NEW INTEGRALS INVOLVING BESSEL FUNCTIONS

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The formula [T. M. MacRobert, Functions of a Complex Variable, 4th ed., Glasgow 1954, p. 406, (1)] namely

$$K_m(x)\,K_n(x)=\frac{\sqrt{\pi}}{4\pi x}\sum_{i,-i}\frac{1}{i}\,E\left(\frac{1+m+n}{2},\frac{1+m-n}{2},\frac{1+n-m}{2},\frac{1-n-m}{2}:\frac{1}{2}:e^{i\pi}x^2\right),\quad (1)$$

will be used to evaluate a large number of integrals involving Bessel Functions which are all entirely new.

Thus from (1), *loc. cit.*, and formula (1)

$$\int_0^\infty e^{-\lambda}\lambda^{k-1}E\left(p;\alpha_r:q;\varrho_s:\frac{x}{\lambda^n}\right)d\lambda=(2\pi)^{\frac{1}{2}-\frac{1}{2}n}\cdot n^{k-\frac{1}{2}}\,E\left(p+n;\alpha_r:q;\varrho_s:\frac{x}{n^n}\right),\quad (2)$$

where $$R(k)>0,\ \alpha_{p+\nu+1}=\frac{k+\nu}{n},\qquad \nu=0,1,2,\ldots,n-1,$$

one gets, if x is real and positive,

$$\int_0^\infty e^{-\lambda}\lambda^{k-1}K_m\left(\frac{x}{\lambda}\right)K_n\left(\frac{x}{\lambda}\right)d\lambda$$

$$=\frac{2^{k-2}}{\pi x}\sum_{i,-i}\frac{1}{i}\,E\left(\frac{1+m+n}{2},\frac{1+m-n}{2},\frac{1+n-m}{2},\frac{1-n-m}{2},\frac{k+1}{2},\frac{k+2}{2}:\frac{1}{2}:\frac{e^{i\pi}x^2}{4}\right).\quad (3)$$

Also from (1) and formula (2)

$$\int_0^\infty e^{-\lambda}\lambda^{k-1}E(p;\alpha_r:q;\varrho_s:\lambda^m x)\,d\lambda=\pi\operatorname{cosec}(k\pi)\,(2\pi)^{\frac{1}{2}m-\frac{1}{2}}\,m^{k-\frac{1}{2}}\times$$

$$\times E\left(p;\alpha_r:1-\frac{k}{m},1-\frac{k+1}{m},\ldots,1-\frac{k+m-1}{m},\varrho_1,\ldots,\varrho_q:e^{\pm m\pi i}m^m z\right)+$$

$$+2^{\frac{1}{2}-\frac{m}{2}}\pi^{\frac{1}{2}+\frac{m}{2}}\sum_{\nu=0}^{m-1}\frac{(-1)^{\nu+1}m^{-\frac{1}{2}-\nu}z^{-(k+\nu)/m}}{\sin\left(\frac{k+\nu}{m}\pi\right)\prod_{s=1}^{\nu}\sin\frac{s\pi}{m}\prod_{t=1}^{m-\nu-1}\sin\frac{t\pi}{m}}\times$$

$$\times E\left(\begin{matrix}p;\ \alpha_r+(k+\nu)/m:\ e^{\pm m\pi i}m^m z\\ 1+\frac{k+\nu}{m},\ 1+\frac{1}{m},\ldots,1+\frac{\nu}{m},\ 1-\frac{1}{m},\ldots,1-\frac{m-\nu-1}{m},\ \varrho_1+\frac{k+\nu}{m},\ldots,\varrho_q+\frac{k+\nu}{m}\end{matrix}\right), \quad (4)$$

where m is a positive integer (equals 2 in this case),

$$R(m\alpha_r+k)>0,\quad r=1,2,3,\ldots,p,\ |\operatorname{amp} z|<\pi,$$

one gets

$$\int_0^\infty e^{-\lambda}\lambda^{k-1}K_m(x\lambda)K_n(x\lambda)\,d\lambda$$

$$=\frac{\sqrt{\pi}\,\Gamma\left(\frac{m+n+k}{2}\right)\Gamma\left(\frac{m-n+k}{2}\right)\Gamma\left(\frac{n-m+k}{2}\right)\Gamma\left(\frac{-n-m+k}{2}\right)}{4\,\Gamma\left(\frac{1+k}{2}\right)\Gamma\left(\frac{k}{2}\right)}x^{-k}\times$$

$$\times{}_4F_3\left(\frac{m+n+k}{2},\ \frac{m-n+k}{2},\ \frac{n-m+k}{2},\ \frac{-n-n+k}{2};\ \frac{1}{2},\ \frac{k}{2},\ \frac{k+1}{2};\ \frac{1}{4x^2}\right)-$$

$$-\frac{\pi\Gamma\left(\frac{m+n+1+k}{2}\right)\Gamma\left(\frac{m-n+k+1}{2}\right)\Gamma\left(\frac{n-m+k+1}{2}\right)\Gamma\left(\frac{-n-m+k+1}{2}\right)}{8\,\Gamma\left(\frac{3}{2}\right)\Gamma\left(\frac{k+1}{2}\right)\Gamma\left(\frac{k+2}{2}\right)}x^{-k-1}\times$$

$$\times{}_4F_3\left(\frac{1+m+n+k}{2},\ \frac{1+m-n+k}{2},\ \frac{1+n-m+k}{2},\ \frac{1-n-m+k}{2};\ \frac{3}{2},\ \frac{1+k}{2},\ \frac{2+k}{2};\ \frac{1}{4x^2}\right), \quad (5)$$

where $\quad R(x)>0,\ R(k\pm m\pm n)>0.$

In particular when $n=m$ the last formula becomes

$$\int_0^\infty e^{-\lambda}\lambda^{k-1}\{K_n(x\lambda)\}^2\,d\lambda$$

$$=\frac{\sqrt{\pi}\,\Gamma\left(n+\frac{k}{2}\right)\Gamma\left(-n+\frac{k}{2}\right)\Gamma\left(\frac{k}{2}\right)}{4\,\Gamma\left(\frac{1+k}{2}\right)}x^{-k}{}_3F_2\left(\begin{matrix}n+\frac{k}{2},\ -n+\frac{k}{2},\ \frac{k}{2};\ \frac{1}{4x^2}\\ \frac{1}{2},\ \frac{1}{2}+\frac{1}{2}k\end{matrix}\right)-$$

$$-\frac{\pi\Gamma\left(n+\frac{k+1}{2}\right)\Gamma\left(-n+\frac{k+1}{2}\right)\Gamma\left(\frac{k+1}{2}\right)}{8\,\Gamma\left(\frac{3}{2}\right)\Gamma\left(1+\frac{1}{2}k\right)}x^{-k-1}{}_3F_2\left(\begin{matrix}n+\frac{k+1}{2},\ -n+\frac{k+1}{2},\ \frac{k+1}{2};\ \frac{1}{4x^2}\\ \frac{3}{2},\ 1+\frac{k}{2}\end{matrix}\right), \quad (6)$$

where

$$R(k) > 0,\ R(k \pm 2n) > 0,\ R(x) > 0.$$

From (1) and formula (3)

$$4 \int_0^\infty \lambda^{m-1} K_n(2\lambda)\, E\left(p;\ \alpha_r\colon q;\ \varrho_s\colon \frac{x}{\lambda^2}\right) d\lambda = E(p+2;\ \alpha_r\colon q;\ \varrho_s\colon x), \tag{7}$$

where

$$\alpha_{p+1} = \frac{m+n}{2},\ \alpha_{p+2} = \frac{m-n}{2},\ R(m \pm n) > 0,$$

one gets

$$\int_0^\infty \lambda^{k-1} K_l(2\lambda)\, K_m\left(\frac{x}{\lambda}\right) K_n\left(\frac{x}{\lambda}\right) d\lambda = \frac{1}{16\sqrt{\pi}\cdot x} \times$$

$$\times \sum_{i,-i} \frac{1}{i} E\left(\frac{m+n+1}{2}, \frac{m-n+1}{2}, \frac{n-m+1}{2}, \frac{-n-m+1}{2}, \frac{k+l+1}{2}, \frac{k-l+1}{2} \colon \frac{1}{2} \colon e^{i\pi} x^2\right), \tag{8}$$

where $R(x) > 0$ and the restrictions necessary for (7) can be removed by analytical continuation.

When $n = m$, (8) gives, if $R(x) > 0$,

$$\int_0^\infty \lambda^{k-1} K_l(2\lambda) \left\{K_n\left(\frac{x}{\lambda}\right)\right\}^2 d\lambda$$

$$= \frac{1}{16\sqrt{\pi}\cdot x} \sum_{i,-i} \frac{1}{i} E\left(\frac{1}{2}+n,\ \frac{1}{2}-n,\ \frac{1}{2},\ \frac{k+l+1}{2},\ \frac{k-l+1}{2} \colon \colon e^{i\pi} x^2\right). \tag{9}$$

From (1) and formula (4)

$$4 i \pi \int_0^\infty \lambda^{m-1} J_n(2\lambda)\, E\left(p;\ \alpha_r\colon q;\ \varrho_s\colon \frac{z}{\lambda^2}\right) d\lambda$$

$$= i^{m-n} E(p+2;\ \alpha_r\colon q;\ \varrho_s\colon z e^{-i\pi}) - i^{n-m} E(p+2;\ \alpha_r\colon q;\ \varrho_s\colon z e^{i\pi}), \tag{10}$$

where

$$R(m+n) > 0,\ R\left(\frac{3}{2} - m + 2\alpha_r\right) > 0,\ r = 1, 2, 3, \ldots, p,\quad \alpha_{p+1} = \frac{m+n}{2},\ \alpha_{p+2} = \frac{m-n}{2}$$

and z is real and positive, one gets

$$\int_0^\infty \lambda^{k-1} J_l(2\lambda)\, K_m\left(\frac{x}{\lambda}\right) K_n\left(\frac{x}{\lambda}\right) d\lambda$$

$$= \frac{e^{\frac{i\pi}{2}(k-l-1)}}{16\pi^{\frac{3}{2}}\cdot x}\left[\begin{array}{l} E\left(\frac{1+m+n}{2}, \frac{1+m-n}{2}, \frac{1+n-m}{2}, \frac{1-n-m}{2}, \frac{1+k+l}{2}, \frac{1+k-l}{2}: \frac{1}{2}: x^2\right) - \\ -E\left(\frac{1+m+n}{2}, \frac{1+m-n}{2}, \frac{1+n-m}{2}, \frac{1-n-m}{2}, \frac{1+k+l}{2}, \frac{1+k-l}{2}; \frac{1}{2}: e^{-2i\pi} x^2\right)\end{array}\right] -$$

$$- \frac{e^{\frac{i\pi}{2}(l-k-3)}}{16\pi^{\frac{3}{2}}\cdot x}\left[\begin{array}{l} E\left(\frac{1+m+n}{2}, \frac{1+m-n}{2}, \frac{1+n-m}{2}, \frac{1-n-m}{2}, \frac{1+k+l}{2}, \frac{1+k-l}{2}: \frac{1}{2}: e^{2i\pi} x^2\right) - \\ -E\left(\frac{1+m+n}{2}, \frac{1+m-n}{2}, \frac{1+n-m}{2}, \frac{1-n-m}{2}, \frac{1+k+l}{2}, \frac{1+k-l}{2}: \frac{1}{2}: x^2\right)\end{array}\right], \quad (11)$$

where $R(x) > 0,\ R\left(\frac{3}{2} - k \pm m \pm n\right) > 0.$

When $n = m$, the last formula becomes

$$\int_0^\infty \lambda^{k-1} J_l(2\lambda)\left\{K_n\left(\frac{x}{\lambda}\right)\right\}^2 d\lambda$$

$$= \frac{e^{\frac{i\pi}{2}(k-l-1)}}{16\pi^{\frac{3}{2}} x}\left[\begin{array}{l} E\left(\frac{1}{2}, \frac{1}{2}+n, \frac{1}{2}-n, \frac{1+k+l}{2}, \frac{1+k-l}{2} : : x^2\right) \\ -E\left(\frac{1}{2}, \frac{1}{2}+n, \frac{1}{2}-n, \frac{1+k+l}{2}, \frac{1+k-l}{2} : : e^{-2i\pi} x^2\right)\end{array}\right] -$$

$$- \frac{e^{\frac{i\pi}{2}(l-k-3)}}{16\pi^{\frac{3}{2}}\cdot x}\left[\begin{array}{l} E\left(\frac{1}{2}, \frac{1}{2}+n, \frac{1}{2}-n, \frac{1+k+l}{2}, \frac{1+k-l}{2} : : e^{2i\pi} x^2\right) \\ -E\left(\frac{1}{2}, \frac{1}{2}+n, \frac{1}{2}-n, \frac{1+k+l}{2}, \frac{1+k-l}{2} : : x^2\right)\end{array}\right], \quad (12)$$

where $R(x) > 0,\ R\left(\frac{3}{2} - k\right) > 0,\ R\left(\frac{3}{2} - k \pm 2n\right) > 0.$

From (1) and formula (5)

$$\int_0^\infty \lambda^{k-1} K_n(\lambda)\, E(p; \alpha_r: q; \varrho_s: z\lambda^2)\, d\lambda$$

$$= 2^{k-2}\frac{\pi^2}{\sin\left(\frac{k+n}{2}\pi\right)\sin\left(\frac{k-n}{2}\pi\right)} E\left(p; \alpha_r: 1 - \frac{k+n}{2}, 1 - \frac{k-n}{2}, \varrho_1, \ldots, \varrho_q: 4z\right)$$

$$+ \sum_{n,-n}\frac{\pi^2\, 2^{-n-2}}{\sin\left(\frac{k+n}{2}\pi\right)\sin(n\pi)} z^{-\left(\frac{k+n}{2}\right)} E\left(\begin{array}{l}\alpha_1 + \frac{k+n}{2}, \ldots, \alpha_p + \frac{k+n}{2}: 4z \\ 1 + \frac{k+n}{2}, 1+n, \varrho_1 + \frac{k+n}{2}, \ldots, \varrho_q + \frac{k+n}{2}\end{array}\right), \quad (13)$$

where $p \geq q+1,\ R(k \pm n + 2\alpha_r) \geq 0,\ r = 1, 2, 3, \ldots, p,\ |\mathrm{amp}\, z| < \pi,$ one gets

$$\int_0^\infty \lambda^{k-1} K_l(\lambda) K_m(x\lambda) K_n(x\lambda)\, d\lambda$$

$$= -\sum_{l,-l} \left[\frac{2^{-l-3}\pi^{\frac{3}{2}} \Gamma\left(\frac{k+l+m+n}{2}\right)\Gamma\left(\frac{k+l+m-n}{2}\right)\Gamma\left(\frac{k+l+n-m}{2}\right)\Gamma\left(\frac{k+l-m-n}{2}\right)}{\sin(l\pi)\Gamma\left(\frac{1+k+l}{2}\right)\Gamma\left(\frac{k+l}{2}\right)\Gamma(1+l)\cdot x} x^{-(k+l-1)} \times {}_4F_3\left(\begin{matrix}\frac{k+l+m+n}{2}, \frac{k+l+m-n}{2}, \frac{k+l+n-m}{2}, \frac{k+l-n-m}{2}; \frac{1}{4x^2}\\ \frac{1+k+l}{2}, \frac{k+l}{2}, 1+l\end{matrix}\right)\right], \quad (14)$$

where $R(x) > 0,\ R(k \pm l \pm m \pm n) > 0.$

In particular when $n = m_1$ the last formula gives

$$\int_0^\infty \lambda^{k-1} K_l(\lambda) \{K_n(x\lambda)\}^2\, d\lambda$$

$$= -\sum_{l,-l} \left[\frac{2^{-l-3}\pi^{\frac{3}{2}} \Gamma\left(\frac{1}{2}k+\frac{1}{2}l+n\right)\Gamma\left(\frac{1}{2}k+\frac{1}{2}l-n\right)\Gamma\left(\frac{1}{2}k+\frac{1}{2}l\right)}{\sin(l\pi)\Gamma\left(\frac{1+k+l}{2}\right)\cdot x\cdot\Gamma(1+l)} x^{-(k+l-1)} \times {}_3F_2\left(\frac{1}{2}k+\frac{1}{2}l+n, \frac{1}{2}k+\frac{1}{2}l-n, \frac{k+l}{2}; \frac{1+k+l}{2}, 1+l; \frac{1}{4x^2}\right)\right], \quad (15)$$

where $R(x) > 0,\ R(k \pm l \pm 2n) > 0,\ R(k \pm l) > 0.$

From (1) and formula (6)

$$\int_0^\infty e^{-\mu} I_n(\mu)\, \mu^{m-1} E(p; \alpha_r; q; \varrho_s: z/\mu^2)\, d\mu$$

$$= \frac{1}{2\sqrt{2}} \frac{\sin\pi(m-n)}{\cos(\pi m)} E\left\{\begin{matrix}\alpha_1, \ldots, \alpha_p, \frac{m+n}{2}, \frac{m-n}{2}, \frac{m-n+1}{2}, \frac{m+n+1}{2} : z\\ \frac{3}{4}+\frac{1}{2}m, \frac{1}{4}+\frac{1}{2}m, \varrho_1, \ldots, \varrho_q\end{matrix}\right\} -$$

$$- \frac{1}{4\sqrt{2}\cdot\pi} \frac{\cos(n\pi)}{\sin\pi\left(\frac{m}{2}-\frac{1}{4}\right)} z^{\frac{m}{2}-\frac{1}{4}} E\left\{\begin{matrix}\alpha_1+\frac{1}{4}-\frac{m}{2}, \ldots, \alpha_p+\frac{1}{4}-\frac{m}{2}, \frac{1}{4}+\frac{n}{2}, \frac{3}{4}+\frac{n}{2}, \frac{1}{4}-\frac{n}{2}, \frac{3}{4}-\frac{n}{2} : z\\ \frac{5}{4}-\frac{m}{2}, \frac{1}{2}, \varrho_1+\frac{1}{4}-\frac{m}{2}, \ldots, \varrho_q+\frac{1}{4}-\frac{m}{2}\end{matrix}\right\} -$$

$$- \frac{1}{4\sqrt{2}\cdot\pi} \frac{\cos(n\pi)}{\sin\pi\left(\frac{3}{4}-\frac{m}{2}\right)} z^{\frac{m}{2}-\frac{3}{4}} E\left\{\begin{matrix}\alpha_1+\frac{3}{4}-\frac{m}{2}, \ldots, \alpha_p+\frac{3}{4}-\frac{m}{2}, \frac{3}{4}+\frac{n}{2}, \frac{5}{4}+\frac{n}{2}, \frac{3}{4}-\frac{n}{2}, \frac{5}{4}-\frac{n}{2} : z\\ \frac{7}{4}-\frac{m}{2}, \frac{3}{2}, \varrho_1+\frac{3}{4}-\frac{m}{2}, \ldots, \varrho_q+\frac{3}{4}-\frac{m}{2}\end{matrix}\right\}, \quad (16)$$

where

$$R(n+m)>0,\ \ R\left(2\alpha_r-m+\frac{1}{2}\right)>0,\ \ r=1,2,3,\ldots,p,\ |\operatorname{amp} z|<\pi,$$

one gets the following formula

$$\int_0^\infty e^{-\lambda}\lambda^{k-1}I_l(\lambda)K_m\left(\frac{x}{\lambda}\right)K_n\left(\frac{x}{\lambda}\right)d\lambda=\frac{1}{4x\sqrt{\pi}}\times$$

$$\times\sum_{i,-i}\frac{1}{i}\left[\begin{array}{l}\dfrac{\sin\pi(k-l)}{2^{\frac{3}{2}}\cdot\pi\cos(\pi k)}\times\\ \times E\left\{\begin{array}{l}\dfrac{1+m+n}{2},\dfrac{1+m-n}{2},\dfrac{1+n-m}{2},\dfrac{1-n-m}{2},\dfrac{1+k+l}{2},\dfrac{2+k+l}{2},\dfrac{1+k-l}{2},\dfrac{2+k-l}{2}:e^{i\pi}x^2\\ \dfrac{5}{4}+\dfrac{1}{2}k,\ \dfrac{3}{4}+\dfrac{1}{2}k,\ \dfrac{1}{2}\end{array}\right\}+\\ +\dfrac{\cos(\pi l)}{2^{5/2}\cdot\pi\cdot\sin\pi\left(\dfrac{k}{2}+\dfrac{1}{4}\right)}(x^2e^{i\pi})^{\frac{k}{2}+\frac{1}{4}}\times\\ \times E\left\{\begin{array}{l}\dfrac{1}{4}+\dfrac{m+n-k}{2},\dfrac{1}{4}+\dfrac{m-n-k}{2},\dfrac{1}{4}+\dfrac{n-m-k}{2},\dfrac{1}{4}+\dfrac{-n-m-k}{2},\dfrac{1}{4}+\dfrac{l}{2},\dfrac{3}{4}+\dfrac{l}{2},\dfrac{1}{4}-\dfrac{l}{2},\dfrac{3}{4}-\dfrac{l}{2}:e^{i\pi}x^2\\ \dfrac{3}{4}-\dfrac{1}{2}k,\ \dfrac{1}{4}-\dfrac{1}{2}k,\ \dfrac{1}{2}\end{array}\right\}-\\ -\dfrac{\cos(\pi l)}{2^{5/2}\cdot\pi\cdot\sin\pi\left(\dfrac{1}{4}-\dfrac{1}{2}k\right)}(x^2e^{i\pi})^{\frac{k}{2}-\frac{1}{4}}\times\\ \times E\left\{\begin{array}{l}\dfrac{3}{4}+\dfrac{m+n-k}{2},\dfrac{3}{4}+\dfrac{m-n-k}{2},\dfrac{3}{4}+\dfrac{n-m-k}{2},\dfrac{3}{4}+\dfrac{-n-m-k}{2},\dfrac{3}{4}+\dfrac{l}{2},\dfrac{5}{4}+\dfrac{l}{2},\dfrac{3}{4}-\dfrac{l}{2},\dfrac{5}{4}-\dfrac{l}{2}:e^{i\pi}x^2\\ \dfrac{5}{4}-\dfrac{1}{2}k,\ \dfrac{3}{4}-\dfrac{1}{2}k,\ \dfrac{3}{2}\end{array}\right\}\end{array}\right],\quad (17)$$

where

$$R(x)>0,\ \ R\left(\frac{1}{2}-k\pm m\pm n\right)>0.$$

When $n=m$, the last formula becomes

$$\int_0^\infty e^{-\lambda}\lambda^{k-1} I_l(\lambda)\left\{K_n\left(\frac{x}{\lambda}\right)\right\}^2 d\lambda = \frac{1}{4\sqrt{\pi}\cdot x}\times$$

$$\times\sum_{i,-i}\frac{1}{i}\left[\begin{aligned}&\frac{\sin\pi(k-l)}{2^{\frac{3}{2}}\pi\cos(\pi k)}E\left\{\begin{matrix}\frac{1}{2},\ \frac{1}{2}+n,\ \frac{1}{2}-n,\ \frac{k+l+1}{2},\ \frac{k+l+2}{2},\ \frac{k-l+1}{2},\ \frac{k-l+2}{2}:\ e^{i\pi}x^2\\ \frac{5}{4}+\frac{1}{2}k,\ \frac{3}{4}+\frac{1}{2}k\end{matrix}\right\}+\\ &+\frac{\cos(\pi l)\cdot(x^2e^{i\pi})^{\frac{k}{2}+\frac{1}{4}}}{2^{\frac{5}{2}}\pi\cdot\sin\pi\left(\frac{k}{2}+\frac{1}{4}\right)}E\left\{\begin{matrix}\frac{1}{4}-\frac{k}{2},\ \frac{1}{4}+n-\frac{k}{2},\ \frac{1}{4}-n-\frac{k}{2},\ \frac{1}{4}+\frac{l}{2},\ \frac{3}{4}+\frac{l}{2},\ \frac{1}{4}-\frac{l}{2},\ \frac{3}{4}-\frac{l}{2}:\ e^{i\pi}x^2\\ \frac{3}{4}-\frac{1}{2}k,\ \frac{1}{2}\end{matrix}\right\}-\\ &-\frac{\cos(\pi l)\cdot(x^2e^{i\pi})^{\frac{k}{2}-\frac{1}{4}}}{2^{\frac{5}{2}}\pi\cdot\sin\pi\left(\frac{1}{4}-\frac{k}{2}\right)}E\left\{\begin{matrix}\frac{3}{4}-\frac{k}{2},\ \frac{3}{4}+n-\frac{k}{2},\ \frac{3}{4}-n-\frac{k}{2},\ \frac{3}{4}+\frac{l}{2},\ \frac{5}{4}+\frac{l}{2},\ \frac{3}{4}-\frac{l}{2},\ \frac{5}{4}-\frac{l}{2}:\ e^{i\pi}x^2\\ \frac{5}{4}-\frac{1}{2}k,\ \frac{3}{2}\end{matrix}\right\}\end{aligned}\right],\qquad(18)$$

where

$$R(x)>0,\ R\left(\frac{1}{2}-k\right)>0,\ R\left(\frac{1}{2}-k\pm 2n\right)>0.$$

Finally from (1) and formula (7)

$$\int_0^\infty e^{-\mu}\,\mathrm{I}_n(\mu)\,\mu^{m-1}E(p;\alpha_r:q;\varrho_s:z\mu^2)\,d\mu$$

$$=\frac{\pi}{\sqrt{2}\sin\pi(m+n)}E\left\{\begin{matrix}\alpha_1,\ \ldots,\ \alpha_r,\ \frac{1}{4}-\frac{m}{2},\ \frac{3}{4}-\frac{m}{2}:\ z\\ 1-\frac{m+n}{2},\ \frac{1}{2}-\frac{m+n}{2},\ \varrho_1,\ \ldots,\ \varrho_q,\ 1+\frac{n-m}{2},\ \frac{1}{2}+\frac{n-m}{2}\end{matrix}\right\}-$$

$$-\frac{\pi\cdot z^{-\left(\frac{m+n}{2}\right)}}{2\sqrt{2}\sin\pi\left(\frac{m+n}{2}\right)}E\left\{\begin{matrix}\alpha_1+\frac{m+n}{2},\ \ldots,\ \alpha_p+\frac{m+n}{2},\ \frac{1}{4}+\frac{n}{2},\ \frac{3}{4}+\frac{n}{2}:\ z\\ 1+\frac{m+n}{2},\ \frac{1}{2},\ \varrho_1+\frac{m+n}{2},\ \ldots,\ \varrho_q+\frac{m+n}{2},\ 1+n,\ \frac{1}{2}+n\end{matrix}\right\}-$$

$$-\frac{\pi\cdot z^{-\left(\frac{m+n+1}{2}\right)}}{2\sqrt{2}\cos\pi\left(\frac{m+n}{2}\right)}E\left\{\begin{matrix}\alpha_1+\frac{m+n+1}{2},\ \ldots,\ \alpha_p+\frac{m+n+1}{2},\ \frac{3}{4}+\frac{n}{2},\ \frac{5}{4}+\frac{n}{2}:\ z\\ 1+\frac{m+n+1}{2},\ \frac{3}{2},\ \varrho_1+\frac{m+n+1}{2},\ \ldots,\ \varrho_q+\frac{m+n+1}{2},\ \frac{3}{2}+n,\ 1+n\end{matrix}\right\},\qquad(19)$$

where

$$R(n+m+2\alpha_r)>0,\ r=1,2,3,\ldots,p,\ R\left(\frac{1}{2}-m\right)>0,\ |\operatorname{amp} z|<\pi,$$

one gets

$$\int_0^\infty e^{-\lambda} I_l(\lambda) K_m(x\lambda) K_n(x\lambda)\, d\lambda$$

$$= \frac{\sqrt{\pi}\,\Gamma\left(\frac{k+l+m+n}{2}\right)\Gamma\left(\frac{k+l+m-n}{2}\right)\Gamma\left(\frac{k+l+n-m}{2}\right)\Gamma\left(\frac{k+l-m-n}{2}\right)\Gamma\left(\frac{1}{4}+\frac{l}{2}\right)\Gamma\left(\frac{3}{4}+\frac{l}{2}\right)}{2^{\frac{5}{2}}\Gamma\left(\frac{1}{2}+\frac{1}{2}k+\frac{1}{2}l\right)\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2}k+\frac{1}{2}l\right)\Gamma(1+l))\,\Gamma\left(\frac{1}{2}+l\right)\cdot x} x^{-(k+l-1)} \times$$

$$\times\, {}_6F_5\left(\begin{matrix}\frac{k+l+m+n}{2}, \frac{k+l+m-n}{2}, \frac{k+l+n-m}{2}, \frac{k+l-n-m}{2}, \frac{1}{4}+\frac{l}{2}, \frac{3}{4}+\frac{l}{2}, \frac{1}{x^2}\\ \frac{k+l+1}{2}, \frac{k+l}{2}, \frac{1}{2}, 1+l, \frac{1}{2}+l\end{matrix}\right) -$$

$$- \frac{\sqrt{\pi}\,\Gamma\left(\frac{k+l+m+n+1}{2}\right)\Gamma\left(\frac{k+l+m-n+1}{2}\right)\Gamma\left(\frac{k+l+n-m+1}{2}\right)\Gamma\left(\frac{k+l-n-m+1}{2}\right)\Gamma\left(\frac{3}{4}+\frac{l}{2}\right)\Gamma\left(\frac{5}{4}+\frac{l}{2}\right)}{2^{\frac{5}{2}}\Gamma\left(1+\frac{k+l}{2}\right)\Gamma\left(\frac{3}{2}\right)\Gamma\left(\frac{1+k+l}{2}\right)\Gamma(1+l)\,\Gamma\left(\frac{3}{2}+l\right)\cdot x} x^{-(k+l)} \times$$

$$\times\, {}_6F_5\left(\begin{matrix}\frac{k+l+m+n+1}{2}, \frac{k+l+m-n+1}{2}, \frac{k+l+n-m+1}{2}, \frac{k+l-n-m+1}{2}, \frac{3}{4}+\frac{l}{2}, \frac{5}{4}+\frac{l}{2}; \frac{1}{x^2}\\ \frac{2+k+l}{2}, \frac{3}{2}, \frac{1+k+l}{2}, 1+l, \frac{3}{2}+l\end{matrix}\right), \quad (20)$$

where $R(x) > 0,\ R(k+l\pm m\pm n) > 0.$

Addendum.—It may be noted that the last formulae may serve as a basis for discussion of the asymptotic behaviour of the integrals for large values of $|z|$ since some of the above integrals were evaluated in terms of MacRobert's E function whose asymptotic expansion was given by him (C. V., p. 358)[1] and the rest were evaluated in terms of ordinary generalized hypergeometric function whose asymptotic expansion has been investigated by several writers [E. W. Barnes, *Proc. Lond. Math. Soc.*, (2) 5 (1906), 249–297; E. M. Wright, *Journ. Lond. Math. Soc.*, 10 (1935), 286–295; and C. S. Meyer, *Proc. K. Akad. v. Wetenschappen*, Amsterdam, XLIX (1946), 1165–1175)].

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[1] C. V. denotes the book by MacRobert referred to in the beginning of this paper.