THE MINIMUM OF A BINARY QUARTIC FORM (I).

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I. Introduction.

1. The question of the lower bound or minimum of an algebraic form $\varphi(x_1, \ldots, x_m)$ for integer values, not all zero, of the variables $x_1, \ldots, x_m$ is an important one and has attracted a great deal of attention for many years. The problem is a difficult one, however, and relatively few results are known.

Confining our attention to the case of forms with real coefficients in the two variables x, y, the results for the binary quadratic form are classical. If

$$\varphi(x, y) = ax^2+bxy+cy^2$$

is a binary quadratic of discriminant $D = b^2-4ac$, the result is that there exist integers x, y, not both zero, such that

$$|\varphi(x, y)| \leq k\sqrt{|D|}\,, \tag{1.1}$$

where $k = \dfrac{1}{\sqrt{3}}$ when $D < 0$ (that is, when the quadratic has complex roots[1]), and $k = \dfrac{1}{\sqrt{5}}$ when $D > 0$ (that is, when the quadratic has real roots). When $D = 0$ the lower bound is trivially found to be zero. These results are best possible, in the sense that the inequality (1.1) is no longer true for all forms of the type specified if the constant k is replaced by a smaller number.

Estimates for the lower bound of a binary cubic form were given many years ago by ARNDT (1) and HERMITE (2), but the best possible results were obtained only recently, by MORDELL (3). If now

[1] By the "roots" of a binary form $\varphi(x, y)$ we mean the roots of $\varphi(x, 1) = 0$.

$$\varphi(x, y) = ax^3+bx^2y+cxy^2+dy^3$$

is a binary cubic with determinant, or negative discriminant,

$$D = 27a^2d^2-18abcd-b^2c^2+4ac^3+4db^3 ,$$

the result is that

$$|\varphi(x, y)| \leq k|D|^{\frac{1}{4}} \tag{1.2}$$

for integers $x, y \neq 0, 0$, where $k = 23^{-\frac{1}{4}}$ if $D > 0$ and $k = 49^{-\frac{1}{4}}$ if $D < 0$. These cases correspond to cubics with just one and three real roots respectively. Again the lower bound is zero if $D = 0$, when the cubic has a repeated root.

The purpose of this investigation is to provide estimates for the lower bounds of binary quartic forms. Estimates of this sort have been given by Hermite (4) and Julia (5), but their results are in general very crude. A few best possible results are known, these being due to Mordell (6, 7), Derry (8) and Mahler (9). These results will be mentioned later in the appropriate places. We find here the best possible results in many cases, including all quartic forms with four distinct complex roots, and an infinity of (in fact "most") forms with four distinct real roots. In all cases estimates are given which are, with one exception, better than any previously known.

2. Much light is thrown on these problems by considering them from a rather more general point of view. Instead of seeking directly the lower bound of $\varphi(x, y)$ for integers x, y, we might investigate its lower bound for points x, y of a lattice L of determinant Δ defined by

$$x = \alpha\xi+\beta\eta, \quad y = \gamma\xi+\delta\eta , \tag{2.1}$$

where $\alpha, \beta, \gamma, \delta$ are real numbers, $\Delta = |\alpha\delta-\beta\gamma| \neq 0$, and ξ, η take all integer values. Considering all possible lattices, that is all values of $\alpha, \beta, \gamma, \delta$, with given Δ, we are led to seek a constant k such that

$$|\varphi(x, y)| \leq (k+\varepsilon)\Delta^{\frac{1}{2}n} \tag{2.2}$$

for some point $x, y \neq 0, 0$ of every lattice of determinant Δ, where n is the degree of the form $\varphi(x, y)$ and ε is any positive number.

Now for a given linear transformation (2.1), we may consider the form $\varphi(x, y)$, of discriminant $D_\varphi \neq 0$, say, as a form $\psi(\xi, \eta)$ in the variables ξ, η, with discriminant $D = \Delta^{n(n-1)}D_\varphi$. Then the result (2.2) may be expressed in the form: There exist integers ξ, η, not both zero, such that

$$(2.3) \qquad |\psi(\xi, \eta)| \leq (k+\varepsilon)\left(\frac{D}{D_\varphi}\right)^{\frac{1}{2(n-1)}},$$

where $\psi(\xi, \eta)$ is any form of discriminant D obtainable from $\varphi(x, y)$ by a real linear transformation. Under these circumstances we shall say, for brevity, that $\psi(\xi, \eta)$ and $\varphi(x, y)$ are *transformable* into each other, and we shall refer to $\varphi(x, y)$ as a *standard form* for the class of forms $\psi(\xi, \eta)$.[1] Clearly a standard form is not unique; we may select any convenient member of the class of forms transformable into each other.

We remark here that if the form $\psi(\xi, \eta)$ possesses another invariant, say I of index l, so that $I = \Delta^l I_\varphi$, we may, if $I_\varphi \neq 0$, replace the right hand side of (2.3) by $(k+\varepsilon)(I/I_\varphi)^{\frac{n}{2l}}$. In the following we shall content ourselves with stating the results in one or other form.

In the case of binary quadratics we may take xy or x^2-y^2 as standard form when $D > 0$, and x^2+y^2 when $D < 0$. For binary cubics we might select x^3+y^3 when $D > 0$, and $xy(x+y)$ when $D < 0$. The facts are necessarily more complex for a binary quartic, since the possibilities regarding reality and equality of roots are then more numerous. Further we will find it necessary in some cases to use a standard form containing a parameter.

Our problem is thus reduced to one in the geometry of numbers, namely to find a constant k, preferably best possible, such that there exists a point $x, y \neq 0, 0$ of every lattice L of determinant Δ, with

$$(2.4) \qquad |f(x, y)| \leq (k+\varepsilon)\Delta^2,$$

where $f(x, y)$ is an appropriate standard form. We consider the region $\mathcal{R}$ defined by

$$|f(x, y)| \leq 1,$$

and seek the lower bound Δ^* of the determinants of lattices admissible with respect to $\mathcal{R}$, that is ones with no point other than the origin as an inner point of $\mathcal{R}$. Then the best possible value of k, say k^*, is $1/\Delta^{*2}$, and if the lower bound is attained by some admissible lattice with a point on the boundary of $\mathcal{R}$ we may put $\varepsilon = 0$ in (2.4). An admissible lattice with $\Delta = \Delta^*$ is called a critical lattice, and the corresponding form $\psi(\xi, \eta)$ a critical form. If we cannot determine Δ^*, but can show that every lattice of determinant Δ' (say) has a point other than the origin in $\mathcal{R}$, it follows that $k^* \leq 1/\Delta'^2$.

[1] Note that a canonical form, in the usual algebraic sense, need not be a standard form; for it may not be obtainable by a *real* linear transformation.

3. Consider a binary quartic

$$\psi(\xi, \eta) = a\xi^4+4b\xi^3\eta+6c\xi^2\eta^2+4d\xi\eta^3+e\eta^4$$

with real coefficients a, b, c, d, e. This form has two irreducible invariants $\mathcal{I}$ and $\mathcal{J}$, of indices 4 and 6 respectively, given by

$$\mathcal{I} = ae-4bd+3c^2 ,$$

$$\mathcal{J} = \begin{vmatrix} a & b & c \\ b & c & d \\ c & d & e \end{vmatrix} = ace+2bcd-ad^2-eb^2-c^3 .$$

The discriminant of the form is

$$\mathcal{D} = \mathcal{I}^3-27\mathcal{J}^2 .$$

We must examine the nature of the roots of $\psi(t, 1) = 0$, and for this purpose we need to introduce the further expressions

$$\mathcal{H} = b^2-ac, \quad \mathcal{K} = 2\mathcal{H}\mathcal{I}+3a\mathcal{J} .$$

It appears difficult to give an exact reference to the results we need. The question was dealt with in full detail by ARNDT (10), but he does not employ the invariants and his notation differs from ours. BURNSIDE and PANTON (11) may also be consulted. For the standard forms, and a general discussion of the algebra of the quartic, reference may be made to the works of SALMON (12) or ELLIOTT (13). In giving standard forms we suppose, as we may for our purpose, that $a \geq 0$.

We summarise below the results we need.

$\mathcal{D} < 0$. Two different real roots and two complex roots. Standard form

$$x^4+6mx^2y^2-y^4 .$$

$\mathcal{D} > 0$. (a) $\mathcal{H} > 0$, $\mathcal{K} > 0$. Four distinct real roots.

(b) Otherwise. Four distinct complex roots. Standard form in each case

$$x^4+6mx^2y^2+y^4 .$$

$\mathcal{D} = 0$. If $\mathcal{I} = 0$, ψ has three equal roots and so is of one of the forms $a(\xi-\omega\eta)^4$, $a(\xi-\omega\eta)^3(\xi-\omega'\eta)$, $\omega \neq \omega'$. We need not distinguish between these cases. If $\mathcal{I} \neq 0$ several cases arise.

(a) $\mathcal{K} = 0$. Then $\psi = a(\xi-\omega\eta)^2(\xi-\omega'\eta)^2$, $\omega \neq \omega'$.

(*i*) $\mathcal{H} > 0$. ω, ω' real.

(*ii*) $\mathcal{H} < 0$. ω, ω' complex.

(b) $\mathcal{H} \neq 0$. Then $\psi = a(\xi-\omega\eta)^2(\xi-\omega'\eta)(\xi-\omega''\eta)$, where ω, ω', ω'' are all different.

(*i*) $\mathcal{H} > 0$. ω', ω'' real.
Standard form $x^2(x^2-y^2)$.

(*ii*) $\mathcal{H} < 0$. ω', ω'' complex.
Standard form $x^2(x^2+y^2)$.

II. Quartics with $\mathcal{D} > 0$ (four complex roots).

4. We consider first the case of forms with four distinct complex roots. We find the best possible results for this case; later (in § 22) we discuss the results previously known.

Here we may take the usual canonical form

$$f(x, y) = x^4+6mx^2y^2+y^4 \tag{4.1}$$

as the standard form; the parameter m is a function of the invariants $\mathcal{I}$ and $\mathcal{J}$.[1] The invariants I and J and the discriminant D of this standard form are given by

$$I = 1+3m^2, \quad J = m(1-m^2), \quad D = (1-9m^2)^2 . \tag{4.2}$$

The nature of the roots being unchanged by a real linear substitution, the roots of $f(t, 1) = 0$ will be complex and distinct. Now $f(t, 1) = 0$ gives

$$t^2 = -3m\pm\sqrt{(9m^2-1)} , \tag{4.3}$$

and so this quantity must be negative and two-valued, or complex. Hence $9m^2 \neq 1$. If $9m^2 < 1$, t^2 is complex, while if $9m^2 > 1$, $t^2 < 0$ if, and only if, $m > 0$. Thus $m > -\frac{1}{3}$, $m \neq \frac{1}{3}$.

If we apply the substitution

$$x = \varkappa(X-Y), \quad y = \varkappa(X+Y) ,$$

where $\varkappa^4 = \frac{1}{2}(1+3m)^{-1}$, which is a real substitution for $m > -\frac{1}{3}$, we obtain

$$f(x, y) = X^4+6MX^2Y^2+Y^4 ,$$

where $M = \dfrac{1-m}{1+3m}$. Now if $m > \frac{1}{3}$, we find $-\frac{1}{3} < M < \frac{1}{3}$, so it is enough to take $-\frac{1}{3} < m < \frac{1}{3}$.

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[1] For given $\mathcal{I}$ and $\mathcal{J}$, m in (4.1) may take in general any one of six values; but only two of these arise from real transformations.

Although the reduction of a quartic to its canonical form is discussed at length in the references (12, 13) cited, and elsewhere, it seems impossible to give a reference to any straightforward method of finding the parameter m.[1] Hence we give here a simple method of calculating the m we have distinguished above.

If Δ is the determinant of the substitution transforming $f(x, y)$ into $\psi(\xi, \eta)$, it is known that $\Delta^2 m = z$, say, satisfies

$$4z^3 - \mathcal{I}z + \mathcal{J} = 0 .$$

Further $\mathcal{I} = \Delta^4(1+3m^2)$, so $\Delta^4 = \mathcal{I} - 3z^2$. Thus we have

$$m^2 = \frac{z^2}{\Delta^4} = \frac{z^2}{\mathcal{I} - 3z^2} = \frac{x^2}{3(1-x^2)} ,$$

where $x^2 = \dfrac{3z^2}{\mathcal{I}}$ satisfies

$$4x^3 - 3x + \mathcal{J}\left(\frac{3}{\mathcal{I}}\right)^{\frac{3}{2}} = 0 .$$

If we write

$$\cos\varphi = -\mathcal{J}\left(\frac{3}{\mathcal{I}}\right)^{\frac{3}{2}} ,\text{[2]} \tag{4.4}$$

we have $x = \cos\theta$, where $\theta = \frac{1}{3}\varphi$, $\frac{1}{3}(\varphi \pm 2\pi)$, and so $m^2 = \frac{1}{3}\cot^2\theta$. (This expression exhibits the six values of m.) Note, however, that we require $|m| < \frac{1}{3}$, and also that then, for a real transformation, the signs of m and $\mathcal{J}$ must be the same, since $\mathcal{J} = \Delta^6 m(1-m^2)$. The appropriate value of m is thus given by

$$m = \frac{1}{\sqrt{3}} \cot\frac{\varphi}{3} , \tag{4.5}$$

where φ is chosen to satisfy (4.4) and $\pi < \varphi < 2\pi$.

5. We now define a region $\mathcal{R}$ by $|f(x, y)| \leq 1$, and investigate its shape. Since the roots of f are complex, we have $f(x, y) \geq 0$ for real x, y, and so $\mathcal{R}$ is bounded by the curve $\mathcal{C}$ given by $f(x, y) = 1$. Since

$$f(x, y) = f(-x, y) = f(y, x) ,$$

we see at once that $\mathcal{R}$ is symmetrical about the lines $x = 0$, $y = 0$, $y = \pm x$, and so also symmetrical about the origin O.

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[1] An explicit formula for m^2 is given by Faà de Bruno (14), but it is in an inconvenient form for our application.

[2] This determines a real φ, since $\mathcal{D} = \mathcal{I}^3 - 27\mathcal{J}^2 > 0$.

If (x, y) is a point on $\mathscr{C}$ and we write $t = y/x$, $\varrho^2 = x^2+y^2$, $\varrho \geq 0$, we have $x^4 f(1, t) = 1$ and

$$\varrho^2 = \frac{1+t^2}{\sqrt{f(1, t)}}\,.$$

Since $f(1, t) \neq 0$ for real t, ϱ is everywhere finite. It is also a single-valued function of t. It follows that $\mathscr{R}$ is a bounded star domain. A brief calculation gives

$$\varrho \frac{d\varrho}{dt} = (1-3m)t(1-t^2)f^{-\frac{3}{2}}\,.$$

Since $1-3m > 0$, clearly $t = 0$ gives a minimum and $t = \pm 1$ a maximum value of ϱ, and we deduce

$$1 \leq \varrho^2 \leq \left(\frac{2}{1+3m}\right)^{\frac{1}{2}}. \tag{5.1}$$

The curve $\mathscr{C}$ has no double points, so its points of inflection are given by its intersections with the Hessian

$$mx^4+(1-3m^2)x^2y^2+my^4 = 0\,,$$

that is, with the lines

$$y = \pm\lambda x, \quad y = \pm x/\lambda\,,$$

where λ is any root of

$$m\lambda^4+(1-3m^2)\lambda^2+m = 0\,.$$

If $m \neq 0$, this gives

$$2m\lambda^2 = 3m^2-1+\sqrt{\{(1-m^2)(1-9m^2)\}}\,, \tag{5.2}$$

and λ is real only if $-\frac{1}{3} < m < 0$. If $m = 0$, we have $\lambda = 0$, but the points of intersection, e. g. $(0, 1)$, are then points of undulation, not inflections. Thus the region $\mathscr{R}$ is convex if $0 \leq m < \frac{1}{3}$, and non-convex if $-\frac{1}{3} < m < 0$.

A typical non-convex $\mathscr{R}$ is illustrated in Fig. 1 (for $m = -\frac{3}{10}$).

When the region is non-convex, we shall call an arc of $\mathscr{C}$ lying between two consecutive points of inflection, and including an intersection with one of the axes, a "concave" arc, and the remaining arcs "convex" arcs.

6. When the region $\mathscr{R}$ is convex, the theory of Minkowski (15) concerning convex regions symmetrical about the origin may be applied. The problem is thus reduced to that of finding the parallelogram of minimum area which has one vertex at O and the other three vertices on $\mathscr{C}$. This problem was solved for $m = 0$ by Mordell (7), and later for $0 \leq m \leq \frac{1}{3}$ by Derry (8). We now show that the method

of Minkowski may be generalised to apply to the regions under consideration, even when they are non-convex.

We require first

Lemma 1. *Let $P\colon (x, y)$ be a point on $\mathscr{C}$ with $x \geq 0$, $0 \leq t \leq 1$, where $t = y/x$, and let s be the slope of the tangent to $\mathscr{C}$ at the point P. Then $1/s \leq t$, with equality only when P is the point $(1, 0)$.*

For we have

$$\frac{1}{s}-t = \frac{dx}{dy}-\frac{y}{x} = -\frac{y(y^2+3mx^2)}{x(x^2+3my^2)}-\frac{y}{x} = -\frac{y(1+3m)(x^2+y^2)}{x(x^2+3my^2)} \leq 0 ,$$

since $x^2+3my^2 \geq (1+3m)y^2 \geq 0$. Equality occurs only if $y = 0$, and then $x = 1$.

We now proceed to prove the fundamental

Lemma 2. *Δ^* is the lower bound of the determinants of admissible lattices with six points on $\mathscr{C}$.*

It suffices to prove that any admissible lattice may be deformed into another with six points on $\mathscr{C}$, and with an equal or smaller determinant. Since the region $\mathscr{R}$ is bounded, admissible lattices certainly exist. Let $OABC$ be a cell of an admissible lattice with no point on $\mathscr{C}$, and imagine the lattice deformed so that A moves along OA towards O, whilst AB remains parallel to OC, until a point of the lattice first appears on $\mathscr{C}$. Call this point A'; then by symmetry its image in O also lies on $\mathscr{C}$. Every point of OA' is in $\mathscr{R}$, and so OA' contains no lattice points other than O and A'. Hence there is a point B' such that (A', B') is a basis of the deformed lattice. Repeat the above procedure with the point B' in place of A. In this way we arrive at an admissible lattice of smaller determinant with four points on $\mathscr{C}$.

We now show that, if no other point of this lattice is on $\mathscr{C}$, we may again deform it to bring a third pair of points on to $\mathscr{C}$, at the same time reducing its determinant still further. Let P and Q be two independent[1] points of the lattice on $\mathscr{C}$. Such points exist, since only two lattice points can be collinear with O. Complete the parallelogram $OPRQ$; its area S is a multiple of Δ, the determinant of the lattice. We will assume, without further repetition, that the process to be described below is discontinued as soon as a further point of the lattice appears on $\mathscr{C}$. If either P or Q, say P, lies on a convex arc, it may be moved along $\mathscr{C}$ in such a way that its

[1] That is, not collinear with O.

distance from OQ, and so also Δ, decreases, until it lies on a concave arc at a point where the tangent is parallel to OQ. Hence we may suppose that both P and Q lie on concave arcs, and if P and Q are the points (x_1, y_1), (x_2, y_2) respectively, we may assume without loss of generality that they lie in the first quadrant, with $y_1 \leq x_2$, i. e. that $0 \leq t_1 \leq 1/t_2 \leq 1$, where $t_1 = y_1/x_1$ and $t_2 = y_2/x_2$. Now if $t_1 < 1/t_2$ we may decrease Δ by moving P along $\mathscr{C}$ until $t_1 = 1/t_2$. For, if s_1 is the slope of the tangent at P, we have $s_1 \geq 1/t_1 > t_2$ by Lemma 1, since $s_1 > 0$, and so during this process P moves towards OQ. Thus we may suppose hereafter that $t_1 = 1/t_2$, i. e. $x_2 = y_1$, $y_2 = x_1$. Then $S = x_1y_2 - x_2y_1 = x_1^2 - y_1^2$, and

$$\frac{dS}{dy_1} = 2x_1\left(\frac{dx_1}{dy_1} - \frac{y_1}{x_1}\right) \leq 0 \ ,$$

by Lemma 1. Further, $S = 0$ when $x_1 = y_1$, and it follows that we may reduce Δ steadily to zero by moving P and Q simultaneously, keeping $t_1 = 1/t_2$, i. e. $x_2 = y_1$. Now Δ clearly has a positive lower bound, since $\mathscr{R}$ contains the circle $x^2+y^2 \leq 1$, by (5.1), and so the process of deformation must be terminated at some stage by the appearance of another pair of lattice points on $\mathscr{C}$. This proves the lemma.

We remark here that the lower bound Δ^* cannot be attained by a lattice unless it has six points on $\mathscr{C}$. This follows immediately from the above proof if it is noted that, by Lemma 1, $\dfrac{dS}{dy_1} < 0$ unless $y_1 = 0$.

Our problem is thus reduced to that of finding the lower bound of the determinants of all admissible lattices with six points on $\mathscr{C}$. Since, however, these six points may in general occupy a variety of different positions in the lattice, we have still to solve a number, perhaps a large number, of minimum problems. As these problems are somewhat troublesome, we find it convenient to deal with the question by this direct method only when the number of minimum problems involved is not more than two. This will be so if the dimensions of $\mathscr{R}$ are sufficiently small; the range of values of m that we consider in this way is determined by the following lemma.

Lemma 3. *Let $P = uA+vB$ be a point of a lattice L with a reduced basis*[1] *(A, B). Suppose the points A, B are not inner points of the region $\mathscr{R}$, with $m \geq m_0 = -0{\cdot}259$. Then if $|u| \geq 2$ or $|v| \geq 2$, P is not a point of $\mathscr{R}$.*

[1] Every lattice has at least one *reduced basis* (A, B), which has the property that the angle θ between the vectors OA and OB satisfies $60° \leq \theta \leq 120°$; see BACHMANN (16).

Let $OA = a$, $OB = b$, and let ϱ_0 be the maximum value of the radius vector ϱ of $\mathscr{C}$. Then

$$\varrho_0^2 = \left(\frac{2}{1+3m}\right)^{\frac{1}{2}} \leq \left(\frac{2}{0{\cdot}223}\right)^{\frac{1}{2}} < 3\,.$$

Since $\varrho \geq 1$ by (5.1), we have $a \geq 1$, $b \geq 1$. If $|u| \geq 2$,

$$OP^2 \geq u^2a^2+v^2b^2-uvab = (vb-\tfrac{1}{2}ua)^2+\tfrac{3}{4}u^2a^2 \geq \tfrac{3}{4}u^2 \geq 3 > \varrho_0^2\,;$$

and the same result is clearly true if $|v| \geq 2$.

If $m \geq m_0$, it follows from Lemma 3 that a necessary and sufficient condition for a lattice L with a reduced basis (A, B) to be admissible is that the points $uA+vB$ with $|u| \leq 1$, $|v| \leq 1$, $(u, v) \neq (0, 0)$ should not be inner points of $\mathscr{R}$. Further, if L is admissible and has six points on $\mathscr{C}$, then $|u| \leq 1$, $|v| \leq 1$ for each of these points. Now there are just eight points of L with $|u| \leq 1$, $|v| \leq 1$, excluding the origin $u = v = 0$. These eight points are the vertices and the mid-points of the sides of a parallelogram formed by four cells of the lattice which meet at O. Hence the remaining two points are either

(I) opposite vertices, or

(II) mid-points of opposite sides

of this parallelogram. Correspondingly, the six points of L on $\mathscr{C}$ are of the form P, Q, $P-Q$ and their images in O, or P, $P+Q$, $P-Q$ and their images in O, where in each case P, Q is a basis (not necessarily reduced) of the lattice. We note, however, that the points P, Q, $P-Q$ in the first case must include a reduced basis. We shall refer to admissible lattices whose points on $\mathscr{C}$ are of the two forms mentioned above as lattices of type I and type II respectively.

Thus our problem, for $m \geq m_0$, becomes that of finding the lower bound of the determinants of lattices of type I and type II. Although this provides us directly with the answer to our original problem only when $m \geq m_0$, we require some of the results in any case. In consequence, we will not suppose in the following that $m \geq m_0$ until this is explicitly stated.

7. We now consider a lattice L of determinant Δ defined by

(7.1) $$x = \alpha\xi+\beta\eta, \quad y = \gamma\xi+\delta\eta, \quad \Delta = \alpha\delta-\beta\gamma\,,$$

and having the points P: (α, γ), Q: (β, δ) and $P-Q = R$: $(\alpha-\beta, \gamma-\delta)$ on $\mathscr{C}$. Write

(7.2) $$f(x, y) = \psi(\xi, \eta) = a\xi^4+4b\xi^3\eta+6c\xi^2\eta^2+4d\xi\eta^3+e\eta^4\,.$$

On making the substitution, we find

$$a = \alpha^4+6m\alpha^2\gamma^2+\gamma^4\ , \tag{7.3}$$

$$b = \alpha^3\beta+\gamma^3\delta+3m\alpha\gamma(\alpha\delta+\beta\gamma)\ , \tag{7.4}$$

$$c = \alpha^2\beta^2+\gamma^2\delta^2+m(\alpha^2\delta^2+4\alpha\beta\gamma\delta+\beta^2\gamma^2)\ , \tag{7.5}$$

$$d = \alpha\beta^3+\gamma\delta^3+3m\beta\delta(\alpha\delta+\beta\gamma)\ , \tag{7.6}$$

$$e = \beta^4+6m\beta^2\delta^2+\delta^4\ . \tag{7.7}$$

We proceed to deduce a relation between Δ and the parameter c.[1] We might, of course, obtain a relation between Δ and some more obvious parameter, e. g. γ/α, but the procedure we follow has the great advantage of yielding simple criteria for deciding whether or not a given lattice point is a point of $\mathcal{R}$.

The points $(\xi, \eta) = (1, 0)$, $(0, 1)$ and $(1, -1)$ lie on $\mathcal{C}$, and so

$$\psi(1, 0) = \psi(0, 1) = \psi(1, -1) = 1\ . \tag{7.8}$$

From (7.2) and (7.8) we find

$$a = e = 1\ , \tag{7.9}$$

$$4(b+d) = 6c+1\ . \tag{7.10}$$

From the invariants in (7.2) we have

$$I\Delta^4 = ae-4bd+3c^2\ , \tag{7.11}$$

$$J\Delta^6 = ace+2bcd-ad^2-eb^2-c^3\ . \tag{7.12}$$

Using (7.9) and (7.10), these become

$$I\Delta^4 = 1-4bd+3c^2\ , \tag{7.13}$$

$$J\Delta^6 = 2bd(c+1)+c-c^3-\tfrac{1}{16}(6c+1)^2\ . \tag{7.14}$$

Eliminating bd between the last two equations, we derive

$$8c^3-12c^2+12c+7-8(c+1)I\Delta^4-16J\Delta^6 = 0\ . \tag{7.15}$$

We write $\Delta^2 = \Lambda$ here, and later use the same notation with suffixes. Then (7.15) becomes

$$\Phi(\Lambda, c) = 0\ , \tag{7.16}$$

where

$$\Phi(\Lambda, c) = 8c^3-12c^2+12c+7-8(c+1)I\Lambda^2-16J\Lambda^3\ . \tag{7.17}$$

[1] Our method is a development of that used by MORDELL (7) for the case $m = 0$ (and so $J = 0$).

We suppose now that $m \geq m_0$, and find conditions for the lattice L to be of type I. As noted above, the points P, Q and $P-Q=R$ include a reduced basis (A, B). Hence a necessary and sufficient condition is that the points $uA+vB$ with $|u| \leq 1$, $|v| \leq 1$, $(u, v) \neq (0, 0)$ should not be inner points of $\mathcal{R}$. We identify these points in terms of the basis (P, Q). We have

$$\xi P+\eta Q = (\xi+\eta)Q+\xi R = (\xi+\eta)P-\eta R ,$$

so if any two of $|\xi|$, $|\eta|$, $|\xi+\eta|$ are ≥ 2 it follows that $|u| \geq 2$ or $|v| \geq 2$.

If $|\xi| \geq 3$ we have $|\xi+\eta| \geq |\xi|-|\eta| \geq 3-|\eta|$, so either $|\eta| \geq 2$ or $|\xi+\eta| \geq 2$. Hence $|\xi| \leq 2$, and we may suppose $\xi \geq 0$. Taking $\xi = 0, 1, 2$ in turn, and rejecting the values $(\xi, \eta) = (0, 0)$, $(0, \pm 1)$, $(1, 0)$, $(1, -1)$ which correspond to the origin and the points on $\mathcal{C}$, we find that our condition is that the points $(\xi, \eta) = (1,1)$, $(1, -2)$ and $(2, -1)$ must not be inner points of $\mathcal{R}$. It follows that L is of type I if

$$\psi(1, 1) \geq 1, \ \psi(1, -2) \geq 1, \ \psi(2, -1) \geq 1 . \tag{7.18}$$

Using (7.9) and (7.10), these conditions become

$$6c \geq -1, \ 3(c-b) \leq 1, \ 3(c-d) \leq 1 . \tag{7.19}$$

We remark that equality signs in (7.19) correspond to those in (7.18) and so occur only if the corresponding lattice point is on $\mathcal{C}$.

8. We turn our attention now to lattices of type II. Consider then a lattice with a basis (α, γ), (β, δ) and with the points (α, γ), $(\alpha+\beta, \gamma+\delta)$ and $(\alpha-\beta, \gamma-\delta)$ on $\mathcal{C}$. Again make the transformation (7.1) and write $f(x, y) = \psi(\xi, \eta)$, as before.

Now the points $(\xi, \eta) = (1, 0)$, $(1, 1)$ and $(1, -1)$ lie on $\mathcal{C}$, and we deduce

$$a = 1, \ b+d = 0, \ e = -6c . \tag{8.1}$$

If the lattice is of type II the point (β, δ), i. e. $(\xi, \eta) = (0, 1)$, is not an inner point of $\mathcal{R}$, and so $\psi(0, 1) \geq 1$. This gives the condition

$$6c \leq -1 . \tag{8. 2}$$

Following the same procedure as that we adopted previously, we find

$$I\Delta^4 = 4b^2-6c+3c^2 , \tag{8.3}$$

$$J\Delta^6 = b^2(4c-1)-6c^2-c^3 , \tag{8.4}$$

and thence

$$\Psi(\Delta, c) = 16c^3-3c^2+6c+(1-4c)I\Delta^2+4J\Delta^3 = 0 . \tag{8.5}$$

9. We are now in a position to solve our minimum problems. We first take the case $- < \frac{1}{6}\, m < \frac{1}{3}$, and we will assume, until the contrary is stated, that this relation is satisfied. We must consider lattices both of type I and of type II; we commence with those of type I.

From the symmetry of the region $\mathscr{R}$, certain lattices of the type investigated in § 7 will obviously have maximum or minimum values of $\varDelta$. In particular, this will be so for the lattice, which plainly exists, with the points (α, α), (β, δ) and $(-\delta, -\beta)$ on $\mathscr{C}$, where $\alpha = \beta+\delta > 0$.

If the determinant of this lattice, say L_1, is $\varDelta_1$, we have

$$\varDelta_1 = \alpha(\delta-\beta) = \delta^2-\beta^2 .$$

Since (α, α) and (β, δ) lie on $\mathscr{C}$,

$$2(1+3m)\alpha^4 = 1, \quad \beta^4+6m\beta^2\delta^2+\delta^4 = 1 . \tag{9.1}$$

Now $2\beta = \beta+\delta+\beta-\delta = \alpha-\omega$, where $\omega = \delta-\beta$. Also $2\delta = \alpha+\omega$ and $\varDelta_1 = \alpha\omega$. Then, by (9.1) ,

$$(\alpha-\omega)^4+6m(\alpha^2-\omega^2)^2+(\alpha+\omega)^4 = 16 ,$$

i. e.
$$2(\alpha^4+6\alpha^2\omega^2+\omega^4)+6m(\alpha^4-2\alpha^2\omega^2+\omega^4) = 16 ,$$

i. e.
$$2(1+3m)\omega^4+12(1-m)\alpha^2\omega^2+2(1+3m)\alpha^4-16 = 0 ,$$

which gives

$$4(1+3m)^2\varDelta_1^4+12(1-m)\varDelta_1^2-15 = 0 . \tag{9.2}$$

Hence, remembering $\varDelta_1^2 > 0$, we obtain

$$\varDelta_1 = \varDelta_1^2 = \frac{1}{2(1+3m)^2}\{2\sqrt{[6(6m^2+3m+1)]}-3(1-m)\} . \tag{9.3}$$

Now L_1 has the points (β, δ), $(-\delta, -\beta)$ and $(\beta+\delta, \delta+\beta)$, i. e. (α, α), on $\mathscr{C}$. Put

$$x = \beta\xi-\delta\eta, \quad y = \delta\xi-\beta\eta , \tag{9.4}$$

and write

$$f(x, y) = \varphi_0(\xi, \eta) = \xi^4+4b_0\xi^3\eta+6c_0\xi^2\eta^2+4d_0\xi\eta^3+\eta^4 . \tag{9.5}$$

By (7.5), we have

$$\begin{aligned} c_0 &= 2\beta^2\delta^2+m(\beta^4+4\beta^2\delta^2+\delta^4) \\ &= \beta^4+6m\beta^2\delta^2+\delta^4-(1-m)(\beta^4-2\beta^2\delta^2+\delta^4) \\ &= 1-(1-m)\varDelta_1^2 , \end{aligned}$$

that is

(9.6) $$c_0 = 1-(1-m)\Lambda_1 .$$

Since $\psi_0(\xi, \eta) = f(x, y) = f(-y, -x) = \psi_0(\eta, \xi)$, we have $b_0 = d_0$, and so

(9.7) $$b_0 = d_0 = \tfrac{1}{2}(b_0+d_0) = \tfrac{1}{8}(6c_0+1) = \tfrac{1}{8}\{7-6(1-m)\Lambda_1\} .$$

We now require some estimates for the numbers c_0 and Λ_1.

Lemma 4. *If* $-\frac{1}{6} < m < \frac{1}{3}$, *then* $-\frac{1}{6} < c_0 < \frac{1}{2}$.

From (9.6) we have $(1-m)\Lambda_1 = 1-c_0$, so, using (9.2),

$$4(1+3m)^2\Lambda_1^2 = 15-12(1-c_0) = 3+12c_0 .$$

Thus we have to prove $1 < 2(1+3m)\Lambda_1 < 3$. If we put $m = \dfrac{3-K}{3+3K}$ in (9.3), we find

$$2(1+3m)\Lambda_1 = \sqrt{(K^2+15)}-K = \lambda(K) ,$$

say. Then

$$\lambda'(K) = \frac{K}{\sqrt{(K^2+15)}}-1 < 0 , \quad \frac{dm}{dK} = -\frac{4}{3(1+K)^2} < 0 ,$$

so that λ is a strictly increasing function of m. Finally, when $m = -\frac{1}{6}$, $K = 7$ and $\lambda(K) = 1$; while when $m = \frac{1}{3}$, $K = 1$ and $\lambda(K) = 3$. This proves the lemma.

Lemma 5. *If* $-\frac{1}{6} < m < \frac{1}{3}$, *then* $\Lambda_1 < 1$.

With the same notation as in the last lemma, we find

$$\Lambda_1 = \tfrac{1}{8}(1+K)\{\sqrt{(K^2+15)}-K\} = \mu(K) ,$$

say. Then

$$\mu'(K) = \frac{1}{8\sqrt{(K^2+15)}}\{\sqrt{(K^2+15)}-K\}\{\sqrt{(K^2+15)}-(1+K)\} ,$$

and $\mu'(K) = 0$ only when $K = 7$. The first two factors in the expression for $\mu'(K)$ are essentially positive, and the last factor, with $K = 7+\varepsilon$, is $-\frac{1}{8}\varepsilon+O(\varepsilon^2)$. Hence $K = 7$ is a maximum. But $K = 7$ when $m = -\frac{1}{6}$, and $\mu(7) = 1$; the result follows.

Now from (9.7) and Lemma 4,

$$c_0-b_0 = c_0-d_0 = c_0-\tfrac{1}{8}(6c_0+1) = \tfrac{1}{4}c_0-\tfrac{1}{8} < 0 ,$$

and $6c_0 > -1$. Thus the conditions (7.19) are satisfied by the lattice L_1, which is therefore of type I.

10. We now find the other roots of the cubic $\Phi(\Lambda_1, c) = 0$ given by putting $\Lambda = \Lambda_1$ in (7.17). The lattice L_1 has the points (α, α), (β, δ) and $(\alpha-\beta, \alpha-\delta)$, i. e. (δ, β), on $\mathscr{C}$. Put

(10.1) $$x = \alpha\xi'+\beta\eta', \quad y = \alpha\xi'+\delta\eta' ,$$

and write

(10.2) $$f(x, y) = \psi_1(\xi', \eta') = \xi'^4+4b_1\xi'^3\eta'+6c_1\xi'^2\eta'^2+4d_1\xi'\eta'^3+\eta'^4 .$$

Now if we put

(10.3) $$\xi = \xi'+\eta', \quad \eta = -\xi' ,$$

we find

$$x = \alpha\xi'+\beta\eta' = (\beta+\delta)\xi'+\beta\eta' = \beta(\xi'+\eta')+\delta\xi' = \beta\xi-\delta\eta ,$$

and similarly $y = \alpha\xi'+\delta\eta' = \delta\xi-\beta\eta$. Thus (10.1) is equivalent to (9.4) and (10.3), and it follows that

$$\psi_1(\xi', \eta') = \psi_0(\xi, \eta) .$$

Substituting (10.3) in (9.5) and comparing with (10.2), we find

(10.4) $$b_1 = 1-3b_0+3c_0-d_0 = 1+3c_0-4b_0 = 1+3c_0-\tfrac{1}{2}(6c_0+1) = \tfrac{1}{2} ,$$

(10.5) $$c_1 = 1+c_0-2b_0 = \tfrac{1}{4}\{1+2(1-m)\Lambda_1\} ,$$

(10.6) $$d_1 = 1-b_0 = \tfrac{1}{8}\{1+6(1-m)\Lambda_1\} .$$

It follows from the above that $\Phi(\Lambda_1, c) = 0$ for $c = c_0, c_1$. If the third root of this equation is c_2, we have, by (7.17),

(10.7) $$c_0+c_1+c_2 = \tfrac{3}{2} ,$$

giving

$$c_2 = \tfrac{3}{2}-c_0-c_1 = \tfrac{1}{4}\{1+2(1-m)\Lambda_1\} = c_1 .$$

Thus

(10.8) $$\Phi(\Lambda_1, c) = 8(c-c_0)(c-c_1)^2 .$$

Further, since $c_0 < \tfrac{1}{2}$, (10.7) gives $c_1 > \tfrac{1}{2} > c_0$.

We will prove that no lattice of type I has a determinant $\Delta < \Delta_1$, and that the value Δ_1 is attained only by L_1, and by its image in the y-axis, say L_1'. For this, we first show that we may suppose $c \geq c_0$. For suppose $c < c_0$. First, $\Lambda \neq \Lambda_1$, since $\Phi(\Lambda_1, c) < 0$, by (10.8). Suppose next that $0 < \Lambda < \Lambda_1$. From (7.13) and (9.7) we have

$$I\Lambda_1^2 = 1-4b_0d_0+3c_0^2 = 1+3c_0^2-\tfrac{1}{16}(6c_0+1)^2 = \tfrac{3}{4}\{(\tfrac{1}{2}-c_0)^2+1\}\,.$$

Since $c_0 < \frac{1}{2}$, $\frac{1}{2}-c > \frac{1}{2}-c_0 > 0$ and so

$$\begin{aligned}(b-d)^2 &= (b+d)^2-4bd = \tfrac{1}{16}(6c+1)^2-(1+3c^2)+I\Lambda^2\\ &= I\Lambda^2-\tfrac{3}{4}\{(\tfrac{1}{2}-c)^2+1\} < I\Lambda_1^2-\tfrac{3}{4}\{(\tfrac{1}{2}-c_0)^2+1\} = 0\,,\end{aligned}$$

which is impossible.

We may write

$$\begin{aligned}\Phi(\Lambda, c) &= \Phi(\Lambda_1, c)+\Phi(\Lambda, c)-\Phi(\Lambda_1, c)\\ &= \Phi(\Lambda_1, c)+\Phi(\Lambda, c_0)-\Phi(\Lambda_1, c_0)+8(c-c_0)I(\Lambda_1^2-\Lambda^2)\,,\end{aligned}$$

that is, using (10.8),

$$\Phi(\Lambda, c) = 8(c-c_0)\{(c-c_1)^2+I(\Lambda_1^2-\Lambda^2)\}+\Phi(\Lambda, c_0)\,. \tag{10.9}$$

Now $\Phi(\Lambda, c_0) > 0$ for $0 < \Lambda < \Lambda_1$. For, firstly, $\Phi(\Lambda_1, c_0) = 0$. Then from (7.17) we find, for $c = c_0$,

$$\frac{\partial\Phi}{\partial\Lambda} = -16\Lambda\{3J\Lambda+(c_0+1)I\}\,;$$

we show that this is negative. We note $I = 1+3m^2 \geq 1$ and $J = m(1-m^2)$. If now $m \geq 0$, $J \geq 0$, so

$$3J\Lambda+(c_0+1)I \geq c_0+1 > 0\,,$$

by Lemma 4. Also, if $-\frac{1}{6} < m < 0$, $J < 0$, and $\Lambda_1 = \dfrac{1-c_0}{1-m}$, by (9.6), so

$$\begin{aligned}3J\Lambda+(c_0+1)I &> 3J\Lambda_1+(c_0+1)I = 3m(1+m)(1-c_0)+(1+c_0)I\\ &> 3(-\tfrac{1}{6})(\tfrac{5}{6})(1-c_0)+1+c_0 = \tfrac{1}{12}(7+17c_0) > 0\,.\end{aligned}$$

Thus $\partial\Phi/\partial\Lambda < 0$, and so $\Phi(\Lambda, c_0) > \Phi(\Lambda_1, c_0) = 0$. It follows then from (10.9) that $\Phi(\Lambda, c) > 0$ for $c \geq c_0$, $0 < \Lambda < \Lambda_1$.

This establishes that Δ_1 is the minimum determinant of a lattice of type I.

11. We must now consider lattices of type II. If $m \geq 0$, the region $\mathcal{R}$ is convex and no such lattices can exist, for they would have three collinear points on the boundary $\mathcal{C}$, which is impossible. Suppose then $-\frac{1}{6} < m < 0$. We have $b^2 \geq 0$ and, by (8.2), $c \leq -\frac{1}{6}$. Then, from (8.3), it follows that $I\Lambda^2 \geq -6c+3c^2 \geq 1+\frac{1}{12}$. But $I = 1+3m^2 < 1+\frac{1}{12}$, so $\Lambda > 1 > \Lambda_1$, by Lemma 5. That is, there are no lattices of type II with determinant $\Delta \leq \Delta_1$.

12. We have proved that $\Delta^* = \Delta_1$ when $-\frac{1}{6} < m < \frac{1}{3}$, and, further, that the determinant Δ_1 is attained only for lattices of type I giving $c = c_0, c_1$. We now consider whether there are any other critical lattices than L_1. We require the following lemma.

Lemma 6. *For* $-\frac{1}{3} < m < \frac{1}{3}$, *the only transformations of* $f(x, y)$ *into itself, that is automorphisms of* $\mathcal{R}$, *are the reflections in the axes of symmetry and in the origin, namely,* $x = \pm\xi$, $y = \pm\eta$ *and* $x = \pm\eta$, $y = \pm\xi$, *where all the signs are independent of each other.*

Consider a transformation of the type (7.1) giving

$$f(x, y) = \psi(\xi, \eta) = \xi^4 + 6m\xi^2\eta^2 + \eta^4 .$$

Since the invariants are unchanged, we have

$$\Delta = \alpha\delta - \beta\gamma = \pm 1 . \tag{12.1}$$

Now from (7.5) we find, using (12.1),

$$m = \alpha^2\beta^2 + \gamma^2\delta^2 + m(1 + 6\alpha\beta\gamma\delta) ,$$

that is,

$$\alpha^2\beta^2 + \gamma^2\delta^2 + 6m\alpha\beta\gamma\delta = 0 . \tag{12.2}$$

Hence

$$(\alpha\beta - \gamma\delta)^2 + 2(1+3m)\alpha\beta\gamma\delta = 0 ,$$

and so $\alpha\beta\gamma\delta \leq 0$, since $1+3m > 0$. Similarly

$$(\alpha\beta + \gamma\delta)^2 - 2(1-3m)\alpha\beta\gamma\delta = 0 ,$$

and so $\alpha\beta\gamma\delta \geq 0$, since $1-3m > 0$. It follows that $\alpha\beta\gamma\delta = 0$, and so at least one of $\alpha, \beta, \gamma, \delta$ must be zero.

Suppose, e. g., $\alpha = 0$. Then, from (12.2), $\gamma\delta = 0$, while, from (12.1), $\beta\gamma = \pm 1$, and so $\gamma \neq 0$. Hence $\delta = 0$ and, by (7.3) and (7.7), $\gamma^4 = \beta^4 = 1$. Therefore $\alpha = 0$, $\beta = \pm 1$, $\gamma = \pm 1$, $\delta = 0$, where the signs are independent, giving $x = \pm\eta$, $y = \pm\xi$. The other results arise similarly on assuming $\beta = 0$, and this completes the proof of the lemma.

We now find all the critical lattices. Given Δ and c, the values of b and d are determined, except for order, by (7.10) and (7.13). Then, having found one lattice which gives these values of b, c, d, all such lattices will be given by the transforms of this one by the automorphisms of $\mathcal{R}$. In our problem we have $\Delta = \Delta_1$, $c = c_0, c_1$, and these conditions are satisfied by the lattice L_1. From the symmetry of L_1, the only other lattice produced by the automorphisms of $\mathcal{R}$ is its reflection in the y-axis, L_1'.

We have thus proved the following result.

Theorem 1. *If $-\frac{1}{6} < m < \frac{1}{3}$, there is a point x, y, other than the origin, of every lattice L of determinant Δ, such that*

$$|x^4+6mx^2y^2+y^4| \leq \frac{\Delta^2}{\Lambda_1},$$

where Λ_1 is given by (9.3). *This is the best possible result, the equality sign being required if, and only if, L is proportional*[1] *to one of the lattices L_1 or L_1'.*

We may take as a critical form, corresponding to the lattice L_1 or L_1', the form $\psi_1(\xi, \eta)$ defined by (10.2) with (10.4), (10.5) and (10.6); all other critical forms, e. g. $\psi_0(\xi, \eta)$, are equivalent to this. Writing

$$A = 4d_1 = \tfrac{1}{2}\{1+6(1-m)\Lambda_1\},$$

(10.5) gives $6c_1 = A+1$, and so

$$\psi_1(\xi, \eta) = \xi^4+2\xi^3\eta+(A+1)\xi^2\eta^2+A\xi\eta^3+\eta^4 .$$

Substituting from (9.3), we find

$$A = \frac{1}{(1+3m)^2}\{3(1-m)\sqrt{[6(6m^2+3m+1)]}-4(1-3m)\} . \tag{12.3}$$

Again, using (9.6) and Lemma 4, we find $2 < A < 4$, and, further, these bounds are best possible in the sense that any such value of A corresponds to an m in the range considered.

Recalling our introductory remarks, and effecting a trifling reduction, Theorem 1 leads immediately to

Theorem 2. *Let $\psi(\xi, \eta)$ be a binary quartic form with real coefficients and $\mathcal{D} > 0$, and either $\mathcal{H} \leq 0$ or $\mathcal{K} \leq 0$. Further, let $35^2\mathcal{J}^3 > 39^3\mathcal{J}^2$ if $\mathcal{J} < 0$, so that m given by* (4.5) *satisfies $-\frac{1}{6} < m < \frac{1}{3}$. Then there exist integers ξ, η, not both zero, such that*

[1] By a lattice "proportional" to the lattice $x = \alpha\xi+\beta\eta$, $y = \gamma\xi+\delta\eta$, we mean one defined by $x' = lx$, $y' = ly$, where l is a real non-zero constant.

$$|\psi(\xi, \eta)| \leq \frac{4\sqrt{\{6(6m^2+3m+1)\}}+6(1-m)}{15(1-9m^2)^{1/3}} \mathscr{D}^{\frac{1}{6}} .$$

This is the best possible result, the equality sign being required if, and only if, $\psi(\xi, \eta)$ is equivalent to a multiple of the form

$$\psi_1(\xi, \eta) = \xi^4+2\xi^3\eta+(A+1)\xi^2\eta^2+A\xi\eta^3+\eta^4 ,$$

with $2 < A < 4$. The value of A as a function of m is given by (12.3).

13. We now investigate the case $m \leq -\frac{1}{6}$. When $m < -\frac{1}{6}$ the lattice L_1 is no longer admissible, for the proof of Lemma 4 would then give $c_0 < -\frac{1}{6}$, in contradiction to the conditions (7.19).

We first put the last two inequalities in (7.19) in a different shape, not involving b and d explicitly. We note that both these inequalities cannot be false, since there is only one lattice point (and its image) in question. Then if we put

$$\Theta = 4\{1-3(c-b)\}\{1-3(c-d)\} ,$$

a necessary and sufficient condition for both these inequalities to hold is

$$\Theta \geq 0 .$$

Using (7.10) and (7.13), we find

$$\begin{aligned} \Theta &= 4\{(1-3c)^2+3(b+d)(1-3c)+9bd\} \\ &= 4(1-3c)^2+3(1-3c)(6c+1)+9(1+3c^2-I\Lambda^2) \\ &= 9c^2-15c+16-9I\Lambda^2 . \end{aligned}$$

Further, we note that $\Theta = 0$ only if $3(c-b) = 1$ or $3(c-d) = 1$, and then the lattice L has eight points on $\mathscr{C}$. Changing our notation, if need be, we may suppose these to be the points (α, γ), (β, δ), $(\alpha+\beta, \gamma+\delta)$, $(\alpha-\beta, \gamma-\delta)$ and their images in O.

If $m \geq m_0$, we may now put our problem, for lattices of type I, in the following form. We seek the lower bound, say Λ_0^*, of Λ corresponding to real lattices, with the conditions

$$\Lambda > 0,\ c \geq -\tfrac{1}{6},\ \Phi(\Lambda, c) = 0,\ \Theta \geq 0 . \tag{13.1}$$

We shall simply omit the provision concerning reality, and later verify that the solution thus obtained does in fact satisfy it.

Now the expression $\Phi(\Lambda, c)-\Phi(\Lambda, -\frac{1}{6})$ vanishes when $c = -\frac{1}{6}$, and so has a factor $6c+1$. By equating coefficients, we find that the other factor is $\frac{4}{27}\Theta$. Thus we have

$$27\Phi(\Lambda, c)-27\Phi(\Lambda, -\tfrac{1}{6}) = 4(6c+1)\Theta\ .$$

It follows then from (13.1) that Λ_0^* is the lower bound of $\Lambda > 0$ with

$$\Phi(\Lambda, -\tfrac{1}{6}) \leq 0\ .$$

On putting $c = -\frac{1}{6}$ in (7.17), we find

$$-27\Phi(\Lambda, -\tfrac{1}{6}) = 432J\Lambda^3+180I\Lambda^2-125\ .$$

Substituting $\Lambda = \dfrac{5}{6z}$, the equation $\Phi(\Lambda, -\frac{1}{6}) = 0$ becomes

$$z^3-Iz-2J = 0\ ,$$

the roots of which are $z = m\pm 1,\ -2m$. It follows that

$$\Phi(\Lambda, -\tfrac{1}{6}) = C(\Lambda-\Lambda_2)(\Lambda-\Lambda_2')(\Lambda-\Lambda_2'')\ ,$$

where

$$C = -16J > 0\ ,$$

$$\Lambda_2 = \frac{5}{6(1+m)} > 0, \quad \Lambda_2' = -\frac{5}{6(1-m)} < 0, \quad \Lambda_2'' = -\frac{5}{12m} > 0\ .$$

Further,

$$\Lambda_2''-\Lambda_2 = -\frac{5(1+3m)}{12m(1+m)} > 0\ .$$

Since $\Lambda_2'' > \Lambda_2 > 0 > \Lambda_2'$, we have $\Phi(\Lambda, -\frac{1}{6}) \leq 0$ if, and only if, $\Lambda \leq \Lambda_2' < 0$ or $\Lambda_2'' \geq \Lambda \geq \Lambda_2 > 0$. Thus $\Lambda_0^* = \Lambda_2$, and this value is attained if, and only if, $c = -\frac{1}{6}$ or $\Theta\ (\Lambda_2, c) = 0$; in each of these cases the corresponding lattices have eight points on $\mathscr{C}$. We remark here, for later reference, that the relation $m \geq m_0$ is not used in deriving the result stated in the last sentence, which is consequently true without this restriction.

14. We proceed to show that these values of Λ and c correspond to real lattices by actually determining the lattices.

Lemma 7. *If the points* (α, γ), (β, δ), $(\alpha+\beta,\ \gamma+\delta)$ *and* $(\alpha-\beta,\ \gamma-\delta)$ *lie on* $\mathscr{C}$, *then* $\beta = \pm\gamma$, $\delta = \mp\alpha$.

Make the substitution (7.1) and note that here $c = -\frac{1}{6}$, by the remark after (7.19). Hence $b = -d$ by (7.10), so we have

(14.1) $$f(x, y) = \psi(\xi, \eta) = \xi^4+4b\xi^3\eta-\xi^2\eta^2-4b\xi\eta^3+\eta^4\ .$$

But $\psi(\xi, \eta) = \psi(-\eta, \xi)$, and so

$$f(x, y) = f(\alpha\xi+\beta\eta, \gamma\xi+\delta\eta) = f(\beta\xi-\alpha\eta, \delta\xi-\gamma\eta) = f(x', y') ,$$

say. The transformation from x, y to x', y' is an automorphism of $\mathcal{R}$, and so, by Lemma 6, must be included in $x' = \pm x$, $y' = \pm y$ and $x' = \pm y$, $y' = \pm x$, with any choice of signs. We easily find that $x' = \pm x$, $y' = \pm y$; $x' = \pm x$, $y' = \mp y$ and $x' = \pm y$, $y' = \pm x$ each imply $\alpha = \beta = \gamma = \delta = 0$, which is impossible. Finally, the automorphisms $x' = \pm y$, $y' = \mp x$ give $\beta = \pm\gamma$, $\delta = \mp\alpha$.

Lemma 8. *If $-\frac{1}{3} < m \leq -\frac{1}{6}$, there exist unique numbers p and q, with $p > q \geq 0$, such that*

$$f(p, q) = f(p-q, p+q) = 1 .$$

Further, $q = 0$ only if $m = -\frac{1}{6}$.

We have

$$p^4+6mp^2q^2+q^4 = 1 , \tag{14.2}$$

$$(p-q)^4+6m(p^2-q^2)^2+(p+q)^4 = 1 ,$$

and so

$$\begin{aligned} 0 &= (p-q)^4+6m(p^2-q^2)^2+(p+q)^4-1 \\ &= (2+6m)(p^4+6mp^2q^2+q^4)-12p^2q^2(3m^2+2m-1)-1 , \end{aligned}$$

giving

$$p^2q^2 = \frac{6m+1}{12(3m-1)(1+m)} = K , \tag{14.3}$$

say. If $m = -\frac{1}{6}$, $K = 0$ and $p = 1$, $q = 0$. Excluding this case, we have $K > 0$ and, substituting for p^2 or q^2 from (14.3) in (14.2), we find that p^4 and q^4 are roots of

$$\zeta^2+(6mK-1)\zeta+K^2 = 0 .$$

This gives

$$\zeta = \tfrac{1}{2}\{(1-6mK)\pm\sqrt{[(1-6mK)^2-4K^2]}\} = \frac{3(2-3m)\pm\sqrt{[5(7-3m)(1-3m)]}}{12(1-3m)(1+m)} .$$

The values of ζ are real and distinct, since $(7-3m)(1-3m) > 0$. Also, $2-3m > 0$ and $(1-3m)(1+m) > 0$, so to show that $\zeta > 0$ it suffices to prove that

$$9(2-3m)^2 > 5(7-3m)(1-3m) ,$$

$$\text{i. e. } (6m+1)^2 > 0 ,$$

which is true. This proves the lemma.

We thus have a lattice, say L_2, given by

$$x = p\xi - q\eta, \quad y = q\xi + p\eta, \tag{14.4}$$

which, by Lemma 8, has eight points on $\mathscr{C}$, namely (p, q), $(-q, p)$, $(p-q, p+q)$, $(p+q, p-q)$ and their images in O. The determinant of this lattice is $\Delta = p^2+q^2$ and, from (14.2) and (14.3), we find

$$\Delta^2 = (p^2+q^2)^2 = p^4+6mp^2q^2+q^4+2(1-3m)p^2q^2 = \frac{5}{6(1+m)} = \Lambda_2 . \tag{14.5}$$

The lattice L_2, then, satisfies the conditions we found above for a minimum, and all such lattices will be given by the transforms of L_2 by the automorphisms of $\mathscr{R}$. As before, we find in this way only two distinct lattices, L_2 and its image in the y-axis, say L_2'.

15. It remains to show that there are no lattices of type II, other than L_2 and L_2', with $\Lambda \leq \Lambda_2$. We have, from (8.5),

$$\begin{aligned}\Psi(\Lambda, c)-\Psi(\Lambda, -\tfrac{1}{6}) &= 16c^3-3c^2+6c+\tfrac{125}{108}-\tfrac{2}{3}(6c+1)I\Lambda^2\\ &= \tfrac{1}{3}(6c+1)(8c^2-\tfrac{17}{6}c+\tfrac{125}{36}-2I\Lambda^2)\\ &= \tfrac{1}{3}(6c+1)Z ,\end{aligned}$$

say, and so

$$\Psi(\Lambda, c) = \Psi(\Lambda, -\tfrac{1}{6})+\tfrac{1}{3}(6c+1)Z . \tag{15.1}$$

Now I and Λ_2 are both strictly decreasing functions of m for $m < 0$, and so, since $m > -\frac{1}{3}$,

$$2I\Lambda_2^2 < 2(\tfrac{4}{3})(\tfrac{5}{4})^2 = \tfrac{150}{36} .$$

Recalling that, by (8.2), $6c+1 \leq 0$ for lattices of type II, we have, if $0 < \Lambda \leq \Lambda_2$,

$$\begin{aligned}Z &= 8c^2-\tfrac{17}{36}(6c+1)+\tfrac{142}{36}-2I\Lambda^2\\ &\geq \tfrac{8}{36}+\tfrac{142}{36}-2I\Lambda^2 \geq \tfrac{150}{36}-2I\Lambda_2^2 > 0 .\end{aligned}$$

Again, we find

$$\begin{aligned}108\Psi(\Lambda, -\tfrac{1}{6}) &= 432J\Lambda^3+180I\Lambda^2-125\\ &= 432J(\Lambda-\Lambda_2)(\Lambda-\Lambda_2')(\Lambda-\Lambda_2'')\\ &\leq 0 ,\end{aligned}$$

for $0 < \Lambda \leq \Lambda_2$. Then, from (15.1), we have $\Psi(\Lambda, c) \leq 0$ if $6c+1 \leq 0$, $0 < \Lambda \leq \Lambda_2$, with equality only if $\Lambda = \Lambda_2$, $c = -\frac{1}{6}$, which is the required result.

16. It follows now that $\Lambda^* = \Lambda_2$ when $m_0 \leq m \leq -\frac{1}{6}$, and so we have the following result, which we put in the form of a lemma, as we will later prove it true for the range $-\frac{1}{3} < m \leq -\frac{1}{6}$ and give the complete result as a theorem.

Lemma 9. *If $m_0 \leq m \leq -\frac{1}{6}$, there is a point x, y, other than the origin, of every lattice L of determinant Δ, such that*

$$|x^4+6mx^2y^2+y^4| \leq \tfrac{6}{5}(1+m)\Delta^2 .$$

This is the best possible result, the equality sign being required if, and only if, L is proportional to one of the lattices L_2 or L_2'.

The critical form, say $\psi_2(\xi, \eta)$, corresponding to the transformation (14.4) will be given by (14.1) with the appropriate value of b. Substituting in (7.4), we find

$$b = -(1-3m)pq(p^2-q^2) .$$

Hence

(16.1) $$\psi_2(\xi, \eta) = \xi^4-h\xi^3\eta-\xi^2\eta^2+h\xi\eta^3+\eta^4 ,$$

where

(16.2) $$h = -4b = 4(1-3m)pq(p^2-q^2) \geq 0 .$$

We determine h by putting $h = -4b = 4d$, $\Delta^2 = \Lambda_2$ and $c = -\frac{1}{6}$ in (7.13), giving

(16.3) $$3h^2 = 12I\Lambda_2^2-13 .$$

Thence

$$3h^2 = \frac{25(1+3m^2)-39(1+m)^2}{3(1+m)^2} = \frac{2(6m+1)(3m-7)}{3(1+m)^2} ,$$

and so

(16.4) $$h = \frac{1}{3(1+m)}\sqrt{\{2(6m+1)(3m-7)\}} .$$

We may now interpret the result given by Lemma 9 in the light of our introductory remarks, and we again state the conclusion as a lemma.

Lemma 10. *Let $\psi(\xi, \eta)$ be any binary quartic form of discriminant $\mathscr{D}$, which is transformable into the standard form $x^4+6mx^2y^2+y^4$ with $m_0 \leq m \leq -\frac{1}{6}$. Then there exist integers ξ, η, not both zero, such that*

$$|\psi(\xi, \eta)| \leq \tfrac{6}{5}(1+m)(1-9m^2)^{-\frac{1}{3}}\mathscr{D}^{\frac{1}{6}} .$$

This is the best possible result, the equality sign being required if, and only if, $\psi(\xi, \eta)$

is equivalent to a multiple of the form

$$\psi_2(\xi, \eta) = \xi^4 - h\xi^3\eta - \xi^2\eta^2 + h\xi\eta^3 + \eta^4 ,$$

where h is given by (16.4).

17. To complete the discussion of forms with four distinct complex roots, we must now consider the lattice-point problem for $-\frac{1}{3} < m < m_0$. In this case, the methods used by Mordell (6) to deal with a certain type of non-convex region may be employed to obtain the best possible result. His general theorems cannot be applied directly to this problem, since his boundary curves have by hypothesis no real finite points of inflection, but we shall find that the same ideas are successful here. We shall follow closely the details of Mordell's presentation.

The result we find is that Lemmas 9 and 10 still hold for the complete range $-\frac{1}{3} < m \leq -\frac{1}{6}$, and we prove this by showing that, for $-\frac{1}{3} < m < m_0$, every lattice of determinant Δ_2 has a point, other than the origin, in the region $\mathcal{R}$, and, further, that this is an inner point except when the lattice is L_2 or L_2'.

We first prove some results required later. We suppose, unless the contrary is stated, that $-\frac{1}{3} < m < m_0 = -0{\cdot}259$.

Lemma 11. $1 < p < \frac{3}{2}$.

Write $r = q/p$ and so $p^4 f(1, r) = 1$. By Lemma 8, $p > q > 0$, i. e. $0 < r < 1$, so

$$f(1, r) = 1 + 6mr^2 + r^4 = 1 + r^2(r^2 + 6m) < 1 + r^2(1 + 6m) < 1 ,$$

and hence $p > 1$. Also, $p^4 < (p^2+q^2)^2 = \dfrac{5}{6(1+m)} < \dfrac{5}{4} < \dfrac{81}{16}$, and so $p < \dfrac{3}{2}$.

Lemma 12. $0 < r < \frac{1}{4}$.

As above, $r > 0$. Also $r^2 = K/p^4 < K$, where $K = p^2q^2$ is given by (14.3). But

$$K - \frac{1}{16} = \frac{4(6m+1) - 3(3m-1)(1+m)}{48(3m-1)(1+m)} = \frac{(7-3m)(1+3m)}{48(3m-1)(1+m)} < 0 ,$$

and the result follows.

Lemma 13. *If $-\frac{1}{3} < m \leq -\frac{1}{6}$, h in* (16.4) *is a strictly decreasing function of m. Further, $0 \leq h < 2$, and these are the best possible bounds.*

We have $h \geq 0$ by (16.2), the value 0 being attained when $q = 0$, i. e. when $m = -\frac{1}{6}$. Both I and Δ_2 decrease strictly if $m < 0$, and (16.3) shows that h does the same. When $m = -\frac{1}{3}$, (16.4) gives $h = 2$ and so $h < 2$ for $m > -\frac{1}{3}$.

Lemma 14. *If* $-\frac{1}{3} < m < m_0$, *then* $h > 1+r+r^2$.

After Lemmas 12 and 13, it is sufficient to prove $h(m_0) > 1+\frac{1}{4}+\frac{1}{16} = 1{\cdot}3125$. Putting $m = m_0 = -0{\cdot}259$ in (16.4), we find $h(m_0) = 1{\cdot}320\ldots > 1{\cdot}3125$; this proves the lemma.

18. We must now examine our geometrical configuration more closely. The following discussion will be aided by reference to Fig. 1.

The lattice L_2 has the points (p, q), $(p-q, p+q)$, $(-q, p)$, $(-p-q, p-q)$ and their images in O lying on $\mathscr{C}$. These points define a square $\mathscr{S}_1$, of which the vertices and the mid-points of the sides all lie on $\mathscr{C}$. Further, the square $\mathscr{S}_2$ which is the image of $\mathscr{S}_1$ in the y-axis has the same properties. The area of each of these squares is $4(p^2+q^2) = 4\Delta_2$.

We denote by $\mathscr{R}_1$ the closed region bounded by the line joining the points

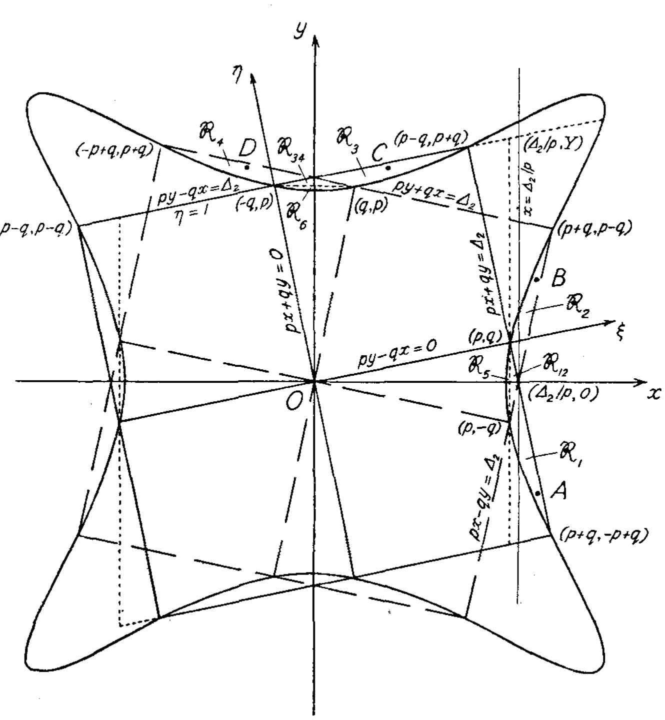

Fig. 1. The region $|x^4-\frac{9}{5}x^2y^2+y^4| \leq 1$.

(p, q) and $(p+q, -p+q)$ and by the arc of $\mathscr{C}$ between these points[1]; and by $\mathscr{R}_2$ the image of this region in the x-axis. Similarly, define $\mathscr{R}_3$ and $\mathscr{R}_4$ in terms of the points $(-q, p)$ and $(p-q, p+q)$, and let $\mathscr{R}_{12}$ and $\mathscr{R}_{34}$ be the regions common to $\mathscr{R}_1$, $\mathscr{R}_2$ and $\mathscr{R}_3$, $\mathscr{R}_4$ respectively. Finally, let $\mathscr{R}_5$ be the closed region bounded by the straight line and the arc of $\mathscr{C}$ each joining the points (p, q) and $(p, -q)$; and let $\mathscr{R}_6$ be the corresponding region for the points $(-q, p)$ and (q, p).

It is convenient for some purposes to transform the variables x, y by the linear substitution (14.4). Then the equation of $\mathscr{C}$ is $\psi_2(\xi, \eta) = 1$, while the square $\mathscr{S}_1$ becomes the square $|\xi| \leq 1$, $|\eta| \leq 1$. The line joining the points $(-q, p)$ and $(p-q, p+q)$, the equation of which is $py-qx = \Delta_2$, transforms into $\eta = 1$. This line meets $\mathscr{C}$ in points with $\xi = 0, \pm 1$, and so in a fourth point with $\xi = h$, by (16.1). Then plainly the segments of this line with $-1 < \xi < 0$ and with $1 < \xi < h$ consist of inner points of $\mathscr{R}$, since, by Lemmas 12 and 14, $h > 1+r+r^2 > 1$; while the segment $0 < \xi < 1$ lies outside $\mathscr{R}$.

We will be concerned later with the parallelogram $\mathscr{P}$ defined by the lines $py-qx = \pm\Delta_2$, $x = \pm p$. This parallelogram lies between the same parallels as the square $\mathscr{S}_1$ and has an equal base, so its area is also $4\Delta_2$. We show now that its vertices are inner points of $\mathscr{R}$, and for this it is sufficient by symmetry to consider the vertices lying on the line $py-qx = \Delta_2$, i. e. $\eta = 1$. The lines $x = \pm p$ transform into $p\xi - q\eta = \pm p$, that is $\xi = \pm 1 + r\eta$, and so the vertices in question are given by $\xi = \pm 1 + r$, $\eta = 1$. Since $1 < 1+r < 1+r+r^2 < h$ and $-1 < -1+r < -1+\frac{1}{4} < 0$, these points lie on the segments of $\eta = 1$ shown above to consist of inner points of $\mathscr{R}$, which is the desired result.

We note here that the region $\mathscr{R}$ contains the square $|x| \leq 1$, $|y| \leq 1$. For if $|x| \leq 1$, $|y| \leq 1$ and, e. g., $x^2 \geq y^2$, we have

$$0 \leq f(x, y) = x^4 + 6mx^2y^2 + y^4 = x^4 + y^2(y^2 + 6mx^2) \leq x^4 \leq 1 ,$$

since $6m < -1$; and the same result clearly holds if $x^2 < y^2$.

19. We require now some classical results in the geometry of numbers, and we state them here, without proof, as lemmas.

Lemma 15. *Any parallelogram with centre at O and area 4Δ contains a point other than O of every lattice of determinant Δ.*

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[1] Since $\mathscr{C}$ is a closed curve, this description is really ambiguous; we shall always mean the shorter arc.

Lemma 16. *If P and Q are any points of a lattice of determinant Δ, then the area of the triangle OPQ is $\frac{1}{2}n\Delta$, where n is an integer.*

Lemma 17. *If a triangle OAB of area $\frac{1}{2}\Delta$ contains two independent points P and Q of a lattice of determinant Δ, then $P = A$ and $Q = B$, or vice versa.*

20. We suppose now that L is any lattice of determinant Δ_2 which has no point other than O as an inner point of $\mathcal{R}$. Our result will follow if we show that L is either L_2 or L_2'.

Lemma 18. *There is a point of L in one of the regions $\mathcal{R}_3$, $\mathcal{R}_5$; and also in one of the regions $\mathcal{R}_4$, $\mathcal{R}_5$.*

The parallelogram $\mathcal{P}$ defined above is of area $4\Delta_2$ and so, by Lemma 15, contains a point of L other than O. But every point of $\mathcal{P}$ is an inner point of $\mathcal{R}$, except for those points in the regions $\mathcal{R}_3$ and $\mathcal{R}_5$ and their images in O. This proves the first part of the lemma, and the second part follows similarly on considering the image of $\mathcal{P}$ in the y-axis.

Lemma 19. *There is a point of L in one of the regions $\mathcal{R}_{12}$, $\mathcal{R}_{34}$.*

Suppose this is false. Then, by Lemma 18, there is a point of L in each of the regions $\mathcal{R}_3$, $\mathcal{R}_4$, and by symmetry also in each of the regions $\mathcal{R}_1$, $\mathcal{R}_2$. Let us denote the points of L in $\mathcal{R}_1$, $\mathcal{R}_2$, $\mathcal{R}_3$, $\mathcal{R}_4$ by A, B, C, D, in that order. Then all these points are distinct, for if, for example, $A = B$, then A is in $\mathcal{R}_{12}$, contrary to hypothesis. Also, plainly no two of the points are collinear with O. Thus we have five points O, A, B, C, D of L, and these will be defined in that lattice by pairs of integers. But a set of integer pairs can have only four different residue pairs modulo 2, and so two of our five points must have the same residue pair. It follows that the mid-point of the join of these two is a point of L. We proceed to show that this is impossible. By symmetry, it is enough to consider the mid-points of the lines OA, AB, AC, AD, BC; denote these points by S, T, U, V, W respectively.

The points A and B lie in the triangle with vertices O, $(p+q,\ -p+q)$ and $(p+q,\ p-q)$, of area $p^2-q^2 < p^2+q^2 = \Delta_2$. Hence, by Lemma 16, the area of the triangle $OAB = \frac{1}{2}\Delta_2$, and it follows that S and T are not points of L.

Again, by Lemma 17, U cannot lie in the triangle with vertices at O, $(p+q,\ -p+q)$ and (p, q); for the area of this triangle is $\frac{1}{2}\Delta_2$ and it contains the lattice point A, so U would lie at (p, q) in $\mathcal{R}_{12}$, contrary to hypothesis. Similarly, U cannot lie in the triangle with vertices O, $(p-q, p+q)$ and $(-q, p)$, which contains C. Thus U must lie in the triangle with vertices O, (p, q) and $(p-q, p+q)$, and not at a vertex. Since it is then an inner point of $\mathcal{R}$ it cannot be a point of L.

In a similar way we see that V must be a point, other than a vertex, of the triangle with vertices O, (p, q) and (q, p); and W a point, not a vertex, of the triangle with vertices O, $(p+q, p-q)$ and $(p-q, p+q)$. It follows that V and W are not points of L.

This completes the proof of the lemma.

21. We may suppose, to fix our ideas, that the lattice L has a point P': (x', y') in the region $\mathcal{R}_{12}$. Our object now is to establish that L then has a point other than O in the interior of $\mathcal{R}$, a contradiction, unless P' is at (p, q) or $(p, -q)$ and L is L_2 or L_2'.

Lemma 20. *There exists a line $Q'R'$ which is equal and parallel to OP' and whose end-points Q', R' lie on the arc of $\mathcal{C}$ joining the points $(-p+q, p+q)$ and $(p-q, p+q)$.*

By symmetry, it is sufficient to prove this for $y' \geq 0$. Take a point R': (x_2', y_2') coinciding initially with the point $(p-q, p+q)$, and through R' draw a line $R'Q'$ equal and parallel to $P'O$. Then the point Q': (x_1', y_1') plainly lies in the triangle formed by the lines $y = p+q$, $py-qx = \Delta_2$ and $px+qy = 0$. (This is really the triangle with vertices O, $(\Delta_2/p, 0)$, (p, q) reflected in the x-axis and displaced). *A fortiori* it lies in the region bounded by the straight line and the arc of $\mathcal{C}$ each joining the points $(-p+q, p+q)$ and $(p-q, p+q)$. Suppose now that R' is moved continuously along the arc of $\mathcal{C}$ to the point $(0, 1)$, the line $Q'R'$ remaining parallel to OP'; during this motion y_2' varies from $p+q$ to 1. Since $y'+y_1' = y_2'$ and $y' \geq 0$, we have $y_1' \leq y_2'$. Hence when R' is at $(0, 1)$, $y_1' \leq 1$. Now, for every point (x, y) on the arc of $\mathcal{C}$ concerned, $y \geq 1$, and so the point Q' has left the interior of the lune. It follows that Q' lies on the arc at some stage of the motion described. This position of the line $Q'R'$ satisfies all the conditions of the lemma.

Lemma 21. *If a line parallel to $Q'R'$, and nearer to the origin than it, meets $\mathcal{C}$ in points between Q' and R', then the length of the intercept so formed is less than $Q'R'$.*

Since $\mathcal{R}$ is a star domain, the arc of $\mathcal{C}$ joining the points Q', R' lies entirely within the angle $Q'OR'$. Hence the intercept by $\mathcal{C}$ on a line parallel to $Q'R'$, and nearer to O than it, is less than the intercept made by the arms of this angle, which is less than $Q'R'$.

Let OP' meet $\mathscr{C}$ in the point P, and let Q, R correspond to P as Q', R' do to P'. Further, let $\varDelta$ be the area of the parallelogram $OPRQ$.

Lemma 22. $\varDelta \geq \varDelta_2$,

with equality only when P is at (p, q) or $(p, -q)$.

The coordinates of Q and R are clearly continuous functions of the coordinates of P, and hence so is the value of $\varDelta$. The lattice generated by the points R and P on $\mathscr{C}$ also has the point $R-P = Q$ on $\mathscr{C}$. Hence (§ 7) its determinant $\varDelta$ satisfies $\Phi(\Lambda, c) = 0$, with $\Lambda = \varDelta^2$. If $\varDelta = \varDelta_2$, we have shown in §§ 13, 14 that the lattice must be L_2 or L_2'; that is, $\varDelta = \varDelta_2$ when, and only when, P is at (p, q) or $(p, -q)$. Hence $\varDelta - \varDelta_2$ has the same sign for all P in the range. We determine this sign when P is the point $(1, 0)$ and $\varDelta = \varDelta_0$, say. By symmetry, R is then the point $(\frac{1}{2}, \varDelta_0)$, and so

$$\varDelta_0^4 + 6m\tfrac{1}{4}\varDelta_0^2 + \tfrac{1}{16} - 1 = 0 ,$$

giving

$$\varDelta_0^2 = \tfrac{1}{4}\{\sqrt{(9m^2+15)} - 3m\} .$$

It remains then to show

$$\tfrac{1}{4}\{\sqrt{(9m^2+15)} - 3m\} > \frac{5}{6(1+m)} ,$$

i. e.

$$3(1+m)\sqrt{(9m^2+15)} > 9m(1+m)+10 ,$$

i. e.

$$9(1+m)^2(9m^2+15) - (9m^2+9m+10)^2 > 0 ,$$

i. e.

$$5(7-3m)(1+3m) > 0 ,$$

which is true.

Now let l' be the line

(21.1) $$x'y - y'x = \varDelta_2 ,$$

parallel to OP' and distant $\varDelta_2/OP'$ from it.

Lemma 23. *The line l' meets $\mathscr{C}$ in points E', G', H', F' in this order, with $E'F' > 2OP'$.*

It is sufficient to show that the intersections of l' and the lines $x = \pm\varDelta_2/p$ are inner points of $\mathscr{R}$. For then the x-component of $E'F' > 2\varDelta_2/p \geq 2x'$, since $\varDelta_2/p$ is the maximum abscissa of the region $\mathscr{R}_{12}$. By symmetry, we need consider only the intersection of l' and $x = \varDelta_2/p$, say the point $(\varDelta_2/p, Y)$. We show that the maximum and minimum values of Y are positive. It is then enough to prove that

$(\Delta_2/p,\ Y)$ is an inner point of $\mathcal{R}$ for these two values of Y, since, by symmetry, the line $x = \Delta_2/p$ can meet $\mathcal{C}$ in at most two points above the x-axis.

We consider first the maximum value of Y. From (21.1) we find

$$Y = \frac{\Delta_2(p+y')}{px'} = \Delta_2\left(\frac{1}{x'}+\frac{t'}{p}\right),$$

where $t' = y'/x'$. Hence, for a fixed value of t', Y is greatest when x' is least, that is when P' is on $\mathcal{C}$. Again, for given x' the greater value of Y arises from $t' \geq 0$. Then

$$\frac{dY}{dx'} = \frac{\Delta_2}{px'}\left(\frac{dy'}{dx'}-\frac{p+y'}{x'}\right).$$

Now

$$\frac{dy'}{dx'} = -\frac{1+3mt'^2}{t'(t'^2+3m)} > 0\,,$$

for $0 \leq t' \leq q/p = r < \frac{1}{4}$, since $1+3mt'^2 > 1+3m > 0$ and $t'^2+3m \leq r^2+3m < \frac{1}{16}-\frac{3}{6} < 0$. Hence $dy'/dx' \geq x'/y'$, by Lemma 1. Since the region $\mathcal{R}_{12}$ containing P' lies below the line $py-qx = 0$, we have $x'/y' \geq p/q = 1/r > 4$. Again, since $x' \geq 1$ and $y' \leq q$, $(p+y')/x' \leq p+q < 2p < 3$. It follows that $dY/dx' > 0$, and so Y is a maximum when $x' = p$, $y' = q$, that is when l' is the line $py-qx = \Delta_2$. Transforming to ξ, η coordinates, this line is $\eta = 1$, and $x = \Delta_2/p$ becomes $p\xi-q\eta = \Delta_2/p$. Thus for the point of intersection we have

$$\xi = \frac{q+\Delta_2/p}{p} = \frac{pq+p^2+q^2}{p} = 1+r+r^2\,.$$

But by Lemmas 12 and 14, we have $1 < 1+r+r^2 < h$, and so this point is an inner point of $\mathcal{R}$.

Next, the value of Y for given t' is least when x' is greatest; and then, for given x', for the negative value of y'. Hence we may suppose P' lies on the line $px-qy=\Delta_2$, and then $dy'/dx' = p/q$. Thus $dY/dx' > 0$, as before, and so the minimum value of Y occurs when P' is the point $(p, -q)$ and l' the line $py+qx = \Delta_2$. It is convenient to consider, instead of $(\Delta_2/p,\ Y)$, the point $(-\Delta_2/p,\ Y)$, which will here be the intersection of the lines $py-qx = \Delta_2$, $x = -\Delta_2/p$, that is of $\eta = 1$ and $p\xi-q\eta = -\Delta_2/p$. For this point we find, as above, $\xi = -1+r-r^2$. But $-1+r-r^2 = -1+r(1-r) > -1$ and $-1+r-r^2 < -1+r < 0$, so we have $-1 < \xi < 0$, and it follows that this point, too, is an inner point of $\mathcal{R}$. Clearly $Y > 0$ for this point, and so also for the maximum Y. This completes the proof that $E'F' > 2OP'$.

Since the line l' meets $x = 0$ in a point with $y = \Delta_2/x' \geq \Delta_2/(\Delta_2/p) = p > 1$, and so outside $\mathcal{R}$, it follows from the above that l' meets $\mathcal{C}$ in two other points, say G', H', and that the segment $G'H'$, except for the endpoints, lies outside $\mathcal{R}$.

Let the elements l, G, H correspond to the point P as the dashed elements do to P'.

Lemma 24. $G'H' \leq OP'$,

with equality only when P' is at (p, q) or $(p, -q)$.

It is clear that $G'H' = OP'$ when P' is at (p, q) or $(p, -q)$. Suppose then that P' is not either of these points. The line QR is distant Δ/OP from the line OP' and, by Lemma 22, we have $\Delta/OP > \Delta_2/OP \geq \Delta_2/OP'$. Then, using Lemma 21, $OP' \geq OP = QR > GH \geq G'H'$, the required result.

Lemma 25. *Every lattice L of determinant Δ_2, other than L_2 or L_2', has a point other than O in the interior of the region $\mathcal{R}$.*

This will follow if we show that the lattice L, which by definition has no point other than O in the interior of $\mathcal{R}$, can only be L_2 or L_2'.

Now the lattice L, which contains the point P', will have lattice points on the line l' distributed at intervals equal to OP'. The intercept $E'F'$ will contain in its interior at least two points of L, since $E'F' > 2OP'$, by Lemma 23. If L is not L_2 or L_2', Lemma 24 gives $G'H' < OP'$, and so both these points cannot lie in the intercept $G'H'$. Thus at least one of these lattice points is an inner point of $\mathcal{R}$, contrary to the definition of L, and the result follows.

Lemma 26. *The lattices L_2 and L_2' have no point except O in the interior of $\mathcal{R}$.*

It is sufficient by symmetry to prove this for the lattice L_2. This is equivalent to showing that $\psi_2(\xi, \eta) \geq 1$ for all integers ξ, η except $\xi = \eta = 0$. Then we have

$$\begin{aligned}
\psi_2(\xi, \eta) &= \xi^4 - h\xi^3\eta - \xi^2\eta^2 + h\xi\eta^3 + \eta^4 \\
&= \xi^4 - 2\xi^3\eta - \xi^2\eta^2 + 2\xi\eta^3 + \eta^4 + (2-h)\xi\eta(\xi^2 - \eta^2) \\
&= (\xi^2 - \xi\eta - \eta^2)^2 + (2-h)\xi\eta(\xi^2 - \eta^2) \\
&\geq (\xi^2 - \xi\eta - \eta^2)^2 \\
&\geq 1 ,
\end{aligned}$$

if $\xi\eta(\xi^2-\eta^2) \geq 0$, since $2-h > 0$ by Lemma 13. If $\xi\eta(\xi^2-\eta^2) < 0$, we have

$$\begin{aligned}\psi_2(\xi, \eta) &= \xi^4-\xi^2\eta^2+\eta^4-h\xi\eta(\xi^2-\eta^2)\\ &\geq \xi^4-\xi^2\eta^2+\eta^4\\ &\geq 1,\end{aligned}$$

since $h \geq 0$ by Lemma 13.

We have now completed the proof that Lemma 9 is valid for the extended range, and we state the result formally as

Theorem 3. *If $-\frac{1}{3} < m \leq -\frac{1}{6}$, there is a point x, y, other than the origin, of every lattice L of determinant Δ, such that*

$$|x^4+6mx^2y^2+y^4| \leq \tfrac{6}{5}(1+m)\Delta^2 .$$

This is the best possible result, the equality sign being required if, and only if, L is proportional to one of the lattices L_2 or L_2'.

As before, this at once yields

Theorem 4. *Let $\psi(\xi, \eta)$ be a binary quartic form with real coefficients and $\mathcal{D} > 0$, and either $\mathcal{H} \leq 0$ or $\mathcal{K} \leq 0$. Further, let $\mathcal{J} < 0$, $35^2\mathcal{J}^3 \leq 39^3\mathcal{J}^2$, so that m given by* (4.5) *satisfies $-\frac{1}{3} < m \leq -\frac{1}{6}$. Then there exist integers ξ, η, not both zero, such that*

$$|\psi(\xi, \eta)| \leq \frac{6(1+m)}{5(1-9m^2)^{1/3}}\mathcal{D}^{\frac{1}{6}} .$$

This is the best possible result, the equality sign being required if, and only if, $\psi(\xi, \eta)$ is equivalent to a multiple of the form

$$\psi_2(\xi, \eta) = \xi^4-h\xi^3\eta-\xi^2\eta^2+h\xi\eta^3+\eta^4 ,$$

with $0 \leq h < 2$. The value of h as a function of m is given by (16.4).

22. We add here a note on the results previously known for quartics with four distinct complex roots. We have remarked already that the best possible lattice constant for the corresponding region $\mathcal{R}$ was given in the case $m = 0$ by Mordell (7) and for $0 \leq m \leq \frac{1}{3}$ by Derry (8); in these cases the region is convex. In neither work is the arithmetical consequence explicitly stated.

The result given by Julia (5) is essentially that a quartic form

$$\psi(\xi, \eta) = a(\xi-\beta_1\eta)(\xi-\beta_1'\eta)(\xi-\beta_2\eta)(\xi-\beta_2'\eta) ,$$

where β_1, β_2 are different complex numbers and β_1', β_2' their conjugates, is equivalent to one whose first coefficient A_0 satisfies

$$|A_0| < \tfrac{1}{3}|a|(\beta_1-\beta_2)(\beta_1'-\beta_2') .$$

In order to compare this with our result we must put it in the same shape. Let $f(x, y) = x^4+6mx^2y^2+y^4$ be the standard form with $|m| < \frac{1}{3}$ which gives $\pm\psi(\xi, \eta)$ by means of a transformation of determinant Δ. Now the expression $a(\beta_1-\beta_2)(\beta_1'-\beta_2')$ is an irrational invariant of ψ of index 2. For the standard form its value is $|\beta_1-\beta_2|^2= 4|\beta_1|^2 = 4$, so we have

$$|a|(\beta_1-\beta_2)(\beta_1'-\beta_2') = 4\Delta^2 .$$

It follows that Julia's result is equivalent to the statement that every lattice of determinant Δ has a point, other than O, in the region

$$|f(x, y)| < \tfrac{4}{3}\Delta^2 .$$

This is the result we would obtain immediately by noting that the region $\mathcal{R}$ contains the circle $x^2+y^2 \leq 1$, and it is naturally a very crude estimate for the general quartic form of the type considered.

Reference might appropriately be made here to a paper by MAHLER (17). In this he finds an asymptotic expression for a quantity $M_\varepsilon(J)$ as $J \to \infty$. In our notation $M_\varepsilon(J) = k^*(m)(1-9m^2)^{-\frac{1}{6}}$, and $J \to \infty$ is equivalent to $m \to \frac{1}{3}$ if $\varepsilon = +1$ and $m \to -\frac{1}{3}$ if $\varepsilon = -1$. His results can thus be deduced immediately from our values of $k^*(m)$. It may be noted that the limits of $M_\varepsilon(J)/J^{\frac{1}{6}}$ he gives are in fact those appropriate for the limit regions (that is, really those dealt with later in Theorem 6), namely $\sqrt{\frac{4}{3}}$, $\sqrt{\frac{12}{25}}$.

He states further (without proof) that $M_\varepsilon(J) \geq \frac{1}{5}(432)^{\frac{1}{6}}$, and suggests that the exact lower bound may be attained for $J = 1$. A little calculation[1] shows that the exact lower bound is attained for $m = -0{\cdot}063\ldots$ ($J = 1$ is $m = 0$) and its value is $1{\cdot}034\ldots$, about twice the above estimate.

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[1] See Appendix II.

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APPENDIX I.

We might, of course, have taken as standard form for a quartic with four distinct complex roots the canonical form with $m > \frac{1}{3}$. If this is done, some of the details are rather lighter (for example, in Lemmas 4 and 5 we really made a transformation to this case). It is perhaps worth remarking that the results given in Theorems 1 and 2 then take a slightly simpler shape, although Theorems 3 and 4 remain unaltered. We state below the result corresponding to Theorem 2.

We define the new value of m by

$$\cos\varphi = -\mathcal{J}\left(\frac{3}{\mathcal{G}}\right)^{\frac{3}{2}}, \quad 0 < \varphi < \pi,$$

$$m = \frac{1}{\sqrt{3}} \cot\frac{\varphi}{3}. \tag{4.5'}$$

Our result then becomes

Theorem 2′. *Let* $\psi(\xi, \eta)$ *be a binary quartic form with real coefficients and* $\mathcal{D} > 0$, *and either* $\mathcal{H} \leq 0$ *or* $\mathcal{K} \leq 0$. *Further, let* $35^2\mathcal{G}^3 > 39^3\mathcal{J}^2$ *if* $\mathcal{J} < 0$, *so that* m *given by* (4.5′) *satisfies* $\frac{1}{3} < m < \frac{7}{3}$. *Then there exist integers* ξ, η, *not both zero, such that*

$$|\psi(\xi, \eta)| \leq \frac{4\{\sqrt{(9m^2+15)}+3m\}}{15(9m^2-1)^{1/3}} \mathcal{D}^{1/6}.$$

This is the best possible result, the equality sign being required if, and only if, $\psi(\xi, \eta)$ *is equivalent to a multiple of the form*

$$\psi_1(\xi, \eta) = \xi^4+2\xi^3\eta+(A+1)\xi^2\eta^2+A\xi\eta^3+\eta^4,$$

with $2 < A < 4$. *The value of* A *as a function of* m *is given by*

$$A = \tfrac{1}{2}\{1-9m^2+3m\sqrt{(9m^2+15)}\}.$$

APPENDIX II.

Since the calculation of the minimum of $k^*(m)(1-9m^2)^{-\frac{1}{3}}$ (§ 22) is rather tedious, we give the details here. We make the substitution

$$m = \frac{3-K}{3+3K},$$

and write

$$k^*(m)(1-9m^2)^{-\frac{1}{3}} = C(K).$$

Then we find (cf. Theorem 2′)

$$C(K) = C_1(K) = \tfrac{4}{15}\{\sqrt{(K^2+15)}+K\}(K^2-1)^{-\frac{1}{3}},$$

for $1 < K \leq 7$; and

$$C(K) = C_2(K) = \tfrac{2}{5}(K+3)(K^2-1)^{-\frac{1}{3}},$$

for $K \geq 7$.

Differentiating, we have

$$\begin{aligned} C_2'(K) &= \tfrac{2}{15}(K^2-1)^{-\frac{4}{3}}\{3(K^2-1)-2K\} \\ &= \tfrac{2}{5}(K^2-1)^{-\frac{4}{3}}\{(K-\tfrac{1}{3})^2-\tfrac{10}{9}\} \\ &> 0, \text{ for } K \geq 7. \end{aligned}$$

Again,

$$C_1'(K) = \tfrac{4}{45}(K^2-1)^{-\frac{4}{3}}\frac{\sqrt{(K^2+15)}+K}{\sqrt{(K^2+15)}}\{3(K^2-1)-2K\sqrt{(K^2+15)}\},$$

and so $C_1'(K) = 0$ if, and only if,

$$3(K^2-1) = 2K\sqrt{(K^2+15)}\ ,$$

i. e.

$$5K^4-78K^2+9 = 0\ ,$$

i. e.

$$5K^2 = 39 \pm 6\sqrt{41}\ .$$

Since $K > 1$ we must take the upper sign, and we find $K^2 = 15{\cdot}4837\ldots$, so $K = 3{\cdot}9349\ldots$ and $m = -0{\cdot}0631\ldots$. We see without calculation that this is a minimum, since $C_1(K) \to \infty$ as $K \to 1$ and $C_2(K) \to \infty$ as $K \to \infty$.

For this value of K we have

$$\begin{aligned} C_1(K) &= \frac{4}{15}\,\frac{2K\sqrt{(K^2+15)}+2K^2}{2K(K^2-1)^{1/3}} \\ &= \frac{2(5K^2-3)}{15K(K^2-1)^{1/3}} \\ &= 1{\cdot}0344\ldots\ . \end{aligned}$$