired matrix is

$$\begin{pmatrix}\varphi(\omega) & h(\omega)\varphi'(\omega)\\ 0 & \varphi(\omega)\end{pmatrix}$$

Thus we obtain as

COROLLARY 1.25. *For T in $\mathcal{B}_1(\Omega)$ a necessary condition that φ in $H^\infty(\Omega)$ define a bounded operator in $(T)'$ is for the function*

$$|\varphi'(\omega)|^2\operatorname{trace}(N_\omega^*N_\omega)=-\frac{|\varphi'(\omega)|^2}{\mathcal{K}_T(\omega)}$$

to be bounded.

1.26. In certain instances one knows that the commutant of T in $\mathcal{B}_1(\Omega)$ is all of $H^\infty(\Omega)$ in which case the preceding corollary can be used to obtain inequalities on the growth of the derivative of bounded holomorphic functions on Ω. However, by applying the corollary to the Dirichlet operator, for example, one obtains necessary conditions for a function to be a multiplier.

The Dirichlet space $D^2(\mathbf{D})$ consists of the holomorphic functions f on $\mathbf{D}$ for which $f(0)=0$ and f' lies in $L^2(m^2)$ where $\|f\|_D^2=\sum_{n=1}^\infty n|a_n|^2$ for $f(\omega)=\sum_{n=1}^\infty a_n\omega^n$. Again D_+^*, the adjoint of multiplication z, lies in $\mathcal{B}_1(\mathbf{D})$. However, in this case $\Gamma_{D_+^*}$ is not onto [23]. One can calculate the curvature of $E_{D_+^*}$ using the kernel function

$$k(z,\omega)=\log\frac{1}{(1-z\bar{\omega})^2}$$

for $D^2(\mathbf{D})$ [19] by Theorem 1.17 and then apply the corollary. After some work one obtains that a necessary condition for a bounded holomorphic function φ on $\mathbf{D}$ to be a multiplier on $D^2(\mathbf{D})$ or equivalently to be in the range of $\Gamma_{D_+^*}$ is for the function

$$\frac{|\varphi'(\omega)|^2}{-\log(1-|\omega|^2)}$$

to be bounded on $\mathbf{D}$. Shields informs us that this necessary condition was known to him and that it is not sufficient. The necessary conditions involving higher derivatives are quite complicated and it is difficult to decide the nature of the further restriction imposed.

We obtain a complete characterization of the commutant only when the 0th order condition is sufficient. Although we could appeal to general results, we give a complete proof hoping that the techniques will be useful in future generalization.

Let us call an Ω *reasonable*, if for φ in $H^\infty(\Omega)$ there is a sequence of rational functions r_n with poles outside the closure $\bar{\Omega}$ of Ω such that $\|r_n\|_{\bar{\Omega}} \leq \|\varphi\|_{\bar{\Omega}}$ and $r_n(\omega) \to \varphi(\omega)$ for ω in Ω. If the interior of $\bar{\Omega}$ is finitely-connected, then Ω is reasonable; there are other properties implying this (cf. [10], VIII § 11).

PROPOSITION 1.27. *If Ω is reasonable and $\bar{\Omega}$ is a spectral set for T in $\mathcal{B}_1(\Omega)$, then Γ_T is an isometric isomorphism from $(T)'$ onto $H^\infty(\Omega)$.*

Proof. By definition the map $r \to r(T)$ for rational functions with poles outside $\bar{\Omega}$ is contractive. For φ in $H^\infty(\Omega)$ let $\{r_n\}_{n=1}^\infty$ be a sequence of rational functions which converge pointwise to φ and satisfy $\|r_n\|_{\bar{\Omega}} \leq \|\varphi\|_\Omega$. We may assume that $\{r_n(T)\}_{n=1}^\infty$ converges weakly to some X in $(T)'$. Moreover, since

$$\varphi(\omega) = \lim_{n\to\infty} r_n(\omega) = \lim_{n\to\infty} (r_n(T)\, e_\omega, e_\omega) = (Xe_\omega, e_\omega)$$

for e_ω a unit vector in $\ker(T-\omega)$, we see that $\Gamma_T X = \varphi$ and $\|X\| \leq \|\varphi\|_\Omega$. Hence Γ_T is onto and isometric.

With a similar proof we can also establish

PROPOSITION 1.28. *If Ω has the property that φ in $H^\infty(\Omega)$ can be pointwise boundedly approximated by rational functions and $\bar{\Omega}$ is a K-spectral set for T in $\mathcal{B}_1(\Omega)$, then Γ_T is an isomorphism.*

We conclude this section with one geometricial implication of the assumption that $\bar{\Omega}$ is a K-spectral set. Analogous results with different rates of growth are undoubtedly true when $\bar{\Omega}$ is a "C^n-spectral set".

PROPOSITION 1.29. *If $\partial\Omega$ consists of finitely many C^1-smooth simple closed curves and T in $\mathcal{B}_1(\Omega)$ has $\bar{\Omega}$ as a K-spectral set, then $\lim_{\operatorname{dist}(\omega,\partial\Omega)\to 0} \mathcal{K}_T(\omega) = -\infty$, where $\mathcal{K}_T$ is defined by (1.17.1).*

Proof. For ω_0 not in $\bar{\Omega}$ we have by Corollary 1.25, the definition of K-spectral set, and the fact that $(T-\omega_0)^{-1}$ is in $(T)'$ that

$$\left|\left(\frac{1}{\omega-\omega_0}\right)'\right| h(\omega) \leq K \left\|\frac{1}{\omega-\omega_0}\right\|_{\bar{\Omega}}$$

which implies

$$h(\omega) \leqslant \frac{K|\omega - \omega_0|^2}{\operatorname{dist}(\omega_0, \partial\Omega)}$$

for ω in Ω.

Now for ω sufficiently close to $\partial\Omega$, there exists ω_0 exterior to $\overline{\Omega}$ such that $|\omega - \omega_0| = 2 \operatorname{dist}(\omega_0, \partial\Omega)$ and hence we obtain $h(\omega) \leqslant 4K \operatorname{dist}(\omega, \partial\Omega)$ which implies using the compactness of $\overline{\Omega}$ that $\lim_{\operatorname{dist}(\omega, \partial\Omega)\to 0} \mathcal{K}_T(\omega) = -\infty$.

§ 2. The rigidity theorem and the canonical connection

2.1. In the first chapter we showed that every operator in $\mathcal{B}_n(\Omega)$ gives rise to a Hermitian holomorphic vector bundle over Ω. In order to obtain the results stated in Chapter 1 we must begin a rather detailed study of certain aspects of complex geometry. In this chapter and the next we concentrate on complex geometry. In Chapter 4 we finally draw it all together.

For $\mathcal{H}$ a separable complex Hilbert space and n a positive integer, let $\mathcal{G}r(n, \mathcal{H})$ denote the *Grassmann manifold*, the set of all n-dimensional subspaces of $\mathcal{H}$. When the dimension of $\mathcal{H}$ is finite, $\mathcal{G}r(n, \mathcal{H})$ is a complex manifold.

For Λ an open connected subset of $\mathbf{C}^n$ we shall say that a map $f: \Lambda \to \mathcal{G}r(n, \mathcal{H})$ is *holomorphic* at λ_0 in Λ if there exists a neighborhood Δ of λ_0 and n holomorphic $\mathcal{H}$-valued functions $\gamma_1, ..., \gamma_n$ on Δ such that $f(\lambda) = \bigvee\{\gamma_1(\lambda), ..., \gamma_n(\lambda)\}$ for λ in Δ. If $f: \Lambda \to \mathcal{G}r(n, \mathcal{H})$ is a holomorphic map, then a natural n-dimensional Hermitian holomorphic vector bundle E_f is induced over Λ, that is,

$$E_f = \{(x, \lambda) \in \mathcal{H} \times \Lambda: x \in f(\lambda)\} \quad \text{and } \pi: E_f \to \Lambda \quad \text{where } \pi(x, \lambda) = \lambda.$$

Equivalently, E_f is the pull-back $f^*S(n, \mathcal{H})$ of the tautological bundle $S(n, \mathcal{H})$ defined over $\mathcal{G}r(n, \mathcal{H})$, where

$$S(n, \mathcal{H}) = \{(x, V) \in \mathcal{H} \times \mathcal{G}r(n, \mathcal{H}): x \in V\}$$

and

$$\pi: S(n, \mathcal{H}) \to \mathcal{G}r(n, \mathcal{H}) \quad \text{such that } \pi(x, V) = V.$$

Our interest in such bundles is obvious since E_T arises as the pull-back of the map $t: \Omega \to \mathcal{G}r(n, \mathcal{H})$ defined by $t(\omega) = \ker(T - \omega)$ for T in $\mathcal{B}_n(\Omega)$. A map $f: \Omega \to \mathcal{G}r(n, \mathcal{H})$ for Ω an open set in $\mathbf{C}$ is said to be a *holomorphic curve*.

Now given two holomorphic maps f and $\tilde{f}: \Lambda \to \mathcal{G}r(n, \mathcal{H})$, we have two vector bundles E_f and $E_{\tilde{f}}$ over Λ. If there exists a unitary operator U on $\mathcal{H}$ such that $\tilde{f} = Uf$, then f and $\tilde{f}$ are said to be *congruent* and E_f and $E_{\tilde{f}}$ are obviously equivalent. The Rigidity Theorem states that is the only way they can be equivalent.

THEOREM 2.2. (Rigidity.) *Let Λ be an open connected subset of $\mathbf{C}^k$ and f and $\tilde{f}$ be holomorphic maps from Λ to $Gr(n, \mathcal{H})$ such that $\bigvee_{\lambda \in \Lambda} f(\lambda) = \bigvee_{\lambda \in \Lambda} \tilde{f}(\lambda) = \mathcal{H}$. Then f and $\tilde{f}$ are congruent if and only if E_f and $E_{\tilde{f}}$ are locally equivalent Hermitian holomorphic vector bundles over Λ.*

Proof. Congruent maps obviously define equivalent bundles. Thus suppose E_f and $E_{\tilde{f}}$ are locally equivalent and that Φ is a holomorphic isometric bundle map from $E_f|\Delta$ onto $E_{\tilde{f}}|\Delta$ for some open set Δ in Λ. Assume that Δ is an open ball in Λ on which both E_f and $E_{\tilde{f}}$ are trivial. If $\{\gamma_1, ..., \gamma_n\}$ are holomorphic cross-sections of E_f defined on Δ which form a frame on Δ and we set $\tilde{\gamma}_i = \Phi\gamma_i$, $i = 1, 2, ..., n$, then $\{\tilde{\gamma}_1, ..., \tilde{\gamma}_n\}$ is a frame for $E_{\tilde{f}}$ on Δ. Moreover, we have

$$(\gamma_i(\lambda), \gamma_j(\lambda)) = (\tilde{\gamma}_i(\lambda), \tilde{\gamma}_j(\lambda))$$

for λ in Δ and

$$1 \leqslant i, \quad j \leqslant n.$$

Further, since the γ_i are holomorphic, we have

$$(\gamma_i^{(p)}(\lambda), \gamma_j^{(q)}(\lambda)) = \frac{\partial^{p+q}}{\partial z^p \partial \bar{z}^q} (\gamma_i(\lambda), \gamma_j(\lambda))$$

for λ in Δ, where p and q are multi-indices $(p_1, p_2, ..., p_k)$ and $(q_1, q_2, ..., q_k)$.

$$\frac{\partial^{p+q}}{\partial z^p \partial \bar{z}^q} = \frac{\partial^{p_1 + \dots + p_k + q_1 + \dots + q_k}}{\partial z_1^{p_1} \dots \partial z_k^{p_k} \partial \bar{z}_1^{q_1} \dots \partial \bar{z}_k^{q_k}},$$

and

$$\gamma_i^{(p)} = \frac{\partial^p}{\partial z^p} \gamma_i.$$

Therefore, we have that

$$(\gamma_i^{(p)}(\lambda), \gamma_j^{(q)}(\lambda)) = (\tilde{\gamma}_i^{(p)}(\lambda), \tilde{\gamma}_j^{(q)}(\lambda))$$

for λ in Δ, $1 \leqslant i$, $j \leqslant n$ and all p and q.

Thus we can define an operator U_λ from $\bigvee_p \bigvee_{i=1}^n \gamma_i^{(p)}(\lambda)$ to $\bigvee_p \bigvee_{i=1}^n \tilde{\gamma}_i^{(p)}(\lambda)$ by

$$U_\lambda \gamma_i^{(p)}(\lambda) = \tilde{\gamma}_i^{(p)}(\lambda)$$

for $1 \leqslant i \leqslant n$ and all p. Moreover, U_λ is isometric, since

$$(U_\lambda \gamma_i^{(p)}(\lambda), U_\lambda \gamma_j^{(q)}(\lambda)) = (\gamma_i^{(p)}(\lambda), \gamma_j^{(q)}(\lambda))$$

for $1 \leqslant i$, $j \leqslant n$ and all p and q, and thus is well-defined. Further, since γ_i and $\tilde{\gamma}$ can be expanded in a Taylor series in some ball Δ_{λ_0} about each λ_0 in Δ, and using the proof of Corollary 1.13 we see that

$$\bigvee_p \bigvee_{i=1}^n \gamma_i^{(p)}(\lambda_0) = \bigvee_{\lambda \in \Delta_{\lambda_0}} \bigvee_{i=1}^n \gamma_i(\lambda) = \bigvee_{\lambda \in \Delta} f(\lambda) = \mathcal{H}$$

and

$$\bigvee_p \bigvee_{i=1}^n \tilde{\gamma}_i^{(p)}(\lambda_0) = \bigvee_{\lambda \in \Delta_{\lambda_0}} \bigvee_{i=1}^n \tilde{\gamma}_i(\lambda) = \bigvee_{\lambda \in \Delta} \tilde{f}(\lambda) = \mathcal{H}.$$

Therefore, U_λ is a unitary operator defined on $\mathcal{H}$ for each λ in Δ. Now for λ in Δ_{λ_0} we have

$$\gamma_i^{(p)}(\lambda) = \sum_r \frac{(\lambda - \lambda_0)^r}{r!} \gamma_i^{(p+r)}(\lambda_0),$$

where $(\lambda - \lambda_0)^r = \prod_{j=1}^k (\lambda_j - \lambda_{j,0})^{r_j}$ and $r! = \prod_{j=1}^k r_j!$. Since U_{λ_0} is bounded, we have

$$\begin{aligned} U_{\lambda_0}(\gamma_i^{(p)}(\lambda)) &= \sum_r \frac{(\lambda - \lambda_0)^r}{r!} U_{\lambda_0}(\gamma_i^{(p+r)}(\lambda_0)) \\ &= \sum_r \frac{(\lambda - \lambda_0)^r}{r!} \tilde{\gamma}_i^{(p+r)}(\lambda_0) = \tilde{\gamma}_i^{(p)}(\lambda) \\ &= U_\lambda(\gamma_i^{(p)}(\lambda)). \end{aligned}$$

and hence $U_{\lambda_0} = U_\lambda$. If we set $U = U_\lambda$, then $\tilde{f}(\lambda) = Uf(\lambda)$ for λ in Δ. Since the subset of Λ on which $\tilde{f} = Uf$ is open and closed we see that f and $\tilde{f}$ are congruent.

The first instance of a Rigidity Theorem for curves in $\mathcal{G}r(n, \mathcal{H})$ is due to Calabi [2], in the case $n = 1$.

2.3. Although we stated the Rigidity Theorem in the context of several complex variables, we now deal exclusively with the one-variable case. Much of what follows makes sense in the several variables context and we hope to consider that in future work.

Here our principal concern is deciding when two pull-back bundles over an open subset of the plane are equivalent. In § 3 we show that this is the same as "equivalence up to finite order". For this we need the canonical connection on such bundles which we define later in this section. We now define when two holomorphic curves have finite order of contact and relate that to the operator-theoretic invariants introduced in the first chapter.

For Ω a connected open subset of $\mathbf{C}$ we follow Griffiths [13] in saying that two holomorphic curves f and $\tilde{f}$: $\Omega \to \mathcal{G}r(n, \mathcal{H})$ have *order of contact k* if for each ω_0 in Ω there exists a unitary U on $\mathcal{H}$ such that Uf and $\tilde{f}$ agree to order k at ω_0, that is, if $\gamma_1, \gamma_2, \ldots, \gamma_n$ are holomorphic spanning cross-sections for $E_{\tilde f}$ at ω_0, then there exists holomorphic spanning cross-sections $\tilde\gamma_1, \tilde\gamma_2, \ldots, \tilde\gamma_n$ for $E_{\tilde f}$ at ω_0 such that

$$U\gamma_i^{(j)}(\omega_0) = \tilde\gamma_i^{(j)}(\omega_0)$$

for $1 \leqslant i \leqslant n$ and $0 \leqslant j \leqslant k$, where the choice of the $\tilde\gamma$ may depend on ω_0.

Before relating this to the local operators defined in § 1 we need the following technical lemma.

LEMMA 2.4. *If $f: \Omega \to \mathcal{G}r(n, \mathcal{H})$ is a holomorphic curve and $\gamma_1, \gamma_2, \ldots, \gamma_n$ are holomorphic cross-sections of the vector bundle E_f defined over Ω such that $\gamma_1(\omega_0), \ldots, \gamma_n(\omega_0)$ is an orthonormal basis for $f(\omega_0)$, then there exist holomorphic cross-sections $\hat\gamma_1, \ldots, \hat\gamma_n$ of E_f defined on some open set Δ about ω_0 such that $\hat\gamma_i(\omega_0) = \gamma_i(\omega_0)$ for $i = 1, 2, \ldots, n$ and*

$$(\hat\gamma_i^{(k)}(\omega_0), \hat\gamma_j(\omega_0)) = 0 \quad \text{for } 1 \leqslant i, j \leqslant n \quad \text{and } k = 1, 2, \ldots.$$

Proof. The matrix function $((\gamma_i(\omega), \gamma_j(\omega_0)))_{i,j=1}^n$ is invertible on some open set Δ containing ω_0 with inverse $(C_{ij}(\omega))_{i,j=1}^n$. If we set $\hat\gamma_i(\omega) = \sum_{j=1}^n C_{ij}(\omega)\gamma_j(\omega)$, then $\hat\gamma_1, \ldots, \hat\gamma_n$ are holomorphic cross-sections of E_f which satisfy $\hat\gamma_i(\omega_0) = \gamma_i(\omega_0)$.

Moreover, since $((\hat\gamma_i(\omega), \hat\gamma_i(\omega_0)))$ is the identity matrix, we see that $(\hat\gamma_i^{(k)}(\omega_0), \hat\gamma_i(\omega_0)) = 0$ for $k > 0$.

PROPOSITION 2.5. *For T and $\tilde T$ in $\mathcal{B}_n(\Omega)$ define $t, \tilde t: \Omega \to \mathcal{G}r(n, \mathcal{H})$ by $t(\omega) = \ker(T - \omega)$, $\tilde t(\omega) = \ker(\tilde T - \omega)$ and $N_\omega^{(k)} = (T - \omega)|\ker(T - \omega)^{k+1}$, $\tilde N_\omega^{(k)} = (\tilde T - \omega)|\ker(\tilde T - \omega)^{k+1}$. Then t and $\tilde t$ have contact of order k if and only if $N_\omega^{(k)}$ and $\tilde N_\omega^{(k)}$ are unitarily equivalent for each ω in Ω.*

Proof. Assume t and $\tilde t$ have contact of order k on Ω. For ω_0 in Ω there exists a unitary U on $\mathcal{H}$, holomorphic spanning cross-sections $\gamma_1, \ldots, \gamma_n$ for $\ker(T - \omega_0)$ and holomorphic spanning cross-sections $\tilde\gamma_1, \ldots, \tilde\gamma_n$ for $\ker(\tilde T - \omega_0)$, such that

$$U(\gamma_i^{(j)}(\omega_0)) = \tilde\gamma_i^{(j)}(\omega_0) \quad \text{for } i = 1, 2, \ldots, n \text{ and } 0 \leqslant j \leqslant k.$$

By Lemma 1.22 we have

$$UN_{\omega_0}^{(k)}(\gamma_i^{(j)}(\omega_0)) = U(j\gamma_i^{(j-1)}(\omega_0)) = \tilde N_{\omega_0}^{(k)} U(\gamma_i^{(j)}(\omega_0))$$

and hence $UN_{\omega_0}^{(k)} = \tilde N_{\omega_0}^{(k)} U$. Since ω_0 is an arbitrary point in Ω, this completes one half of the proof.

Conversely, suppose $N_\omega^{(k)}$ and $\tilde N_\omega^{(k)}$ are unitarily equivalent for each ω in Ω. Moreover, let U be a unitary defined on $\mathcal{H}$ such that $U\{\ker(T - \omega_0)^{k+1}\} = \ker(\tilde T - \omega_0)^{k+1}$ and satisfying $UN^{(k)}_{\omega_0}=\tilde{N}^{(k)}_{\omega_0}U$. If $\gamma_1(\omega_0), \ldots, \gamma_n(\omega_0)$ is an orthonormal basis for $\ker(T-\omega_0)$, then $\tilde{\gamma}_i(\omega_0)=U\gamma_i(\omega_0)$ for $i=1, 2, \ldots, n$ defines an orthornormal basis for $\ker(\tilde{T}-\omega_0)$. Moreover, using the preceding lemma we can choose holomorphic cross-sections $\gamma_1, \gamma_2, \ldots, \gamma_n$ for E_T and $\tilde{\gamma}_1, \tilde{\gamma}_2, \ldots, \tilde{\gamma}_n$ for $E_{\tilde{T}}$ defined on some open set Δ containing ω_0, and satisfying

$$(2.5.1) \qquad (\gamma_i^{(j)}(\omega_0), \gamma_l(\omega_0)) = (\tilde{\gamma}_i^{(j)}(\omega_0), \tilde{\gamma}_l(\omega_0)) = 0$$

for $1 \leqslant i$, $l \leqslant n$ and $j \geqslant 1$. We claim that

$$(2.5.2) \qquad U(\gamma_i^{(j)}(\omega_0)) = \tilde{\gamma}_i^{(j)}(\omega_0) \quad \text{for } i=1, 2, \ldots, n \text{ and } 0 \leqslant j \leqslant k.$$

Statement (2.5.2) is valid for $j=0$ and we assume it holds for some $j<k$. Then

$$\tilde{N}^{(k)}_{\omega_0}\{U\gamma_i^{(j+1)}(\omega_0) - \tilde{\gamma}_i^{(j+1)}(\omega_0)\} = U\{(j+1)\gamma_i^{(j)}(\omega_0)\} - (j+1)\tilde{\gamma}_i^{(j)}(\omega_0) = 0$$

by hypothesis which implies $U\gamma_i^{(j+1)}(\omega_0) - \tilde{\gamma}_i^{(j+1)}(\omega_0)$ is in $\ker(\tilde{T}-\omega_0)$. But $\ker(\tilde{T}-\omega_0)$ is spanned by $\tilde{\gamma}_1(\omega_0), \ldots, \tilde{\gamma}_n(\omega_0)$, while

$$(U(\gamma_i^{(j+1)}(\omega_0)) - \tilde{\gamma}_i^{(j+1)}(\omega_0), \tilde{\gamma}_l(\omega_0)) = (U(\gamma_i^{(j+1)}(\omega_0)), U\gamma_l(\omega_0)) - (\tilde{\gamma}_i^{(j+1)}(\omega_0), \tilde{\gamma}_l(\omega_0)) = 0$$

since U is unitary and (2.5.1) holds. Therefore (2.5.2) holds for $j+1$ which completes the proof since ω_0 is an arbitrary point in Ω.

2.6. Thus the proof of Theorem 1.6 is reduced to showing that pullback bundles which agree to sufficiently high order of contact are equivalent. Unfortunately, we are unable to prove this without involving the geometry already implicit in this context. The reason is that our analysis of two Hermitian holomorphic vector bundles to decide whether or not they are equivalent takes us out of the category. In particular, we consider non-holomorphic sub-bundles. Hence we show that the equivalence problem for Hermitian holomorphic vector bundles can be replaced by a standard equivalence problem in differential geometry which we then proceed to solve.

2.7. Let E and $\tilde{E}$ be n-dimensional Hermitian holomorphic vector bundles over the connected open subset Ω of $\mathbf{C}$. We are interested in finding invariants for E and $\tilde{E}$ which will determine, at least locally, when the bundles are isometrically and holomorphically equivalent. We do this by making explicit the geometry already implicit in this context. By geometry we are referring to vector bundles equipped with a compatible connection which then allows one to compare vectors in fibres over different points of Ω. In general, there exist many connections on a Hermitian bundle which are compatible with the metric. However, for Hermitian holomorphic bundles there is a natural or a canonical choice. We briefly describe this in our context. For more details on these matters the reader is advised to consult [26, Chap. III], whose notation we shall follow.

2.8. For Ω a connected open subset of $\mathbf{C}$ let z and $\bar{z}$ denote the co-ordinate functions defined by $z=x+iy$ and $\bar{z}=x-iy$. If $T^*(\Omega)$ denotes the co-tangent bundle of Ω, then a basis for each fibre is given by dz and $d\bar{z}$. Thus a basis for the fibre of the exterior algebra bundle $\Lambda^p(T^*(\Omega))$ is 1 for $\Lambda^0(T^*(\Omega))$, dz and $d\bar{z}$ for $\Lambda^1(T^*(\Omega))$, and $dz d\bar{z}$ for $\Lambda^2(T^*(\Omega))$. We let $\mathcal{E}(\Omega)$ denote the algebra of complex-valued C^∞ functions on Ω and $\mathcal{O}(\Omega)$ the subalgebra of holomorphic functions on Ω. For F a C^∞ vector bundle over Ω, we let $\mathcal{E}(\Omega, F)$ denote the C^∞ cross-sections of F and if F is holomorphic, then $\mathcal{O}(\Omega, F)$ denotes the holomorphic cross-sections. We let $\mathcal{E}^p(\Omega)=\mathcal{E}(\Omega, \Lambda^p(T^*(\Omega)))$ denote the C^∞ differential forms of degree p on Ω. Thus we have $\mathcal{E}^0(\Omega)=\mathcal{E}(\Omega)$, $\mathcal{E}^1(\Omega)=\{f dz+g d\bar{z}: f, g\in\mathcal{E}(\Omega)\}$ and $\mathcal{E}^2(\Omega)=\{f dz d\bar{z}: f\in\mathcal{E}(\Omega)\}$. We further decompose $\mathcal{E}^1(\Omega)=\mathcal{E}^{1,0}(\Omega)\oplus\mathcal{E}^{0,1}(\Omega)$ into (1, 0)- and (0, 1)-forms such that $\mathcal{E}^{1,0}(\Omega)=\{f dz: f\in\mathcal{E}(\Omega)\}$ and $\mathcal{E}^{0,1}(\Omega)=\{g d\bar{z}: g\in\mathcal{E}(\Omega)\}$ and set $\mathcal{E}^{0,0}(\Omega)=\mathcal{E}(\Omega)$, $\mathcal{E}^{1,1}(\Omega)=\mathcal{E}^2(\Omega)$, and $\mathcal{E}^{2,0}(\Omega)=\mathcal{E}^{0,2}(\Omega)=0$. Then the exterior derivative d: $\mathcal{E}^p(\Omega)\to\mathcal{E}^{p+1}(\Omega)$ defined for $p=0, 1, 2$ can be decomposed to obtain ∂: $\mathcal{E}^{p,q}(\Omega)\to\mathcal{E}^{p+1,q}(\Omega)$ and $\bar{\partial}$: $\mathcal{E}^{p,q}(\Omega)\to\mathcal{E}^{p,q+1}(\Omega)$. Observe that f in $\mathcal{E}(\Omega)$ is holomorphic if and only if $\bar{\partial}f=0$.

Now for F a C^∞ vector bundle over Ω let $\mathcal{E}^p(\Omega, F)=\mathcal{E}(\Omega, \Lambda^p(T^*(\Omega))\otimes F)$ be the differential forms of degree p on Ω with coefficients in F, that is, ξ is in $\mathcal{E}^p(\Omega, F)$ if it can be written $\xi=\sum_{i=1}^k \eta_i\otimes\sigma_i$ for $\eta_1, \ldots, \eta_k$ in $\Lambda^p(T^*(\Omega))$ and $\sigma_1, \ldots, \sigma_k$ in $\mathcal{E}(\Omega, F)$. Again $\mathcal{E}^0(\Omega, F)=\mathcal{E}(\Omega, F)$, $\mathcal{E}^1(\Omega, F)=\{dz\otimes\sigma_1+d\bar{z}\otimes\sigma_2: \sigma_1, \sigma_2\in\mathcal{E}(\Omega, F)\}$, and $\mathcal{E}^2(\Omega, F)=\{dz d\bar{z}\otimes\sigma: \sigma\in\mathcal{E}(\Omega, F)\}$. Using the Hermitian structure on F we can define a map $\mathcal{E}^p(\Omega, F)\times\mathcal{E}^q(\Omega, F)\to\mathcal{E}^{p+q}(\Omega)$ such that, for example for ξ in $\mathcal{E}^p(\Omega)$, η in $\mathcal{E}^q(\Omega)$ and σ_1, σ_2 in $\mathcal{E}(\Omega, F)$ we have

$$(\xi\otimes\sigma_1, \eta\otimes\sigma_2)(\omega)=\xi(\omega)\wedge\overline{\eta(\omega)}(\sigma_1(\omega), \sigma_2(\omega)), \tag{2.8.1}$$

where the bar denotes complex conjugation of forms which satisfies

$$\begin{cases} \overline{dz}=(\overline{dx+idy})=dx-idy=d\bar{z} \\ \overline{d\bar{z}}=(\overline{dx-idy})=dx+idy=dz \\ \overline{dz d\bar{z}}=d\bar{z}dz=-dz d\bar{z} \end{cases} \tag{2.8.2}$$

2.9. Let E be a Hermitian holomorphic bundle over Ω. A *connection* on E is a first-order differential operator D: $\mathcal{E}^0(\Omega, E)\to\mathcal{E}^1(\Omega, E)$ such that

$$D(f\sigma)=df\otimes\sigma+fD\sigma \quad \text{for } f \text{ in } \mathcal{E}(\Omega) \text{ and } \sigma \text{ in } \mathcal{E}^0(\Omega, E). \tag{2.9.1}$$

On any Hermitian holomorphic vector bundle E over Ω there is a unique *canonical connection* D which preserves both the Hermitian and holomorphic structures, that is, such that

(2.9.2) D is *metric-preserving* or $d(\sigma, \tilde{\sigma})=(D\sigma, \tilde{\sigma})+(\sigma, D\tilde{\sigma})$

for σ and $\tilde{\sigma}$ in $\mathcal{E}^0(\Omega, E)$, and

$$D''\sigma = 0 \quad \text{for } \sigma \text{ in } \mathcal{O}(\Omega, E), \text{ where } D = D' + D'' \tag{2.9.3}$$

is the decomposition of D into (1, 0)- and (0, 1)-form valued operators, or equivalently *D'' extends $\bar{\partial}$.*

Since all holomorphic bundles over Ω are holomorphically trivial by Grauert's Theorem [12], there exist cross-sections $\sigma_1, \ldots, \sigma_n$ in $\mathcal{O}(\Omega, E)$ which form a frame for E on Ω. The metric on E defines a positive-definite $n \times n$ matrix function $\{h_{ij}(\omega)\} = (\sigma_j(\omega), \sigma_i(\omega))$ which depends on $\{\sigma_i\}_{i=1}^n$. If we define $\theta = h^{-1}\partial h$ or $\theta_{ij}(\omega) = \sum_{k=1}^n g_{ik}(\omega)(\partial h_{kj}/\partial z)dz$ where $g_{ik}(\omega) = (h(\omega)^{-1})_{ik}$, then θ is the matrix of connection 1-forms for E and the canonical connection D is defined by

$$D\left(\sum_{i=1}^n f_i \sigma_i\right) = \sum_{i=1}^n df_i \otimes \sigma_i + \sum_{i=1}^n \sum_{j=1}^n f_i \theta_{ji} \sigma_j \quad \text{for } \sum_{i=1}^n f_i \sigma_i \text{ in } \mathcal{E}^0(\Omega, E). \tag{2.9.4}$$

Now although D is $\mathbb{C}$-linear, it is not $\mathcal{E}(\Omega)$-linear and hence is not a bundle map. However, the commutator of D with a bundle map is also a bundle map. Before proving this we observe that if φ is a bundle map between bundles E and $\tilde{E}$ over Ω, then φ induces a bundle map from $E \otimes \Lambda^p(T^*(\Omega))$ to $\tilde{E} \otimes \Lambda^p(T^*(\Omega))$ which we also denote by φ.

LEMMA 2.10. *Let E and $\tilde{E}$ be C^∞ vector bundles over Ω with connections D and $\tilde{D}$, respectively. If φ is a C^∞ bundle map from E to $\tilde{E}$, then $\tilde{D}\varphi - \varphi D$ is a C^∞ bundle map from E to $\tilde{E} \otimes T^*(\Omega)$.*

Proof. It is enough to prove that $\tilde{D}\varphi - \varphi D$ is $\mathcal{E}(\Omega)$-linear, or equivalently that

$$(\tilde{D}\varphi - \varphi D)(f\eta) = f[(\tilde{D}\varphi - \varphi D)\eta] \quad \text{for } f \text{ in } \mathcal{E}(\Omega) \quad \text{and } \eta \text{ in } \mathcal{E}^0(\Omega, E).$$

Using (2.9.1) we obtain

$$\begin{aligned}(\tilde{D}\varphi - \varphi D)(f\eta) &= \tilde{D}(f(\varphi\eta)) - \varphi D(f\eta) \\ &= df \otimes \varphi\eta + f\tilde{D}(\varphi\eta) - \varphi(df \otimes \eta) - \varphi(fD\eta) \\ &= f\tilde{D}(\varphi\eta) - f\varphi(D\eta) = f[\tilde{D}\varphi - \varphi D]\eta\end{aligned}$$

which completes the proof.

We use this lemma to define partial derivatives of bundle maps in this context.

Definition 2.11. For C^∞ bundles E and $\tilde{E}$ over Ω with connections D, $\tilde{D}$ and φ a C^∞ bundle map from E to $\tilde{E}$ we define the covariant partial derivatives φ_z and $\varphi_{\bar{z}}$ to be the bundle maps from E to $\tilde{E}$ defined by

$$\tilde{D}\varphi - \varphi D = \varphi_z \otimes dz + \varphi_{\bar{z}} \otimes d\bar{z}.$$

Moreover, we define $\varphi_{z^r} = (\varphi_{z^{r-1}})_z$, $\varphi_{\bar{z}^s} = (\varphi_{\bar{z}^{s-1}})_{\bar{z}}$ and $\varphi_{z^r\bar{z}^s} = (\varphi_{z^r})_{\bar{z}^s}$ for $1 \leq r, s$. Note that we include the possibility that $E = \tilde{E}$, $D = \tilde{D}$. Further φ_z is always relative to the connections on the domain and range of φ, which should be clear from the context.

If E, $\tilde{E}$, and $\tilde{\tilde{E}}$ are C^∞ bundles with connections D, $\tilde{D}$, and $\tilde{\tilde{D}}$, and $\varphi\colon E \to \tilde{E}$ and $\psi\colon \tilde{E} \to \tilde{\tilde{E}}$ are bundle maps, then

$$\tilde{\tilde{D}}(\psi \circ \varphi) - (\psi \circ \varphi) D = (\tilde{\tilde{D}}\psi - \psi\tilde{D})\varphi + \psi(\tilde{D}\varphi - \varphi D)$$

which gives the Leibnitz rule

(2.11.1) $$(\psi\varphi)_z = \psi_z\varphi + \psi\varphi_z \quad \text{and } (\psi\varphi)_{\bar{z}} = \psi_{\bar{z}}\varphi + \psi\varphi_{\bar{z}}.$$

If E and $\tilde{E}$ are Hermitian holomorphic vector bundles and D and $\tilde{D}$ are the canonical connections, then the matrix for $\varphi_{\bar{z}}$ relative to holomorphic frames is just the usual $\bar{z}$-derivative of the matrix of φ relative to the same frames. Moreover there is a simple relation between our z- and $\bar{z}$-partial derivatives.

Lemma 2.12. *If E and $\tilde{E}$ are C^∞ Hermitian vector bundles, D and $\tilde{D}$ are metric preserving connections (2.9.2) and φ is a bundle map from E to $\tilde{E}$, then $(\varphi_z)^* = (\varphi^*)_{\bar{z}}$ and $(\varphi_{\bar{z}})^* = (\varphi^*)_z$.*

Proof. For σ and $\tilde{\sigma}$ in $\mathcal{E}(\Omega, E)$ and $\mathcal{E}(\Omega, \tilde{E})$, respectively we have using (2.8.1) and (2.9.2) that

(2.12.1) $$\begin{aligned}((\tilde{D}\varphi - \varphi D)\sigma, \tilde{\sigma}) - (\sigma, (D\varphi^* - \varphi^*\tilde{D})\tilde{\sigma}) \\ = [(\tilde{D}\varphi\sigma, \tilde{\sigma}) + (\varphi\sigma, \tilde{D}\tilde{\sigma})] - [(D\sigma, \varphi^*\tilde{\sigma}) + (\sigma, D\varphi^*\tilde{\sigma})] \\ = d[(\varphi\sigma, \tilde{\sigma}) - (\sigma, \varphi^*\tilde{\sigma})] = 0.\end{aligned}$$

Therefore appealing to Definition 2.11 we have

$$(\varphi_z\sigma, \tilde{\sigma})\,dz + (\varphi_{\bar{z}}\sigma, \tilde{\sigma})\,d\bar{z} = (\sigma, (\varphi^*)_{\bar{z}}\tilde{\sigma})\,dz + (\sigma, (\varphi^*)_z\tilde{\sigma})\,d\bar{z}$$

and hence $(\varphi_z)^* = (\varphi^*)_{\bar{z}}$ and $(\varphi_{\bar{z}})^* = (\varphi^*)_z$.

We can now show that the equivalence problem for Hermitian holomorphic bundles equipped with this canonical connection is just the standard equivalence problem in differential geometry. Although this is well-known, we include a proof for completeness.

Lemma 2.13. *Let E and $\tilde{E}$ be Hermitian holomorphic vector bundles over Ω with the canonical connections D and $\tilde{D}$, respectively, and let $\varphi\colon E \to \tilde{E}$ be a C^∞ isometric bundle map. Then φ is holomorphic if and only if φ is connection-preserving, that is, if and only if*

(2.13.1) $$\tilde{D} \circ \varphi = \varphi \circ D.$$

Proof. The bundle map φ is holomorphic if and only $\varphi_{\bar{z}}=0$. Analogously, φ holomorphic implies $\varphi^*=\varphi^{-1}$ is holomorphic and hence $(\varphi_z)^*=(\varphi^*)_{\bar{z}}=0$. Thus $\tilde{D}\varphi-\varphi D=\varphi_z dz+\varphi_{\bar{z}}d\bar{z}=0$ or φ is connection-preserving. The converse follows similarly.

Thus by the lemma we may generalize the equivalence problem for Hermitian holomorphic vector bundles as follows:

Let E and $\tilde{E}$ be n-dimensional C^∞-Hermitian vector bundles over Ω with metric-preserving connections D and $\tilde{D}$. We wish to find invariants which will determine when there exists a connection-preserving isometric bundle map between E and $\tilde{E}$.

Remark 2.14. If there is a connection-preserving isometry φ, it is unique up to the action of the group $\mathcal{J}(E)$ of connection-preserving isometries of E, and $\mathcal{J}(E)$ may be identified with a Lie-subgroup of $U(n, \mathbf{C})$. In particular, if E and $\tilde{E}$ are irreducible as Hermitian vector bundles with connection, then φ is unique up to multiplication by a scalar of absolute value one.

2.15. The connection D: $\mathcal{E}^0(\Omega, E)\to\mathcal{E}^1(\Omega, E)$ can be extended in a natural way as a derivation of $\mathcal{E}^p(\Omega, E)$ to $\mathcal{E}^{p+1}(\Omega, E)$ for $p=0, 1, 2$ so that

$$D(\sigma\otimes\alpha) = D\sigma\otimes\alpha+\sigma\otimes d\alpha \quad \text{for } \sigma \text{ in } \mathcal{E}(\Omega, E) \text{ and } \alpha \text{ in } \mathcal{E}^p(\Omega). \tag{2.15.1}$$

One checks easily that $D^2(f\sigma)=f(D^2\sigma)$ for σ in $\mathcal{E}(E)$ and f in $\mathcal{E}(\Omega)$. Thus D^2 is a bundle map from E to $E\otimes\Lambda^2(T^*(\Omega))$ and we define the curvature $K(E, D)$ as the C^∞ cross-section of Hom $(E, E\otimes\Lambda^2(T^*(\Omega)))$ by

$$K = K(E, D) = D^2. \tag{2.15.2}$$

It is also useful to define a related "curvature" $\mathcal{K}$ (more precisely $\mathcal{K}(E, D)$) as a C^∞ cross-section of Hom (E, E) such that

$$K\sigma = \mathcal{K}\sigma dz d\bar{z} \quad \text{for } \sigma \text{ in } \mathcal{E}(\Omega, E). \tag{2.15.3}$$

Using (2.8.1), (2.9.2), and (2.15.1) we obtain

$$\begin{aligned}(D^2\sigma, \tilde{\sigma})+(\sigma, D^2\tilde{\sigma}) &= (D\sigma, D\tilde{\sigma})+d(D\sigma, \tilde{\sigma})-(D\sigma, D\tilde{\sigma})+d(\sigma, D\tilde{\sigma})\\ &= d^2(\sigma, \tilde{\sigma}) = 0 \quad \text{for } \sigma, \tilde{\sigma} \text{ in } \mathcal{E}(\Omega, E).\end{aligned}$$

Therefore we have

$$(\mathcal{K}\sigma, \tilde{\sigma})dzd\bar{z} = (K\sigma, \tilde{\sigma}) = -(\sigma, K\tilde{\sigma}) = -(\sigma, \mathcal{K}\tilde{\sigma})d\bar{z}dz = (\sigma, \mathcal{K}\tilde{\sigma})dzd\bar{z} \tag{2.15.4}$$

and hence $\mathcal{K}$ is self-adjoint.

An identity involving the curvature and partial derivatives is

$$(2.15.5)\qquad [\mathcal{K}, \varphi] = (\varphi_{\bar z})_z - (\varphi_z)_{\bar z} \quad \text{for } \varphi\colon E \to E \text{ a bundle map.}$$

To verify this we write

$$\begin{aligned}\{(\varphi_{\bar z})_z - (\varphi_z)_{\bar z}\}\, dz\, d\bar z &= [D', \varphi_{\bar z}]\, d\bar z + [D'', \varphi_z]\, dz \\ &= D'[D'', \varphi] + [D'', \varphi] D' + D''[D', \varphi] + [D', \varphi] D'' \\ &= [D'D'' + D''D', \varphi] = [K, \varphi] = [\mathcal{K}, \varphi]\, dz\, d\bar z\end{aligned}$$

where $(D')^2 = (D'')^2 = 0$ since there are no non-zero (2, 0)- or (0, 2)-forms on Ω.

Remark. 2.16. If E and $\tilde E$ are C^∞ vector bundles over Ω with connections D and $\tilde D$ and $\varphi\colon E \to \tilde E$ is a connection preserving bundle map (2.13.1) then $\tilde D^2 \varphi = \varphi D^2$, so $\tilde{\mathcal{K}} \varphi = \varphi \mathcal{K}$. Thus $[\tilde D, \tilde{\mathcal{K}}]\varphi = \varphi[D, \mathcal{K}]$, which implies (2.11) that $\tilde{\mathcal{K}}_z \varphi = \varphi \mathcal{K}_z$, $\tilde{\mathcal{K}}_{\bar z} \varphi = \varphi \mathcal{K}_{\bar z}$. Continuing in this fashion, we clearly have

$$(2.16.1)\qquad \tilde{\mathcal{K}}_{z^i \bar z^j} \varphi = \varphi \mathcal{K}_{z^i \bar z^j}, \quad \text{all } 0 \leqslant i, j < \infty.$$

We can now state our definition of local equivalence.

Definition 2.17. Given n-dimensional Hermitian vector bundles E and $\tilde E$ over Ω with metric-preserving connections D and $\tilde D$, we say that E and $\tilde E$ are *equivalent to order k at ω* in Ω for some $k \geqslant 1$ if and only if there exists an isometry V from E_ω onto $\tilde E_\omega$ such that

$$(2.17.1)\qquad V \mathcal{K}_{z^i \bar z^j} = \tilde{\mathcal{K}}_{z^i \bar z^j} V \quad \text{for } 0 \leqslant i, j \leqslant i + j \leqslant k,\ (i, j) \neq (0, k), (k, 0).$$

Note that since $\mathcal{K}$ and $\tilde{\mathcal{K}}$ are self-adjoint, by Lemma 2.12 it suffices for (2.17.1) to hold for $i \leqslant j$.

In the next section we show there exists $k \leqslant n$ such that if E and $\tilde E$ are equivalent to order k at each ω in Ω, then E and $\tilde E$ are locally equivalent off some closed nowhere dense subset of Ω and that k is 2 in the generic case. Before beginning the proof of this, however, we connect this notion to that of finite order of contact we introduced for holomorphic curves earlier in this chapter.

PROPOSITION 2.18. *Let f and $\tilde f$ be holomorphic maps from Ω into $Gr(n, \mathcal{H})$ and E_f and $E_{\tilde f}$ be the associated Hermitian holomorphic vector bundles with canonical connections D and $\tilde D$, respectively. The holomorphic curves f and $\tilde f$ have contact of order k at ω_0 in Ω if and only if there exists an isometry $V\colon f(\omega_0) \to \tilde f(\omega_0)$ such that*

$$(2.18.1)\qquad V \mathcal{K}_{z^p \bar z^q}(\omega_0) = \tilde{\mathcal{K}}_{z^p \bar z^q}(\omega_0) V \quad \text{for } 0 \leqslant p, q \leqslant k - 1.$$

Proof. Let f and $\tilde{f}$ be holomorphic curves over Ω having contact of order $k \geq 1$ at ω_0 and let $\{\sigma_1, \ldots, \sigma_n\}$ and $\{\tilde{\sigma}_1, \ldots, \tilde{\sigma}_n\}$ be frames for E_f and $E_{\tilde{f}}$ and U a unitary operator on $\mathcal{H}$ such that

$$U(\sigma_i^{(j)}(\omega_0)) = \tilde{\sigma}_i^{(j)}(\omega_0) \quad \text{for } 0 \leq j \leq k. \tag{2.18.2}$$

Obviously, we can assume that $\sigma_1(\omega_0), \ldots, \sigma_n(\omega_0)$ are orthonormal which in turn implies that $\tilde{\sigma}_1(\omega_0), \ldots, \tilde{\sigma}_n(\omega_0)$ are orthonormal. Let $h = ((\sigma_j, \sigma_i))$ and $\tilde{h} = ((\tilde{\sigma}_j, \tilde{\sigma}_i))$ be the matrix of inner products for E_f and $E_{\tilde{f}}$, respectively. Note that $h(\omega_0) = I = \tilde{h}(\omega_0)$. Further, for φ a bundle map on E_f let $\varphi(\sigma)$ denote the matrix for φ with respect to the frame $\sigma_1, \ldots, \sigma_n$. By 2.15.2, 2.9.4 and 2.15.1 we have

$$\begin{aligned} K(\sigma) = K(E_f, D)(\sigma) &= \theta \wedge \theta + d\theta \\ &= h^{-1}\partial h\, h^{-1}\partial h + \partial(h^{-1}\partial h) + \bar{\partial}(h^{-1}\partial h) \\ &= \bar{\partial}(h^{-1}\partial h) \end{aligned}$$

which implies

$$\mathcal{K}(\sigma) = -\frac{\partial}{\partial \bar{z}}\left(h^{-1}\frac{\partial h}{\partial z}\right) \quad \text{for all } \omega \text{ in } \Omega. \tag{2.18.3}$$

Moreover, from Definition 2.11 it follows that for any C^∞ bundle map φ on E we have

$$\begin{cases} \varphi_{\bar{z}}(\sigma) = \dfrac{\partial}{\partial \bar{z}}(\varphi(\sigma)) \\[2ex] \varphi_z(\sigma) = \dfrac{\partial}{\partial z}(\varphi(\sigma)) + \left[h^{-1}\dfrac{\partial h}{\partial z}, \varphi(\sigma)\right]. \end{cases} \tag{2.18.4}$$

Thus by (2.18.3) and (2.18.4) we can express $\mathcal{K}_{z^p\bar{z}^q}(\sigma)$ in terms of h^{-1} and $\partial^{r+s}h/\partial z^r \partial \bar{z}^s$ for $0 \leq r \leq p+1$ and $0 \leq s \leq q+1$. Similarly, we can express $\tilde{\mathcal{K}}_{z^p\bar{z}^q}(\tilde{\sigma})$ in terms of $\tilde{h}^{-1}$ and $\partial^{r+s}\tilde{h}/\partial z^r \partial \bar{z}^s$. Since

$$\frac{\partial^{r+s}h}{\partial z^r \partial \bar{z}^s}(\omega_0) = (\sigma_i^{(r)}(\omega_0), \sigma_j^{(s)}(\omega_0)) = (\tilde{\sigma}_i^{(r)}(\omega_0), \tilde{\sigma}_j^{(s)}(\omega_0)) = \frac{\partial^{r+s}\tilde{h}}{\partial z^r \partial \bar{z}^s}(\omega_0)$$

for $0 \leq r$, $s \leq k$ by (2.18.2), this implies $\mathcal{K}_{z^p\bar{z}^q}(\sigma) = \tilde{\mathcal{K}}_{z^p\bar{z}^q}(\sigma)$ at ω, for $0 \leq p$, $q \leq k-1$. But $\mathcal{K}_{z^p\bar{z}^q}$ and $\tilde{\mathcal{K}}_{z^p\bar{z}^q}$ are bundle maps and hence (2.18.1) holds where $V\colon f(\omega_0) \to \tilde{f}(\omega_0)$ is defined by $Vx = Ux$ for x in $f(\omega_0)$.

Now assume that V is an isometry defined from $f(\omega_0)$ to $\tilde{f}(\omega_0)$ satisfying (2.18.1). If $\sigma_1, \ldots, \sigma_n$ is any frame for E_f, then from (2.18.3) we obtain

$$\frac{\partial^2 h}{\partial z \partial \bar{z}} = \frac{\partial h}{\partial \bar{z}} h^{-1} \frac{\partial h}{\partial z} - h\mathcal{K}(\sigma). \tag{2.18.5}$$

We claim that $\partial^{r+s}h/\partial z^r\partial\bar{z}^s$ can be expressed in terms of h^{-1}, $\partial^i h/\partial z^i$ for $0\leqslant i\leqslant r$, $\partial^j h/\partial\bar{z}^j$ for $0\leqslant j\leqslant s$ and $\mathcal{K}_{z^p\bar{z}^q}(\sigma)$ for $0\leqslant p\leqslant r-1$ and $0\leqslant q\leqslant s-1$. This is trivially valid for $r+s\leqslant 1$. Assume it is true for $r+s\leqslant m$. Then if $r+s=m$, $\partial^{m+1}h/\partial z^{r+1}\partial\bar{z}^s=\partial/\partial z\{\partial^m h/\partial z^r\partial\bar{z}^s\}$ can be expressed in the specified terms, except possibly when $s=m$. But $\partial^{m+1}h/\partial z\partial\bar{z}^m=\partial^{m-1}/\partial\bar{z}^{m-1}(\partial^2 h/\partial z\partial\bar{z})$, which by (2.18.5) and the induction hypothesis can be expressed in the specified terms. Similarly for $\partial^{m+1}h/\partial z^r\partial z^{s+1}$, and hence the claim is true for $r+s\leqslant m+1$.

If (2.18.1) holds, then we can choose frames $\sigma_1,\ldots,\sigma_n$ for E_f and $\tilde{\sigma}_1,\ldots,\tilde{\sigma}_n$ for $E_{\tilde{f}}$ such that the $\sigma_1(\omega_0),\ldots,\sigma_n(\omega_0)$ and the $\tilde{\sigma}_1(\omega_0),\ldots,\tilde{\sigma}_n(\omega_0)$ are orthonormal and

$$\mathcal{K}_{z^p\bar{z}^q}(\sigma)=\tilde{\mathcal{K}}_{z^p\bar{z}^q}(\tilde{\sigma})\quad \text{at } \omega_0 \text{ for } 0\leqslant p,\, q\leqslant k-1. \tag{2.18.6}$$

We can further normalize the σ_i and $\tilde{\sigma}_i$ as in Lemma 2.4, so that

$$(\sigma_j^{(l)}(\omega_0),\sigma_i(\omega_0))=0=(\tilde{\sigma}_j^{(l)}(\omega_0),\tilde{\sigma}_i(\omega_0))\quad \text{for } l\geqslant 1.$$

Note that (2.18.6) still holds because the σ and $\tilde{\sigma}$ are unchanged at ω_0. But since $\partial^i h/\partial z^i=\partial^i h/\partial\bar{z}^i=0$ at ω_0 for this frame, we can express $\partial^{r+s}h/\partial z^r\bar{z}^s(\omega_0)$ in terms of h^{-1} and $\mathcal{K}_{z^p\bar{z}^q}(\sigma)$ at ω_0 for $0\leqslant p\leqslant r-1$ and $0\leqslant q\leqslant s-1$. Therefore we have $(\sigma_j^{(p)}(\omega_0),\sigma_i^{(q)}(\omega_0))=\partial^{p+q}h/\partial z^p\partial\bar{z}^q(\omega_0)=\partial^{p+q}\tilde{h}/\partial z^p\partial\bar{z}^q(\omega_0)=(\tilde{\sigma}_j^{(p)}(\omega_0),\tilde{\sigma}_i^{(q)}(\omega_0))$ for $0\leqslant p,\, q\leqslant k$ which implies f and $\tilde{f}$ have contact of order k at ω_0.

2.19. If T is in $\mathcal{B}_n(\Omega)$, then Propositions 2.5 and 2.18 show that there is some relationship among the $N_\omega^{(k)}$ and the $\mathcal{K}_{z^p\bar{z}^q}$, where $\mathcal{K}$ is the curvature of E_T. We shall see (§ 3) that the $\mathcal{K}_{z^p\bar{z}^q}$ are fairly tractable, but we have been unable to put the $N_\omega^{(k)}$ into any reasonable canonical form. To illustrate the difficulties in going from the $\mathcal{K}_{z^p\bar{z}^q}$ to $N_\omega^{(k)}$ we give the following generalization of Theorem 1.17 which shows that the relationship between $\mathcal{K}(\omega)$ and $N_\omega^{(1)}$ is non-linear.

PROPOSITION 2.20. *Let T be in $\mathcal{B}_n(\Omega)$ and let $\mathcal{K}$ be the "curvature" of the bundle E_T with the canonical connection. Fix ω_0 in Ω, and let*

$$\lambda_1\leqslant\ldots\leqslant\lambda_n$$

be the eigenvalues of $\mathcal{K}(\omega_0)$. Then the λ_i are strictly negative and there is an orthonormal basis of $\ker(T-\omega_0)^2$ *such that relative to this basis $N_{\omega_0}^{(1)}$ has the matrix*

$$\left(\begin{array}{c|c} 0 & \begin{matrix}\mu_1 & & 0\\ & \ddots & \\ 0 & & \mu_n\end{matrix} \\ \hline 0 & 0\end{array}\right)$$

where $\mu_i=(-\lambda_i)^{-\frac{1}{2}}$.

Proof. Since $\mathcal{K}(\omega_0)$ is self-adjoint, let $v_1, \ldots, v_n$ be an orthonormal basis of $(E_T)_{\omega_0}$ such that $\mathcal{K}(\omega_0)(v_i)=\lambda_i v_i$, $1\leqslant i\leqslant n$. Let $\sigma_1, \ldots, \sigma_n$ be holomorphic sections of E_T such that at ω_0, $\sigma_i(\omega_0)=v_i$ for all i. By Lemma 2.4 we can assume that $(\sigma_i^{(k)}(\omega_0), \sigma_j(\omega_0))=0$ for $1\leqslant i$, $j\leqslant n$ and $k=1, 2, \ldots$. By (2.18.5), *at* ω_0 we have

$$\mathcal{K}(\sigma)=-((\sigma_j'(\omega_0), \sigma_i'(\omega_0)))$$

which implies that the $\sigma_j'(\omega_0)$ are orthogonal, and $\|\sigma_j'(\omega_0)\|=(-\lambda_j)^{\frac{1}{2}}$. By Lemma 1.22, if we let $v_{i+k}=\sigma_i'(\omega_0)$, $i=1, \ldots, n$, then $v_1, \ldots, v_{2n}$ form a basis for $\ker(T-\omega_0)^2$, so λ_j is less than 0 for each i, and $N_{\omega_0}^{(1)}(v_i)=0$ and $N_{\omega_0}^{(1)}(v_{i+n})=v_i$ for $i=1, \ldots, n$. Thus relative to the orthonormal basis $v_1, \ldots, v_n, v_{n+1}/\|v_{n+1}\|, \ldots, v_{2n}/\|v_{2n}\|$ for $\ker(T-\omega_0)^2$, $N_{\omega_0}^{(1)}$ has the required form.

§ 3. Invariants of C^∞ hermitian vector bundles with metric preserving connections

3.1. Let E and $\tilde{E}$ be C^∞ Hermitian vector bundles of dimension n over the open subset Ω of $\mathbf{C}$, with metric preserving connections D and $\tilde{D}$. We show in this section that pointwise equivalence of E and $\tilde{E}$ over Ω to some finite order not greater than n determines local equivalence of E and $\tilde{E}$ at any point in the complement of a closed nowhere dense subset of Ω. One case where this is easy is when the scalarized "curvatures" $\mathcal{K}$ and $\tilde{\mathcal{K}}$ of E and $\tilde{E}$ are scalar multiples of the identity. This we call the 0-umbilic case. For the general case we first put E and $\tilde{E}$ into a canonical form, where we can reduce the equivalence problem to equivalence of direct sums of 0-umbilic bundles with some auxiliary conditions. We solve this problem, and relate the solution to equivalence of order k (Definition 2.17).

The basic idea of the proof is not difficult, but there are two complicating factors. One is the introduction of a certain closed nowhere dense subset of Ω, which is unavoidable in the case of C^∞ bundles. For the Hermitian holomorphic case with real analytic metric (which is all we need for the applications to operator theory) we show that local equivalence holds at every point of Ω. Nonetheless, the subset enters into the proof even in that case.

The other complicating factor is the indexing necessary to effect the reduction of a bundle to its canonical form, and the bookkeeping involved in obtaining the bound of n on the order. Indeed we show that the order of pointwise equivalence necessary for local equivalence depends on the canonical form of the bundles, and that *equivalence to order two* suffices for "generic" bundles.

Equivalence problems have of course been much studied in geometry. Classical arguments of Veblen [25] and E. Cartan [16] seem to indicate that equivalence to some finite order implies local equivalence in our situation, at least for the real analytic case.

However, we have been unable to find a reference which gives an upper bound on the order for equivalence of holomorphic Hermitian vector bundles. Furthermore, although our bound of n appears to be dictated by the geometry, we are unable to show that the bound is sharp. Examples which require equivalence to order n would necessarily be highly non-generic and thus presumably quite complicated.

We have not attempted to deal with global equivalence (but cf. (3.27)), which has a topological aspect. Our results are purely local and suffice for the applications to operator theory due to the uniqueness of analytic continuation. The proof also uses the one complex dimensional nature of Ω in an essential fashion. We hope to deal with domains in $\mathbf{C}^n$ at a later date (cf. § 4).

We begin with the case of 0-umbilic bundles or bundles for which the curvature is essentially a scalar function. What we need follows more or less directly from the fact that such a bundle is a flat bundle tensored with a line bundle. That is essentially the content of the following Lemma.

LEMMA 3.2. *If E is a Hermitian vector bundle over an open subset Ω of $\mathbf{C}$ with metric preserving connection D such that $\mathcal{K}(E, D)=\lambda I$, then there exists (locally) an orthonormal frame $\sigma_1, \ldots, \sigma_n$ for E over Ω and η in $\mathcal{E}^1(\Omega)$ such that $\eta=-\bar{\eta}$ and $D\sigma_i=\eta\sigma_i$, where the choice of η depends only on λ.*

Proof. Since $\lambda dz d\bar{z}=(2\lambda/i)dx dy$ is a closed, pure imaginary two-form, there exists a pure imaginary η in $\mathcal{E}^1(\Omega)$ such that $d\eta=\lambda dz d\bar{z}$. If $\tau_1, \ldots, \tau_n$ is an orthonormal frame for E over Ω, which is possible since E is trivial, then there exists a matrix $\theta=\{\theta_{ij}\}$ of connection one-forms such that $D\tau_j=\sum \theta_{ij}\tau_i$. We seek a C^∞ unitary matrix function $U=\{U_{ij}\}$ such that $\sigma_j=\sum U_{ij}\tau_i$ satisfies

$$\eta(\sum_i U_{ij}\tau_i)=\eta\sigma_j=D\sigma_j=D(\sum_i U_{ij}\tau_i)=\sum_i dU_{ij}\tau_i+\sum_k U_{kj}\theta_{ik}\tau_i$$

or equivalently such that

$$dU=-(\theta-\eta I)U.$$

If we define a new connection $\hat{D}$ on E by setting $\hat{D}\tau_j=\sum_i\hat{\theta}_{ij}\tau_i$, where $\hat{\theta}=\theta-\eta I$, then we seek U such that $dU=-\hat{\theta}U$ or equivalently such that $\hat{D}\sigma_j=0$. Since an easy computation shows that the matrix for $\hat{K}$ is given by $d\hat{\theta}+\hat{\theta}\wedge\hat{\theta}$ (cf. (2.18.3)), we have

$$\hat{K}=d\theta-d\eta I+\theta\wedge\theta=K-d\eta I=(\lambda dz d\bar{z}-d\eta)I=0,$$

and hence $\hat{D}$ is *flat*.

The existence of U follows from the Frobenius Theorem as follows (cf. [9, p. 102]):

Let L be the one-form on $\Omega \times M_n(\mathbf{C})$ defined by $L = dZ + \hat{\theta}(z)Z$. Then

$$dL = d\hat{\theta}Z - \hat{\theta}dZ = d\hat{\theta}Z + \hat{\theta} \wedge \hat{\theta}Z - \hat{\theta}L = -\hat{\theta}L$$

and hence by the Frobenius Theorem, there exists a C^∞ matrix function $A(z)$ with prescribed initial value at z_0 such that $Z = A$ is an integral submanifold of $L = 0$, that is, such that $dA = -\hat{\theta}A$. If $A(z_0) = I$, then we put $B = (A^*)^{-1}$, so that $B(z_0) = I$ and

$$dB = -A^{*-1}dA^*A^{*-1} = -(A^{-1}dA\,A^{-1})^* = (A^{-1}\hat{\theta})^* = -\hat{\theta}B,$$

since $\hat{\theta}$ is skew-adjoint (since $\hat{D}$ is metric preserving). Thus by uniqueness $B = A$, that is, A is unitary.

Thus we can find the desired U and defining $\sigma_j = \sum_i U_{ij}\tau_i$ completes the proof.

We now show that equivalence to first order at every point of Ω implies equivalence for 0-umbilic bundles. This is well known and follows in standard fashion from Lemma 3.2.

PROPOSITION 3.3. *Let E and $\tilde{E}$ be C^∞ n-dimensional Hermitian vector bundles over the open subset Ω of $\mathbf{C}$ with metric-preserving connections D and $\tilde{D}$. If E and $\tilde{E}$ are 0-umbilic, that is, if $\mathcal{K} = \lambda \cdot I$ and $\tilde{\mathcal{K}} = \tilde{\lambda} \cdot I$ for C^∞ real-valued functions λ and $\tilde{\lambda}$ on Ω, then there exists (locally) an isometric connection-preserving bundle map φ: $E \to \tilde{E}$ if and only if $\lambda = \tilde{\lambda}$ or equivalently, if and only if* trace $(\mathcal{K})$ = trace $(\tilde{\mathcal{K}})$.

Proof. If such a φ exists, then obviously $\lambda = \tilde{\lambda}$. Conversely, if both E and $\tilde{E}$ are 0-umbilic, then by the lemma there exist frames $\sigma_1, \ldots, \sigma_n$ and $\tilde{\sigma}_1, \ldots, \tilde{\sigma}_n$ for E and $\tilde{E}$ respectively such that $D\sigma_i = \eta\sigma_i$ and $\tilde{D}\tilde{\sigma}_i = \tilde{\eta}\tilde{\sigma}_i$, where $d\eta = \lambda dz d\bar{z}$ and $d\tilde{\eta} = \tilde{\lambda} dz d\bar{z}$. Further, if $\lambda = \tilde{\lambda}$, we can take $\eta = \tilde{\eta}$ and hence if we define φ: $E \to \tilde{E}$ such that $\varphi\sigma_i = \tilde{\sigma}_i$, then φ has the desired properties.

Note that by Definition 2.17, equivalence to first order means precisely that λ equals $\tilde{\lambda}$, in the 0-umbilic case. In the general case, E and $\tilde{E}$ are equivalent to first order if and only if $\mathcal{K}$ and $\tilde{\mathcal{K}}$ are unitarily equivalent at each point. By (2.15.4), $\mathcal{K}$ and $\tilde{\mathcal{K}}$ are self-adjoint. In view of Proposition 3.3, it seems very natural to diagonalize $\mathcal{K}$ and $\tilde{\mathcal{K}}$ and to investigate the decomposition of E and $\tilde{E}$ into eigen sub-bundles, with the hope that this decomposition will simplify the equivalence problem, and it does.

To analyze the function $\mathcal{K}$ we need some results on self-adjoint matrix functions (cf. [17]; [28], Chap. II, § 6):

LEMMA 3.4. *If H: $\Lambda \to M_n(\mathbf{C})$ is a C^∞ self-adjoint $n \times n$ matrix valued function defined on the open subset Λ of R^m, then there exists a closed nowhere dense subset Z_H of Λ such that if*

Λ_1 is a component of $\Lambda - Z_H$, then there exist C^∞ functions $\lambda_1 < \dots < \lambda_k$ and C^∞ (orthogonal) projection valued functions $P_1, \dots, P_k$ defined on Λ_1 such that

$$\lambda_1(\omega), \dots, \lambda_k(\omega) \tag{3.4.1}$$

are the distinct eigenvalues of $H(\omega)$ and $P_i(\omega)$ is the orthogonal projection onto the eigenspace of $H(\omega)$ for the eigenvalue $\lambda_i(\omega)$ for ω in Λ_1.

Proof. Let $f_\omega(z)$ be the characteristic polynomial of $H(\omega)$, k the maximal number of distinct roots of f_ω for ω in Λ, and $\tilde{\Lambda}$ the set of ω such that f_ω has k distinct roots: $\lambda_1(\omega) < \dots < \lambda_k(\omega)$. Fix ω_0 a point in $\tilde{\Lambda}$ and let m_j be the multiplicity of $\lambda_j(\omega_0)$. Let $\Delta_1, \dots, \Delta_k$ be disjoint discs centered at $\lambda_1(\omega_0), \dots, \lambda_k(\omega_0)$. For ω close enough to ω_0,

$$\frac{1}{2\pi i}\int_{\partial\Delta_j}\frac{f'_\omega(z)}{f_\omega(z)}\,dz$$

is continuous in ω, hence is equal to m_j, and thus ω is in $\tilde{\Lambda}$ by the argument principle. This implies by pigeon-holing that $\lambda_j(\omega)$ is in Δ_j, that the multiplicity of $\lambda_j(\omega)$ is m_j, and since

$$\lambda_j(\omega) = \frac{1}{2\pi i m_j}\int_{\partial\Delta_j}\frac{f'_\omega(z)}{f_\omega(z)}\,z\,dz,$$

it follows that the λ_j are differentiable.

To show that $P_i(\omega)$ is a C^∞ function on $\tilde{\Lambda}$ we recall that

$$P_i(\omega) = \prod_{j \neq i}\frac{1}{\lambda_i(\omega) - \lambda_j(\omega)}\,(H(\omega) - \lambda_j(\omega)I). \tag{3.4.2}$$

We can continue this procedure on the interior of $\Lambda - \tilde{\Lambda}$, absorbing the boundary of $\tilde{\Lambda}$ into the set Z_H and etc., which completes the proof.

We now proceed to analyze a C^∞ Hermitian vector bundle E over Ω, an open connected subset of $\mathbf{C}$, where E is equipped with a metric-preserving connection D.

Definition 3.5. An open connected subset Λ of Ω has a *regular* 1-*eigenvalue structure* for E if there exist C^∞ real-valued functions

$$\lambda_1(\omega) < \dots < \lambda_{n(E)}(\omega) \tag{3.5.1}$$

defined on Λ which are the distinct eigenvalues of $\mathcal{K}(\omega)$. Each λ_i has multiplicity m_i (necessarily constant on Λ) and a corresponding eigen sub-bundle E_i of E restricted to Λ such that

$$E = E_1 \oplus \dots \oplus E_{n(E)}. \tag{3.5.2}$$

The index set $\{1, \dots, n(E)\}$ is denoted by $\mathfrak{J}_1(\Lambda)$.

Throughout this section we will use the notation $\lambda_i(E, \omega)$, $m_i(E)$, $\mathfrak{J}_i(\Lambda, E)$ and so forth when we wish to emphasize dependence on the bundle E.

By the previous lemma there is a closed nowhere dense subset Z_1 of Ω such that each component of $\Omega - Z_1$ has a regular 1-eigenvalue structure for E.

Note that if E is a *holomorphic* Hermitian vector bundle and D the canonical connection, the E_i's will not in general be holomorphic sub-bundles of E, since $\mathcal{K}(\omega)$ is not at all holomorphic (it is self-adjoint). Nonetheless if P_i denotes the orthogonal projection of the C^∞ Hermitian bundle E onto E_i, then

$$D_i = P_i D P_i \tag{3.5.3}$$

gives a metric preserving connection on E_i.

Let Λ have a regular 1-eigenvalue structure for E. We say that Λ has a *regular* 2-*eigenvalue structure* for E if Λ has a regular 1-eigenvalue structure for each E_i for i in $\mathfrak{J}_1(\Lambda)$. We decompose E_i into eigen sub-bundles $E_{i1}, \ldots, E_{in(E_i)}$ corresponding to the eigenvalues $\lambda_{i1}(\omega) < \ldots < \lambda_{in(E_i)}(\omega)$ of $\mathcal{K}_i = \mathcal{K}(E_i, D_i)$, the "curvature" of E_i with respect to the connection D_i. Then $\mathfrak{J}_2(\Lambda)$ is the set of all $(i, 1), \ldots, (i, n(E_i))$ for i in $\mathfrak{J}_1(\Lambda)$. Continuing in this fashion, we say that Λ has a *regular k-eigenvalue structure* for E if Λ has a regular $(k-1)$-eigenvalue structure for E and has a regular 1-eigenvalue structure for each E_I with its connection D_I and "curvature" $\mathcal{K}_I = \mathcal{K}(E_I, D_I)$, $I = (i_1, \ldots, i_{k-1})$ in $\mathfrak{J}_{k-1}(\Lambda)$. We put λ_J equal to $\lambda_{i_k}(E_I)$ for $J = (i_1, \ldots, i_k)$, where $1 \leqslant i_k \leqslant n(E_I)$ and let $\mathfrak{J}_k$ be the set of all J of this form. Then m_J denotes the multiplicity of λ_J, E_J the eigen sub-bundle of E_I corresponding to λ_J, and P_J the orthogonal projection of E onto E_J. Define

$$D_J = P_J D P_J \tag{3.5.4}$$

which gives a metric preserving connection on E_J, for J in $\mathfrak{J}_k(\Lambda)$.

To keep our notation consistant we put $\mathfrak{J}_0(\Lambda)$ equal to $\{\varnothing\}$, $E_\varnothing$ equal to E, $D_\varnothing$ equal to D, and $\mathcal{K}_\varnothing$ equal to $\mathcal{K}$.

By Lemma 3.4 there exists a closed nowhere dense subset Z_k of Ω such that the components of $\Omega - Z_k$ have a regular k-eigenvalue structure for E. Note that if E is 0-umbilic on Ω, then it trivially has a k-eigenvalue structure for all k.

Let Λ have a regular n-eigenvalue structure for E, where the dimension of E is n. If I is in $\mathfrak{J}_k(\Lambda)$ for $0 \leqslant k \leqslant n-1$, then $n(E_I)$, the number of eigen sub-bundles of E_I, is at least one and hence the dimension of each eigen sub-bundle of E_I is strictly smaller than the dimension of E_I, unless E_I is 0-umbilic, that is, unless $\mathcal{K}_I = \lambda_{(I,1)} \cdot$ identity. Thus there exists m, $0 \leqslant m \leqslant n-1$ such that E_I is 0-umbilic for all I in $\mathfrak{J}_m(\Lambda)$. Let $M(\Lambda, E)$ be the smallest integer m for which this is true.

Definition 3.6. A C^∞ Hermitian vector bundle E of dimension n over the open subset Ω of $\mathbf{C}$ with metric preserving connection D is said to be *M-umbilic* on an open connected subset Λ of Ω, if Λ has a regular n-eigenvalue structure for E and

$$M = M(\Lambda, E). \tag{3.6.1}$$

3.7. Let E and $\tilde{E}$ be n-dimensional Hermitian vector bundles over Ω with metric preserving connections D and $\tilde{D}$. If φ: $E \to \tilde{E}$ is a C^∞ isometric bundle map which is connection preserving, then $\varphi \circ \mathcal{K} = \tilde{\mathcal{K}} \circ \varphi$ (2.16.1), which implies that the eigenvalues of $\mathcal{K}$ and $\tilde{\mathcal{K}}$ are equal. Thus if Λ, contained in Ω, has a 1-eigenvalue structure for E, then it has a 1-eigenvalue structure for $\tilde{E}$. In addition $\mathfrak{J}_1(\Lambda, E)$ equals $\mathfrak{J}_1(\Lambda, \tilde{E})$, $\lambda_i(E)$ equals $\lambda_i(\tilde{E})$, the multiplicities agree, and $\varphi \circ P_i = \tilde{P}_i \circ \varphi$. If φ_i: $E_i \to \tilde{E}_i$ is φ restricted to E_i, then φ_i preserves the connections D_i and $\tilde{D}_i$ on E_i and $\tilde{E}_i$, and hence $\varphi_i \circ \mathcal{K}_i$ is $\tilde{\mathcal{K}}_i \circ \varphi_i$. Continuing in this fashion we see that if Λ has a regular n-eigenvalue structure for E, it has one for $\tilde{E}$, and the following hold for all $1 \leqslant k \leqslant n$:

$$\mathfrak{J}_k(\Lambda, E) = \mathfrak{J}_k(\Lambda, \tilde{E}) \tag{3.7.1}$$

$$\lambda_I(E) = \lambda_I(\tilde{E}) \quad \text{and } m_I(E) = m_I(\tilde{E}) \quad \text{for each } I \text{ in } \mathfrak{J}_k(\Lambda). \tag{3.7.2}$$

Furthermore, $\varphi \circ P_I = \tilde{P}_I \circ \varphi$ for I in $\mathfrak{J}_k(\Lambda)$, so if we let φ_I: $E_I \to \tilde{E}_I$ be the restriction of φ to E_I, then φ_I is an isometry and

$$\varphi_I \circ D_I = \tilde{D}_I \circ \varphi_I. \tag{3.7.3}$$

Thus we have shown that the $\mathfrak{J}_k(\Lambda, E)$, $\lambda_I(E, \omega)$, and $m_I(E)$ for I in $\mathfrak{J}_k(\Lambda, E)$ and $1 \leqslant k \leqslant n$ are invariants of E and we call them the *eigenvalue structure* of E on Λ. If Λ has a regular n-eigenvalue structure for both E and $\tilde{E}$, we say that E and $\tilde{E}$ have the same eigenvalue structure if (3.7.1) and (3.7.2) hold.

Moreover, if for I and J in $\mathfrak{J}_k(\Lambda)$, $I \neq J$, we define

$$\Psi_{IJ} = P_I D P_J \tag{3.7.4}$$

then it is easy to check that Ψ_{IJ} is function-linear on the sections of E_J, so it is a C^∞ bundle map from E_J to $E_I \otimes T^*(\Lambda)$, and

$$\varphi_I \Psi_{IJ} = \tilde{\Psi}_{IJ} \varphi_J \tag{3.7.5}$$

3.8. If E and $\tilde{E}$ have the same eigenvalue structure on Λ, and for fixed k, $1 \leqslant k \leqslant n$, there are isometries φ_I from E_I onto $\tilde{E}_I$ for each I in $\mathfrak{J}_k(\Lambda)$ which are connection preserving

(3.7.3) and satisfy the auxiliary conditions (3.7.5) for all I and J in $\mathfrak{J}_k(\Lambda)$, $I \neq J$, then E and $\tilde{E}$ are equivalent, since

$$D = \sum_{I \in \mathfrak{J}_k} D_I + \sum_{\substack{I, J \in \mathfrak{J}_k \\ I \neq J}} \Psi_{IJ}. \tag{3.8.1}$$

This observation is not very useful for general k, but if we let k be the integer M such that E is M-umbilic, then $\tilde{E}$ is M-umbilic and the E_I's and $\tilde{E}_I$'s are all 0-umbilic for I in $\mathfrak{J}_M(\Lambda)$. By Proposition 3.3 we can completely determine which φ_I are connection preserving isometries of E_I onto $\tilde{E}_I$, and thus we need only check when the conditions (3.7.5) can also be satisfied. These are conditions on bundle maps (or equivalently on matrix valued functions) rather than on connections, which makes them much easier to handle.

In order to analyze the auxiliary conditions, we need a technical lemma which we prove after giving the following well known result from differential geometry.

LEMMA 3.9. *Let Δ be a connected open set in $\mathbf{R}^k$ and $\sigma_1, \ldots, \sigma_m$: $\Delta \to \mathbf{C}^n$ be C^∞ functions such that $\sigma_1(x), \ldots, \sigma_m(x)$ are independent for each x in Δ and $(\partial\sigma_i/\partial x_j)(x)$ is in $\bigvee\{\sigma_1(x), \ldots, \sigma_m(x)\} = \mathfrak{V}(x)$, for all x in Δ, $i = 1, \ldots, m$, and $j = 1, \ldots, k$. Then $\mathfrak{V}(x)$ is constant.*

Proof. Fix x_0 in Δ. Then there exist C^∞ maps $\tau_1, \ldots, \tau_m$: $\Delta \to \mathfrak{V}(x_0)^\perp$ and a C^∞ matrix function (α_{ij}) with $\sigma_j(x) = \sum_i \alpha_{ij}(x)\sigma_i(x_0) + \tau_j(x)$, for $j = 1, \ldots, m$. In some neighborhood Δ_0 of x_0, (α_{ij}) is invertible with inverse (β_{ij}). Setting $\tilde{\sigma}_i(x) = \sum \beta_{ji}(x)\sigma_j(x)$ we obtain C^∞ maps such that $\tilde{\sigma}_i(x) = \sigma_i(x_0) + \tilde{\tau}_i(x)$, where $\tilde{\tau}_i(x)$ lies in $\mathfrak{V}(x_0)^\perp$. But then $\partial\tilde{\sigma}_i(x)/\partial x_j = \partial\tilde{\tau}_i(x)/\partial x_j$ is in $\mathfrak{V}(x_0)^\perp$, while at the same time $\partial\tilde{\sigma}_i(x)/\partial x_j$ is in $\mathfrak{V}(x) = \bigvee\{\sigma_1(x_0) + \tilde{\tau}_1(x), \ldots, \sigma_m(x_0) + \tilde{\tau}_m(x)\}$, which implies $\partial\tilde{\sigma}_i/\partial x_j \equiv 0$. Thus $\tilde{\sigma}_1, \ldots, \tilde{\sigma}_m$ are constant, so $\mathfrak{V}(x)$ is also.

Remark 3.10. Let f: $\mathbf{R} \to \mathbf{R}$ be a C^∞ function such that $f^{(i)}(0)$ is 0 for $i = 0, 1, \ldots$, but $f(x)$ is never zero for x non-zero. Define σ_1 from $\mathbf{R}$ into $\mathbf{C}^2$ by

$$\sigma_1(x) = \begin{cases} (f(x), 0) & \text{for } x \leq 0 \\ (0, f(x)) & \text{for } x \geq 0. \end{cases}$$

Then $\sigma_1'(x)$ is in $\mathfrak{V}(x)$ for all x, but $\mathfrak{V}(x)$ is not constant. Thus the independence of the σ is a necessary hypothesis in Lemma 3.9.

LEMMA 3.11. *For Ω an open subset of $\mathbf{C}$ and positive integers m and n let x: $\Omega \to \mathbf{C}^{m+n}$ be a C^∞ function. For each ω in Ω, let α_ω be in $S^{2m-1} \times S^{2n-1} \subset \mathbf{C}^m \times \mathbf{C}^n$ such that*

$$\alpha_\omega \cdot \frac{\partial^{r+s} x}{\partial z^r \partial \bar{z}^s}(\omega) = 0$$

for $r, s \geq 0$, $r + s \leq m + n - 2$ and for $r = 0$, $s = m + n - 1$.

where $(a_1, \ldots, a_{m+n}) \cdot (b_1, \ldots, b_{m+n}) = \sum a_i b_i$. *Then there exists a closed nowhere dense subset* S_x *of* Ω *such that* $\alpha \cdot x \equiv 0$ *in each component of* $\Omega - S_x$, *where* $\alpha = \alpha_\omega$ *for any* ω *in the component.*

Moreover, if each α_ω *is just assumed to lie in* C^{m+n}, *then the same conclusion holds if we require* $r = m+n-1$, $s = 0$ *as well.*

Proof. If $\Omega^1 = \{\omega : x(\omega) \neq 0\}$ and Ω^2 is the interior of $\{\omega : x(\omega) = 0\}$, then $\Omega - (\Omega^1 \cup \Omega^2)$ is a closed nowhere dense subset of Ω. On Ω^2 the lemma is trivial so we may assume by deleting a closed nowhere dense subset that x never vanishes on Ω. Now do the same for $x \wedge \partial x/\partial \bar{z}$, that is, we may assume (neglecting a closed nowhere dense subset) that $x \wedge \partial x/\partial \bar{z}$ never vanishes or vanishes identically. Continuing in this manner we set $x_1 = x$ and having chosen x_i, we choose x_{i+1} to be the first of x, $\partial x/\partial \bar{z}$, $\partial x/\partial z$, $\partial^2 x/\partial \bar{z}^2$, $\partial^2 x/\partial z \partial \bar{z}$, $\partial^2 x/\partial z^2, \ldots$ after x_i such that $x_1 \wedge x_2 \ldots \wedge x_{i+1}$ is never zero. Let x_p be the last one which can be chosen. Let $\mathfrak{V}(\omega) = \mathfrak{V}\{x_1(\omega), \ldots, x_p(\omega)\}$. Since the partial derivative of any vector function in $\mathfrak{V}(\omega)$ remains in $\mathfrak{V}(\omega)$, then $\mathfrak{V}(\omega) = \mathfrak{V}_0$ is a constant subspace of $\mathbf{C}^{m+n}$ by the previous lemma. We consider two cases (1) $x \wedge \partial x/\partial \bar{z}$ never vanishes and (2) $x \wedge \partial x/\partial \bar{z} \equiv 0$.

In (1) if $x \wedge \partial x/\partial \bar{z} \wedge \partial x/\partial z$ is never zero (so $m+n \geq 3$), then since the dimension of $\mathfrak{V}_0$ is not greater than $m+n$, we see that

$$\mathfrak{V}_0 = \bigvee \left\{ \frac{\partial^{r+s} x}{\partial z^r \partial \bar{z}^s} : r+s \leq m+n-2 \right\},$$

where we use the fact that if all the partials of total order k are dependent on the lower order derivatives, then $\mathfrak{V}_0$ is spanned by the lower order derivatives. Hence $\alpha_\omega \perp \mathfrak{V}_0$ for all ω in the component, from which the result follows. On the other hand if $x \wedge \partial x/\partial \bar{z} \wedge \partial x/\partial z \equiv 0$, then there exist C^∞ functions f_1 and f_2 such that $\partial x/\partial z = f_1 x + f_2(\partial x/\partial \bar{z})$ and $\partial^{r+s}x/(\partial z^r \partial \bar{z}^s)$ is in the span of $x, \partial x/\partial \bar{z}, \ldots, \partial^{m+n-1}x/\partial \bar{z}^{m+n-1}$. Therefore $\alpha_\omega \cdot (\partial^{r+s}x/\partial z^r \partial \bar{z}^s) = 0$ for all $0 \leq r+s \leq m+n-1$, hence $\alpha_\omega \perp \mathfrak{V}$ from which the conclusion follows.

Now suppose $x \wedge \partial x/\partial \bar{z} \equiv 0$ and let f be the non-vanishing C^∞ function such that $\partial x/\partial \bar{z} = fx$. If we set $\tilde{x} = gx$, where $\partial \log g/\partial \bar{z} = -f$, then $\partial \tilde{x}/\partial \bar{z} = 0$. Thus without loss of generality (replacing x by $\tilde{x}$) we may assume $\partial x/\partial \bar{z} = 0$ or that x is holomorphic which implies $x_i = \partial^{i-1}x/\partial z^{i-1}$. If p, the dimension of $\mathfrak{V}_0$, is not greater than $m+n-1$, then we proceed as in (1) and we are done. If not, then $x(\omega)$ is a holomorphic curve from $\tilde{\Omega} \subset \Omega$ into $\mathbf{C}^{n+m}$ such that $\{x_i(\omega)\}$ $i = 1, \ldots, m+n$, form a basis for each ω. Let $y_1, \ldots, y_{m+n}$ be the dual basis, that is, the basis which satisfies $y_j \cdot x_i = \delta_{ij}$. If $x_j(\omega) = \sum A_{ij}(\omega) e_i$, where $\{e_i\}$ is a basis for $\mathbf{C}^{m+n}$ then $y_j = \sum B_{ij}(\omega) \delta_i$, where δ_i is the basis dual to e_i and $B = (A^{-1})^t$ is holomorphic. Thus the y_j's are holomorphic. Since α_ω is necessarily a non-zero multiple of y_{m+n}, and α_ω is in $S^{2m-1} \times S^{2n-1}$, we have $y_{m+n} = V(\omega) \oplus \tilde{V}(\omega)$ where $V(\omega)$, $\tilde{V}(\omega)$ are holomorphic curves in

$\mathbf{C}^m$, $\mathbf{C}^n$ respectively, and $\|V(\omega)\| \equiv \|\tilde{V}(\omega)\|$. Let $\mathcal{H} = \bigvee_{\omega \in \tilde{\Omega}}\{V(\omega)\}$ and $\tilde{\mathcal{H}} = \bigvee_{\omega \in \tilde{\Omega}}\{\tilde{V}(\omega)\}$. By an obvious variation of the rigidity theorem (2.2), there exists U: $\mathcal{H} \to \tilde{\mathcal{H}}$ an isomorphism such that $U(V(\omega)) \equiv \tilde{V}(\omega)$. Let $P_{\mathcal{H}}$ denote orthogonal projection on $\mathcal{H}$. Define T in End $(\mathbf{C}^{m+n})$ by $T(e, \tilde{e}) = (0, UP_{\mathcal{H}}(e) - \tilde{e})$. Then $T(y_{m+n}(\omega)) = (0, U(V(\omega)) - \tilde{V}(\omega)) \equiv 0$, and $T \neq 0$. Since $T(y_{m+n}) \cdot e_i = y_{m+n} \cdot T^t e_i \equiv 0$, then $T^t(\mathbf{C}^{m+n})$ is contained in $\bigvee_{i=1,\dots,m+n-1}\{x_i(\omega)\}$. Let ξ in $T^t(\mathbf{C}^{m+n})$ be non-zero; then there exist holomorphic functions ξ_i such that $\xi = \sum_{i=1}^r \xi_i(\omega)x_i(\omega)$, $r \leq m+n-1$, and ξ_r not identically zero. Since ξ is constant, $\xi' = 0 = \sum_{i=1}^r \xi_i'(\omega)x_i(\omega) + \sum_{i=1}^r \xi_i(\omega)x_{i+1}$ implies x_{r+1} is dependent on $x_1, \dots, x_r$, which is a contradiction; that is, p is indeed less than $m+n$.

If α_ω is just in $\mathbf{C}^{m-n}$, then $\alpha_\omega \cdot \partial^{m+n-1}x/\partial z^{m+n-1}(\omega) = 0$ implies x_{m+n} is dependent on $x_1, \dots, x_{m+n-1}$, so again p is less than $m+n$.

3.12. We can now state our theorem for M-umbilic bundles. Let E be M-umbilic over Ω (Definition 3.6). For I and J distinct in $\mathfrak{J}_k(\Omega)$, the map Ψ_{IJ}(3.7.4) induces C^∞ bundle maps Ψ'_{IJ} and Ψ''_{IJ} from E_J to E_I by decomposing Ψ_{IJ},

$$\Psi_{IJ} = \Psi'_{IJ}dz + \Psi''_{IJ}d\bar{z}. \tag{3.12.1}$$

We recall Definition 2.11, so that, for example,

$$(\Psi'_{IJ})_z dz = D'_I \Psi'_{IJ} - \Psi'_{IJ} D'_J.$$

Note that by (2.12.1), with $D = \tilde{D}$, $\varphi = P_J$, σ in $\mathcal{E}(\Omega, E_J)$ and $\tilde{\sigma}$ in $\mathcal{E}(\Omega, E_I)$, we obtain

$$(\Psi'_{IJ})^* = -\Psi''_{JI}. \tag{3.12.2}$$

Theorem 3.13. *Let E and $\tilde{E}$ be C^∞ n-dimensional Hermitian vector bundles over the open connected subset Ω of $\mathbf{C}$, with metric preserving connections D and $\tilde{D}$. If E and $\tilde{E}$ are M-umbilic on Ω, with the same eigenvalue structure, then there exists a closed nowhere dense subset $Z_{E,\tilde{E}}$ of Ω such that E and $\tilde{E}$ are locally equivalent on $\Omega - Z_{E,\tilde{E}}$ if and only if for each ω in Ω and I in $\mathfrak{J}_M(\Omega)$ there exists an isometry $V_{I,\omega}$: $(E_I)_\omega \to (\tilde{E}_I)_\omega$, where $(E_I)_\omega$ is the fibre of E_I at ω, such that*

$$V_{I,\omega} \circ (\Psi'_{IJ})_{z^r\bar{z}^s}(\omega) = (\tilde{\Psi}'_{IJ})_{z^r\bar{z}^s}(\omega) \circ V_{J,\omega} \tag{3.13.1}$$

for all I and J distinct in $\mathfrak{J}_M(\Omega)$ and all r and s satisfying either $r+s \leq M_I + M_J - 2$ or $r = 0$ and $s = m_I + m_J - 1$.

Note that if $m_I = 1$ for all I in $\mathfrak{J}_M(\Omega)$, then (3.13.1) must hold for $r = 0$ and $s = 0$ and 1. When r and s are both zero we obtain $\|\Psi'_{IJ}\| = \|\tilde{\Psi}'_{IJ}\|$. Then (3.13.1) holds for $r = 0$ and

$s=1$ if and only if it holds for $r=1$ and $s=0$, since $\Psi_{IJ}'^{*}=\|\Psi_{IJ}'\|^2\Psi_{IJ}'^{-1}$ and $(\Psi_{IJ}'^{*})_{\bar z}=(\Psi_{IJ}')_z^{*}$ by Lemma 2.12. It is a generalization of this technical fact that we use .

Proof. If E and $\tilde{E}$ are locally equivalent on Ω minus a closed nowhere dense subset, then the existence of the $V_{I,\omega}$ is immediate ((3.7.5) with $V_{I,\omega}=\varphi_I(\omega)$).

Conversely, suppose that (3.13.1) holds, and let $Z_{E,\tilde{E}}$ be the complement of the set in Ω on which E and $\tilde{E}$ are locally equivalent. Since E and $\tilde{E}$ are M-umbilic, E_I and $\tilde{E}_I$ are 0-umbilic for I in $\mathfrak{J}_M(\Omega)$. Furthermore, since E and $\tilde{E}$ have the same eigen-value structure, we have trace $\mathcal{K}_I$ equals trace $\tilde{\mathcal{K}}_I$. Thus if there exists a non-empty open set Ω_0 contained in $Z_{E,\tilde{E}}$ we can apply Lemma 3.2 to obtain orthonormal frames $\sigma_I=\{\sigma_I^1, \ldots, \sigma_I^{m_I}\}$ for E_I and $\tilde{\sigma}_I=\{\tilde{\sigma}_I^1, \ldots, \tilde{\sigma}_I^{m_I}\}$ for $\tilde{E}_I$, defined on Ω_0 such that

$$D_I(\sigma_I^i)=\eta_I\sigma_I^i \quad \text{and} \quad \tilde{D}_I(\tilde{\sigma}_I^i)=\eta_I\tilde{\sigma}_I^i \quad \text{for } \eta_I \quad \text{in } \mathcal{E}^1(\Omega_0). \tag{3.13.2}$$

We wish to show that there exists no such set Ω_0.

We seek to construct a C^∞ isometric bundle map φ_I: $E_I\to\tilde{E}_I$ which will satisfy the conditions (3.7.3) and (3.7.5). If U_I is the matrix for φ_I relative to the frames σ_I and $\tilde{\sigma}_I$, then by (3.13.2) φ_I is isometric and connection-preserving if and only if $dU_I=0$, that is if and only if U_I is a constant unitary.

Let Ψ_{IJ}' have the matrix A_{IJ} with respect to the frames σ_I and σ_J and similarly let $\tilde{A}_{IJ}$ be the matrix for $\tilde{\Psi}_{IJ}'$ with respect to the frames $\tilde{\sigma}_I$, $\tilde{\sigma}_J$ for I, J in $\mathfrak{J}_M(\Omega)$. Then for φ_I to satisfy (3.7.5) by (3.12.2) it suffices to find constant unitary matrices $\{U_I\}_{I\in\mathfrak{J}_M(\Omega)}$ such that

$$U_IA_{IJ}=\tilde{A}_{IJ}U_J \quad \text{for } I\neq J. \tag{3.13.3}$$

If $\eta_I=\eta_I'dz+\eta_I''d\bar z$, then the matrix of $(\Psi_{IJ}')_z$ is $(\eta_I'-\eta_J')A_{IJ}+\partial A_{IJ}/\partial z$, and in general the matrix of $(\Psi_{IJ}')_{z^r\bar z^s}$ is just $\partial^{r+s}A_{IJ}/(\partial z^r\partial\bar z^s)$ plus terms involving lower order derivatives of A_{IJ} multiplied by derivatives of $\eta_I'-\eta_J'$ and $\eta_I''-\eta_J''$. So (3.13.1) implies that for each ω in Ω_0, there exists a unitary matrix $U_{I,\omega}$ such that

$$U_{I,\omega}\frac{\partial^{r+s}A_{IJ}}{\partial z^r\partial\bar z^s}=\frac{\partial^{r+s}\tilde{A}_{IJ}}{\partial z^r\partial\bar z^s}U_{J,\omega} \tag{3.13.4}$$

for $0\leqslant r, s\leqslant r+s\leqslant m_I+m_J-2$ and $r=0$, $s=m_I+m_J-1$. The proof now follows from Lemma 3.11. If for fixed i and j we set

$$x(\omega)=((A)_{IJ})_{1j}, \ldots, (A_{IJ})_{m_Ij}, (\tilde{A}_{IJ})_{i1}, \ldots, (\tilde{A}_{IJ})_{im_J})$$

and

$$\alpha_\omega=((U_{I,\omega})_{i1}, \ldots, (U_{I,\omega})_{im_I}, -(U_{J,\omega})_{1j}, \ldots, -(U_{J,\omega})_{m_Jj})$$

then x: $\Omega_0 \to \mathbf{C}^{m_I} \times \mathbf{C}^{m_J}$, α_ω lies in $S^{2m_I-1} \times S^{2m_J-1}$, and $\alpha_\omega \cdot \partial^{r+s}x/\partial z^r \partial \bar{z}^s = 0$ by (3.13.4) for the appropriate r and s. Thus there exists a closed nowhere dense subset S of Ω_0 such that on any component Δ of $\Omega_0 - S$, we may use the $\{U_{I,\omega_0}\}$ for any ω_0 in Δ to define the φ_I and hence an isometric connection preserving map φ between $E|_\Delta$ and $\tilde{E}|_\Delta$. But then Δ is not contained in $Z_{E,\tilde{E}}$ and hence $Z_{E,\tilde{E}}$ is nowhere dense which completes the proof.

3.14. Although Theorem 3.13 gives a geometrically plausible condition for two bundles to be equivalent, its usefulness is severely limited in practice. One must not only show that E and $\tilde{E}$ have the same eigenvalue structure, but also one must exhibit the structure in order to find the Ψ'_{IJ}'s and check (3.13.1). Now for $n \geqslant 5$, it is not possible to find the eigenvalues of $\mathcal{K}$ and $\tilde{\mathcal{K}}$, much less the whole eigenvalue structure. Yet one can in principle compute whether or not $\mathcal{K}$ and $\tilde{\mathcal{K}}$ have the same eigenvalues (without finding them) simply by finding the traces of $\mathcal{K}$ and $\tilde{\mathcal{K}}$ raised to powers less than or equal to n. Thus we would like to state Theorem 3.13 in a form which is computationally more convenient, in terms of pointwise equivalence (Definition 2.17).

An additional reason for doing this is that in the operator theoretic case (§ 1) we can give an interpretation of the eigenvalues of the curvature $\mathcal{K}(\omega)$ for the bundle E_T in terms of the nilpotent $N_\omega^{(1)}$, by Proposition 2.20. However we have not been able to put $N_\omega^{(k)}$ into any canonical form when k is greater than 1 and we have no interpretation of the whole eigenvalue structure of E_T in terms of the nilpotents. We do have a relationship between the nilpotents and pointwise equivalence (Propositions 2.5 and 2.18) and in § 4 we will use this to prove Theorem 1.6.

3.15. In order to reformulate Theorem 3.13 we first find the relationship of the $\mathcal{K}_I$ and $\Psi_{I,J}$ to the "partial derivatives" of $\mathcal{K}$. So let E be a Hermitian bundle with metric-preserving connection D over Ω, where Ω has a regular n-eigenvalue structure for E and dimension E equals n. Fix $I = (i_1, i_2, \ldots, i_l)$ in $\mathcal{J}_l(\Omega)$, $0 \leqslant l \leqslant n-1$, and let $1 \leqslant i, j \leqslant n(E_I)$. Since

$$P_{I,i}[D_I, \mathcal{K}_I]P_{I,j} = \sum_{k=1}^{n(E_I)} P_{I,i}[D_I, \lambda_{I,k} P_{I,k}]P_{I,j}$$
$$= \sum_{k=1}^{n(E_I)} \{P_{I,i}P_{I,k}P_{I,j}d\lambda_{I,k} + \lambda_{I,k}P_{I,i}D_IP_{I,k}P_{I,j} - \lambda_{I,k}P_{I,i}P_{I,k}D_IP_{I,j}\},$$

then

$$P_{I,i}[D_I, \mathcal{K}_I]P_{I,i} = d\lambda_{I,i}P_{I,i} \quad \text{and} \tag{3.15.1}$$

$$\Psi_{(I,i)(I,j)} = \frac{1}{\lambda_{I,j} - \lambda_{I,i}} P_{I,i}[D_I, \mathcal{K}_I]P_{I,j} \tag{3.15.2}$$
$$= \frac{1}{\lambda_{I,j} - \lambda_{I,i}} P_{I,i}\{\mathcal{K}_{Iz}dz + \mathcal{K}_{I\bar{z}}d\bar{z}\}P_{I,j}.$$

Further, the identity

$$P_{I,i}\mathcal{K}_I P_{I,j}\,dz\,d\bar z = P_{I,i}D_I^2 P_{I,j} = \sum_{k=1}^{n(E_I)} P_{I,i}D_I P_{I,k} D_I P_{I,j}$$

implies

(3.15.3)

$$P_{I,i}\mathcal{K}_I P_{I,j}\,dz\,d\bar z = \begin{cases} \sum_{k\neq i,j} \Psi_{(I,i)(I,k)}\Psi_{(I,k)(I,j)} + D_{I,i}\Psi_{(I,i)(I,j)} + \Psi_{(I,i)(I,j)}D_{I,j} & i\neq j \\ \sum_{k\neq i} \Psi_{(I,i)(I,k)}\Psi_{(I,k)(I,i)} + \mathcal{K}_{I,i}\,dz\,d\bar z & i=j \end{cases}$$

or by (3.15.2),

$$\mathcal{K}_{I,i} = P_{I,i}\left\{\mathcal{K}_I + \sum_{j\neq i}\frac{1}{(\lambda_{I,i}-\lambda_{I,j})^2}\mathcal{K}_{Iz}P_{I,j}\mathcal{K}_{I\bar z} - \mathcal{K}_{I\bar z}P_{I,j}\mathcal{K}_{Iz})\right\}P_{I,i}. \tag{3.15.4}$$

Note that when i and j are not equal, the left hand side of (3.15.3) is zero, so there is a relationship among the Ψ's and their derivatives. We do not make use of this relationship, but it has proved to be an obstruction to producing examples.

Now for I and J distinct in $\mathcal{J}_k(\Omega)$, $I=(i_1, \ldots, i_k)$ and $J=(j_1, \ldots, j_k)$ let l be such that $i_\alpha = j_\alpha$ for $\alpha = 1, \ldots, l$ and $i_{l+1} \neq j_{l+1}$. If $L=(i_1, \ldots, i_l)$, then L is in $\mathcal{J}_l(\Omega)$, $E_I \subset E_{L,i_{l+1}} \subset E_L$ and

$$\Psi_{IJ} = P_I D P_J = P_I P_{L,i_{l+1}} P_L D P_L P_{L,j_{l+1}} P_J = P_I P_{L,i_{l+1}} D_L P_{L,j_{l+1}} P_J.$$

Thus by (3.15.2) we obtain

$$\Psi'_{IJ} = \frac{1}{\lambda_{L,j_{l+1}} - \lambda_{L,i_{l+1}}} P_I(\mathcal{K}_L)_z P_J. \tag{3.15.5}$$

We note that if I and J are in $\mathcal{J}_k(\Omega)$, I and J not necessarily distinct, and if $\varphi_{IJ}\colon E_J \to E_I$ is a C^∞ bundle map, then

$$D_I\varphi_{IJ} - \varphi_{IJ}D_J = P_I(DP_I\varphi_{IJ}P_J - P_I\varphi_{IJ}P_J D)P_J$$

which implies

$$(\varphi_{IJ})_z = P_I(P_I\varphi_{IJ}P_J)_z P_J \quad\text{and}\quad (\varphi_{IJ})_{\bar z} = P_I(P_I\varphi_{IJ}P_J)_{\bar z}P_J, \tag{3.15.6}$$

where $P_I\varphi_{IJ}P_z$ is considered as a bundle map of E into itself. We emphasize that the z and $\bar z$ subscripts refer to covariant differentiation with respect to different connections on the two sides of the equations (3.15.6).

THEOREM 3.16. *Let E and $\tilde E$ be C^∞ n-dimensional Hermitian vector bundles over the open connected subset Ω of $\mathbf{C}$ with metric-preserving connections D and $\tilde{D}$. If E is M-umbilic* *on Ω, $M \geqslant 1$, then there exists a closed nowhere-dense subset $Z_{E,\tilde{E}}$ of Ω such that E and $\tilde{E}$ are locally equivalent on $\Omega - Z_{E,\tilde{E}}$ if and only if E and $\tilde{E}$ are equivalent to order k_E, where*

$$k_E = M - 1 + \max_{\substack{I, J \in \mathfrak{J}_M(\Omega) \\ I \neq J}} \{m_I + m_J\} \tag{3.16.1}$$

and thus

$$M + 1 \leqslant k_E \leqslant n. \tag{3.16.2}$$

Proof. The one direction is obvious (Remark 2.16). For the other, suppose E and $\tilde{E}$ are equivalent to order k_E at each ω in Ω and let V_ω: $E_\omega \to \tilde{E}_\omega$ be an isometry effecting this. Since $M \geqslant 1$, E is not 0-umbilic, and thus $\mathfrak{J}_M(\Omega)$ has at least two distinct elements. Since each m_I is greater than 0, k_E is at least $M+1$. Since E and $\tilde{E}$ are equivalent to order 1, it follows that $\lambda_i(\omega) = \tilde{\lambda}_i(\omega)$ for $i = 1, \ldots, n(E)$, and that Ω has a regular 1-eigenvalue structure for $\tilde{E}$ (which agrees with the one for E) and $V_\omega P_i = \tilde{P}_i V_\omega$. Let $V_{i,\omega}$: $E_i \to \tilde{E}_i$ be V_ω restricted to E_i.

Now we claim that if E and $\tilde{E}$ are equivalent to order $k+1$, then E_i and $\tilde{E}_i$ are equivalent to order k via $V_{i,\omega}$ for each i in $\mathfrak{J}_1(\Omega)$. This follows from (3.15.4), (3.4.2), (3.15.6) and the Leibnitz rule (2.11.1).

Thus by induction, Ω has a regular k_E-eigenvalue structure for $\tilde{E}$ which agrees with the one for E, and for each I in $\mathfrak{J}_k(\Omega)$, $0 \leqslant k \leqslant k_E - 1$, E_I and $\tilde{E}_I$ are equivalent to order $k_E - k$, via $V_{I,\omega}$: $E_I \to \tilde{E}_I$, where $V_{I,\omega}$ is the restriction of V_ω to E_I. Since for an M-umbilic bundle the $(M+1)$-eigenvalue structure contains all the information of the n-eigenvalue structure, and since k_E is at least $M+1$, then $\tilde{E}$ is M-umbilic, with the same eigenvalue structure as E. Note, we need the $(M+1)$-eigenvalue structures of E and $\tilde{E}$ to show that the eigenvalues for the 0-umbilic bundles E_I and $\tilde{E}_I$, I in $\mathfrak{J}_M(\Omega)$, are the same.

In order to apply Theorem 3.13 we need to show that (3.13.1) holds. Now by (3.15.5), for I and J distinct in $\mathfrak{J}_M(\Omega)$, $(\Psi'_{IJ})_{z^r\bar{z}^s}$ can be expressed in terms of (i) the ordinary partial derivatives of $\lambda_{L,i_{l+1}}$ and $\lambda_{L,j_{l+1}}$, (ii) $(P_I)_{z^r\bar{z}^s}$ and $(P_J)_{z^r\bar{z}^s}$ and covariant partial derivatives of total order less than $r+s$, and (iii) $(\mathcal{K}_L)_{z^{r+1}\bar{z}^s}$ and covariant partial derivatives of total order less than $r+s+1$. Thus (3.13.1) holds if

$$V_{I,\omega}(P_I)_{z^r\bar{z}^s} = (\tilde{P}_I)_{z^r\bar{z}^s} V_{I,\omega} \tag{3.16.3}$$

and

$$V_{J,\omega}(P_J)_{z^r\bar{z}^s} = (\tilde{P}_J)_{z^r\bar{z}^s} V_{J,\omega}$$

for each ω in Ω and r and s such that either $r+s \leqslant m_I + m_J - 2$ or $r = 0$ and $s = m_I + m_J - 1$, and

$$V_{L,\omega}(\mathcal{K}_L)_{z^r\bar{z}^s} = (\tilde{\mathcal{K}}_L)_{z^r\bar{z}^s} V_{L,\omega} \tag{3.16.4}$$

for each ω in Ω and r and s such that either $r+s \leq m_I+m_J-1$ or $r=1$ and $s=m_I+m_J-1$. Note that by the technical result Lemma 3.11 we were able to avoid requiring (3.13.1) to hold for $r=m_I+m_J-1$ and $s=0$ and thus we do not require (3.16.4) to hold for $r=m_I+m_J$ and $s=0$.

Now E_I and $\tilde{E}_I$ are equivalent to order k_E-M, so by definition of k_E we obtain (3.16.3) for I, and similarly for J. Since l is less than M, E_L and $\tilde{E}_L$ are equivalent to order k_E-l which is at least m_I+m_J, and thus (3.16.4) holds. We have therefore shown that (3.13.1) is satisfied.

To complete the proof, note that the total number of E_I's for I in $\mathfrak{J}_M(\Omega)$ is at least $M+1$. Since m_I is the dimension of E_I, we have

$$M-1+m_I+m_J \leq \sum_{K \in \mathfrak{J}_M(\Omega)} \dim E_K = n$$

which proves (3.16.2).

Of course to determine whether a bundle is M-umbilic requires that we know the eigenvalue structure. Using Theorem 3.16 we can now give an equivalence result which requires no knowledge of the eigenvalue structure.

THEOREM 3.17. *Let E and $\tilde{E}$ be n-dimensional Hermitian vector bundles over the open subset Ω of $\mathbf{C}$ with metric-preserving connections D and $\tilde{D}$. Then there exists a closed nowhere-dense subset $Z_{E,\tilde{E}}$ of Ω such that E and $\tilde{E}$ are locally equivalent on $\Omega-Z_{E,\tilde{E}}$ if and only if E and $\tilde{E}$ are equivalent to order n on Ω.*

Proof. There exists a closed nowhere dense subset Z_n of Ω such that if $\{\Omega_\alpha\}$ are the components of $\Omega-Z_n$, then Ω_α has an n-eigenvalue structure for E for each α. Thus E restricted to Ω_α is M-umbilic for some M. By Theorem 3.16, if E and $\tilde{E}$ are equivalent to order n on Ω_α, there exists a closed nowhere dense subset $Z^\alpha_{E,\tilde{E}}$ of Ω_α such that E and $\tilde{E}$ are locally equivalent on $\Omega_\alpha-Z^\alpha_{E,\tilde{E}}$. If $Z_{E,\tilde{E}}$ is the union of Z_n and all the $Z^\alpha_{E,\tilde{E}}$, then $Z_{E,\tilde{E}}$ is closed and nowhere dense in Ω.

Although in general n will suffice, a much better estimate can be given in the generic case.

Definition 3.18. A C^∞ Hermitian vector bundle E over the open subset Ω of $\mathbf{C}$ with metric-preserving connection D is said to be *generic* if $\mathcal{K}$ has distinct eigenvalues of multiplicity one at each point of Ω.

COROLLARY 3.19. *Let E and $\tilde{E}$ be generic C^∞ Hermitian vector bundles over the open subset Ω of $\mathbf{C}$ with metric-preserving connections D and $\tilde{D}$. Then E and $\tilde{E}$ are locally equivalent off a closed nowhere dense subset $Z_{E,\tilde{E}}$ of Ω if and only if E and $\tilde{E}$ are equivalent to* *order two on Ω, that is, if and only if $\mathcal{K}$, $\mathcal{K}_{\bar z}$, $\mathcal{K}_{z\bar z}$ are simultaneously unitarily equivalent to $\tilde{\mathcal{K}}$, $\tilde{\mathcal{K}}_{\bar z}$, $\tilde{\mathcal{K}}_{z\bar z}$ at each point of $\Omega - Z_{E,\tilde E}$.*

Proof. If E and $\tilde E$ are generic, then they are both 1-umbilic and $m_i = 1$ for each i in $\mathcal{J}_1(\Omega)$. Thus k_E is 2.

COROLLARY 3.20. *Let E and $\tilde E$ be generic C^∞ Hermitian vector bundles over the open subset Ω of $\mathbf{C}$ with metric-preserving connections D and $\tilde D$. Then E and $\tilde E$ are locally equivalent off a closed nowhere dense subset of Ω if and only if the following conditions hold:*

(3.20.1) trace $\mathcal{K}^i$ = trace $\tilde{\mathcal{K}}^i$ *for $i = 1, 2, ..., n$ or equivalently the eigenvalues of $\mathcal{K}$ are equal to those of $\tilde{\mathcal{K}}$; and*

(3.20.2) trace $\mathcal{K}^i\mathcal{K}_z\mathcal{K}^j\mathcal{K}_{\bar z}\mathcal{K}^l$, trace $\mathcal{K}^i\mathcal{K}_z\mathcal{K}^j\mathcal{K}_{z\bar z}\mathcal{K}^l$, *and* trace $\mathcal{K}^{i_1}\mathcal{K}_z\mathcal{K}^{i_2} \dots \mathcal{K}^{i_p}\mathcal{K}_z\mathcal{K}^{i_1}$ *are equal to the corresponding traces in $\tilde{\mathcal{K}}$ for $0 \leqslant i, j, l \leqslant n-1$ and distinct $0 \leqslant i_1, ..., i_p \leqslant n-1$, where $p \geqslant 2$.*

Proof. Let σ_i be a unit section of E_i for each i in $\mathcal{J}_1(\Omega)$, put $P_i\mathcal{K}_zP_j\sigma_j = a_{ij}\sigma_i$ and $P_i\mathcal{K}_{z\bar z}P_j\sigma_j = b_{ij}\sigma_i$; similarly for $\tilde\sigma_i$, $\tilde a_{ij}$ and $\tilde b_{ij}$. Then E and $\tilde E$ are locally equivalent if and only if for each ω in Ω there exist $\varrho_1, \varrho_2, ..., \varrho_n$, $|\varrho_i| = 1$ such that

$$\varrho_i a_{ij} = \tilde a_{ij}\varrho_j, \quad i \neq j \quad \text{and} \tag{3.20.3}$$

$$\varrho_i b_{ij} = \tilde b_{ij}\varrho_j, \quad \text{whenever } a_{ij} \neq 0, \tag{3.20.4}$$

since by (3.15.5) a_{ij} vanishes identically in an open set if and only if Ψ'_{IJ} vanishes identically. Thus if (3.20.3) holds, then (3.20.4) is equivalent to

$$a_{ij}\bar b_{ij} = \tilde a_{ij}\bar{\tilde b}_{ij}. \tag{3.20.5}$$

Now (3.20.3) holds if and only if

$$a_{ij}\bar a_{ij} = \tilde a_{ij}\bar{\tilde a}_{ij} \quad \text{and } a_{i_1 i_2}a_{i_2 i_3} \dots a_{i_p i_1} = \tilde a_{i_1 i_2}\tilde a_{i_2 i_3} \dots \tilde a_{i_p i_1} \tag{3.20.6}$$

for all distinct $i_1, ..., i_p$, and $p \geqslant 2$ ($p > 2$ is needed since some a_{ij}'s could be 0).

Note by Lemma 2.12 and by (2.15.5) applied to $\mathcal{K}: E \to E$, $\mathcal{K}_{z\bar z}$ is self-adjoint. Thus the b_{ij}'s are real; (3.20.5) and (3.20.6) hold if and only if

$$\begin{aligned}
&\operatorname{trace} P_i\mathcal{K}_zP_j\mathcal{K}_{z\bar z}P_i = \operatorname{trace} \tilde P_i\tilde{\mathcal{K}}_z\tilde P_j\tilde{\mathcal{K}}_{z\bar z}\tilde P_i \quad \text{for } 1 \leqslant i, j \leqslant n, \ i \neq j,\\
&\operatorname{trace} P_i\mathcal{K}_zP_j\mathcal{K}_{\bar z}P_i = \operatorname{trace} \tilde P_i\tilde{\mathcal{K}}_z\tilde P_j\tilde{\mathcal{K}}_{\bar z}\tilde P_i, \quad \text{for } 1 \leqslant i, j \leqslant n, \ i \neq j \quad \text{and}\\
&\operatorname{trace} P_{i_1}\mathcal{K}_zP_{i_2} \dots P_{i_p}\mathcal{K}_zP_{i_1} = \operatorname{trace} \tilde P_{i_1}\tilde{\mathcal{K}}_z\tilde P_{i_2} \dots \tilde P_{i_p}\tilde{\mathcal{K}}_z\tilde P_{i_1}\\
&\qquad\qquad \text{for distinct } 1 \leqslant i_2, ..., i_p \leqslant n, \ p \geqslant 2.
\end{aligned}$$

Now (3.20.2) follows from (3.4.2).

3.21. Corollary 3.20 is slightly neater than the formulation given in [6], though it in fact involves more traces. Note further that the genericity of E and $\tilde{E}$ can be determined by the non-vanishing of a polynomial in the traces of $\mathcal{K}^i$. Thus the corollary gives a complete set of invariants for local equivalence of generic bundles.

To give similar invariants for non-generic bundles would be difficult but the following lemma shows that a finite set does exist. We will also use this result to strengthen our equivalence results for bundles with real analytic metric by omitting mention of the closed nowhere dense subset. The following lemma seems to be well-known but we have been unable to find a suitable reference.

LEMMA 3.22. *If $L_1, ..., L_p$ and $\tilde{L}_1, ..., \tilde{L}_p$ are complex $n\times n$ matrices, then there exists a unitary matrix U such that $UL_i=\tilde{L}_iU$ for $i=1, ..., p$ if and only if the trace of a finite number of words in $L_1, ..., L_p, L_1^*, ..., L_p^*$ agree with the trace of the corresponding words in $\tilde{L}_1, ..., \tilde{L}_p, \tilde{L}_1^*, ..., \tilde{L}_p^*$.*

Proof. Setting $M_j=L_j+L_j^*+\alpha I$ and $N_j=i(-L_j+L_j^*)+\alpha I$ and the same for the $\tilde{L}_j$ where α is a real number, we have the L_j's and the $\tilde{L}_j$'s simultaneously unitarily equivalent if and only if the self-adjoint matrices $\{M_j\}\cup\{N_j\}$ and $\{\tilde{M}_j\}\cup\{\tilde{N}_j\}$ are simultaneously unitarily equivalent. Moreover by choosing α sufficiently large we can assume that all are positive definite matrices.

Let $\mathcal{L}$ be the $(2p+1)n\times(2p+1)n$ nilpotent matrix with all entries zero except the entries above the diagonal which consists of $G_1, G_2, ..., G_{2p}$, where $G_i=M_i$ and $G_{i+p}=N_i$ for $i=1, 2, ..., p$, and similarly for $\tilde{\mathcal{L}}$. If the two collections are simultaneously unitarily equivalent, then $\mathcal{L}$ and $\tilde{\mathcal{L}}$ are unitarily equivalent. Conversely, if $\mathcal{L}$ and $\tilde{\mathcal{L}}$ are unitarily equivalent via the unitary $\mathcal{U}$, then $\mathcal{U}$ must be block diagonal with entries $U_1, ..., U_{2p+1}$. A calculation yields $U_iG_i=\tilde{G}_iU_{i+1}$ for $i=1, \dots 2p$ which implies $U_iG_i^2U_i^*=\tilde{G}_i^2$. By uniqueness of the positive square root, we have $U_iG_iU_i^*=\tilde{G}_i$ which implies $U_i=U_{i+1}$ and hence setting $U=U_i$, we have $UL_i=\tilde{L}_iU$ for $i=1, ..., p$.

Now the result of Pearcy [18] and Specht [21] completes the proof since the trace of a word in $\mathcal{L}$ and $\mathcal{L}^*$ will be a polynomial in α with coefficients the trace of a word in the L_i and L_i^*.

Of course, the number of traces required to show unitary equivalence of two $m\times m$ matrices might be enormous (Pearcy requires 4^{m^2} traces).

3.23. We emphasize that the preceeding results, on equivalence of Hermitian bundles with metric-preserving connections, apply to holomorphic Hermitian bundles where equivalence refers to both the holomorphic and metric structures. The converse of this is also true, that is, given a Hermitian bundle E, over Ω contained in $\mathbf{C}$, with metric preserving connection D, E can be given a complex structure so that it is a holomorphic Hermitian vector bundle and D is the canonical connection. Indeed, if $\sigma_1, \ldots, \sigma_n$ is a frame for E in a neighborhood Λ of the point ω_0 in Ω, and θ the matrix of connection 1-forms relative to σ, then we can find a C^∞ $n \times n$ matrix A which solves the differential equation

$$\bar{\partial} A + \theta'' A = 0 \tag{3.23.1}$$

in a neighborhood of ω_0 with initial condition $A(\omega_0) = I$, where we have written $\theta = \theta' dz + \theta'' d\bar{z}$. To see this, assume that ω_0 is the origin and let $\varrho(\omega)$ be a C^∞ function with compact support in the disk $\mathbf{D}_r$ of radius r such that ϱ is identically 1 on $\mathbf{D}_{r_0}$, $r_0 < r$. Let A_0 be I and define A_{m+1} by

$$A_{m+1}(\omega) = I - \frac{1}{2\pi i} \int_{\mathbf{D}_r} \varrho(\xi)\, \theta''(\xi)\, A_m(\xi) \left\{ \frac{1}{\xi - \omega} - \frac{1}{\xi} \right\} d\xi\, d\bar{\xi}. \tag{3.23.2}$$

Then $A_{m+1}(0) = I$, $(\partial A_{m+1}/\partial \bar{z}) = -\varrho\theta'' A_m$ on $\mathbf{D}_r$, and thus

$$\frac{\partial A_{m+1}}{\partial \bar{z}} = -\theta'' A_n \text{ on } \mathbf{D}_{r_0}. \tag{3.23.3}$$

For r_0 small enough, the A_m's converge uniformly to A, a continuous matrix valued function such that $A(0) = I$. It is easy to check that A gives a distribution solution to (3.23.1) on $\mathbf{D}_{r_0}$, and thus is C^∞ by [29, p. 86]. Moreover, if $\{\Lambda_\alpha\}$ is a covering of Λ by open sets, and A_α is a non-singular C^∞ matrix which solves (3.23.1) on Λ_α, then on $\Lambda_\alpha \cap \Lambda_\beta$,

$$\begin{aligned} \bar{\partial}(A_\alpha^{-1} A_\beta) &= -A_\alpha^{-1} \bar{\partial} A_\alpha A_\alpha^{-1} A_\beta + A_\alpha^{-1} \bar{\partial} A_\beta \\ &= A_\alpha^{-1} \theta'' A_\beta - A_\alpha^{-1} \theta'' A_\beta = 0. \end{aligned} \tag{3.23.4}$$

Thus the $A_\alpha^{-1} A_\beta$ are holomorphic and can be considered transition data for a holomorphic bundle on Λ which by Grauert's Theorem [12] is trivial. So there exist holomorphic matrices Φ_α on Λ_α such that $A_\alpha \Phi_\alpha$ equals $A_\beta \Phi_\beta$ on $\Lambda_\alpha \cap \Lambda_\beta$ and this defines a non-singular solution A of (3.23.1) on all of Λ.

If $\tilde{\sigma}_j = \sum A_{ij} \sigma_i$, on Λ, then $\tilde{\sigma}_1, \ldots, \tilde{\sigma}_n$ form a frame for E and $D'' \tilde{\sigma}_i = 0$, $i = 1, \ldots, n$. Thus if we trivialize E restricted to Λ using the frame $\tilde{\sigma}_1, \ldots, \tilde{\sigma}_n$ and give E the complex structure of the trivial bundle, then we may consider the $\tilde{\sigma}_1, \ldots, \tilde{\sigma}_n$ as holomorphic sections. Relative to this structure D satisfies (2.9.2) and (2.9.3) so it is the canonical connection, (cf. [30, p. 62] for another proof).

We want to show that for Hermitian holomorphic vector bundles with real analytic metric we can get equivalence on all of Ω. To do this we first give the following undoubtedly well-known lemmas.

LEMMA 3.24. *Let E and $\tilde{E}$ be holomorphic Hermitian vector bundles with real analytic metrics, over an open subset Ω contained in $\mathbf{C}$. If E and $\tilde{E}$ are equivalent to infinite order at a point ω_0 in Ω, that is if E and $\tilde{E}$ are equivalent to order k for all $k=1, 2, \ldots$ at ω_0, then E and $\tilde{E}$ are equivalent in a neighborhood of ω_0.*

Proof. Note that by the compactness of the $n\times n$ unitary group, we can find an isometry $V_{\omega_0}: E_{\omega_0}\to \tilde{E}_{\omega_0}$ such that $V_{\omega_0}\mathcal{K}_{z^i\bar{z}^j} = \tilde{\mathcal{K}}_{z^i\bar{z}^j} V_{\omega_0}$ for all i and j.

Now let $\sigma=\{\sigma_1, \ldots, \sigma_n\}$ be a holomorphic frame for E in a neighborhood of ω_0 and let $h(\sigma)=((\sigma_j, \sigma_i))$ be the matrix of inner products. Then h is real analytic and if for simplicity we assume that ω_0 is the origin then $h(\sigma)=\sum_{i,j=0}^{\infty} h^{ij}z^i\bar{z}^j$. If we let $B=(\sum_{i=0}^{\infty} h^{i0}z^i)^{-1}$, then B exists, is holomorphic, and is non-singular in a neighborhood of 0. Let $\gamma_j=\sum B_{ij}\sigma_i$ so that $\gamma=\{\gamma_1, \ldots, \gamma_n\}$ is a holomorphic frame for E in a neighborhood of 0. Since $h(\gamma)$ equals $B^*h(\sigma)B$, we have

$$\frac{\partial^i h(\gamma)}{\partial z^i}(0)=0 \quad \text{for all } i=1, 2, \ldots. \tag{3.24.1}$$

Since $h(\gamma)$ is self-adjoint, (3.24.1) holds for the $\bar{z}$ derivative as well. In a similar fashion we can find a holomorphic frame $\tilde{\gamma}$ for $\tilde{E}$ such that (3.24.1) is satisfied for $\tilde{h}(\tilde{\gamma})$. Using (2.18.5), as in the proof of Proposition 2.18, we obtain for U, the matrix of V_0 relative to $\gamma_1, \ldots, \gamma_n$ and $\tilde{\gamma}_1, \ldots, \tilde{\gamma}_n$,

$$Uh^{ij}=\tilde{h}^{ij}U \quad \text{for all } i, j$$

which implies that $Uh(\gamma)=\tilde{h}(\tilde{\gamma})U$ in a neighborhood of 0. Thus if we define $\varphi: E\to\tilde{E}$ in a neighborhood of 0 by letting the matrix of φ relative to the frames $\gamma_1, \ldots, \gamma_n$ and $\tilde{\gamma}_1, \ldots, \tilde{\gamma}_n$ be the constant matrix U, then φ is a holomorphic isometry, that is, E and $\tilde{E}$ are equivalent.

LEMMA 3.25. *Let E and $\tilde{E}$ be holomorphic Hermitian vector bundles with real analytic metrics, over an open connected subset Ω contained in $\mathbf{C}$. If E and $\tilde{E}$ are equivalent in any non-empty open subset of Ω, then they are locally equivalent on Ω.*

Proof. By Lemma 3.22, equivalence to order k is given by the equality of certain traces in the covariant partial derivatives of $\mathcal{K}$ and $\tilde{\mathcal{K}}$, which are real analytic since the matrices relative to holomorphic frames are real analytic. Thus if E and $\tilde{E}$ are equivalent to order k on any open subset of Ω, they are equivalent to order k on Ω. Since they are equivalent to order k for all k on an open subset, they are equivalent to infinite order at each point of Ω and the lemma follows from the previous lemma.

3.26. If E and $\tilde{E}$ are any C^∞ Hermitian bundles with metric-preserving connections such that locally there are frames relative to which the matrix of inner products and the connection 1-forms are real analytic, then we can solve (3.23.1) for a real analytic A, so we can consider E and $\tilde{E}$ locally as holomorphic vector bundles with real analytic metrics. Then by using Lemma 3.25 we can delete the set $Z_{E,\tilde{E}}$ from Theorems 3.16 and 3.17, and Corollary 3.20.

3.27. We conclude this chapter with an example of two Hermitian bundles on $\mathbf{C}$ with metric-preserving connections which are locally equivalent on $\mathbf{C}$ minus the y-axis but are not equivalent in any neighborhood of a point on the y-axis. A similar example gives two bundles which are locally equivalent on $\mathbf{C}$ but not globally equivalent.

We let E be the trivial 2-dimensional Hermitian bundle over $\mathbf{C}$, with the global orthonormal frame $\sigma=\{\sigma_1, \sigma_2\}$. Let $f(x)$ be a real C^∞ function of the real variable x which vanishes to infinite order at $x=\pm c$, c a non-negative constant, and such that f and f' are non-zero for $|x|>c$ and identically zero for $|x|\leqslant c$. Define real C^∞ functions a and b of the complex variable $z=x+iy$ by

$$a(z) = \tfrac{1}{2}(f(x)+f'(x)), \quad b(z) = \tfrac{1}{2}(f(x)-f'(x)).$$

Define the connection D on E by letting the matrix θ of connection 1-forms relative to the frame σ be

$$\theta=\begin{pmatrix} \frac{1}{2}(dz-d\bar{z}) & a\,dz-b\,d\bar{z} \\ b\,dz-a\,d\bar{z} & 0 \end{pmatrix}.$$

Since θ is skew-adjoint, D is metric preserving. The matrix, $\mathcal{K}(\sigma)$, of $\mathcal{K}$ relative to the frame σ is then given by

$$\mathcal{K}(\sigma)=\begin{pmatrix} b^2-a^2 & -\dfrac{\partial a}{\partial \bar{z}}-\dfrac{\partial b}{\partial z}+\tfrac{1}{2}(a-b) \\ -\dfrac{\partial a}{\partial z}-\dfrac{\partial b}{\partial \bar{z}}+\tfrac{1}{2}(a-b) & a^2-b^2 \end{pmatrix}$$
$$=\begin{pmatrix} -f(x)f'(x) & 0 \\ 0 & f(x)f'(x) \end{pmatrix}.$$

We construct the bundle $\tilde{E}$ and connection $\tilde{D}$ in exactly the same manner using the function $\tilde{f}(x)$ to define $\tilde{a}$, $\tilde{b}$, and $\tilde{\theta}$, where

$$\tilde{f}(x)=\begin{cases} f(x) & x\geqslant 0 \\ -f(x) & x\leqslant 0. \end{cases}$$

Note that θ equals $\tilde{\theta}$ on Ω_c^+, the set of z such that $x>-c$, and

$$\tilde{\theta}=\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}^{-1}\theta\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

on Ω_c^-, the set where $x<c$. Thus E and $\tilde{E}$ are equivalent on Ω_c^+ and Ω_c^- via φ_+ and φ_- respectively, where $\varphi_+(\sigma_1)=\tilde{\sigma}_1$, $\varphi_+(\sigma_2)=\tilde{\sigma}_2$ and $\varphi_-(\sigma_1)=\tilde{\sigma}_1$, $\varphi_-(\sigma_2)=-\tilde{\sigma}_2$.

Let $\tilde{\Omega}$ be a connected open subset of $\mathbf{C}$ and φ: $E\to\tilde{E}$ an equivalence on $\tilde{\Omega}$ represented by the 2×2 unitary matrix $U=(u_{ij})$ relative to the frames σ and $\tilde{\sigma}$. Then since $U\mathcal{K}(\sigma)=\tilde{\mathcal{K}}(\tilde{\sigma})U$, and $\mathcal{K}(\sigma)=\tilde{\mathcal{K}}(\tilde{\sigma})$ is diagonal with unequal entries except when $|x|\leqslant c$, we have $u_{12}=u_{21}=0$ on the intersection of $\tilde{\Omega}$ with the set of z for which $|x|\geqslant c$. Thus $u_{12}a$, etc. are identically 0. Now φ preserves the connection if and only if $U\theta=\tilde{\theta}U+dU$, that is if and only if

$$du_{ii}=0 \quad \text{for } i=1,2 \quad \text{and} \tag{3.27.1}$$

$$u_{22}(b\,dz-a\,d\bar{z})=(\tilde{b}\,dz-\tilde{a}\,d\bar{z})u_{11}. \tag{3.27.2}$$

Thus by (3.27.1) u_{ii} is constant on $\tilde{\Omega}$ for $i=1$ and 2. Since a equals $\tilde{a}$ and b equals $\tilde{b}$ on $x>c$ and a or b is always non-zero there, (3.27.2) implies that u_{11} equals u_{22} on $\tilde{\Omega}\cap\{x>c\}$. Similarly, since a equals $-\tilde{a}$ and b equals $-\tilde{b}$ on $x<-c$, we have u_{11} equal to $-u_{22}$ on $\tilde{\Omega}\cap\{x<-c\}$. Thus if $\tilde{\Omega}$ is not contained in Ω_c^+ or Ω_c^-, then since the u_{ii} are constant on $\tilde{\Omega}$, we must have $u_{11}=u_{22}=0$, so U is 0 on $\tilde{\Omega}\cap\{|x|\geqslant c\}$, so φ would not be an equivalence. Hence $\tilde{\Omega}$ is contained in either Ω_c^+ or Ω_c^-.

When c is positive, this gives an example of bundles which are locally equivalent but not globally. When c is 0, E and $\tilde{E}$ are not equivalent on any open set Ω which contain points on the y-axis.

Note that by continuity of the traces in $\mathcal{K}_{z^r\bar{z}^s}$ and $\tilde{\mathcal{K}}_{z^r\bar{z}^s}$, since the bundles E and $\tilde{E}$ are equivalent to infinite order at every point of Ω_0^+ and Ω_0^-, they are equivalent to infinite order at each point of $\mathbf{C}$.

We emphasize finally that by the discussion in § 3.23 we could consider our bundles E and $\tilde{E}$ to be holomorphic Hermitian bundles over $\mathbf{C}$ and thus obtain equivalence to infinite order at every point, but local equivalence only off the y-axis. Therefore the bad set cannot be eliminated even for Hermitian holomorphic vector bundles with canonical connection if the metric is not real analytic. Similarly, local equivalence for Hermitian holomorphic vector bundles does not imply global equivalence even when the domain is simply connected.

§ 4. Conclusion and open problems

4.1. In the first section we introduced the class of operators $\mathcal{B}_n(\Omega)$ and indicated how their study could be related to that of Hermitian holomorphic vector bundles over Ω and stated various theorems without proof. Now having obtained the necessary results in complex geometry in § 2 and § 3, we put it all together.

Recall given an operator T in $\mathcal{B}_n(\Omega)$ we have the Hermitian holomorphic vector bundle E_T defined over Ω as the pullback of the map t defined by $t(\omega) = \ker(T-\omega)$. Consequently, if T and $\tilde{T}$ are unitarily equivalent operators in $\mathcal{B}_n(\Omega)$ and W is a unitary operator such that $T = W^*\tilde{T}W$, then W defines a congruence on $\mathcal{G}r(n, \mathcal{H})$ taking t onto $\tilde{t}$. Therefore, E_T and $E_{\tilde{T}}$ are equivalent as Hermitian holomorphic vector bundles. Conversely, if E_T and $E_{\tilde{T}}$ are equivalent bundles, then by Theorem 2.2 it follows that there exists a unitary W on $\mathcal{H}$ such that

$$W \ker(T-\omega) = \ker(\tilde{T}-\omega) \quad \text{for } \omega \text{ in } \Omega.$$

But then for x in $\ker(T-\omega)$ we have

$$W^*\tilde{T}Wx - Tx = W^*(\tilde{T}-\omega)Wx = 0$$

and hence $W^*\tilde{T}W = T$ since $\bigvee_{\omega\in\Omega} \ker(T-\omega) = \mathcal{H}$. Thus we have proved Theorem 1.14 showing that the study of operators in $\mathcal{B}_n(\Omega)$ up to unitary equivalence can be reduced to the study of the associated Hermitian holomorphic vector bundles.

4.2. Now recall that the local operator N_ω is defined by

$$N_\omega = (T-\omega)\,|\ker(T-\omega)^{n+1} \quad \text{for } \omega \text{ in } \Omega.$$

Thus if T and $\tilde{T}$ in $\mathcal{B}_n(\Omega)$ are unitarily equivalent and W is a unitary which satisfies $T = W^*\tilde{T}W$ then

$$W \ker(T-\omega)^{n+1} = \ker(\tilde{T}-\omega)^{n+1} \quad \text{and } V_\omega N_\omega = \tilde{N}_\omega V_\omega,$$

where V_ω: $\ker(T-\omega)^{n+1} \to \ker(\tilde{T}-\omega)^{n+1}$ is the isometric restriction of W. Thus the local operators are unitarily equivalent. Conversely, suppose T and $\tilde{T}$ are operators in $\mathcal{B}_n(\Omega)$ such that N_ω and $\tilde{N}_\omega$ are unitarily equivalent for each ω in Ω. Then the associated maps $t, \tilde{t}$: $\Omega \to \mathcal{G}r(n, \mathcal{H})$ have contact of order n by Proposition 2.5. Further, by Proposition 2.18 we have that for each ω_0 in Ω, there exists an isometry V: $\ker(T-\omega_0) \to \ker(\tilde{T}-\omega_0)$ such that

$$V\mathcal{K}_{z^p\bar{z}^q}(\omega_0) = \tilde{\mathcal{K}}_{z^p\bar{z}^q}(\omega_0)V \quad \text{for } 0 \leqslant p, q \leqslant n-1,$$

where $\mathcal{K}$ and $\tilde{\mathcal{K}}$ denote the "curvatures" relative to the canonical connections on E_T and $E_{\tilde{T}}$, respectively. Finally, by Theorem 3.17 there exists an open subset Ω_0 of Ω on which the restrictions of E_T and $E_{\tilde{T}}$, are equivalent. Now using Corollary 1.13 and Theorem 1.14

we see by viewing the operators T and $\tilde{T}$ as being in $\mathcal{B}_n(\Omega_0)$ that they are unitarily equivalent. This completes the proof of our main Theorem 1.6.

4.3. It is now clear why we can restrict our attention to local equivalence. As proved in the previous section, if bundles E_T and $E_{\tilde{T}}$ are locally equivalent, they are globally equivalent. This is not true for Hermitian holomorphic vector bundles, in general (cf. 3.27) except those induced as pullbacks of maps into a Grassmanian.

4.4. If the bundle E_T corresponding to T is generic, then the same argument given in § 4.2 but with Corollary 3.19 replacing Theorem 3.17 shows that "local operators" for T could be defined as $T|\ker(T-\omega)^3$. Moreover, if we write $T|\ker(T-\omega)^2$ as a 2×2 matrix with $n\times n$ blocks as in Proposition 2.20, then E_T is generic if and only if the eigenvalues of the upper right hand block are distinct.

4.5. There are several ways in which we could generalize our results. First, we could allow the operator T to have both a kernel and cokernel, for example, in the generality covered by Proposition 1.11. If we do, we obtain two bundles, E_T^+ defined by $\ker(T-\omega)$ which is holomorphic and E_T^- defined by $\ker(T-\omega)^*$ which is anti-holomorphic. Now while our results apply to each bundle separately, there is an ingredient missing in order to construct T. Since $\bigvee_{\omega\in\Delta}\ker(T-\omega)=\mathcal{M}$ and $\bigvee_{\omega\in\Delta}\ker(T-\omega)^*=\mathcal{N}$ are orthogonal and span $\mathcal{H}$, the matrix for T relative to $\mathcal{H}=\mathcal{M}\oplus\mathcal{N}$ is

$$\begin{pmatrix} T_1 & X \\ 0 & T_2 \end{pmatrix}$$

where E_T^+ determines T_1 and E_T^- determines T_2. However, the operator X is not determined by T_1 and T_2. We have succeeded in classifying these operators (and in fact a larger class) by considering the local operators in two parameters ω_1 and ω_2 obtained by compressing T to $\ker(T-\omega_1)^{n_1}\oplus\ker(T-\omega_2)^{*n_2}$ for sufficiently large integers n_1 and n_2. Details will be given in a sequel.

A different situation arises if $\partial\Omega$ contains isolated points. In that case it may be impossible to resolve the bundle at such a point. From the operator theoretic point of view, what may happen is that $T-\omega_0$ fails to be onto or the dimension of $\ker(T-\omega_0)$ increases.

4.6. A different generalization would be to the context of several variables. Although we have made some progress on this, it is complicated at both ends, both in the operator theory and in the complex geometry. For the operator theory one would consider a commuting m-tuple of operators and use the homological analysis introduced by J. L. Taylor

[24] to study the notion of joint spectrum. Assuming the m-tuple is onto on an open subset of $\mathbf{C}^m$ but not invertible, one can show that the joint kernels form a holomorphic vector bundle. (This argument was shown to us by D. Voiculescu.) Thus one can define an analog of the class $\mathcal{B}_n(\Omega)$ for open subsets of $\mathbf{C}^m$. Moreover, one would like to allow complex submanifolds of lower dimension but how is an open problem. Now although the notion of local operator is replaced naturally by that of a local m-tuple, we would need to know much more about such m-tuples than we do.

For the geometry, the rigidity theorem holds without restriction. Thus the operator theoretic study can be reduced to that of geometry. The equivalence problem, however, now poses real difficulties. The curvature is no longer a single operator but one must either consider sectional curvatures or some other algebraic combination. We have obtained one result, however, in case the sectional curvatures generate the full matrix algebra at each point. This corresponds to Proposition 3.3 where $n=1$. We believe it should be possible to generalize most of our results to this context but that this will require a more conceptual understanding of our equivalence proofs.

4.7. In § 1 we associated a Hermitian holomorphic vector bundle to an operator in $\mathcal{B}_n(\Omega)$ and more generally in § 2 to a holomorphic curve in $\mathcal{G}r(n, \mathcal{H})$. We now analyze the problem of which bundles can arise in this manner. We first discuss generalizations of the Frenet formulas ([3], [7], [13]) for a Hermitian holomorphic vector bundle, and the obstructions they give to inducing the bundle from a holomorphic map into a Grassmannian (Propositions 4.9 and 4.17). The reader interested in applications to operator theory can proceed to § 4.21 without loss of continuity.

Let E be a Hermitian holomorphic bundle of dimension n over the open subset Ω of $\mathbf{C}$. For each $k=0, 1, \ldots$ we associate to E a $(k+1)n$-dimensional holomorphic bundle $J_k(E)$, the holomorphic *k-jet bundle* of E (1) as follows:

If $\sigma=\{\sigma_1, \ldots, \sigma_n\}$ is a holomorphic frame for E, on an open subset Λ contained in Ω, then $J_k(E)$ has an associated holomorphic frame $J_k(\sigma)=\{\sigma_{10}, \ldots, \sigma_{n0}, \ldots, \sigma_{1k}, \ldots, \sigma_{nk}\}$ defined on Λ. If $\tilde\sigma$ is another holomorphic frame for E defined on $\tilde\Lambda$, then on $\Lambda \cap \tilde\Lambda$, we have $\tilde\sigma_j=\sum a_{ij}\sigma_i$, where $A=(a_{ij})$ is a holomorphic, $n\times n$, non-singular matrix, and we denote this symbolically by

$$\tilde\sigma=\sigma A.$$

Let $J_k(A)$ be the $(k+1)n\times(k+1)n$, non-singular, holomorphic matrix

(1) Although unnecessary for what follows, the reader may consult [11] for information on jet bundles and their uses in other contexts.

$$
(4.7.1)\qquad J_k(A)=\begin{bmatrix} A & A' & A'' & \dots & \binom{k}{k}A^{(k)} \\ \cdot & A & 2A' & \dots & \binom{k}{k-1}A^{(k-1)} \\ \cdot & & A & \dots & \binom{k}{k-2}A^{(k-2)} \\ \cdot & & & \ddots & \vdots \\ 0 & \cdot & \cdot & \cdot & A \end{bmatrix}.
$$

Then by definition the frames $J_k(\sigma)$ and $J_k(\tilde{\sigma})$ are related on $\Lambda\cap\tilde{\Lambda}$ by

$$
(4.7.2)\qquad J_k(\tilde{\sigma})=J_k(\sigma)J_k(A).
$$

A straightforward computation yields that if A and $\tilde{A}$ are holomorphic $n\times n$ matrices, then

$$
J_k(A\tilde{A})=J_k(A)J_k(\tilde{A})
$$

so the bundle $J_k(E)$ is well-defined (cf. [26], p. 14, for a discussion of constructing a bundle from the transition functions).

The Hermitian metric h on E induces a Hermitian form $J_k(h)$ on $J_k(E)$ such that if $h(\sigma)$ is the matrix of inner products $((\sigma_j, \sigma_i))$, then

$$
(4.7.3)\qquad J_k(h)\,(J_k(\sigma))=\begin{pmatrix} h(\sigma) & \dots & \dfrac{\partial^k h(\sigma)}{\partial z^k} \\ \vdots & & \vdots \\ \dfrac{\partial^k h(\sigma)}{\partial \bar{z}^k} & \dots & \dfrac{\partial^{2k} h(\sigma)}{\partial z^k \partial \bar{z}^k} \end{pmatrix}
$$

is the matrix of $J_k(h)$ relative to the frame $J_k(\sigma)$. To see that $J_k(h)$ is well-defined, we need

$$
(4.7.4)\qquad J_k(h)\,(J_k(\tilde{\sigma}))=J_k(A)^*\{J_k(h)\,(J_k(\sigma))\}J_k(A)
$$

which follows from an easy computation.

In a natural way E is $J_0(E)$ and $J_k(E)$ is a holomorphic sub-bundle of $J_l(E)$ for $l\geqslant k$, with $J_k(h)$ the restriction of $J_l(h)$ to $J_k(E)$, and $J_0(h)=h$.

Note that in general $J_k(h)$ is not positive on $J_k(E)$, so $J_k(E)$ has no natural Hermitian metric, just a Hermitian form.

4.8. If $f\colon \Omega\to \mathcal{G}r(n, \mathcal{H})$ is a holomorphic curve and σ is a holomorphic frame for E_f on Λ contained in Ω, then each σ_i is a holomorphic function $\sigma_i\colon \Lambda\to\mathcal{H}$. We say that f is *k-nondegenerate* if $\sigma_1(\omega), \dots, \sigma_n(\omega), \dots, \sigma_1^{(k)}(\omega), \dots, \sigma_n^{(k)}(\omega)$ are independent for each ω in the open set Λ, and for all such Λ. If this holds for all $k=0, 1, \ldots$, then f is *infinitely nondegenerate*.

If f is k-nondegenerate, then f induces a holomorphic map $j_k(f)$: $\Omega \to \mathcal{G}r((k+1)n, \mathcal{H})$ such that $j_k(f)(\omega)$ is the span of $\sigma_1(\omega), \ldots, \sigma_n^{(k)}(\omega)$. If σ is a frame for E_f on Λ, let $j_k(\sigma) = \{\sigma_1, \ldots, \sigma_n, \ldots, \sigma_1^{(k)}, \ldots, \sigma_n^{(k)}\}$ be the induced frame for $E_{j_k(f)}$. Then $J_k(E_f)$ and $E_{j_k(f)}$ are naturally equivalent Hermitian holomorphic vector bundles by identifying σ_{ir} with $\sigma_i^{(r)}$, since $(\sigma_{ir}, \sigma_{js}) = \partial^{r+s}(\sigma_i, \sigma_j)/\partial z^r \partial \bar{z}^s = (\sigma_i^{(r)}, \sigma_j^{(s)})$. We emphasize that in this case $J_k(h)$ is a Hermitian metric for $J_k(E_f)$, that is, $J_k(h)$ is positive.

Note that if T is in $\mathcal{B}_n(\Omega)$ and t: $\Omega \to \mathcal{G}r(n, \mathcal{H})$ is the induced holomorphic curve, then if σ is a frame for E_T, by Lemma 1.22 the $\sigma_i^{(r)}$ are independent at each point, so t is infinitely nondegenerate.

PROPOSITION 4.9. *Let E be an n-dimensional Hermitian holomorphic vector bundle over Ω contained in $\mathbf{C}$. Then locally E is equivalent to a bundle E_f for f a k-nondegenerate holomorphic curve $(0 \leqslant k < \infty)$ in $\mathcal{G}r(n, \mathbf{C}^{(k+1)n})$ if and only if $J_k(h)$ is a (positive) metric on E and the curvature, induced by the canonical connection on $J_k(E)$, is zero on Ω.*

Proof. If f is a holomorphic, k-nondegenerate curve in $\mathcal{G}r(n, \mathbf{C}^{(k-1)n})$ then $j_k(f)$ maps to the one point space $\mathcal{G}r((k+1)n, \mathbf{C}^{(k+1)n})$ and $E_{j_k(f)}$ is the trivial Hermitian holomorphic $(k+1)n$-dimensional bundle, so the curvature is zero, and the same holds for any bundle equivalent to E_f.

Conversely, if the curvature induced by $J_k(h)$ is zero, then by Lemma 3.2 there exists a (local) frame $\gamma = \{\gamma_1, \ldots, \gamma_{(k-1)n}\}$ for $J_k(E)$ such that γ is orthonormal and $D\gamma_i = 0$ for each i, that is, γ_i is holomorphic (since D'' extends $\bar{\partial}$ (2.9.3)). Let $\sigma_j = \sigma_{j0} = \sum_{i=1}^{(k-1)n} \tilde{\sigma}_{ij}\gamma_i$ define holomorphic maps $\tilde{\sigma}_j = (\tilde{\sigma}_{1j}, \ldots, \tilde{\sigma}_{(k+1)n,j})$ into $\mathbf{C}^{(k+1)n}$, where σ is a frame for E. Then

$$(\sigma_j, \sigma_i) = (\tilde{\sigma}_j, \tilde{\sigma}_i)$$

since the γ_i are orthonormal. Thus E is equivalent to E_f, where f is the span of $\{\tilde{\sigma}_1, \ldots, \tilde{\sigma}_n\}$, so f is a curve in $\mathcal{G}r(n, \mathbf{C}^{(k+1)n})$. Since the matrix $J_k(h)\,(J_k(\sigma))$ is the matrix of inner products of $j_k(\tilde{\sigma})$, then f is k-nondegenerate.

4.10. Note that there need be no globally defined f: $\Omega \to \mathcal{G}r(n, \mathbf{C}^{(k-1)n})$ such that E and E_f are equivalent or even locally equivalent. For example, let E be the 1-dimensional bundle defined on $\Omega = \{z \mid 0 < |z| < 1\}$, with global frame given by one section σ such that $\|\sigma\|^2 = 1 + |z|$. Then E is locally equivalent to E_f, where $f(z)$ is the span of $(\sqrt{z}, 1)$ in $\mathbf{C}^2$, so f is only locally a well-defined holomorphic curve. Now if there is a holomorphic curve $\tilde{f}$: $\Omega \to \mathcal{G}r(1, \mathbf{C}^2)$ such that E and $E_{\tilde{f}}$ were locally equivalent, then by the Rigidity Theorem

(2.2), $\tilde{f}$ is locally congruent to f. By the uniqueness of analytic continuation (cf. § 4.2) $\tilde{f}$ is globally congruent to f, and hence is multiple valued.

4.11. Let $f\colon \Omega \to \mathcal{G}r(n, \mathcal{H})$ be a holomorphic curve. Then just as in Proposition 2.20 we can find a frame σ for E_f such that at a fixed point ω in Ω the σ_i are orthonormal, the σ_i' are orthogonal, and the $\sigma_i^{(r)}$ are perpendicular to the σ_j for all $r \geqslant 1$. Thus by (2.18.3), at ω we have

$$\mathcal{K}(\sigma) = \begin{pmatrix} -|\sigma_1'|^2 \dots & 0 \\ \vdots \ \ddots & \vdots \\ 0 & -|\sigma_n'|^2 \end{pmatrix} \tag{4.11.1}$$

so $\mathcal{K}$ is negative semi-definite. Applying this to the curve $j_k(f)$ we find that the curvature of $J_k(E_f)$ is negative semidefinite. Thus a necessary condition for a bundle E to be equivalent to a bundle of the form E_f for f a k-nondegenerate curve in $\mathcal{G}r(n, \mathcal{H})$, is that the curvature induced by the metric $J_k(h)$ on $J_k(E)$ be negative semi-definite.

Since $J_k(h)$ is induced by a metric on an n-dimensional bundle, its curvature involves much cancellation, as we see from the following Lemma.

LEMMA 4.12. *Let E be an n-dimensional Hermitian holomorphic vector bundle over Ω in $\mathbf{C}$. If for $k \geqslant 1$, $J_k(h)$ is a metric on $J_k(E)$, then $J_{k-1}(E)$ is contained in the kernel of the curvature induced by $J_k(h)$, which thus has rank at most n.*

Proof. For simplicity we assume 0 is in Ω and find the rank of the curvature at 0. Let σ be a holomorphic frame for E in a neighborhood of 0, which is orthonormal at 0. If we set

$$B = \left(\sum_{r=0}^{(k+1)n} \frac{1}{r!} \frac{\partial^r h(\sigma)}{\partial z^r}(0) z^r \right)^{-1}$$

then B is $n \times n$, holomorphic, and non-singular in a neighborhood of 0. Let $\tilde{\sigma}$ be another frame defined by $\tilde{\sigma} = \sigma B$. Then $h(\tilde{\sigma}) = B^* h(\sigma) B$, and

$$\frac{\partial^r h(\tilde{\sigma})}{\partial z^r}(0) = 0 \quad \text{for } r = 1, \dots, (k+1)n.$$

There is nothing to prove if $k = 0$. If k is at least 1, let $\tilde{h}$ be the Hermitian form on E determined by

$$\tilde{h}(\tilde{\sigma}_i, \tilde{\sigma}_j) = \frac{\partial^2 h(\tilde{\sigma}_i, \tilde{\sigma}_j)}{\partial z \partial \bar{z}}. \tag{4.12.1}$$

Then at 0,

$$J_k(h)\,(J_k(\tilde\sigma)) = \begin{pmatrix} I & 0 \\ 0 & J_{k-1}(\tilde h)\,(J_{k-1}(\tilde\sigma)) \end{pmatrix}. \tag{4.12.2}$$

Since the upper left block of $J_{k-1}(\tilde h)(J_{k-1}(\tilde\sigma))$ is $\tilde h(\tilde\sigma)$, the positivity of $J_k(h)$ implies that $\tilde h$ is positive definite at 0, and hence $\tilde h$ is a Hermitian metric on E in a neighborhood of 0. Let $\tilde H$ denote $J_{k-1}(\tilde h)(J_{k-1}(\tilde\sigma))$. Then by (2.18.3), at 0 we have

$$\mathcal{K}(J_k(\tilde\sigma)) = \begin{pmatrix} I & 0 \\ 0 & \tilde H \end{pmatrix}^{-1} \left\{ \begin{pmatrix} 0 & \dfrac{\partial H_I}{\partial \bar z} \\ 0 & \dfrac{\partial \tilde H}{\partial \bar z} \end{pmatrix} \begin{pmatrix} I & 0 \\ 0 & \tilde H \end{pmatrix}^{-1} \begin{pmatrix} 0 & 0 \\ \dfrac{\partial H_{II}}{\partial z} & \dfrac{\partial \tilde H}{\partial z} \end{pmatrix} - \begin{pmatrix} \tilde h(\tilde\sigma) & \dfrac{\partial^2 H_I}{\partial z \partial \bar z} \\ \dfrac{\partial^2 H_{II}}{\partial z \partial \bar z} & \dfrac{\partial^2 \tilde H}{\partial z \partial \bar z} \end{pmatrix} \right\}$$

where

$$J_k(h)\,(J_k(\tilde\sigma)) = \begin{pmatrix} h(\tilde\sigma) & H_I \\ H_{II} & \tilde H \end{pmatrix}.$$

Thus

$$\mathcal{K}(J_k(\tilde\sigma)) = \begin{pmatrix} \dfrac{\partial H_I}{\partial \bar z} \tilde H^{-1} \dfrac{\partial H_{II}}{\partial z} - \tilde h(\tilde\sigma) & \dfrac{\partial H_I}{\partial \bar z} \tilde H^{-1} \dfrac{\partial \tilde H}{\partial z} - \dfrac{\partial^2 H_I}{\partial z \partial \bar z} \\ \tilde H^{-1} \left\{ \dfrac{\partial \tilde H}{\partial \bar z} \tilde H^{-1} \dfrac{\partial H_{II}}{\partial z} - \dfrac{\partial^2 H_{II}}{\partial z \partial \bar z} \right\} & \tilde H^{-1} \left\{ \dfrac{\partial \tilde H}{\partial \bar z} \tilde H^{-1} \dfrac{\partial \tilde H}{\partial z} - \dfrac{\partial^2 \tilde H}{\partial z \partial \bar z} \right\} \end{pmatrix}.$$

Now $\partial H_I/\partial \bar z$ is just the first "row" of $\tilde H$ and $\partial H_{II}/\partial z$ is just the first "column". Thus

$$\frac{\partial H_I}{\partial \bar z} \tilde H^{-1} = (I\ 0 \dots 0) \text{ and } \tilde H^{-1} \frac{\partial H_{II}}{\partial z} = \begin{pmatrix} I \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

So

$$\frac{\partial H_I}{\partial \bar z} \tilde H^{-1} \frac{\partial H_{II}}{\partial z} - \tilde h(\tilde\sigma) = \frac{\partial^2 \tilde h(\tilde\sigma)}{\partial z \partial \bar z} - \tilde h(\tilde\sigma) = 0,$$

$$\frac{\partial \tilde H}{\partial \bar z} \tilde H^{-1} \frac{\partial H_{II}}{\partial z} - \frac{\partial^2 H_{II}}{\partial z \partial \bar z} = 0, \quad \text{and}$$

$$\frac{\partial H_I}{\partial \bar z} \tilde H^{-1} \frac{\partial \tilde H}{\partial z} - \frac{\partial^2 H_I}{\partial z \partial \bar z} = 0.$$

Thus if we let $\tilde{\mathcal{K}}$ denote the curvature of $J_{k-1}(E)$ induced by the metric $J_{k-1}(\tilde h)$, then we have shown that at 0,

$$\mathcal{K}(J_k(\tilde\sigma)) = \begin{pmatrix} 0 & 0 \\ 0 & \tilde{\mathcal{K}}(J_{k-1}(\tilde\sigma)) \end{pmatrix} \tag{4.12.3}$$

and the Lemma follows by induction on k.

4.13. Let W be a finite dimensional complex vector space, with a Hermitian linear form $(\ ,\ )$. Then there is a Hermitian form induced on $\wedge^p W$, the p-fold wedge product of W, by

$$(w_1 \wedge \ldots \wedge w_p, \tilde{w}_1 \wedge \ldots \wedge \tilde{w}_p) = \det((w_i, \tilde{w}_j)).$$

Furthermore, if V is a proper, non-zero subspace of W and the Hermitian form is positive definite on V, then it induces a Hermitian form on W/V by

$$([w], [\tilde{w}]) = \|v_1 \wedge \ldots \wedge v_n\|^{-2}(v_1 \wedge \ldots v_n \wedge w, v_1 \wedge \ldots \wedge v_n \wedge \tilde{w}), \tag{4.13.1}$$

where $[w]$ denotes the equivalence class of w in W/V and $v_1, \ldots, v_n$ is a basis for V. It is easy to check that this is well-defined on equivalence classes and independent of the choise of basis for V.

If $v_1, \ldots, v_n$ is chosen to be an orthonormal basis, then

$$\|[\omega]\|^2 = \det \begin{pmatrix} 1 & & 0 & (v_1, w) \\ & \ddots & & \vdots \\ 0 & & 1 & (v_n, w) \\ (w, v_1) & \ldots & (w, v_n) & \|w\|^2 \end{pmatrix} = \|w\|^2 - \sum |(w, v_i)|^2,$$

and thus the Hermitian form is positive on W if and only if the induced form is positive on W/V.

4.14. Let $[\ ]_k$ denote equivalence class in $J_k(E)/J_{k-1}(E)$, where we put $J_{-1}(E) = 0$. Then there is a *natural* isomorphism π_k of the holomorphic bundles $J_{k+1}(E)/J_k(E)$ and $J_k(E)/J_{k-1}(E)$ given by

$$\pi_k([\sigma_{ik+1}]_{k+1}) = [\sigma_{ik}]_k. \tag{4.14.1}$$

By (4.7.2) this is well-defined and independent of the choice of frame.

Let $\mathcal{K}_{J_k(h)}$ denote the curvature on $J_k(E)$ induced by $J_k(h)$ if $J_k(h)$ is positive. By Lemma 4.12, since the kernel of $\mathcal{K}_{J_k(h)}$ contains $J_{k-1}(E)$, $\mathcal{K}_{J_k(h)}$ induces an endomorphism $J_k(\mathcal{K})$ of $J_k(E)/J_{k-1}(E)$ defined by

$$\{J_k(\mathcal{K})\}([s]_k) = [\mathcal{K}_{J_k(h)}(s)]_k, \quad \text{for } s \text{ in } J_k(E).$$

Since $\mathcal{K}_{J_k(h)}$ is self-adjoint relative to $J_k(h)$, $\mathcal{K}_{J_k(h)}$ is completely determined by $J_k(\mathcal{K})$.

PROPOSITION 4.15. *Let E by an n-dimensional holomorphic vector bundle over Ω contained in $\mathbf{C}$, with a C^∞ Hermitian metric h. If $J_k(h)$ is positive on $J_k(E)$, and $(\ ,\ )_{k+1}$ denotes* *the Hermitian form induced on* $J_{k+1}(E)/J_k(E)$ *by* $J_{k+1}(h)$, *then for* s *and* $\tilde{s}$ *in the same fibre of* $J_{k+1}(E)$,

$$([s]_{k+1}, [\tilde{s}]_{k+1})_{k+1} = -(\{J_k(\mathcal{K})\}(\pi_k([s]_{k+1})), \pi_k([\tilde{s}]_{k+1}))_k. \tag{4.15.1}$$

Note in particular that $J_{k+1}(h)$ is positive on $J_{k+1}(E)$ if and only if the curvature of $J_k(h)$ is negative semi-definite and has rank n, and that $J_k(h)$ is flat if and only if the induced Hermitian form on $J_{k+1}(E)/J_k(E)$ is zero, by § 4.13.

Proof. When k is 0, the metric induced by $J_1(h)$ on $J_1(E)/J_0(E)$ satisfies at 0

$$\begin{aligned}([\tilde{\sigma}_{j1}], [\tilde{\sigma}_{i1}]) &= (\tilde{\sigma}_1 \wedge \dots \wedge \tilde{\sigma}_n \wedge \tilde{\sigma}_{j1}, \tilde{\sigma}_1 \wedge \dots \wedge \tilde{\sigma}_n \wedge \tilde{\sigma}_{i1})\|\tilde{\sigma}_1 \wedge \dots \wedge \tilde{\sigma}_n\|^{-2} \\ &= \frac{\partial^2 h(\tilde{\sigma}_j, \tilde{\sigma}_i)}{\partial z \partial \bar{z}} \quad \text{by (4.12.2)} \\ &= -(\mathcal{K}(\tilde{\sigma}_j), \tilde{\sigma}_i),\end{aligned} \tag{4.15.2}$$

where σ is the frame for E defined in Lemma 4.12, and thus proves (4.15.1) in that case.

For k bigger than 0, assume the proposition is true for $k-1$. Consider the metric $\tilde{h}$ defined by (4.12.1). Then by (4.12.3) and the induction hypothesis, we need only find the Hermitian form induced on $J_k(E)/J_{k-1}(E)$ by $J_k(\tilde{h})$. But

$$\begin{aligned}&(\tilde{\sigma}_{10} \wedge \dots \wedge \tilde{\sigma}_{n0} \wedge \dots \wedge \tilde{\sigma}_{1k} \wedge \dots \wedge \tilde{\sigma}_{nk} \wedge \tilde{\sigma}_{jk+1}, \tilde{\sigma}_{10} \wedge \dots \wedge \tilde{\sigma}_{n0} \wedge \dots \wedge \tilde{\sigma}_{1k} \wedge \dots \wedge \tilde{\sigma}_{nk} \wedge \tilde{\sigma}_{ik+1})_h \\ &\quad = (\tilde{\sigma}_{11} \wedge \dots \wedge \tilde{\sigma}_{n1} \wedge \dots \wedge \tilde{\sigma}_{1k} \wedge \dots \wedge \tilde{\sigma}_{nk} \wedge \tilde{\sigma}_{jk+1}, \tilde{\sigma}_{11} \wedge \dots \wedge \tilde{\sigma}_{n1} \wedge \dots \tilde{\sigma}_{1k} \wedge \dots \wedge \tilde{\sigma}_{nk} \wedge \tilde{\sigma}_{ik+1})_h \\ &\quad = (\tilde{\sigma}_{10} \wedge \dots \wedge \tilde{\sigma}_{n0} \wedge \dots \wedge \tilde{\sigma}_{1k-1} \wedge \dots \wedge \tilde{\sigma}_{nk-1} \wedge \tilde{\sigma}_{jk}, \tilde{\sigma}_{10} \wedge \dots \wedge \tilde{\sigma}_{n0} \wedge \dots \wedge \tilde{\sigma}_{1k-1} \wedge \dots \wedge \tilde{\sigma}_{nk-1} \wedge \tilde{\sigma}_{ik})_{\tilde{h}}\end{aligned}$$

where $(\ ,\ )_h$ and $(\ ,\ )_{\tilde{h}}$ denote the Hermitian forms with respect to h and $\tilde{h}$. Similarly $\|\tilde{\sigma}_{10} \wedge \dots \wedge \tilde{\sigma}_{nk}\|_h = \|\tilde{\sigma}_{10} \wedge \dots \wedge \tilde{\sigma}_{nk-1}\|_{\tilde{h}}$. Thus $J_k(E)/J_{k-1}(E)$ with the Hermitian form induced by $J_k(\tilde{h})$ and $J_{k+1}(E)/J_k(E)$ with the Hermitian form induced by $J_{k+1}(h)$ are isomorphic at 0, and the proposition follows.

4.16. The bundle $J_k(E)/J_{k-1}(E)$ is *naturally* isomorphic (as a holomorphic bundle) to E, where $[\sigma_{ik}]$ is sent to σ_i. This is just the composition of $\pi_{k-1}, \dots, \pi_0$.

It might be of interest to find the induced Hermitian form, $(\ ,\ )_k$ on E identified with $J_k(E)/J_{k-1}(E)$, in terms of the covariant derivatives of $\mathcal{K}$ on E. For example, by (4.15.1), if s and $\tilde{s}$ are sections of E, then under this identification

$$(s, \tilde{s})_1 = -(\mathcal{K}(s), \tilde{s}). \tag{4.16.1}$$

Similarly, if $\mathcal{K}$ is negative definite, then at 0

$$\begin{aligned}(s, \bar{s})_2 &= -(\tilde{\mathcal{K}}(s), \bar{s})_{\tilde{h}} \quad \text{by (4.16.1) applied to } \tilde{h} \\ &= -(\tilde{\mathcal{K}}(s), \bar{s})_1 \\ &= (\mathcal{K} \circ \tilde{\mathcal{K}}(s), \bar{s}), \quad \text{by (4.16.1).}\end{aligned}$$

We claim that

$$\tilde{\mathcal{K}} = 2\mathcal{K} + \mathcal{K}^{-1}\{\mathcal{K}_{\bar{z}}\mathcal{K}^{-1}\mathcal{K}_z - \mathcal{K}_{z\bar{z}}\} \tag{4.16.2}$$

so we have

$$(s, \bar{s})_2 = (\{2\mathcal{K}^2 + \mathcal{K}_{\bar{z}}\mathcal{K}^{-1}\mathcal{K}_z - \mathcal{K}_{z\bar{z}}\}(s), \bar{s}). \tag{4.16.3}$$

To prove (4.16.2) we compute at 0 using the frame $\tilde{\sigma}$ of Lemma 4.12. Now

$$\mathcal{K}(\tilde{\sigma}) = h^{-1}\left\{\frac{\partial h}{\partial \bar{z}} h^{-1} \frac{\partial h}{\partial z} - \frac{\partial^2 h}{\partial z \partial \bar{z}}\right\}$$

where we abbreviate $h(\tilde{\sigma})$ to h, so

$$\begin{aligned}\frac{\partial \mathcal{K}(\tilde{\sigma})}{\partial z} = &-h^{-1}\frac{\partial h}{\partial z} h^{-1}\left\{\frac{\partial h}{\partial \bar{z}} h^{-1}\frac{\partial h}{\partial z} - \frac{\partial^2 h}{\partial z \partial \bar{z}}\right\} \\ &+ h^{-1}\left\{\frac{\partial^2 h}{\partial z \partial \bar{z}} h^{-1}\frac{\partial h}{\partial z} - \frac{\partial h}{\partial \bar{z}} h^{-1}\frac{\partial h}{\partial z} h^{-1}\frac{\partial h}{\partial \bar{z}} + \frac{\partial h}{\partial \bar{z}} h^{-1}\frac{\partial^2 h}{\partial z^2} - \frac{\partial^3 h}{\partial z^2 \partial \bar{z}}\right\}.\end{aligned}$$

Thus at 0,

$$\mathcal{K}(\tilde{\sigma}) = -\frac{\partial^2 h}{\partial z \partial \bar{z}}, \quad \frac{\partial \mathcal{K}(\tilde{\sigma})}{\partial z} = -\frac{\partial^3 h}{\partial z^2 \partial \bar{z}}, \quad \frac{\partial \mathcal{K}(\tilde{\sigma})}{\partial \bar{z}} = -\frac{\partial^3 h}{\partial z \partial \bar{z}^2}, \tag{4.16.4}$$

and

$$\frac{\partial^2 \mathcal{K}(\tilde{\sigma})}{\partial z \partial \bar{z}} = 2\left(\frac{\partial^2 h}{\partial z \partial \bar{z}}\right)^2 - \frac{\partial^4 h}{\partial z^2 \partial \bar{z}^2}.$$

Now since $\tilde{\sigma}$ is a holomorphic frame,

$$\mathcal{K}_{\bar{z}}(\tilde{\sigma}) = \frac{\partial \mathcal{K}(\tilde{\sigma})}{\partial \bar{z}},$$

and since $\tilde{\sigma}$ is orthonormal at 0, then at 0

$$\mathcal{K}_z(\tilde{\sigma}) = \left(\frac{\partial \mathcal{K}(\tilde{\sigma})}{\partial \bar{z}}\right)^* = \frac{\partial \mathcal{K}(\tilde{\sigma})}{\partial z}.$$

Furthermore,

$$\begin{aligned}\mathcal{K}_{z\bar z}(\tilde\sigma) &= \mathcal{K}_{\bar z z}(\tilde\sigma) \quad \text{by (2.15.5)}\\ &= \left[h^{-1}\frac{\partial h}{\partial z}, \mathcal{K}_{\bar z}(\tilde\sigma)\right] + \frac{\partial}{\partial z}\{\mathcal{K}_{\bar z}(\tilde\sigma)\}\\ &= \frac{\partial^2 \mathcal{K}(\tilde\sigma)}{\partial z\partial\bar z} \quad \text{at } 0.\end{aligned}$$

Thus

$$\begin{aligned}\tilde{\mathcal{K}}(\tilde\sigma) &= h^{-1}\left\{\frac{\partial h}{\partial \bar z}h^{-1}\frac{\partial h}{\partial z} - \frac{\partial^2 h}{\partial z\partial\bar z}\right\}\\ &= (-\mathcal{K}(\tilde\sigma))^{-1}\{(-\mathcal{K}_{\bar z}(\tilde\sigma))(-\mathcal{K}(\tilde\sigma))^{-1}(-\mathcal{K}_z(\tilde\sigma)) - 2(\mathcal{K}(\tilde\sigma))^2 + \mathcal{K}_{z\bar z}(\tilde\sigma)\}\end{aligned}$$

by (4.12.1) and (4.16.4), which proves (4.16.2).

By Proposition 4.15 and the computations (4.16.1) and (4.16.3) we have shown

PROPOSITION 4.17. *Let E be a Hermitian holomorphic bundle of dimension n over Ω contained in $\mathbf{C}$. Then locally E is equivalent to a bundle E_f, for f a 1-nondegenerate curve into $\mathcal{G}r(n, \mathbf{C}^{2n})$ if and only if*

(4.17.1) $\qquad \mathcal{K}$ *is negative definite, and* $2\mathcal{K}^2 + \mathcal{K}_{\bar z}\mathcal{K}^{-1}\mathcal{K}_z - \mathcal{K}_{z\bar z} = 0.$

4.18. As we see from (4.16.3) the formulas for $(\ ,\ )_k$ can be very complicated and we don't compute them when k is bigger than 2. We have done some related computations which generalize the "Frenet formulas" which are used in Value Distribution Theory (cf. [3], [7]).

Let $f\colon \Omega \to \mathcal{G}r(n, \mathcal{H})$ be a holomorphic curve. Let $\sigma = \{\sigma_1, \ldots, \sigma_n\}$ be a holomorphic frame for E_f and define F_i^k for each $0 \leqslant k < \infty$ by

(4.18.1) $$F_i^k(\sigma) = \sigma_1 \wedge \ldots \wedge \sigma_n \wedge \ldots \wedge \sigma_1^{(k-1)} \wedge \ldots \wedge \sigma_n^{(k-1)} \wedge \sigma_i^{(k)}$$

and let $h_k(\sigma)$ be the matrix

(4.18.2) $$h_k(\sigma) = ((F_j^k(\sigma), F_i^k(\sigma))).$$

Note that $(\det h_{k-1}(\sigma))^{-1} h_k(\sigma)$ is the matrix of inner products of the induced metric $(\ ,\ )_k$ of § 4.16.

Define $\mathcal{K}^{(k)}(\sigma)$ by

$$\mathcal{K}^{(k)}(\sigma) = -\frac{\partial}{\partial \bar z}\left\{h_k(\sigma)^{-1}\frac{\partial h_k(\sigma)}{\partial z}\right\}$$

whenever $h_k(\sigma)$ is non-singular.

PROPOSITION 4.19. *If $h_0(\sigma), \ldots, h_k(\sigma)$ are all positive definite for $0 \leqslant k < \infty$, then $h_{k+1}(\sigma)$ is determined by $\mathcal{K}^{(k)}(\sigma)$ as follows:*

(4.19.1) $$\mathcal{K}(\sigma) = -h_0(\sigma)^{-1} h_1(\sigma)\,(\det h_0(\sigma))^{-1},$$

and for k bigger than 0,

(4.19.2)
$$\mathcal{K}^{(k)}(\sigma) = h_k(\sigma)^{-1} h_{k+1}(\sigma) \prod_{i=0}^{k} (\det h_i(\sigma))^{\alpha_{k-i}}$$
$$+ \{h_{k-1}(\sigma)^{-1} h_k(\sigma) - [\operatorname{trace}(h_{k-1}(\sigma)^{-1} h_k(\sigma))] I\} \prod_{i=0}^{k-1} (\det h_i(\sigma))^{\alpha_{k-i-1}}$$

where

$$\alpha_0 = -1 \quad \textit{and } \alpha_j = n(1-n)^{j-1} \quad \textit{for } j \geqslant 1.$$

In particular, when $n = 1$

$$\mathcal{K}(\sigma) = h_1(\sigma)/h(\sigma)^2$$
$$\mathcal{K}^{(k)}(\sigma) = \{h_{k-1}(\sigma) h_{k+1}(\sigma)\}/h_k(\sigma)^2$$

(*which are the Frenet equations*).

We sketch the proof. First, (4.19.1) follows immediately from (4.16.1). Then one can check that both sides of (4.19.2) transform in the same manner under change of frame. Then (4.19.2) can be computed relative to the frame $\tilde{\sigma}$ of Lemma 4.12 by induction, as in Proposition 4.15, with the induction step provided by (4.16.4).

4.20. The proof of Proposition 4.19 would seem to go through, almost word for word, for any Hermitian holomorphic bundle, where we replace the $\sigma_i^{(r)}$ by σ_{ir}, the section of the jet bundle, in (4.18.1). This gives necessary conditions for embedding Ω in $\mathcal{G}r(n, \mathcal{H})$. For $\mathcal{G}r(1, \mathcal{H}) = P(\mathcal{H})$ Griffiths ([13], p. 794) showed these conditions were also sufficient.

4.21. Finally, we characterize those Hermitian holomorphic vector bundles with real analytic metric which are (locally) equivalent to the pullback of a holomorphic curve in a Grassmanian. In case $n = 1$ this is due to Calabi [2] and our proof is essentially his. Further, although we state the result for bundles over open subsets of $\mathbf{C}$, the same result with basically the same proof holds for bundles over open subsets of $\mathbf{C}^m$.

THEOREM 4.22. *If E is an n-dimensional Hermitian holomorphic vector bundle over Ω with real-analytic metric h then E is locally equivalent to E_f at ω_0 for some holomorphic curve $f\colon \Omega \to \mathcal{G}r(n, \mathcal{H})$ where $\mathcal{H}$ has dimension N, $1 \leqslant N \leqslant \aleph_0$, if and only if $J_k(h)(\omega_0)$ is non-negative and has rank at most N for $k = 1, 2, 3, \ldots$.*

Proof. Suppose $\sigma_1, \sigma_2, \ldots, \sigma_n$ are holomorphic functions from Ω to $\mathcal{H}$ which form a frame σ for E_f and that h is defined by

$$h(\omega) = ((\sigma_j(\omega), \sigma_i(\omega))_{i,j=1}^n.$$

If we let $j_k(\sigma) = \{\sigma_1, \ldots, \sigma_n, \ldots, \sigma_1^{(k)}, \ldots, \sigma_n^{(k)}\}$, then $J_k(h)$ is the matrix of inner products of $j_k(\sigma)$ and is thus non-negative. Since at most N of the elements of $j_k(\sigma)$ are independent, $J_k(h)$ has rank at most N.

Conversely, suppose that E is an n-dimensional Hermitian holomorphic vector bundle over Ω with metric h such that each $J_k(h)$ is non-negative and has rank at most N, $1 \leqslant N \leqslant \aleph_0$. Since h is real analytic we can expand it about ω_0 in Ω in a Taylor series

$$h(\omega) = \sum_{l,m=0}^{\infty} \frac{M_{lm}}{l!\,m!} (\omega - \omega_0)^l (\bar{\omega} - \bar{\omega}_0)^m$$

which converges for $|\omega - \omega_0| < 2\delta$ for some $\delta > 0$, where M_{lm} is an $n \times n$ matrix. Hence, there exists a constant C such that

$$\|M_{lm}\| \leqslant C l!\, m!\, (\tfrac{3}{2}\delta)^{-(l+m)} \quad \text{for } 0 \leqslant l, m < \infty. \tag{4.22.1}$$

If we multiply the infinite block matrix $J_\infty(h)$ on each side by the block diagonal matrix with lth diagonal block $\delta^l/l!$ times the identity, we obtain the block matrix

$$\mathcal{J} = \left(\frac{M_{lm}\delta^{(l+m)}}{l!\,m!} \right).$$

Moreover, the upper left hand $kn \times kn$ submatrix of $\mathcal{J}$ is non-negative since $J_k(h)$ is non-negative. Further, by (4.22.1) the matrix entries of $\mathcal{J}$ are square-summable and hence $\mathcal{J}$ defines a Hilbert–Schmidt operator on the Hilbert space $\mathcal{E} = l^2_{E_{\omega_0}}$ of sequences $(x_0, x_1, \ldots)$, where x_k is in E_{ω_0} and $\|(x_0, x_1, \ldots)\|^2 = \sum_{k=0}^{\infty} \|x_k\|^2$. Since $\mathcal{J}$ defines a bounded non-negative operator on $\mathcal{E}$ of rank at most N, there exists a bounded operator A from $\mathcal{E}$ to a Hilbert space $\mathcal{H}$ of dimension N such that

$$\mathcal{J} = A^* A. \tag{4.22.2}$$

We choose an orthonormal basis for $\mathcal{H}$ and express A as a matrix to obtain

$$A = \begin{pmatrix} a_{11}^1 \ldots a_{n1}^1 \ldots a_{1k}^1 \ldots a_{nk}^1 \ldots \\ \vdots \\ a_{11}^N \ldots a_{n1}^N \ldots a_{1k}^N \ldots a_{nk}^N \ldots \end{pmatrix}.$$

Let $a_{il} = (a_{il}^1, a_{il}^2, \ldots, a_{il}^N)$ denote the vector in $\mathcal{H}$ and set $\sigma_i(\omega) = \sum_{l=0}^{\infty} a_{il}\delta^{-l}(\omega - \omega_0)^l$ for $i = 1, 2, \ldots, n$. Since

$$\frac{M_{lm}^{ij}\,\delta^{(l+m)}}{l!\,m!} = \sum_{k=1}^{N} a_{jl}^{k}\bar{a}_{im}^{k} = (a_{jl}, a_{im}) \tag{4.22.3}$$

by (4.22.2), it follows from (4.22.3) that $\|a_{il}\|^2/\delta^{2l} \leqslant |M_{ll}^{ii}|/(l!)^2 \leqslant C(2/3\delta)^{2l}$ and therefore the Taylor series for $\sigma_i(\omega)$ converges for $|\omega-\omega_0|<\delta$. Further, we see by (4.22.3) that

$$h(\omega) = \sum_{l,m=0}^{\infty} \frac{M_{lm}}{l!\,m!}(\omega-\omega_0)^l(\bar{\omega}-\bar{\omega}_0)^m = ((\sigma_j(\omega), \sigma_i(\omega)))_{i,j=1}^{n}$$

for $|\omega-\omega_0|<\delta$. Therefore, $f=(\sigma_1, \ldots, \sigma_n)$ defines a holomorphic curve in $\mathcal{G}r(n, \mathcal{H})$ such that E_f is equivalent to the restriction of E to $\{\omega: |\omega-\omega_0|<\delta\}$ which completes the proof.

4.23. There is an appealing way to conceptualize the preceding proof, at least heuristically. If we let $J_\omega(E)$ denote the inductive union of the jet bundles, we obtain the bundle of formal Taylor polynomials for E and we can use $J_\infty(h)$ to define a metric on $J_\omega(E)$. Completing $J_\omega(E)$ we obtain a Hilbert bundle $J_\infty(E)$ which contains E and the preceding proof can be regarded as showing that $J_\infty(E)$ is flat by pushing the non-zero part of the curvature off to infinity in Lemma 4.12. If we identify $J_\infty(E)$ with $\mathcal{H}\times\Omega$, we obtain the embedding.

4.24. The preceding proof required that E have real-analytic metric. It is possible, however, that in the C^∞ case the hypothesis that the $J_k(h)$ are all non-negative implies real-analyticity (cf. Proposition 4.9).

4.25. Now not every pullback bundle is associated with an operator. For example, a necessary condition for E_f to be associated to an operator is for the subspaces $f(\omega_1), f(\omega_2), \ldots, f(\omega_k)$ to be independent for each finite subset $\omega_1, \omega_2, \ldots, \omega_k$ of distinct points in Ω. Although the condition is not very tractable we can obtain necessary and sufficient conditions for a pullback bundle to be associated with an operator. We continue the notation of the last few sections and introduce a little more. For $k\geqslant 1$ let $J_k^0(h)$ denote the matrix obtained from that for $J_k(h)$ in which the left column and top row have been replaced by 0. Further let S_k denote the $(k+1)\times(k+1)$ matrix with $n\times n$ blocks defined by

$$S_k = \begin{bmatrix} 0 & 1 & 0 \ldots 0 & 0 \\ 0 & 0 & 2 \ldots 0 & 0 \\ 0 & 0 & 0 \ldots 0 & 0 \\ \vdots & \vdots & \vdots \ddots \vdots & \vdots \\ 0 & 0 & 0 \ldots 0 & k-1 \\ 0 & 0 & 0 \ldots 0 & 0 \end{bmatrix}$$

Our result is:

THEOREM 4.26. *If $f: \Omega \to \mathcal{G}r(n, \mathcal{H})$ is a holomorphic curve over Ω such that $\bigvee_{\omega \in \Omega} f(\omega) = \mathcal{H}$ and ω_0 is a point in Ω, then E_f is equivalent to E_T for some operator T in $\mathcal{B}_n(\Delta)$ for some neighborhood Δ of ω_0 if and only if there exist constants $0 < m \leq M$ such that*

$$MJ_k(h) \geqslant S^* J_{k-1}(h) S \geqslant m J_k^0(h) \quad \textit{for } k = 1, 2, 3, \ldots. \tag{4.26.1}$$

Proof. For simplicity we give the proof for n equal to 1. Then a dense collection of vectors in $\mathcal{H}$ can be written in the form $\sum_{l=0}^{k} \alpha_l \sigma^{(l)}(\omega_0)$, where $\alpha_0, \alpha_1, \ldots, \alpha_k$ are complex and $k = 1, 2, 3, \ldots$. Moreover, using Lemma 1.22 we see that if there exists T, then

$$(T - \omega_0)\left(\sum_{l=0}^{k} \alpha_l \sigma^{(l)}(\omega_0)\right) = \sum_{l=0}^{k-1} (l+1)\, \alpha_{l+1} \sigma^{(l)}(\omega_0). \tag{4.26.2}$$

Using the fact that $T - \omega_0$ is bounded and is bounded below on the orthogonal complement of its finite dimensional kernel, inequalities (4.26.1) follow. Conversely, if the left hand inequalities (4.26.1) hold, then defining $(T - \omega_0)$ by (4.26.2) yields an operator X with bound at most M and after setting $T = X + \omega_0$, the right hand inequalities (4.26.1) imply that $T - \omega$ has closed range for ω in some sufficiently small neighborhood Δ of ω_0. Since the range of $T - \omega_0$ is obviously dense in $\mathcal{H}$, we see that T is in $\mathcal{B}_1(\Delta)$ and we can easily check that the restriction of E_f to Δ is equivalent to E_T which completes the proof.

4.27. Thus far we have dealt solely with *unitary* equivalence for operators in $\mathcal{B}_n(\Omega)$ and the related complex differential geometry. We now present some results pertaining to similarity for such operators, directions in which further investigations might proceed, and some open questions. Our results are of two types. The first is a necessary condition on the curvature for similarity in the general case. The second is a theory of similarity for curves in finite dimensional Grassmannians, for which we have no generalization to curves in $\mathcal{G}r(n, \mathcal{H})$ when $\mathcal{H}$ is infinite dimensional. Both types of results suggest that additional hypotheses will be necessary, particularly on the maximality of Ω (cf. § 1.3), in order to formulate a comprehensive theory.

Definition 4.28. Let f and $\tilde{f}$ be two holomorphic curves from the open subset Ω of $\mathbf{C}$ into $\mathcal{G}r(n, \mathcal{H})$. Then f and $\tilde{f}$ are *similar* if and only if there exists a bounded invertible operator S on $\mathcal{H}$ such that

$$\tilde{f}(\omega) = S(f(\omega)) \tag{4.28.1}$$

for all ω in Ω. If for each ω there exists such an S_ω so that (4.28.1) holds to kth order at ω, then f and $\tilde{f}$ are *similar to kth order at* ω.

Just as in the case of unitary equivalence, if t and $\tilde{t}$ are curves in $\mathcal{Gr}(n, \mathcal{H})$ induced by T and $\tilde{T}$ in $\mathcal{B}_n(\Omega)$, then T and $\tilde{T}$ are similar if and only if t and $\tilde{t}$ are similar.

PROPOSITION 4.29. *Let f and $\tilde{f}$ from Ω into $\mathcal{Gr}(n, \mathcal{H})$ be holomorphic curves which are similar via the bounded invertible operator S (4.28.1). Define a bundle isomorphism Φ_S: $E_f \to E_{\tilde{f}}$ by restricting S to each fibre. Then the negative semi-definite (4.11) endomorphisms $\mathcal{K}$ and $\tilde{\mathcal{K}}$ satisfy*

$$\|S\|^2(-\mathcal{K}(\omega)) \geqslant \Phi_S(\omega)^*(-\tilde{\mathcal{K}}(\omega))\Phi_S(\omega) \geqslant \|S^{-1}\|^{-2}(-\mathcal{K}(\omega)).$$

Proof. For fixed ω in Ω choose frames σ for E_f and $\tilde{\sigma}$ for $E_{\tilde{f}}$ as in (4.11). Since f and $\tilde{f}$ are similar, there exist holomorphic functions s_{ij} such that

$$S(\sigma_j) = \sum_i s_{ij}\tilde{\sigma}_i$$

and the matrix of $\Phi_S(\omega)$ relative to the frames σ and $\tilde{\sigma}$, is just $(s_{ij}(\omega))$.

Now

$$S(\sigma_j') = \sum_i (s_{ij}'\tilde{\sigma}_i + s_{ij}\tilde{\sigma}_i').$$

Thus if $(\alpha_1, \ldots, \alpha_n)$ is in $\mathbf{C}^n$, then at the point ω

$$\|S(\sum_j \alpha_j\sigma_j'(\omega))\|^2 = \|\sum_{i,j}\alpha_j s_{ij}'(\omega)\tilde{\sigma}_i(\omega)\|^2 + \|\sum_{i,j}\alpha_j s_{ij}(\omega)\tilde{\sigma}_i'(\omega)\|^2$$

which implies that

$$\|S\|^2\|\sum_j \alpha_j\sigma_j'(\omega)\|^2 \geqslant \|\sum_{i,j}\alpha_j s_{ij}(\omega)\tilde{\sigma}_i'(\omega)\|^2.$$

Thus by (4.11.1)

$$\|S\|^2(-\mathcal{K}(\omega)) \geqslant \Phi_S(\omega)^*(-\tilde{\mathcal{K}}(\omega))\Phi_S(\omega) \tag{4.29.1}$$

holds at ω, which is an arbitrary point in Ω. The proposition follows by applying (4.29.1) to S^{-1} and by

$$\Phi_{S^{-1}} = \Phi_S^{-1}. \tag{4.29.2}$$

We now obtain a necessary condition for two curves to be similar.

COROLLARY 4.30. *Let f and $\tilde{f}$ from Ω into $\mathcal{Gr}(n, \mathcal{H})$ be holomorphic curves which are similar via the bounded invertible operator S. Then*

$$\|S^{-1}\|^2\|S\|^2\|\mathcal{K}\| \geqslant \|\tilde{\mathcal{K}}\| \geqslant \|S^{-1}\|^{-2}\|S\|^{-2}\|\mathcal{K}\|,$$

at each point of Ω.

Proof. Since Φ_S is the restriction of S,

(4.30.1)
$$\|S\|^2 I \geqslant \|\Phi_S\|^2 I \geqslant \Phi_S^* \Phi_S.$$

By Proposition 4.29,

$$\|S\|^2 \|\mathcal{K}\| I \geqslant \Phi_S^*(-\tilde{\mathcal{K}})\Phi_S$$

so that by (4.29.2)

$$\begin{aligned} -\tilde{\mathcal{K}} &\leqslant \|S\|^2 \|\mathcal{K}\| \Phi_{S^{-1}}^* \Phi_{S^{-1}} \\ &\leqslant \|S\|^2 \|S^{-1}\|^2 \|\mathcal{K}\| I \end{aligned}$$

by (4.30.1), and the corollary follows.

Note that

(4.30.2)
$$\|S\| \, \|S^{-1}\| \geqslant 1$$

and that equality holds if and only if $\|S\|^{-1} S$ is unitary, that is to say if and only if f and $\tilde{f}$ are unitarily equivalent.

4.31. The necessary condition in Corollary 4.30 is not very strong. Indeed if we let $\mathcal{K}$ be the curvature induced by the backward shift U_+^* and $\tilde{\mathcal{K}}$ the curvature induced by the adjoint B_+^* of the Bergman shift, then by Theorem 1.17

$$\frac{\|\mathcal{K}\|}{\|\tilde{\mathcal{K}}\|} = 1/2$$

although U_+^* and B_+^* are not similar. The problem is that this is a pointwise criterion whereas similarity is obviously a global phenomena.

In case n equals 1, we can strengthen Corollary 4.30 by means of standard techniques in Value Distribution Theory (Generalized Nevanlinna Theory) [27], [14]. Let $\mathbf{D}_R$ denote the disc of radius R in $\mathbf{C}$, and let $f\colon \mathbf{D}_R \to \mathcal{G}r(1, \mathcal{H})$ be a holomorphic curve. Note that $\mathcal{G}r(1, \mathcal{H})$ is the *projective space* of $\mathcal{H}$. Define the *order function* T_f for f by

(4.31.1)
$$T_f(r) = \int_{r_0}^{r} \left(\int_{\mathbf{D}_\varrho} dd^c \log \|f\|^2 \right) \frac{d\varrho}{\varrho}, \quad 0 < r_0 \leqslant r < R \leqslant \infty$$

where we decompose d into operators of type $(1, 0)$ and $(0, 1)$

$$d = \partial + \bar{\partial}$$

and d^c is the *real* differential operator

$$d^c = \frac{i}{4\pi} (\bar{\partial} - \partial).$$

Note that $\|f\|$ means $\|\sigma_1\|$ locally, where $\sigma = \{\sigma_1\}$ is holomorphic frame for E_f. If $\tilde{\sigma}$ is another frame, then $\tilde{\sigma}_1 = \theta\sigma_1$ for θ a non-zero holomorphic function. Since $dd^c \log|\theta|^2$ is just

$$\frac{i}{2\pi}\frac{\partial^2 \log|\theta|^2}{\partial z\partial\bar{z}}\,dz\,d\bar{z},$$

which is zero since $\log|\theta|^2$ is harmonic, T_f is well-defined. Indeed,

$$\begin{aligned} dd^c \log\|\sigma_1\|^2 &= \frac{i}{2\pi}\frac{\partial}{\partial\bar{z}}\left(\|\sigma_1\|^{-2}\frac{\partial\|\sigma_1\|^2}{\partial z}\right)dz\,d\bar{z} \\ &= \frac{i}{2\pi}(-\mathcal{K}(\sigma))\,dz\,d\bar{z} \quad \text{by (2.18.3)} \\ &= \frac{i}{2\pi}\|\mathcal{K}\|\,dz\,d\bar{z} \end{aligned}$$

since the matrix $\mathcal{K}(\sigma)$ is 1×1. Thus from our point of view, we have

$$T_f(r) = \int_{r_0}^{r}\left(\int_{\mathbf{D}_\varrho}\frac{i}{2\pi}\|\mathcal{K}\|\,dz\,d\bar{z}\right)\frac{d\varrho}{\varrho}, \quad 0 < r_0 \leqslant r < R \leqslant \infty. \tag{4.31.2}$$

The order function measures the rate of growth of a holomorphic curve; in fact, the integral $\int_{\mathbf{D}_t} dd^c \log\|f\|^2$ is the area of $f(\mathbf{D}_t)$ in $\mathcal{G}r(1, \mathcal{H})$. When the dimension of $\mathcal{K}$ is finite, the rate of growth of a curve is independent of the metric on the underlying vector space of $\mathcal{H}$ ([27], [14]). The standard proof of this fact (see for example [14, p. 19]) gives us a stronger necessary condition for similarity.

PROPOSITION 4.32. *Let f and $\tilde{f}$ from $\mathbf{D}_R$ into $\mathcal{G}r(1, \mathcal{H})$ be holomorphic curves which are similar via the bounded invertible operator S on $\mathcal{H}$. Then for $r_0 \leqslant r < R$,*

$$|T_f(r) - T_{\tilde{f}}(r)| \leqslant \log(\|S\|\,\|S^{-1}\|).$$

Proof. We first compute d^c in polar coordinates. If $z = \varrho e^{i\theta}$, then

$$d\varrho = \tfrac{1}{2}\{e^{-i\theta}dz + e^{i\theta}d\bar{z}\} \text{ and } d\theta = \frac{i}{2\varrho}\{e^{i\theta}d\bar{z} - e^{-i\theta}dz\}$$

so that if g is a C^∞ function defined on an open subset of $\mathbf{C}$, then

$$\begin{aligned} dg &= \frac{\partial g}{\partial\varrho}d\varrho + \frac{\partial g}{\partial\theta}d\theta \\ &= \tfrac{1}{2}e^{-i\theta}\left\{\frac{\partial g}{\partial\varrho} - \frac{i}{\varrho}\frac{\partial g}{\partial\theta}\right\}dz + \tfrac{1}{2}e^{i\theta}\left\{\frac{\partial g}{\partial\varrho} + \frac{i}{\varrho}\frac{\partial g}{\partial\theta}\right\}d\bar{z} \end{aligned}$$

which implies that

$$d^c g = \frac{i}{4\pi}\left(\tfrac{1}{2}e^{i\theta}\left\{\frac{\partial g}{\partial \varrho} + \frac{i}{\varrho}\frac{\partial g}{\partial \theta}\right\}d\bar{z} - \tfrac{1}{2}e^{-i\theta}\left\{\frac{\partial g}{\partial \varrho} - \frac{i}{\varrho}\frac{\partial g}{\partial \theta}\right\}dz\right) \tag{4.32.1}$$

$$= \frac{1}{4\pi}\left(\varrho\frac{\partial g}{\partial \varrho}d\theta - \frac{1}{\varrho}\frac{\partial g}{\partial \theta}d\varrho\right).$$

Thus

$$T_{\tilde{f}} - T_f = \int_{r_0}^{r}\left(\int_{\mathbf{D}_\varrho} dd^c \log\frac{\|S(f)\|^2}{\|f\|^2}\right)\frac{d\varrho}{\varrho} \tag{4.32.2}$$

$$= \int_{r_0}^{r}\left(\int_{\partial\mathbf{D}_\varrho} d^c \log\frac{\|S(f)\|^2}{\|f\|^2}\right)\frac{d\varrho}{\varrho} \quad \text{(by Stoke's Theorem)}$$

$$= \frac{1}{4\pi}\int_{r_0}^{r}\left(\int_{\partial\mathbf{D}_\varrho}\frac{\partial}{\partial\varrho}\left\{\log\frac{\|S(f)\|^2}{\|f\|^2}\right\}d\theta\right)d\varrho \quad \text{(by (4.32.1))}$$

$$= \frac{1}{4\pi}\int_0^{2\pi}\left(\log\frac{\|S(f)\|^2}{\|f\|^2}\Bigg|_{r_0}^{r}\right)d\theta.$$

The proposition follows from (4.32.2) and the inequality

$$\log\|S\|^2 \geqslant \log\frac{\|S(f)\|^2}{\|f\|^2} \geqslant \log\|S^{-1}\|^{-2}$$

used at the end points r_0 and r.

4.33. For the backward shift U_+^*, $\mathcal{K}$ is $\partial^2 \log(1-|z|^2)/\partial z\partial\bar{z}$ and for the adjoint B_-^* of the Bergman shift, the curvature $\tilde{\mathcal{K}}$ is $\partial^2\log(1-|z|^2)^2/\partial z\partial\bar{z}$. Thus by (4.32.2), the difference of the order functions is

$$\frac{1}{4\pi}\int_0^{2\pi}\left(\log\frac{(1-|z|^2)^{-2}}{(1-|z|^2)^{-1}}\Bigg|_{r_0}^{r}\right)d\theta = \tfrac{1}{2}\log\frac{1-r_0^2}{1-r^2}$$

which is unbounded as r approaches 1, thus giving a purely computational proof that U_+^* and B_+^* are not similar. We emphasize that this follows from the behaviour of the curvature at the boundary of Ω (in this case the unit disc).

4.34. Of course the order function defined by (4.31.2) could be defined when n is bigger than 1. Unfortunately, the Value Distribution Theory for curves in $Gr(n, \mathcal{H})$, n greater than 1, is not well understood, and we don't know whether Proposition 4.32 holds for curves in $Gr(n, \mathcal{H})$, with T_f defined by (4.13.2).

If f is a curve in $Gr(n, \mathcal{H})$, then f induces a curve $\wedge(f)$: $\Omega \to Gr(1, \Lambda^n\mathcal{H})$, where if $\sigma = \{\sigma_1, \ldots, \sigma_n\}$ is a frame for E_f, then $\wedge(\sigma) = \{\sigma_1 \wedge \ldots \wedge \sigma_n\}$ gives a frame for $\wedge(f)$. If ω is a fixed point in Ω, then by normalizing the frame σ at ω as in (4.11), it can be shown that

$$\mathcal{K}(\wedge(\sigma)) = \operatorname{trace}\mathcal{K}(\sigma)$$

and Proposition 4.32 applied to $\wedge(f)$ and $\wedge(\tilde{f})$ gives a necessary condition in terms of an integral of the traces of $\mathcal{K}$ (instead of $\|\mathcal{K}\|$). Perhaps this could be generalized to traces of powers of $\mathcal{K}$, or to the elementary symmetric functions of $\mathcal{K}$.

If the curves f and $\tilde{f}$ from Ω into $\mathcal{G}r(n, \mathcal{H})$ are similar to order 1 (Definition 4.28), where the norms of the operators S_ω and S_ω^{-1} are uniformly bounded, then Corollary 4.30 holds with obvious alterations in the bounds. Results of this type are presumably the first in a series of necessary inequalities involving the covariant partial derivatives of the curvature.

4.35. As we mentioned previously similarity as opposed to equivalence is a global phenomena. Thus by assuming completeness of the holomorphic curves in an appropriate sense it may be possible to obtain deeper results on similarity. We discuss one kind of conjecture in the context of $\mathcal{B}_1(\mathbf{D})$: if T, $\tilde{T}$ are operators in $\mathcal{B}_1(\mathbf{D})$ each having $\overline{\mathbf{D}}$ as a K-spectral set, then T and $\tilde{T}$ are similar if and only if

$$\lim_{|\omega|\to 1} \frac{\mathcal{K}(\omega)}{\tilde{\mathcal{K}}(\omega)} = 1.$$

We are unable to prove or disprove either implication although Proposition 4.32 makes the one direction seem very plausible. Also the limit equals 1 in the few examples we can check.

4.36. For curves in $\mathcal{G}r(n, \mathcal{H})$ we showed that equivalence to order n implies congruence. We now investigate similarity for curves in certain finite dimensional Grassmannians, and show that there is *no* finite number $N(n)$ such that similarity to order $N(n)$, for two curves in $\mathcal{G}r(n, \mathcal{H})$, implies similarity.

Let Ω be an open subset of $\mathbf{C}$ and $f: \Omega \to \mathcal{G}r(n, \mathbf{C}^{(k+1)n})$ a holomorphic curve which is k-nondegenerate (4.8). Let $\sigma = \{\sigma_1, \ldots, \sigma_n\}$ be a holomorphic frame for E_f, defined on an open subset Λ contained in Ω. Represent f by the $(k+1)n \times n$ holomorphic matrix $\mathfrak{F}$ whose columns are the coordinates of the σ_i relative to an orthonormal basis for $\mathbf{C}^{(k+1)n}$. Then by the k-nondegeneracy assumption, the columns of $\mathfrak{F}(\omega), \ldots, \mathfrak{F}^{(k)}(\omega)$ span $\mathbf{C}^{(k+1)n}$ for each ω in Λ, so there exist holomorphic $n \times n$ matrics $S_0, \ldots, S_k$ such that

$$\mathfrak{F}^{(k+1)}(\omega) = \sum_{i=0}^{k} \mathfrak{F}^{(i)}(\omega) S_i(\omega). \tag{4.36.1}$$

If $\tilde{\sigma}$ is another frame for E_f on Λ, then there is a holomorphic $n \times n$ non-singular matrix A such that $\tilde{\sigma} = \sigma A$ or equivalently

$$\tilde{\mathfrak{F}} = \mathfrak{F} A,$$

where $\tilde{\mathfrak{F}}$ represents f via the $\tilde{\sigma}_i$. Thus by (4.36.1)

$$\tilde{\mathfrak{F}}^{(k+1)} = \sum_{j=0}^{k+1} \binom{k+1}{j} \mathfrak{F}^{(j)} A^{(k+1-j)} \tag{4.36.2}$$

$$= \sum_{j=0}^{k} \mathfrak{F}^{(j)} \left\{ \binom{k+1}{j} A^{(k+1-j)} + S_j A \right\}.$$

Now locally in Λ we can solve for A such that

$$(k+1)A' + S_k A = 0. \tag{4.36.3}$$

In that case, $\tilde{\mathfrak{F}}^{(k+1)}$ is in the span of $\mathfrak{F}, \ldots, \mathfrak{F}^{(k-1)}$ and thus in the span of $\tilde{\mathfrak{F}}, \ldots, \tilde{\mathfrak{F}}^{(k-1)}$, which is to say that $\tilde{S}_k$ is identically zero.

Definition 4.37. A representative $\mathfrak{F}$ for a k-nondegenerate curve f into $\mathcal{G}r(n, \mathbf{C}^{(k+1)n})$ is a *Schwarzian representative* if and only if S_k is identically 0. The $n \times n$ matrices $S_0, \ldots, S_{k-1}$ are the *generalized Schwarzian derivatives* of f.

We have shown that we may cover Ω by open sets, on each of which there is a Schwarzian representative.

4.38. A Schwarzian representative is unique up to multiplication on the right by a (constant) element of $\mathcal{G}l(n, \mathbf{C})$, since if $\tilde{\mathfrak{F}}$ is another such representative then $\tilde{\mathfrak{F}}$ equals $\mathfrak{F}A$, for A holomorphic, non-singular, and $n \times n$. By (4.36.3) A' is zero, so A is constant.

4.39. For example, if k is 1, we may always choose an orthonormal basis for $\mathbf{C}^{2n}$ such that f is represented by $\mathfrak{F}$, where

$$\mathfrak{F} = \begin{pmatrix} I \\ F \end{pmatrix}$$

for F an $n \times n$ holomorphic matrix, with F' non-singular (cf. 4.44). Then for $\mathfrak{F}$,

$$S_0 = 0 \quad \text{and } S_1 = (F')^{-1} F''.$$

A Schwarzian representative $\tilde{\mathfrak{F}}$ must be of the form $\mathfrak{F}A$, where by (4.36.3)

$$2A' + (F')^{-1} F'' A = 0.$$

Then from (4.36.2) we get that

$$\tilde{S}_0 = A^{-1} A''$$
$$= -\tfrac{1}{2} A^{-1} \{ (F')^{-1} F''' - \tfrac{3}{2} (F')^{-1} F'' (F')^{-1} F'' \} A$$

by taking the derivative of (4.36.3).

Note that the expression in braces is the non-commutative version of the classical Schwarzian derivative.

PROPOSITION 4.40. *Let f and $\tilde{f}$ be k-nondegenerate holomorphic curves from Ω, a connected open subset of $\mathbf{C}$, into $\mathcal{G}r(n, \mathbf{C}^{(k+1)n})$. Then f and $\tilde{f}$ are similar if and only if there exist an open subset Λ of Ω, Schwarzian representatives $\mathfrak{J}$ and $\tilde{\mathfrak{J}}$ for f and $\tilde{f}$ on Λ, and a non-singular $n\times n$ constant matrix A such that*

$$AS_i = \tilde{S}_i A \quad \text{for } i=0, ..., k-1. \tag{4.40.1}$$

Proof. If f and $\tilde{f}$ are similar, then there exist an element S of $\mathcal{G}l((k+1)n, \mathbf{C})$ and an $n\times n$, holomorphic, non-singular matrix A such that

$$S\mathfrak{J} = \tilde{\mathfrak{J}}A. \tag{4.40.2}$$

But then

$$(S\mathfrak{J})^{(k+1)} = S\mathfrak{J}^{(k+1)} = S\sum_{i=0}^{k-1}\mathfrak{J}^{(i)}S_i = \sum_{i=0}^{k-1}(S\mathfrak{J})^{(i)}S_i \tag{4.40.3}$$

which implies that $S\mathfrak{J}$ is a Schwarzian representative for $\tilde{f}$. By section 4.38, A is constant and (4.40.1) follows from (4.40.3), since

$$(S\mathfrak{J})^{(i)} = \tilde{\mathfrak{J}}^{(i)}A.$$

Conversely, if there is a matrix A in $\mathcal{G}l(n, \mathbf{C})$ so that (4.40.1) holds, define a holomorphic $(k+1)n\times(k+1)n$ non-singular matrix S by

$$S = (\tilde{\mathfrak{J}} \dots \tilde{\mathfrak{J}}^{(k)})\,\mathcal{A}(\mathfrak{J} \dots \mathfrak{J}^{(k)})^{-1}, \tag{4.40.4}$$

where

$$\mathcal{A} = \begin{pmatrix} A & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & A \end{pmatrix}$$

and $(\mathfrak{J} \dots \mathfrak{J}^{(k-1)})$ is the matrix whose columns are the coordinates of $\sigma_1, ..., \sigma_n, ..., \sigma_1^{(k)}, ..., \sigma_n^{(k)}$ relative to the orthonormal basis, and is thus non-singular.

Now by (4.36.1)

$$(\mathfrak{J} \dots \mathfrak{J}^{(k)})' = (\mathfrak{J} \dots \mathfrak{J}^{(k)})\,\mathcal{S}$$

where

$$\mathcal{S} = \begin{pmatrix} 0 & \dots & 0 & S_0 \\ I & & 0 & \vdots \\ \vdots & \ddots & & S_k \\ 0 & \dots & I & 0 \end{pmatrix}$$

Thus

$$\begin{aligned} S' &= (\tilde{\mathfrak{J}} \dots \tilde{\mathfrak{J}}^{(k)})' \mathcal{A} (\mathfrak{J} \dots \mathfrak{J}^{(k)})^{-1} \\ &\quad - (\tilde{\mathfrak{J}} \dots \tilde{\mathfrak{J}}^{(k)}) \mathcal{A} (\mathfrak{J} \dots \mathfrak{J}^{(k)})^{-1} (\mathfrak{J} \dots \mathfrak{J}^{(k)})' (\mathfrak{J} \dots \mathfrak{J}^{(k)})^{-1} \\ &= (\tilde{\mathfrak{J}} \dots \tilde{\mathfrak{J}}^{(k)}) (S\mathcal{A} - \mathcal{A}S) (\mathfrak{J} \dots \mathfrak{J}^{(k)})^{-1} = 0 \end{aligned}$$

by (4.40.1). Thus S is constant and satisfies (4.40.2) by (4.40.4). This implies that f and $\tilde{f}$ are similar on Λ. By the uniqueness of analytic continuation, f and $\tilde{f}$ are similar on Ω (cf. § 4.2).

4.41. Just as we proved (3.13.3) from (3.13.4), we may use Lemma 3.11 and the uniqueness of analytic continuation to show that (4.40.1) is equivalent to

$$S_0^{(i)}(\omega), \dots, S_{k-1}^{(i)}(\omega) \quad \text{for } i = 0, \dots, 2n-1 \tag{4.41.1}$$

are simultaneously similar to

$$\tilde{S}_0^{(i)}(\omega), \dots, \tilde{S}_{k-1}^{(i)}(\omega) \quad \text{for } i = 0, \dots, 2n-1,$$

for each ω in Λ, where the similarity can depend on ω. There does not seem to be any easy method to show simultaneous similarity, as there is for simultaneous unitary equivalence (Lemma 3.22).

Note that when n is 1, (4.40.1) is equivalent to

$$S_i = \tilde{S}_i \quad \text{for } i = 0, \dots, k-1. \tag{4.41.2}$$

4.42. Let $S_0, \dots, S_{k-1}$ be $n \times n$ holomorphic matrices defined in a neighborhood of 0 in $\mathbf{C}$. Then by the existence and uniqueness theorem for holomorphic differential equations, the system

$$\mathfrak{J}^{(k+1)} = \sum_{i=0}^{k-1} \mathfrak{J}^{(i)} S_i$$

has, in a smaller neighborhood of 0, a holomorphic solution $\mathfrak{J}$, a $(k+1)n \times n$ matrix, such that at 0, $(\mathfrak{J} \dots \mathfrak{J}^{(k)})$ is the identity. If f is the curve in $\mathcal{G}r(n, \mathbf{C}^{(k+1)n})$ spanned by the columns of $\mathfrak{J}$, then f is a holomorphic k-nondegenerate curve in a small enough neighborhood of the origin, $\mathfrak{J}$ is a Schwarzian representative for f, and $S_0, \dots, S_{k-1}$ are the generalized Schwarzian derivatives.

Let $\tilde{S}_0, \dots, \tilde{S}_{k-1}$ be another such collection of holomorphic matrices, such that $S_0(0), \dots, S_{k+1}(0)$ are not simultaneously similar to $\tilde{S}_0(0), \dots, \tilde{S}_{k-1}(0)$. Let $\tilde{f}$ be a k-nondegenerate curve in $\mathcal{G}r(n, \mathbf{C}^{(k+1)n})$ defined in a neighborhood of 0, with Schwarzian representative $\tilde{\mathfrak{J}}$

and Schwarzian derivatives the $\tilde{S}_i$. Let A be any fixed matrix in $\mathcal{G}l(n, \mathbf{C})$ and define S by (4.40.4). Then at any point ω, $S(\omega)$ satisfies

$$S(\omega)\,\mathfrak{F}^{(i)}(\omega) = \tilde{\mathfrak{F}}^{(i)}(\omega)\,A, \quad i = 0, \ldots, k$$

and thus f and $\tilde{f}$ are similar to order k in neighborhood of 0, but are not similar, since they don't satisfy (4.40.1).

Thus we have shown that for each n and k, there exists a finite dimensional Hilbert space $\mathcal{H}$ such that similarity to order k is *not* sufficient for similarity of holomorphic curves in $\mathcal{G}r(n, \mathcal{H})$.

4.43. We conclude with a discussion of the sharpness of our results, and some examples of curves in $\mathcal{G}r(n, \mathbf{C}^{2n})$.

In a beautiful paper [13], Griffiths states that second order contact implies equivalence for 1-nondegenerate holomorphic curves in $\mathcal{G}r(n, \mathbf{C}^{2n})$. He gave a proof only for what we call the 0-umbilic and generic cases. We have been unable to show via our methods that second order (rather than nth order) always suffices in this case, or to give what would necessarily be a non-generic counter-example. Furthermore, we have no operator theoretic examples which require contact of order greater than two, or indeed even greater than 1. It would be valuable to find some non-generic examples, which are not direct sums of generic curves.

Griffiths conjectured that first order contact was not sufficient in his case, and we show this is indeed true.

4.44. We first compute the curvature for a curve in $\mathcal{G}r(n, \mathbf{C}^{n+m})$. Let $f\colon \Omega \to \mathcal{G}r(n, \mathbf{C}^{n-m})$ be holomorphic, and let σ be a holomorphic frame for E_f, such that σ is normalized at a fixed point ω in Ω as in section 4.11. Let $v_1, \ldots, v_{n+m}$ be an orthonormal basis for $\mathbf{C}^{n+m}$, such that

$$v_i = \sigma_i(\omega), \quad \text{for } i = 1, \ldots, n.$$

Since $\sigma_i^{(r)}(\omega)$ is perpendicular to $v_1, \ldots, v_n$ for all r, then

$$\sigma_j = v_j + \sum_{i=1}^{m} f_{ij} v_{n+i},$$

where $F = (f_{ij})$ is a holomorphic $n \times n$ matrix. Thus

$$h(\sigma) = I + F^* F$$

and

$$\mathcal{K}(\sigma) = h^{-1}\frac{\partial h}{\partial \bar z} h^{-1}\frac{\partial h}{\partial z} - h^{-1}\frac{\partial^2 h}{\partial z \partial \bar z} \tag{4.44.1}$$

$$= (I + F^*F)^{-1}\{F'^*F(I + F^*F)^{-1}F^*F' - F'^*F'\}$$

$$= (I + F^*F)^{-1}F'^*\{F(I + F^*F)^{-1}F^* - I\} F'.$$

Now

$$\{F(I + F^*F)^{-1}F^* - I\}\{I + FF^*\} = F(I + F^*F)^{-1}(I + F^*F)F^* - FF^* - I = -I.$$

Thus by (4.44.1) we obtain

$$\mathcal{K}(\sigma) = -(I + F^*F)^{-1}F'^*(I + FF^*)^{-1}F'. \tag{4.44.2}$$

4.45. Let g: $\mathbf{C} \to \mathcal{G}r(n, \mathbf{C}^{2n})$ be the holomorphic curve represented by the $n \times n$ matrix

$$\mathcal{G}(z) = \begin{pmatrix} e^{\Gamma z} \\ e^{-\Gamma z} \end{pmatrix} \tag{4.45.1}$$

where Γ is the diagonal matrix

$$\Gamma = \begin{pmatrix} \gamma_1 & \dots & 0 \\ \vdots & & \\ 0 & \dots & \gamma_n \end{pmatrix}$$

with γ_i non-zero and constant. Note that

$$(\mathcal{G}, \mathcal{G}') = \begin{pmatrix} e^{\Gamma z} & 0 \\ 0 & e^{-\Gamma z} \end{pmatrix}\begin{pmatrix} I & \Gamma \\ I & -\Gamma \end{pmatrix}$$

and

$$\det\begin{pmatrix} I & \Gamma \\ I & -\Gamma \end{pmatrix} = \det\begin{pmatrix} 2I & 0 \\ I & -\Gamma \end{pmatrix}$$

thus $(\mathcal{G}\mathcal{G}')$ is always non-singular, and g is 1-nondegenerate.

Let T and $\tilde T$ be elements of $\mathcal{G}l(2n, \mathbf{C})$, and define holomorphic curves f and $\tilde f$ from $\mathbf{C}$ into $\mathcal{G}r(n, \mathbf{C}^{2n})$, representing f by $\mathfrak{F}$ and $\tilde f$ by $\tilde{\mathfrak{F}}$, where

$$\mathfrak{F} = T\mathcal{G}, \quad \tilde{\mathfrak{F}} = \tilde T\mathcal{G}. \tag{4.45.2}$$

The curves f and $\tilde f$ are 1-nondegenerate since g is.

Suppose that f and $\tilde f$ are congruent via a unitary on $\mathbf{C}^{2n}$; then there exists U in $U(2n)$ and a holomorphic $n \times n$ non-singular matrix A such that

$$U\mathfrak{F} = \tilde{\mathfrak{F}}A$$

or equivalently

$$BG = GA, \tag{4.45.3}$$

where

$$B = \tilde{T}^{-1}UT. \tag{4.45.4}$$

Let

$$B = \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix} \tag{4.45.5}$$

where each B_{ij} is an $n \times n$ matrix.

LEMMA 4.46. *If f and $\tilde{f}$ are holomorphic curves in $Gr(n, \mathbf{C}^{2n})$ defined by* (4.45.2), *they are congruent only if there exists a $2n \times 2n$ unitary U such that if the B_{ij} are defined by* (4.45.4) *and* (4.45.5), *then*

$$B_{11} = B_{22} \quad \text{and } B_{12} = B_{21}, \tag{4.46.1}$$

where

$$\Gamma B_{11} = B_{11}\Gamma \quad \text{and } \Gamma B_{12} = -B_{12}\Gamma. \tag{4.46.2}$$

Proof. Now

$$G'' = G\Gamma^2$$

so G is a Schwarzian representative for g and S_0 is Γ^2. By Proposition 4.40, (4.45.3) implies that A is constant.

By (4.45.3), we have

$$\begin{pmatrix} B_{11}e^{\Gamma z} + B_{12}e^{-\Gamma z} \\ B_{21}e^{\Gamma z} + B_{22}e^{-\Gamma z} \end{pmatrix} = \begin{pmatrix} e^{\Gamma z}A \\ e^{-\Gamma z}A \end{pmatrix} \tag{4.46.3}$$

and by taking derivatives (since A is constant) we get

$$\begin{aligned} B_{11}\Gamma e^{\Gamma z} - B_{12}\Gamma e^{-\Gamma z} &= \Gamma e^{\Gamma z}A \\ &= \Gamma B_{11}e^{\Gamma z} + \Gamma B_{12}e^{-\Gamma z} \end{aligned}$$

which implies

$$(B_{11}\Gamma - \Gamma B_{11})e^{2\Gamma z} = \Gamma B_{12} + B_{12}\Gamma.$$

Thus (4.46.2) holds, which implies that

$$B_{11}e^{\Gamma z} = e^{\Gamma z}B_{11} \quad \text{and } B_{12}e^{-\Gamma z} = e^{\Gamma z}B_{12}. \tag{4.46.4}$$

Thus from (4.46.3),

$$A = B_{11} + B_{12}$$

so by (4.46.4)

$$\begin{aligned} e^{-\Gamma z} A &= B_{11} e^{-\Gamma z} + B_{12} e^{\Gamma z} \\ &= B_{21} e^{\Gamma z} + B_{22} e^{-\Gamma z} \quad \text{by (4.46.3)} \end{aligned}$$

and (4.46.1) follows, since

$$(B_{21} - B_{12}) e^{2\Gamma z} = B_{22} - B_{11}.$$

LEMMA 4.47. *Let f and $\tilde{f}$ be curves in $\mathcal{G}r(n, \mathbf{C}^{2n})$ defined by (4.45.2) where we assume in addition that Γ is real diagonal and*

(4.47.1) $$T = \begin{pmatrix} I & 0 \\ 0 & P \end{pmatrix} \quad \text{and} \quad \tilde{T} = \begin{pmatrix} P^{-1} & 0 \\ 0 & I \end{pmatrix}$$

for some positive definite $n \times n$ matrix P. Then f and $\tilde{f}$ have contact of order 1.

Proof. The curve f is represented by

$$\begin{pmatrix} e^{\Gamma z} \\ P e^{-\Gamma z} \end{pmatrix} = \begin{pmatrix} I \\ P e^{-2\Gamma z} \end{pmatrix} e^{\Gamma z}$$

and thus f is also represented by

$$\begin{pmatrix} I \\ P e^{\tilde{\Gamma} z} \end{pmatrix}$$

where $\tilde{\Gamma} = -2\Gamma$. By (4.44.2)

(4.47.2) $$\begin{aligned} \mathcal{K}(\sigma) &= -(I + e^{\tilde{\Gamma}\bar{z}} P^2 e^{\tilde{\Gamma} z})^{-1} e^{\tilde{\Gamma} z} \tilde{\Gamma} P (I + P e^{\tilde{\Gamma} z} e^{\tilde{\Gamma}\bar{z}} P)^{-1} P \tilde{\Gamma} e^{\tilde{\Gamma} z} \\ &= e^{-\tilde{\Gamma} z} \{ -(e^{-\tilde{\Gamma} z} e^{-\tilde{\Gamma}\bar{z}} + P^2)^{-1} \tilde{\Gamma} e^{-\tilde{\Gamma} z} e^{-\tilde{\Gamma}\bar{z}} (e^{-\tilde{\Gamma} z} e^{-\tilde{\Gamma}\bar{z}} + P^2)^{-1} P^2 \tilde{\Gamma} \} e^{\tilde{\Gamma} z}. \end{aligned}$$

In a similar manner $\tilde{f}$ is represented by

$$\begin{pmatrix} P^{-1} e^{\Gamma z} \\ e^{-\Gamma z} \end{pmatrix} = \begin{pmatrix} I \\ e^{\tilde{\Gamma} z} P \end{pmatrix} P^{-1} e^{\Gamma z},$$

so $\tilde{f}$ is represented by

$$\begin{pmatrix} I \\ e^{\tilde{\Gamma} z} P \end{pmatrix}$$

and

(4.47.3) $$\begin{aligned} \tilde{\mathcal{K}}(\sigma) &= -(I + P e^{\tilde{\Gamma}\bar{z}} e^{\tilde{\Gamma} z} P)^{-1} P \tilde{\Gamma} e^{\tilde{\Gamma}\bar{z}} (I + e^{\tilde{\Gamma} z} P^2 e^{\tilde{\Gamma}\bar{z}})^{-1} e^{\tilde{\Gamma} z} \Gamma P \\ &= P^{-1} \tilde{\Gamma}^{-1} (e^{-\tilde{\Gamma} z} e^{-\tilde{\Gamma}\bar{z}} + P^2) e^{\tilde{\Gamma} z} \mathcal{K}(\sigma) e^{-\tilde{\Gamma} z} (e^{-\tilde{\Gamma} z} e^{-\tilde{\Gamma}\bar{z}} + P^2)^{-1} \Gamma P. \end{aligned}$$

Thus $\mathcal{K}(\sigma)$ and $\tilde{\mathcal{K}}(\sigma)$ are similar which implies (since $\mathcal{K}$ and $\tilde{\mathcal{K}}$ are self-adjoint) that $\mathcal{K}$ and $\tilde{\mathcal{K}}$ are unitarily equivalent and f and $\tilde{f}$ have contact of order one by Proposition 2.18.

PROPOSITION 4.48. *Let f and $\tilde{f}$ be defined as in Lemma 4.47, where Γ^2 has distinct entries on the diagonal and P^2 is not diagonal. Then f and $\tilde{f}$ have contact of order one but are not congruent.*

Proof. The curves f and $\tilde{f}$ have contact of order one by the previous Lemma. By Lemma 4.46, if f and $\tilde{f}$ are congruent, then Γ^2 commutes with B_{12}, so B_{12} is diagonal and thus by (4.46.2) B_{12} is 0, and B_{11} is diagonal.

By (4.45.4),

$$U = \begin{pmatrix} P^{-1}B_{11} & 0 \\ 0 & B_{11}P^{-1} \end{pmatrix}$$

is unitary, so

$$P^2 = B_{11}B_{11}^*$$

which is diagonal, a contradiction.

4.49. Note that if n is 2 then f and $\tilde{f}$ are necessarily generic off a closed nowhere dense subset of $\mathbf{C}$. The only other possibility when n is 2 would be for E_f and $E_{\tilde{f}}$ to be 0-umbilic, but then E_f and $E_{\tilde{f}}$ would be equivalent to order one, so equivalent, and thus f and $\tilde{f}$ would be congruent.

For n greater than 2, we are unable to show if there exists a choice of T and $\tilde{T}$ so that f and $\tilde{f}$ have contact of order 2, but are not congruent. By Proposition 4.17 it would suffice to work with C^∞ bundles with metric preserving connections that satisfy (4.17.1) and investigate the eigenvalue structures which could arise, but this seems very difficult.

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