CURVE FAMILIES F^* LOCALLY THE LEVEL CURVES OF A PSEUDOHARMONIC FUNCTION

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Introduction

The family F^* may be defined over an arbitrary open Riemann surface Q. When Q is not simply connected there may exist no single-valued PH [pseudoharmonic] function on Q with F^* as its family of level lines. On the universal covering surface M of Q there do exist PH functions u, single-valued on M and with a family F_M^* of level lines which projects into F^* on Q. While u may not be single-valued on Q it may behave like an integral in that it has branches which differ by a constant, or it may have a real logarithm which has this property. In studying such behavior of u one may focus on the branches of u obtained by continuation of u along a single closed curve k not homotopic to zero on Q.

In this way one is led to the essentially typical case of a family F^* defined on a sphere Σ^* with a north pole N and south pole S removed. Although there may be no single-valued PH function u on Σ^* with F^* as its family of level lines there will in general be multiple-valued functions u satisfying linear relations

$$u[p^{(1)}] = a\,u(p) + b \qquad (a \neq 0) \tag{1.0}$$

where p and $p^{(1)}$ are points on the universal covering surface M of Σ^*, and where p and $p^{(1)}$ in M project into the same point in Σ^*, but on M have longitudes θ and $\theta + 2\pi$ respectively. However the values of the constants a and b for which a relation (1.0) may hold depend in a deep way upon the nature of the family F^*. See MJ 4 and MJ 5.

In the present paper we decompose Σ^* into canonical regions, "primitives," "caps," "annuli," "polar sectors," "cut sectors," etc., whose nature is determined by F^*. With F we associate integral indices $\nu(F)$ and $\mu(F)$ [defined in a later paper]. The existence of PH functions u satisfying prescribed linear relations (1.0) depends upon these indices and upon the character of the decomposition of Σ^*.

When F^* is non-singular Kaplan [3] has given a decomposition of Σ^* in some but not in all of the cases we find essential. Kaplan's results are less general in an *a priori* sense than ours even in the non-singular case in that he requires the curves in F^* to be homeomorphic mappings of intervals and circles. We confirm many of Kaplan's theorems particularly on asymptotes, and add the results necessary for our purposes. Our main theorems are strongly affected by the presence of singular points in F^*. See also Boothby [1, 2] and Tôki [5].

§ 2. Review and extensions

Let Σ be a 2-sphere. Let N and S be diametrically opposite poles in Σ, termed respectively the north and south poles of Σ. Set

$$\Sigma^* = \Sigma - N - S.$$

Let ω be a discrete set of points in Σ^* and set $G = \Sigma^* - \omega$. Consider a family F of open arcs or top [topological] circles in G supposing that F contains a unique element α_p meeting any given point $p \in G$. An open arc in G is understood as the image in a $1-1$ continuous mapping into G of an open interval. The arc is the image and not the mapping. A top circle in G is the homeomorph in G of a circle.

F-neighborhoods X_p. Let D be the open disc $\{|w| < 1\}$ in the complex w-plane. With each point $p \in \Sigma^*$ there shall be associated an F-neighborhood X_p of p with $\bar{X}_p \subset G \cup p$, and a top mapping of $\bar{X}_p$ onto $\bar{D}$ which sends p into $w = 0$ and the maximal open arcs of $F | X_p$ into the maximal open level arcs of $\mathcal{R} w^n$, in $D, n > 0$, [with $w = 0$ excluded when $n > 1$]. We suppose $n = 1$ for $p \in G$ and $n > 1$ for $p \in \omega$. Points in ω are termed *singular* points of F. The value $w \in D$ is termed a canonical parameter of its antecedent in X_p.

The open arcs of $F | (X_p - p)$ incident with p are termed *F-rays* of X_p. These rays divide $X_p - p$ into $2n$ open regions termed *F-sectors* of X_p incident with p.

Right N. With each $p \in G$ we also associate a neighborhood N_p of p in G and a homeomorphic mapping of $\bar{N}_p$ onto a square $K:(-1 \leqq u \leqq 1)(-1 \leqq v \leqq 1)$ such that p goes into the origin in K and the maximal subarcs of $F | \bar{N}_p$ go into arcs $u = c, -1 \leqq v \leqq 1$, where c is a constant in the interval $[-1, 1]$. We refer to N_p as a right neighborhood or right N_p of p and term (u, v) *canonical coordinates* of the antecedent in N_p of (u, v) in K.

An open arc λ in N_p on which $v = \varphi(u)(-1 < u < 1)$, where φ is single-valued and continuous, is called a transversal of N_p. The open arc in N_p on which $v = 0, -1 < u < 1$, is called the *principal transversal* of N_p. More generally a transversal μ shall be any open arc in G each point of which is in an open subarc of μ which is a transversal of some right N_p. A transversal with a closure in Σ which is an arc in G is the principal transversal of a suitably chosen right N_p. A non-singular arc in Σ^* is termed a *transverse* arc if a subarc of some transversal. If Y is an F-sector incident with p, a transversal meeting each element of $F | Y$ and with p as limiting end point is called a *transversal ray* of Y incident with p. Transversal rays λ and μ incident with p, but in different F-sectors of X_p incident with p, define a *transversal cut* $\lambda p \mu$ of X_p.

F-vectors. Any sensed subarc of an $\alpha \in F$ will be called an F-vector. By definition an F-vector is simple and closed in G, and never a top circle. Each F-vector is in some right N_p [MJ 2 § 3].

Coherent sensing. Let each $\alpha \in F$ be given a sense. The resulting family F^s of sensed α will be called a sensed image of F. We shall refer to a continuous deformation Δ of an F-vector A in the space of F-vectors with a Fréchet metric. The sense of an image of A under Δ shall be determined by the Δ-images of the initial and final points of A. We say that F^s is coherently sensed if any continuous deformation Δ of an F-vector A (initially sensed as in F^s) through F-vectors sensed by Δ is necessarily through F-vectors sensed by F^s. We say that F is coherent if it admits a coherently sensed image F^s, otherwise non-coherent. A family F on Σ^* may be coherent or non-coherent as the following examples show.

Examples. Let Σ^* be represented by the z-plane, with $z \neq 0$, S by the origin, and N by the point at infinity. If $z = x + iy$, the level lines of x on Σ^* afford a coherent family. The loci on which

$$y = \frac{x^2}{4a} - a \qquad (a > 0)$$

taken with the open arc on which $x = 0$, $y > 0$ afford a non-coherent family.

We shall establish the following theorem.

Theorem 2.1. *A necessary and sufficient condition that F be coherent over G is that, taken over some neighborhood of N or of S, F admit two distinct coherent sensings.*

This follows as in the proof of Th 4.2 of MJ 2.

The family F^*. The family F^* shall consist of elements $h, k, m, \ldots$ in Σ^* which are top circles or open arcs in Σ^*. If a non-singular point p is in h, h shall contain the open arc $\alpha_p \in F$ meeting p and any limiting end point or points of $\alpha_p \in \Sigma^*$. An $h \in F^*$ comprising just one $\alpha \in F$ is called non-singular.

Positive and negative limit points. Let an open arc $h \in F^*$ be sensed and be given a $1-1$ representation in which $p(t)$ is the $1-1$ continuous image of t, $-\infty < t < \infty$, with t increasing in the positive sense of h. By a positive (negative) limit point of h is meant any point in Σ which is a limit point of a sequence of points $p(t_n)$, where t_n increases (decreases) without limit as $n \uparrow \infty$.

Covering surfaces M and families F_M. Let K be any open, oriented and connected Riemann surface and ω, F, F^* admissibly defined for K as above for Σ^*. Let M be a relatively unbranched, unbordered covering surface of K. Let α_M be any maximal open arc or top circle in M "covering" an element $\alpha \in F$. If α is an open arc, α_M is an open arc. If α is a top circle, α_M is an open arc or a homeomorphic top circle. The totality of the elements α_M covering the elements $\alpha \in F$ forms a family F_M in M. Let ω_M be the set of points in M covering the set ω in Σ^*. The family F_M includes one and only one element meeting each point of $M - \omega_M$. Just as F^* was defined in Σ^* with the aid of F and ω, so here F_M^* is defined in M in terms of F_M and ω_M.

In the remainder of this paper M shall be the universal covering surface of Σ^*.

From MJ 2 we infer the following.

Theorem 2.2. *Let the universal covering surface M of Σ^* be considered as a top sphere H from which one point Z has been removed.*

(1) *Then any open arc $\alpha_M \in F_M$ has limiting end points in H, distinct unless coincident with Z. Each end point of α_M different from Z is a point of ω_M.* Cf. MJ 2 Th 7.1.

(2) *There are no top circles in F_M^*.*

(3) *For any open arc $h_M \in F_M^*$ the positive and negative limit sets reduce to Z.* Cf. MJ 2 Th 7.2.

Corollary 2.1. *No two top circles in F^* can intersect or be joined by a subarc of an element in F^*.*

If two top circles g_1 and g_2 in F^* met there would be a finite sequence of elements of F in $g_1 \cup g_2$ whose closure would carry a closed curve g (not necessarily simple) bounding a region in Σ^*. A suitably chosen closed curve g_M covering g would be simple with carrier in F_M^*, contrary to Th 2.2(2). The second affirmation of the corollary is similarly established.

Theorem 2.3. *If M is the universal covering surface of Σ^* no $h \in F_M^*$ intersects a transversal or a transversal cut in M in more than one point.*

Since M is the homeomorph of a finite z-plane this follows from Cor. 7.5 of MJ 2.

§ 3. *F*-sets and *F*-regions

With F there are naturally associated certain sets and regions which we now define. By a *region* we shall always mean an open, connected set.

F-sets. A set $H \subset \Sigma$ will be termed an F-set, if whenever a non-singular point p is in H, the $\alpha_p \in F$ meeting p is also in H. From the nature of a right N_p of p it is clear that the

complement, closure or boundary of an F-set is an F-set. The intersection, or union of any ensemble of F-sets is also an F-set.

Conditions Θ. An open set R in Σ which is an F-set will be said to satisfy Conditions Θ if the inclusion of a point p of Σ^* in βR implies that any sector of a sufficiently restricted F-neighborhood X_p of p is in R or in $\Sigma - \bar{R}$, and at least two of the F-rays bounding sectors of $X_p \cap R$ are in βR.

F-regions. A simply connected region R in Σ whose boundary consists of more than one point and which satisfies Conditions Θ will be called an F-region.

The boundary βR. If R is an F-region βR is an F-set. Let $p \in \beta R$ be non-singular and suppose that $\alpha_p \in F$ meets p. Any sufficiently small right neighborhood N_p of p is separated by α_p into two regions of which at least one and possibly both are in R.

Up to this point we have not used curves (which are mappings) but rather sets such as open arcs and top circles. We now introduce open curves and closed curves as continuous mappings into Σ of sensed open arcs and circles respectively. Two such curves φ_1 and φ_2 are the *same* or more precisely in the *same curve class* if $\varphi_1 = \varphi_2 T$, where T is a top sense preserving mapping of the domain of φ_1 onto the domain of φ_2. If T is a top mapping of the domain of a curve φ onto itself inverting sense, then φT will be denoted by $\varphi -$. We may denote φ by $\varphi +$.

If φ is a mapping of a domain E into Σ defining a curve, the image $\varphi(E)$ in Σ will be termed the *carrier* $|\varphi|$ of φ. By the intersection of two curves φ and ψ we mean the intersection of $|\varphi|$ and $|\psi|$. By definition a curve φ bounds a set E if $\beta E = |\varphi|$.

R-continuations in βR. Suppose Σ oriented so that the local right (left) sets of any point in an open sensed arc are well defined, cf. MJ 1 § 5. Let R be an F-region and $\alpha \in F$ be in βR. Let α be sensed so that its local right sets are in R. Then R can be continued as a locally simple curve φ, cf. Morse [4], so that its carrier is an F-set, and so that the sensed carrier of a simple open subcurve of φ has its local right sets in R. Continued maximally in this way with carrier in $F^* | \beta R$, φ will be called a *right* R-continuation in βR. *Left* R-continuations in βR are similarly defined. Two R-continuations φ_1 and φ_2 are regarded as the *same* if and only if they are both right or both left continuations, and if φ_1 and φ_2 are in the same curve class.

We need a parameterization of the boundary of an F-region. We shall make use of an open disc $D\{|w| < 1\}$ and suppose that βD is given the *counter-clockwise* sense in the w-plane, so that local left sets of βD are in D.

Theorem 3.1. *An F-region R is the* $1-1$ *image in a directly conformal map f of an open disc* $D\{|w| < 1\}$ *onto R. Any such map admits a continuous extension over* $\bar{D}$ *such that* βD

is mapped onto βR. This mapping is $1-1$ in a sufficiently small neighborhood relative to $\bar{D}$ of any point of βD whose image is in Σ^*. The antecedent in βD of N or of S is a nowhere dense and possibly empty set. The mapping

$$\varphi = f \,|\, \beta D \tag{3.1}$$

defines a closed curve bounding R of which βR is the carrier.

One first defines f over D. By well known theorems one can extend f as stated over a set D_1 such that $f(D_1)$ covers $\bar{R} \cap \Sigma^*$. If βR does not include N or S the proof is complete. To continue consider the case in which βR includes both N and S.

Let D_N and D_S be respectively the f-antecedents of the intersection of $\bar{R} \cap \Sigma^*$ with the northern and southern hemispheres of Σ. Let E be the set of points in βD_N at which f is not yet defined. Set $f(z) = N$ for $z \in E$. We shall show that f as extended is continuous at each point z_0 of E.

Now E is bounded from D_S so that f is continuous at z_0 if $f \,|\, D_N$ is continuous at z_0. Let z_n, $n = 1, 2, \ldots$ then be an arbitrary sequence of points in D_N tending to z_0. Then $f(z_n) \to N$. Otherwise the set of points $f(z_n)$ would have an accumulation point p in $f(\bar{D}_N) - N$ at which $f^{-1}(p)$ is well defined and has a finite set of values $a_1, \ldots, a_m$, not in E and in number at most the number of different F-sectors in an F-neighborhood of p. In a sufficiently small neighborhood of each a_i in $\bar{D}$, f is well defined and z bounded from z_0. From this contradiction we infer that $f(z_n) \to N$ and that f is continuous at z_0.

We similarly extend f over $\bar{D}_S$. The case in which βR includes N or S alone is similar.

If the antecedent of N in βD were dense in βD, $f(z)$ would equal N over some arc of βD and hence be constant.

The theorem follows.

Inner cycles. A Jordan curve φ whose carrier is a top circle in F^* will be called an inner cycle.

N-loops, S-loops, NS-curves, SN-curves. Let g be an open arc in F^*. Let φ be a simple sensed curve whose carrier is g. Suppose that g has a unique negative limit point A and a unique positive limit point B. Then either $A = B = N$, or $A = B = S$, or $A = N$ and $B = S$, or $A = S$ and $B = N$. [Th 2.2(3).] Then φ is called respectively an N-loop, S-loop, NS-curve, SN-curve. By the *exterior* $E\varphi$ of an N-loop [S-loop] with carrier g is meant that region in Σ which is bounded by $\bar{g}$ and contains S [N]. The interior $I\varphi$ of φ is defined as $\Sigma - \mathrm{Cl}\, E\varphi$. We call attention to the fact that an N- or S-loop, NS- or SN-curve is in Σ^*.

Meridians. The carrier g of an NS- or SN-curve will be called a meridian. If M is the universal covering surface of Σ^* one sees that a g_M covering g divides M into two disjoint regions.

Lemma 3.0. *Let φ be an inner cycle or an N- or S-loop in F^*, and let q be a point not in φ if φ is an inner cycle, and in $E\varphi$ if φ is a loop. Then a transversal ray λ incident with q meets φ in at most one point.*

Let H be either one of the two regions bounded by φ if φ is an inner cycle, and $I\varphi$ if φ is a loop. If λ enters H at a point p of φ it leaves H at no other point r. Otherwise the subarcs of λ and $|\varphi|$ with end points p and r would bound a simply connected region in Σ containing neither N nor S. This is impossible by Th 2.3.

Lemma 3.1. *Any set of N-loops (S-loops) with disjoint interiors and diameters exceeding some positive constant is finite.*

If the lemma were false for N-loops there would exist a set of N-loops φ_n, $n = 1, 2, \ldots$ with disjoint interiors $I\varphi_n$ and points $p_n \in \varphi_n$ such that $p_n \to q \in \Sigma^*$ as $n \uparrow \infty$. The number of φ_n which meet q is at most the number of F-sectors in an F-neighborhood of q. We can accordingly suppose that no φ_n meets q. Note that q is in no $I\varphi_n$. We can further suppose the p_n chosen in φ_n so as to be in a transversal ray λ incident with q. Since q is in $E\varphi_n$ for each n, it follows from L 3.0 that φ_n meets λ only in p_n. Hence $I\varphi_n$ includes the open arc λ_n of λ separated from q by p_n. Thus

$$I\varphi_n \cap I\varphi_m \supset \lambda_n \cap \lambda_m.$$

This contradiction to the choice of the φ_n implies the lemma.

To adequately describe the boundary of an F-region we must define N- and S-circuits.

N-circuits, S-circuits. N-circuits are defined as locally simple open curves φ in Σ^* whose carriers are F-sets which have end points at N (i.e., positive and negative limit sets in N) and which intersect themselves without crossing. From this definition of an N-circuit the reader can derive the following decomposition of an N-circuit.

When φ is an N-circuit, $|\varphi|$ carries three open arcs, a, b, c, whose closures are simple F-sets; of which $\bar{a}$ has the initial point N and a terminal point $P \in \Sigma^*$, $\bar{b}$ has the initial point P and terminal point P, $\bar{c}$ has the initial point P and terminal point N. These arcs and end points N and P derive the order

$$NaPbPcN$$

from φ. The top circle $\bar{b}$ separates N from S. Finally $a \cap b = c \cap b = 0$, while $a \cap c$ is the empty set, or a half open arc with limiting end point P, or $a = c$.

The N-circuit φ carries the top circle $\bar{b}$. Let ψ be an inner cycle with carrier $\bar{b}$ and with a sense derived from φ. We term ψ the *inner cycle of the N-circuit* φ. It is uniquely determined by φ. [Cor 2.1.]

When φ is an N-circuit the *interior* $I\varphi$ of φ shall be the region whose boundary is $|\varphi| \cup N$. This region is unique and does not contain S.

The definition and decomposition of an S-circuit are similar.

The sensed closed curve $\varphi = f|\beta R$ *of* Th 3.1. When βR meets N or S let d be any maximal open arc in βD such that $\varphi(d)$ is in Σ^*. Then $\varphi|d$ is a locally simple, open, sensed curve ψ with end points in $N \cup S$, and $|\psi|$ is an F-set. By virtue of the conformality of $f|D$, the continuity of $f|\bar{D}$, and the locally simple character of $\varphi|d$, $\varphi|d$ is a left R-continuation in βR. When βR meets neither N nor S we set $d = \beta D$ and note that $\varphi = \varphi|d$, so that φ is a left R-continuation in βR with carrier βR. In each of the above cases we term $\varphi|d$ a *maximal subcurve* of φ in Σ^*.

The following theorem is basic.

Theorem 3.2. *If the boundary curve of an F-region R is given by* (3.1) *in* Th 3.1 *then any "subcurve" ψ of φ maximal in Σ^* is a left R-continuation in βR.*

(1) *Each maximal subcurve ψ of φ is either* (a) *an N- or S-loop,* (b) *an N-circuit,* (c) *an S-circuit,* (d) *an NS- or SN-curve, or* (e) *an inner cycle.*

(2) *Different types of subcurves* (b) *to* (e) *cannot coexist. There is at most one ψ of types* (b), (c) *or* (e). *When an inner cycle φ occurs $|\varphi| = \beta R$.*

(3) *Type* (d) *occurs if and only if $\beta R \supset N \cup S$. The subcurves ψ then include just one NS-curve ψ_1, and just one SN-curve ψ_2. If $|\psi_1| \neq |\psi_2|$, $|\psi_1| \cap |\psi_2|$ is either empty, a point, an arc, or a half open arc with one limiting end point at N or at S.*

(4) *The carriers of no two ψ intersect at most excepting ψ_1 and ψ_2 in* (3).

Let $\{\varphi\}$ be the set of subcurves ψ of φ maximal in Σ^*.

The first statement in the theorem has been covered.

Proof of (1), *Case* I. The subcurve ψ joins the two poles. Then ψ must be simple. Otherwise $|\psi|$ would carry a top circle separating N from S on $\bar{R}$. This is impossible when ψ joins pole to pole and R is connected. In Case I, ψ is an NS-curve or SN-curve.

Case II. *Not Case* I. If ψ is simple it is clearly an N- or S-loop, or an inner cycle. If not simple ψ is locally simple, has an F-set carrier, intersects itself without crossing itself and joins a pole Z to Z. These are the characteristics defining an N- or S-circuit.

Proof of (2), *Case* I. ψ is an inner cycle. In this case $\beta R = |\psi|$, and all types other than (e) are excluded.

Case II. ψ is an N- or S-circuit. An inner cycle is excluded as just seen. Since $|\psi|$ separates S from N, an N-circuit excludes an S-circuit and vice versa, and any circuit excludes an NS- or SN-curve.

Case III. ψ is an NS- or SN-curve. The exclusion of types (b), (c), (e), has already been established.

Proof of (3). It is immediate that ψ is of type (d), only if βR includes $N \cup S$. Conversely if φ is a closed curve and meets N and S there is at least one NS-curve ψ_1, and one SN-curve ψ_2 in $\{\varphi\}$. These curves ψ_1 and ψ_2 intersect, if at all, as stated in Th 3.2(3). For $|\psi_1| \cup |\psi_2|$ cannot carry a top circle g; since g would either separate N from S, which is impossible when $\beta R \supset N \cup S$, or bound a region not containing N or S, which is impossible by Th 2.2.

Finally ψ_1 and ψ_2 are unique as NS- and SN-curves in $\{\varphi\}$. To see this let R_M cover R on M, and in βR_M let ψ_{1M} and ψ_{2M} cover ψ_1 and ψ_2 respectively. There can be no meridian g in βR which is not covered on M by ψ_{1M} or by ψ_{2M}; otherwise a covering g_M of g in $\bar{R}_M$ would divide R_M. This is impossible since R_M is a homeomorph of R. We infer that ψ_1 and ψ_2 are unique NS- and SN-curves in $\{\varphi\}$. We note that $\psi_{1M} \cap \psi_{2M} = 0$.

Proof of (4). Let η be any element of $\{\varphi\}$ of type (a), (b), (c), or (e). Then the closure of $|\eta|$ separates Σ into at least two open sets of which one, say Ω, contains R. Any element of $\{\varphi\}$ is in $\bar{R} \subset \Omega$. Let p be a point of η. A neighborhood H_p of p relative to Ω may be obtained as a union of a finite number of left sets of η associated with p. Since $\Omega \supset R$, H_p contains a neighborhood of p relative to R. But if sufficiently restricted, the above left sets of η are in R, since η is an R-continuation, so that H_p, if sufficiently restricted, is a neighborhood of p relative to R. We suppose H_p so restricted.

Let ψ be any element of $\{\varphi\}$. Then

$$|\psi| \cap \bar{H}_p \subset (\Omega - R) \cap \bar{H}_p \subset |\eta| \cap \bar{H}_p.$$

If ψ meets η it must then be carried by $|\eta|$. But $|\eta|$ carries no element in $\{\varphi\}$ other than η. Thus intersections of elements of $\{\varphi\}$ can occur only for two elements of type (d). This establishes Th 3.2 (4).

We shall now give certain definitions and lemmas useful in the application of Th 3.2.

F^*-cycles. Let R be any simply connected region in Σ which contains S and whose boundary βR is the union of N and of a finite or countably infinite set of disjoint N-loops. Then R is an F-region whose boundary becomes the carrier of a closed curve defined by the mapping $\varphi = f|\beta D$ of βD onto βR as in Th 3.1. We term the curve φ or $\varphi -$ an N-cycle. S-cycles are similarly defined.

An *N-cycle, S-cycle,* or *inner cycle* will be called an *F^*-cycle.*

Concavity. Let R be a region bounded in part by F^*-cycles, circuits, or open arcs in F^*. Any such element φ will be termed concave toward R if no element of $F^*|R$ has a limiting end point in φ. We term R concave if no element in $F^*|R$ has a limiting end point in βR. If an F^*-cycle φ bounds a region H and is concave toward H, φ will be termed concave toward any set in H. In particular an inner cycle will be termed N-concave [S-concave] if concave toward N [S]. An N-loop φ will be termed S-concave [N-concave] if concave towards $E\varphi$ [$I\varphi$]. An S-loop φ is termed N-concave [S-concave] if concave towards $E\varphi$ [$I\varphi$].

F^*-sets. A set in Σ will be called an F^*-set if it contains each $h \in F^*$ which it meets. In particular an F^*-set which contains an $\alpha \in F$ contains any limiting end point which α may have in Σ^*. The union or intersection of an ensemble of F^*-sets is an F^*-set. Regions bounded by F^*-cycles whose maximal subcurves in Σ^* are non-singular are F^*-sets. The following is particularly noted.

(Δ). *If a region R is an F^*-set and if Y is an F-sector in R incident with a point q in βR, then βR includes the two F-rays in βY incident with q.*

The following lemma is a major tool in applying Th 3.2.

Lemma 3.2. *Let W be any ensemble of regions R each of which is the interior of a non-singular N- or S-loop, or a region bounded by a non-singular inner cycle, and set $U =$* Union *$R \mid (R \in W)$.*

(a). *Then U is an F^*-set.*

(b). *U satisfies Conditions Θ.*

(c). *If the regions of W form a sequence $H_1 \subset H_2 \subset \ldots$, if $\Sigma^* - U \neq 0$, and if βH_n is a top circle in F for each n, then U is concave and $\beta U \mid \Sigma^*$ is simple.*

We discard the trivial case in which $\beta U \cap \Sigma^* = 0$.

Proof of (a). Since each $R \in W$ is an F^*-set, U is an F^*-set.

Proof of (b). Let q be a point of βU in Σ^* and let Y be an arbitrary F-sector incident with q. We prove the following.

(m). *If $\beta(Y \cap U)$ meets q then $Y \subset U$.*

Case I. Some $H \in W$ contains an F-sector in Y incident with q. The closure of a transversal ray μ in Y, incident with q, meets βH in q. Since βH is a non-singular inner cycle or N- or S-loop it cannot meet $\bar{\mu}$ other than in q. [Th 2.3.] Hence Y is in H and so in U.

Case II. Not Case I. Since $\beta(Y \cap U)$ meets q there exists a sequence of points $p_n \in Y \cap U$ such that $p_n \to q$ as $n \uparrow \infty$. Since Case I is excluded, and since q is non-singular if in any $\beta R \mid (R \in W)$, there is at most one $R \in W$ such that q meets βR. Without loss of generality we can then suppose that p_n is in some $R_n \in W$ such that βR_n does not meet q. There accordingly exists a point r_n in $\beta R_n \cap Y$ such that $r_n \to q$ as $n \uparrow \infty$. Let λ be a transversal ray of Y incident with q. We can take r_n in λ. Let λ_n be the open arc in λ separated from q on λ by r_n. Then λ_n is in R_n by L 3.0. Since $r_n \to q$ as $n \uparrow \infty$, λ is in U and hence Y is in U.

Thus (m) is true in Case II as well as in Case I. Statement (b) follows from (m) and (a) of the lemma.

Proof of (c). To show that $\beta U | \Sigma^*$ is simple note that U in (c) is simply connected and hence an F-region. Moreover U contains N or S, and hence by Th 3.2 the components of $\beta U | \Sigma^*$ contain neither a meridian nor the carrier of an N- or S-circuit. Th 3.2 then implies that the remaining components of $\beta U | \Sigma^*$ do not intersect, so that $\beta U | \Sigma^*$ is simple.

If U were not concave there would exist an arc $h \in F^* | U$ with a limiting end point z in βU. Let z_1 be a point of h. For some n, H_n contains z_1. H_n excludes z. Nor does βH_n contain z, since βH_n is non-singular and contains neither N nor S. Thus βH_n separates z from z_1 and so must meet h. Since βH_n is non-singular this is impossible.

The *inner closure* $\hat{E}$ in Σ^* of a set E is defined by the equation

$$\hat{E} = \bar{E} \cap \Sigma^* - \beta \bar{E}. \tag{3.2}$$

Equivalently the inner closure in Σ^* of a set E is the set of all points in Σ^* which possess a neighborhood in which E is everywhere dense.

Lemma 3.3. *If R satisfies Conditions Θ its inner closure in Σ^*, the complement of its closure, and any component of R also satisfy Conditions Θ.*

Consider the F-set $\hat{R}$. We shall show that $\hat{R}$ satisfies Θ with R. To that end let q be a point of βR in Σ^*. The F-rays in βR incident with q are in two classes. Class i ($i = 1, 2$) consists of the F-rays incident with i F-sectors in R incident with q. F-rays in Class 2 in βR are not in $\hat{R}$. The number of F-rays in βR incident with q and in Class 1 is even (possibly zero) so that the number of F-rays in $\beta \hat{R}$ incident with q is even. If this number is zero q is not in $\beta \hat{R}$. Thus $\hat{R}$ satisfies Θ with R.

The remainder of the lemma is readily verified.

§ 4. Asymptotes

Let h be an open arc in F^* and q a point of h. Sense h. The sensed open subarc of h following q on h will be called an F^*-ray π. Let π be given as a $1-1$ continuous image in Σ^* of the interval $0 < t < \infty$, with the point $\pi(t)$ corresponding in π to t. Let φ be an F^*-cycle in Σ given by a mapping of a circle, on which $w = e^{i\theta}$, into Σ, so that $\varphi(\theta)$ corresponds to $w = e^{i\theta}$ and $\varphi(\theta + 2\pi) = \varphi(\theta)$. We say that π and h are asymptotic to φ in the positive sense of π if for some admissible representation of π

$$\operatorname{dist}[\pi(\theta), \varphi(\theta)] \to 0 \quad [\text{as } \theta \uparrow \infty].$$

In discussing asymptotic rays the following lemma is fundamental.

Lemma 4.1. *Let λ be a transversal. If an F^*-ray π meets λ in points p_1, p_2, p_3 which are successive on π in the set of intersections of π and λ, then these points are also successive on λ in one of λ's two senses, and π always crosses λ in the same sense.*

For $i \neq j$ let π_{ij} be the arc of π bounded by p_i and p_j, and let λ_{ij} be the arc of λ bounded by p_i and p_j. We show first that p_1, p_2, p_3 occur in the order written on λ, regarding p_3, p_2, p_1 as the same order on λ.

Case I. *The ray π crosses λ in opposite senses at p_1 and p_2.*

The arcs π_{12} and λ_{21} form a top circle. Let R_{12} be the region bounded by $\pi_{12}\lambda_{21}$ in Σ and not containing $\lambda - \lambda_{12}$. By virtue of Th 2.3, R_{12} includes a pole, say S. The arcs π_{23} and λ_{32} also form a top circle. Let R_{23} be a region bounded by this curve chosen so as not to contain R_{12}. Then R_{23} contains N. Any order of p_1, p_2, p_3 on λ other than that written will be shown to be impossible.

Suppose the order on λ is p_1, p_3, p_2 or p_3, p_1, p_2. These are the only orders which must be excluded. In these cases π_{13} and λ_{31} form a top circle bounding the region

$$R_{13} = \Sigma - (\bar{R}_{12} \cup \bar{R}_{23}).$$

Here $\bar{R}_{13}$ is simply connected, contains no pole, and in βR_{13}, π_{13} meets λ_{13} twice. On M there exists a homeomorphic covering of $\bar{R}_{13}$ on which π_{13M} and λ_{13M} meet twice. This is contrary to Th 2.3. Thus p_1, p_2, p_3 is the only order possible in λ.

Case II. *The ray π crosses λ in the same sense at p_1 and p_2.*

Suppose p_1, p_3, p_2 is the order in λ. Then λ_{12} and π_{12} together bound a region R into whose interior π enters at p_2. Continued in this sense π must leave R by crossing λ_{12} at p_3 in a sense opposite to the crossing at p_1 and p_2. On reversing the sense of π and denoting p_1, p_2, p_3 by p_3, p_2, p_1 respectively, the situation comes under Case I and is impossible.

The order p_3, p_1, p_2 may be excluded in Case II as contrary to the Jordan Separation Theorem. Thus the order on λ must be p_1, p_2, p_3 in any case.

It remains to show that all crossings of λ by π are in the same sense. We first show that Case I is impossible. In Case I, π reversed in sense enters R_{13} at p_1, and continued as an $h \in F^*$ must meet λ_{13} in a point p_0; for there is no pole in the simply connected region R_{13} to which h can tend, cf. MJ 2 Th 7.2. Thus p_0, p_1, p_2 appear in this order on π but in the order p_1, p_0, p_2 or p_1, p_2, p_0 on λ_{13}. Hence Case I is impossible.

In Case II the crossing of λ at p_3 is in the same sense as at p_1 and p_2; for otherwise a reversal of the sense of π would yield Case I again.

This establishes the lemma.

We state an extension whose proof is essentially identical with the preceding.

Lemma 4.2. *Let λ be a transversal cut with vertex q. Let h be an element in F^* such that $h \cap \lambda$ excludes q. If π is an F^*-ray in h the conclusion of Lemma 4.1 holds as stated.*

The exclusion of q from $h \cap \lambda$ insures that each point of $h \cap \lambda$ is an actual crossing of λ by π. All other points of λ are non-singular.

Each of the regions of Σ into which the carrier of an inner cycle divides Σ will be called a *side* of φ or of $|\varphi|$. The two sides of φ may be distinguished as the *north* or *south side* of φ according as the side contains N or S. Given an N- or S-cycle the region H bounded by $|\varphi|$ which contains S or N respectively will be called the south or north side of φ.

The construction of the region J. Let π be an F^*-ray with a positive limit point $p \in \Sigma^*$. We shall construct an F-region J such that π is asymptotic to a curve carried by βJ.

Let λ be a transversal ray incident with p and such that π intersects λ in an infinite sequence of points

$$p_1, p_2, p_3, \ldots \tag{4.1}$$

with limit point p. Suppose that the intersections of π with λ appear on π in the order (4.1). In accordance with L 4.1 the points (4.1) appear on λ in the same order.

For $k = 1, 2, \ldots$, let π_k be the arc $p_k p_{k+1}$ of π and λ_k the arc $p_k p_{k+1}$ of λ. Let J_k be the open region bounded by the top circle $\pi_k \lambda_k$ chosen so as not to contain p. Then

$$J_1 \subset J_2 \subset J_3 \subset \ldots .$$

By virtue of Th 2.3 J_1 must contain a pole, say S. Let

$$J = \text{Union } J_k \quad (k = 1, 2, \ldots).$$

Then J is a simply connected region which contains S. Now $\Sigma - J_k$ meets N and hence

$$N \in \Sigma - J = \bigcap_k (\Sigma - J_k) \quad (k = 1, 2, \ldots).$$

We continue with a proof of the following.

(i) *There is no singular point in π following p_2 in π.*

Suppose z were such a singular point. There then exists an $\alpha \in F$ with z as an initial point and $\alpha \cap \pi = 0$. Let h be an F^*-ray continuing α. Since z is not in the transversal λ and follows p_2 in π, a sufficiently restricted open arc of h with end point z in π is in

$$J_{n+1} - \bar{J}_n = Q_n$$

for some $n > 0$. Observe that

$$\beta Q_n = \text{Union } \pi_n \lambda_n \pi_{n+1} \lambda_{n+1},$$

and that the set $Q_n^* = \bar{Q}_n - p_{n+1}$ is simply connected.

Since Q_n is simply connected and contains neither N nor S, h must meet βQ_n in a point z' following z on h. There are several a priori possibilities for the location of z and z' in βQ_n but in each case one infers the existence of an inner cycle of F^* in Q_n^*, or else of a subarc

$b \subset Q_n^*$ of an element in F^* such that b intersects λ in more than one point. Since this is impossible (i) follows.

(ii) *βJ is the set Ω of positive limit points of π, so that the maximum distance of points of π_n from βJ tends to zero as $n \uparrow \infty$.*

Statement (ii) is a consequence of an elementary lemma, namely that βJ is the set of all points which are limit points of sequences z_n, $n = 1, 2, \ldots$ in which z_n is in βJ_n. This lemma presupposes the inclusion relations $J_1 \subset J_2 \subset J_3 \ldots$ and the disjointness of the top circles βJ_n, $n = 1, 3, 5, \ldots$. One notes that βJ_n separates βJ_{n-2} from βJ_{n+2}. Statement (ii) follows on recalling that βJ_n is the union of λ_n and π_n, and that the diameter of λ_n tends to zero as $n \uparrow \infty$.

(iii) *J is a concave F-region whose curve boundary is an F^*-cycle.*

If the boundaries of the regions J_n were top circles in F, J would satisfy Conditions Θ on an F-region, by L 3.2(c). We can, however, alter F in an F-sector Y containing the transversal ray λ used in constructing J so that βJ_n, $n = 1, 3, 5, \ldots$ becomes a top circle in a new family F' replacing F. This alteration shall leave $F|(\Sigma - Y)$ unaffected, and can be made in Y in an obvious manner leaving Y an F'-sector. Cf. proof of L 7.1 in MJ 2.

We conclude that J satisfies Conditions Θ, is concave, and that $\beta J|\Sigma^*$ is simple [L 3.2]. Since J is simply connected and βJ contains more than one point, it follows that J is an F-region. Since J contains a pole, βJ carries no N- or S-circuits nor NS- or SN-curves. Its curve boundary as given by Th 3.2 reduces to an F^*-cycle.

This establishes (iii).

The principal theorem of this section follows.

Theorem 4.1. *If an F^*-ray π has a positive limit point $p \in \Sigma^*$ then π is asymptotic to an F^*-cycle φ. Moreover, π does not intersect φ, and φ is concave toward the side of φ which contains π.*

The ray π determines an F-region J as just constructed. By (iii) βJ carries an F^*-cycle ψ and we shall show that $\varphi = \psi \pm$ satisfies Th 4.1. In accordance with Th 3.1 ψ maps the circle C ($|w| = 1$) into βJ in a manner which is $1 - 1$ over those open arcs of C whose images are in $\beta J|\Sigma^*$. We suppose that the point $w = e^{i\theta}$ in C corresponds to the point $\psi(\theta)$ in βJ. Identify the point p of Th 4.1 with the point p used in the construction of J and suppose that $p = \psi(2n\pi)$, $n = 0, \pm 1, \pm 2, \ldots$. Recall the transversal ray λ incident with p, and the points $p_1, p_2, \ldots$ on $\lambda \cap \pi$ used in the construction of J.

We state the following.

(η) *The Fréchet distance on Σ of the subarc $p_n p_{n+1}$ of π from the arc $\varphi|(0 \leq \theta \leq 2\pi)$ tends to zero as $n \uparrow \infty$* [for proper choice of φ as $\psi \pm$].

The proof of (η) starts with (ii) and is almost identical with the proof of Th 4.1 of MJ 3. It will be omitted. Granting (η) one can then map the arc $p_n p_{n+1}$ of π in a homeomorphic manner onto the interval $2n\pi \leq \theta \leq 2(n+1)\pi$, $n = 1, 2, \ldots$ with $\pi(\theta)$ a single-valued continuous image of θ for all such θ, and such that

$$\text{dist}\,[\pi(\theta), \varphi(\theta)] \to 0 \quad [\text{as } \theta \uparrow \infty].$$

This representation of π from p_1 on is $1-1$ and so admissible, and π is accordingly asymptotic to φ.

By L 3.2(c), J is concave so that φ is concave towards the side of φ which contains π. This completes the proof of the theorem.

Corollary 4.1. *No point of an open arc $h \in F^*$ is a positive or negative limit point of h.*

This is immediate if h is not an asymptote, and true for asymptotes since asymptotes cannot intersect the concave F^*-cycles which are their positive or negative limit sets. This corollary also follows from L 4.1 by a suitable argument.

§ 5. Concave annuli $A(\varphi_1, \varphi_2)$

We shall consider open annuli $A(\varphi_1, \varphi_2)$ in Σ^* each bounded by two non-intersecting F^*-cycles, φ_1 and φ_2, and, in the case in which the annulus is concave, give a complete description of elements of F^* in the annulus. We proceed with three lemmas.

Lemma 5.1. *A sensed $h \in F^*$ which is asymptotic in its positive sense to an F^*-cycle φ can be asymptotic in its negative sense neither to $\varphi+$ nor to $\varphi-$.*

Let h be divided by a point p into two rays π' and π''. Let λ be a transversal tending to a point q of φ from the side of φ which contains h. If the lemma were false π' and π'' would intersect λ in sequences of points p_n' and p_n'', $n = 1, 2, \ldots$ respectively, tending to q in λ as $n \uparrow \infty$, with the order

$$\ldots, p_2', p_1', q, p_1'', p_2'', \ldots \tag{5.1}$$

on h. These points cannot appear in the order (5.1) on λ as they should by L 4.1, and we infer the truth of L 5.1.

Corollary 5.1. *In the set of F^*-rays issuing from a given point $p \in \Sigma^*$ and asymptotic either to $\varphi+$ or $\varphi-$, where φ is an F^*-cycle, there is at most one F^*-ray.*

Lemma 5.2. *If φ_1 is an inner cycle and φ_2 a second inner cycle or a concave N- or S-cycle, then φ_1 is concave toward φ_2.*

The cycle φ_1 does not intersect φ_2. This follows from Cor 2.1, if φ_2 is an inner cycle, and from the concavity of φ_2, if φ_2 is an N- or S-cycle. For definiteness suppose that φ_2 is on the north side of φ_1.

If the lemma were false there would be an F^*-ray π with initial point r in φ_1, entering $A(\varphi_1, \varphi_2)$ at the point r and not meeting φ_1 again. This ray cannot intersect φ_2 if φ_2 is an inner cycle [Cor 2.1], or if φ_2 is S-concave. It follows from Th 4.1 that π is asymptotic to an F^*-cycle φ_3 (possibly φ_2) in Cl $A(\varphi_1, \varphi_2)$. Let λ be a transversal in the south side of φ_3 tending to a point $p \in \varphi_3$. Then π will intersect λ in an infinite sequence of points $p_1, p_2, \ldots$ tending to p as a limit point and appearing on π in the order written [L 4.1]. Let

$$\pi(r, p_2), \quad \lambda(p_2, p_1), \quad \pi(p_1, r) \tag{5.2}$$

be respectively subarcs of π from r to p_2, of λ from p_2 to p_1, and of π from p_1 tò r, forming a sequence b of arcs joining r to itself. Let b_M be an arc covering b on M. The end points of b_M cover r but are not coincident. They can, however, be joined on M by an arc covering $|\varphi_1|$ a finite number m of times to form a top circle g_M on M. (That $m = 1$ is true, but not necessary for the proof.)

Let λ_M be the covering of λ which meets g_M. Then $g_M - \lambda_M$ admits an extension on M as an element h_M in F_M^*. This is impossible since h_M meets λ_M in two points. [Th 2.3.]

We infer the truth of the lemma.

Lemma 5.3. *In a concave annulus B between two F^*-cycles there can be no singular point P.*

First note that any inner cycle φ in B must be non-singular. For it follows from L 5.2 that φ must be both N- and S-concave and hence non-singular.

Suppose then that L 5.3 is false in that P exists. There then exist at least four F-rays issuing from P. The continuations as elements in F^* of none of these F-rays can carry an inner cycle, since such an inner cycle would be singular. There are thus at least four F^*-rays issuing from P. None of these F^*-rays can meet βB or have a limiting end point at N or S in βB, since B is concave. It follows from Th 4.1 that each such F^*-ray must be asymptotic to an F^*-cycle in $\bar{B}$. Moreover two different F^*-rays issuing from P are asymptotic to F^*-cycles φ_1 and φ_2 with different carriers [Cor 5.1].

The cycles φ_1 and φ_2 are concave toward their respective sides containing P, so that P is in $A(\varphi_1, \varphi_2)$; for if φ_i separated φ_j from P [$i, j = (1, 2)$ or $(2, 1)$] then the asymptotic ray from P to φ_j would intersect φ_i, contrary to the concavity of φ_i toward P. Any two remaining rays issuing from P determine an annulus $A(\varphi_3, \varphi_4) \subset A(\varphi_1, \varphi_2)$. From the reciprocity of the pairs (φ_1, φ_2) and (φ_3, φ_4), we infer that $A(\varphi_1, \varphi_2) = A(\varphi_3, \varphi_4)$. Hence with proper notation $|\varphi_1| = |\varphi_3|$ and $|\varphi_2| = |\varphi_4|$. Two of the four rays must then be asymptotic to two cycles with the same carrier, contrary to Cor 5.1.

Hence P does not exist and the lemma is true.

Types of asymptotes in $A(\varphi_1, \varphi_2)$. Suppose that $A(\varphi_1, \varphi_2)$ is concave and includes no inner cycle. An $h \in F^*$ in $A(\varphi_1, \varphi_2)$ is an asymptote in both its senses since $A(\varphi_1, \varphi_2)$ is concave. It follows from Cor 5.1 that in one of its senses h is asymptotic to $\varphi_1 \pm$ and in the other to $\varphi_2 \pm$. If h is asymptotic to φ_1 in one sense and to φ_2 in its other sense then h is said to be of asymptotic type $[\varphi_1, \varphi_2] = [\varphi_2, \varphi_1]$. The four possible asymptotic types of h are

$$[\varphi_1 +, \varphi_2 +], \quad [\varphi_1 +, \varphi_2 -], \quad [\varphi_1 -, \varphi_2 +], \quad [\varphi_1 -, \varphi_2 -].$$

Two elements h and h' of F^* in $A(\varphi_1, \varphi_2)$ must be of the *same* asymptotic type. Otherwise h and h' would intersect in a point P. The existence of P becomes clear on considering the elements h_M and h'_M covering h and h' respectively on M. The intersection P cannot exist by L 5.3.

We shall complete the analysis of annuli $A(\varphi_1, \varphi_2)$ in § 8.

§ 6. *N*-caps, *S*-caps

In the decomposition of Σ into basic regions of a nature dictated by F^* one comes naturally to N-caps and S-caps. For the purpose of defining these caps and for many other purposes we shall abbreviate the phrase "inner closure in Σ^* of the union" by the word symbol Ûnion. With this understood an N-cap [S-cap] is the Ûnion of all non-singular N-loops [S-loops]. This definition requires justification and elaboration.

There is at most a countably infinite number of elements in F with singular end points. Through each neighborhood in Σ^* there accordingly passes a non-singular $h \in F^*$. There are also examples of families F^* such that through each neighborhood in Σ^* there passes a singular $h \in F^*$. Thus non-singular elements in F^* *are* in fact everywhere dense, while singular elements in F^* *may* be everywhere dense. If in particular a neighborhood is in the interior of an N-loop [S-loop], then each non-singular $h \in F^*$ meeting this neighborhood carries an N-loop [S-loop]. This fact will simplify subsequent proofs.

Beside the question as to the vanishing of N-caps [S-caps] there is the question as to whether N-caps [S-caps] are *bounded* from S [N]. This leads to the natural separation of the cases in which N-loops [S-loops] are or are not bounded from S [N].

A first theorem follows.

Theorem 6.1. *If N-loops are not bounded from S there exists an F-region Q which is an F^*-set, whose curve boundary meets Σ^* in an NS- and an SN-curve and in disjoint N- and S-loops at most countable in number.*

There exist non-singular N-loops bearing points arbitrarily near S. It follows from L 3.1 that there is an infinite sequence $\varphi_1, \varphi_2, \ldots$ of nonsingular N-loops such that

$$I(\varphi_1) \subset I(\varphi_2) \subset \ldots, \tag{6.1}$$

while dist $[S, \varphi_n] \to 0$ as $n \uparrow \infty$. We shall show that the set $Q =$ Union $I(\varphi_n)$, $n = 1, 2, \ldots$ satisfies the theorem.

The set Q is simply connected and βQ contains N and S. Q satisfies Conditions Θ by L 3.2(b), and so is an F-region. It is an F^*-set since a union of F^*-sets. It follows from Th 3.2 that the closed curve boundary of Q meets Σ^* in an NS- and SN-curve, and in disjoint N- and S-loops at most countable in number.

Corollary 6.1. *If N-loops are not bounded from S there exists at least one meridian in F^*.*

Th 6.2 is similar to Th 6.1, with g in Th 6.2 replacing S in Th 6.1. The proof is similar.

Theorem 6.2. *Let g be a top circle in F^*. If N-loops are not bounded from g there exists an F-region which is an F^*-set, which is in the north side of g and whose boundary meets N and g.*

One defines Q as in the proof of Th 6.1, except that here dist $[g, \varphi_n] \to 0$ as $n \uparrow \infty$. One continues as in the proof of Th 6.1.

N-caps, S-caps. We have already defined an N-cap [S-cap]. One can equivalently define an N-cap U_N as the Union of the *interiors* of all non-singular N-loops. If there are no N-loops we understand that $U_N = 0$. S-caps U_S are similarly defined. We term U_N [U_S] *bounded* if bounded from S [N], otherwise *unbounded.* It is possible that U_N or U_S may equal Σ^*.

Maximal N-cycles, S-cycles. By the interior $I\varphi$ of an N-cycle φ is meant the union of the interiors of all the N-loops carried by $|\varphi|$. An N-cycle φ will be termed *maximal* if $I\varphi \supset I\psi$ whenever ψ is an N-cycle. There may be no maximal N-cycle φ, but when one exists the N-cap $U_N \neq 0$ and $\beta U_N = |\varphi|$. Maximal S-cycles are similarly defined.

Corollary 6.2. *If N-loops are not bounded away from an inner cycle φ, then φ is the inner cycle of an N-circuit ψ.*

Let $I_N\varphi$ be the north side of φ. According to Th 6.2 $F^*|I_N\varphi$ includes an open arc h with end points on $|\varphi|$ and N respectively. Then $|\varphi| \cup h$ carries an N-circuit ψ with the sense of φ, so that φ is the inner cycle of ψ.

Cor 6.2 suggests a major theorem.

Theorem 6.3. (i). *If U_N is a bounded, non-empty N-cap, $\beta(U_N \cup N)$ is the minimum carrier either of a maximal N-cycle φ, or of an inner cycle ψ not N-concave.*

(ii). *Conversely, a maximal N-cycle φ or an inner cycle ψ not N-concave bounds $U_N \cup N$, where U_N is a non-empty N-cap bounded from S.*

(iii). *A maximal N-cycle φ and an inner cycle ψ which is not N-concave cannot coexist. When φ exists $\varphi-$ is the only other maximal N-cycle, and when ψ exists $\psi-$ is the only other inner cycle which is not N-concave.*

Proof of (i). We begin by establishing (m) and (n).

(m). *If βU_N carries an inner cycle ψ, $\beta(U_N \cup N) = |\psi|$, and ψ is the inner cycle of an N-circuit.*

The south side $I_S \psi$ of ψ does not meet U_N, since no non-singular N-loop meets ψ. Thus $U_N \subset I_N \psi$. By Cor 6.2 ψ is the inner cycle of an N-circuit η. As such ψ cannot be N-concave. Each non-singular element in $I_N \psi$ is an N-loop since there is an open arc in $F^* | I_N \psi$ joining ψ to N. Hence $U_N \cup N = I_N \psi$ and $\beta(U_N \cup N) = |\psi|$.

(n). *If $U_N \neq 0$ and if βU_N carries no inner cycle, βU_N is the minimum carrier of a maximal N-cycle.*

Set $H = \Sigma - \mathrm{Cl}\, U_N$ and let L be the component of H which includes S. We shall show that L is an F-region. Since U_N is an F-set, L is an F-set. Since $\bar{U}_N$ is connected, L is simply connected. Finally L satisfies Conditions Θ on F-regions with H and U_N [L 3.3]. Now βL carries no inner cycle and hence no N-circuit. By Th 3.2, βL must be the minimum carrier of an N-cycle φ. This N-cycle must be a maximal N-cycle; otherwise L would meet the interior of some N-loop, contrary to its definition. The interior $I\varphi$ of a maximal N-cycle φ is an N-cap U_N. Hence $\beta U_N = |\varphi|$ and (n) is established.

Statement (i) follows from (m) and (n).

Proof of (ii). We first establish (a).

(a). *An inner cycle ψ which is not N-concave is the inner cycle of an N-circuit η.*

If ψ is not N-concave, an F^*-ray π exists in the north side of ψ with an initial point in ψ. This ray must have a limiting end point at N. Otherwise π would be asymptotic to an S-concave F^*-cycle ψ_1 north of ψ. [Th 4.1], and ψ would be N-concave by L 5.2, contrary to hypothesis. It is clear that $|\psi| \cup \pi$ carries an N-circuit η with the sense of ψ, so that ψ is the inner cycle of η. This establishes (a).

If ψ exists any non-singular element in $I(\eta)$ is an N-loop by (a), and all non-singular N-loops are north of ψ. It follows that $\beta(U_N \cup N) = \psi$. If φ exists $\beta(U_N \cup N) = |\varphi|$, as already noted. In both cases $U_N \neq 0$.

Proof of (iii). There is precisely one N-cap U_N, in accordance with its definition. Now $\beta(U_N \cup N) = |\varphi|$ or $|\psi|$ by (ii), when φ or ψ exists. These possibilities are mutually exclusive since βU_N is unique. Statement (iii) follows.

Theorem 6.3 has an obvious counterpart for S-caps.

§ 7. The metric space Φ

Let ψ be an inner cycle or S-cycle. Let R be the region north of ψ. The cycle ψ is sensed. We say that ψ is *positively sensed* in Σ if ψ is in the curve class of the curve $\varphi = f|\beta R$ of Th 3.1. Alternately let ψ be an inner or N-cycle. Let R be the region south of ψ. It is consistent with the preceding to say that ψ is positively sensed in Σ if $\psi -$ is in the curve class of $\varphi = f|\beta R$ of Th 3.1. For S-cycles and N-cycles the two definitions are not overlapping.

Let the latitude θ on Σ be so defined that when θ increases on a parallel g, N is to the left of g. Since a parallel can be homotopically deformed through top circles on Σ^*, nonbounding on Σ^*, into any such top circle, it follows that when an inner cycle ψ is positively sensed in Σ and is traced in its positive sense n times, then $\theta \uparrow \infty$ as $n \uparrow \infty$.

Let $\mathcal{N}$ and $\mathcal{S}$ be degenerate closed curves with N and S, respectively, as carriers. We shall introduce a space Φ consisting of $\mathcal{N}$, $\mathcal{S}$, and all inner, maximal N- and S-cycles, positively sensed in Σ. By a *cycle in* Φ we shall mean any element in Φ other than $\mathcal{N}$ or $\mathcal{S}$. Two cycles φ and ψ in Φ will be ordered by the relation $\varphi < \psi$, or equivalently, $\psi > \varphi$, if $|\varphi|$ separates $|\psi|$ from S. We write $\mathcal{S} < \mathcal{N}$ and for every cycle $\varphi \in \Phi$, $\mathcal{N} < \varphi < \mathcal{S}$. Equality of elements in Φ shall mean that they belong to the same curve class. The order relation $<$ is transitive. No two different cycles in Φ intersect.

Let x, y be any pair of elements in Φ and let p be an arbitrary point in y. Let

$$d(x, y) = \max_{p \in y} \text{dist}\,[x, p].$$

This distance makes Φ a non-symmetric metric space. Set

$$K(y) = d[\mathcal{S}, y]\,|\,(y \in \Phi).$$

Suppose that the radius of Σ is 1.

Lemma 7.1. *The transformation K of Φ into the interval $[0, \pi]$ carries Φ in a $1-1$ manner into a closed subset of $[0, \pi]$.*

Let y_n be a sequence of elements in Φ such that $K(y_n)$ converges to a value b in $[0, \pi]$. It is sufficient to show that for some $a \in \Phi$, $K(a) = b$. Without loss of generality we can suppose that the sequence y_n is decreasing in Φ and that $y_n < \mathcal{N}$. The case of an increasing sequence is similar.

Let R_n be the region in Σ north of y_n and set $R = \text{Union } R_n$. If $R = \Sigma - S$, $b = 0 = K(\mathcal{S})$. Suppose then that $R \neq \Sigma - S$. For $n > 2$ each y_n is a non-singular inner cycle [L 5.3], so that it follows from L 3.2(c) that R is a concave F-region with $\beta R|\Sigma^*$ simple. Since $N \subset R$, $f|\beta R$ in Th 3.1 defines an F^*-cycle φ with carrier βR. If φ is an inner cycle, φ is in Φ. If φ is an S-cycle, the concavity of R implies that φ is a maximal S-cycle, and so in Φ. We shall show that $b = K(\varphi)$. Observe that $0 < K(y_n) - K(\varphi) = d[S, y_n] - d[S, \varphi] \leq d[\varphi, y_n]$, using the triangle axiom. But $d[\varphi, y_n] \to 0$ as $n \uparrow \infty$ so that $K(\varphi) = \lim K(y_n) = b$.

This establishes L 7.1.

Our final metric for Φ shall be one in which the distance between any two elements x and y in Φ shall be the Euclidean distance $|K(x) - K(y)|$ between $K(x)$ and $K(y)$ as points in the interval $[0, \pi]$.

In terms of the order defined in Φ and its metric, sup E and inf E are defined for any non-empty subset E of Φ. We understand thereby that "sup" = L.U.B. and "inf" = G.L.B. Other terms involving limits are similarly defined. It follows from L 7.1 that Φ is compact.

U-type asymptotes. The metricizing of the space Φ and our earlier results concerning the absence of singular points on asymptotic rays make possible the treatment of a problem concerning concave annuli. Such an annulus A, bounded as it is by cycles φ and ψ in Φ, will be said to be of U-type if filled with asymptotes of type $[\varphi +, \psi +]$ or $[\varphi -, \psi -]$. Cf. § 5. The theorem is as follows.

Theorem 7.1. *Let C_e, $e > 0$, be the union of e-neighborhoods of N and S. For fixed e the number of disjoint annuli of U-type which do not meet C_e is finite.*

Suppose the theorem false. There would then exist a sequence A_n, $n = 1, 2, \ldots$ of disjoint annuli of U-type with boundaries $\varphi_n, \psi_n, \varphi_n < \psi_n$, not meeting C_e. Without loss of generality we can suppose that $\varphi_1 > \varphi_2 > \ldots$. Since Φ is complete and since $d[S, \varphi_n] > e$, for some cycle $\eta \in \Phi$, $\varphi_n \to \eta$ as $n \uparrow \infty$.

Let p be a point on η at the maximum distance on $|\eta|$ from $|\eta|$ to S. Let λ be a transversal on the north side of η, incident with η at p. We know that $d[S, \varphi_n]$ decreases monotonically to $d(S, \eta)$ as a limit, and that $d[S, \eta]$ equals the distance of p from S. Hence for n sufficiently large, say $n = m$, λ will intersect $\bar{A}_m$ in a simple arc μ, interior to $\bar{A}_m$ except for end points on φ_m and ψ_m respectively. [L 3.0.]

Let H "cover" A_m on M, with boundaries h and k in F^*_M covering φ_m and ψ_m respectively. The arc μ is covered on M by an unending sequence of disjoint open arcs $\ldots, b_{-1}, b_0, b_1, \ldots$ such that $\theta | b_r$ tends to $\pm\infty$ as $r \to \pm\infty$, while b_r separates H into two regions on one of which $b_{r+1}, b_{r+2}, \ldots$ lie, and on the other $b_{r-1}, b_{r-2}, \ldots$.

Let g on H cover a U-type asymptote in A_m. Let R_n be the region on H bounded on H by b_n and b_{n+1}. If g meets R_n it must cross b_n and b_{n+1} in unique points. [Th 2.3.] We infer that g must meet the open arcs $\ldots, b_{-1}, b_0, b_1, \ldots$ in unique points p_n appearing on g in the order of the indices n. Hence $\theta | g$ tends to $+\infty$ in one sense of g and to $-\infty$ in the other, contrary to our hypothesis that its projection on Σ^* is of U-type.

This establishes Th 7.1. Cf. Kaplan [3].

§ 8. The case of no meridian

We begin with the following lemma.

Lemma 8.1. *The absence of meridians implies the following*:

(a). *There is at least one cycle in* Φ.

(b). *If* Φ *contains precisely one cycle* φ, φ *is* N- *or* S-*concave.*

(c). *An inner cycle,* φ *in* Φ, *is* N- *or* S-*concave.*

(d). *If* $\psi \in \Phi$ *is the immediate successor of* $\varphi \in \Phi$ *and if* φ *and* ψ *are cycles, then one at least of these two cycles is concave toward the other.*

Proof of (a). Suppose that there is no cycle in Φ. There are then no asymptotes in F^* [Th 4.1], and no bounded S- or N-caps [Th 6.3 (i)]. The set K of N-loops is empty; otherwise U_N would be unbounded and a meridian exist [Cor 6.1], contrary to hypothesis. Similarly there are no S-loops. With asymptotes excluded the positive [negative] limit set of an open $h \in F^*$ must reduce to N or to S. [Th 4.1.] Hence each $h \in F^*$ is a meridian. From this contradiction we infer (a).

Proof of (b). Suppose (b) false in that φ is neither N- nor S-concave. If φ is an N-cycle not S-concave, there would exist an element $h \in F^* | I_S \varphi$ (where $I_S \varphi$ is the south side of φ) with a limiting initial point in $|\varphi|$. Such an h could not be asymptotic to an F^*-cycle ψ, because $\psi \pm$ would be in Φ and different from φ. Hence $|\varphi| \cup h$ would carry a meridian contrary to hypothesis. Similarly φ is not an S-cycle. If φ were an inner cycle there would exist rays π_1 and π_2 in the north and south sides of φ respectively, with initial points in $|\varphi|$ and limiting final points in N and S respectively, so that $|\varphi| \cup \pi_1 \cup \pi_2$ would carry a meridian contrary to hypothesis. Thus (b) is true.

Proof of (c). If (c) were false it would follow from (a) of the proof of Th 6.3 (ii) that φ would be the inner cycle of an N-circuit φ_1 and of an S-circuit φ_2. Then $|\varphi| \cup |\varphi_1| \cup |\varphi_2|$ would carry a meridian contrary to hypothesis.

Proof of (d). There are four cases as follows.

(1) φ and ψ inner cycles.

(2) φ an inner cycle, ψ a maximal N-cycle.

(3) φ a maximal S-cycle, ψ an inner cycle.

(4) φ a maximal S-cycle, ψ a maximal N-cycle.

In Case (1), (d) follows from L 5.2.

In Case (2), the falsity of (d) implies, as in the proof of (b), that there exists an $h \in F^*$ which meets φ and ψ; φ is accordingly not N-concave, and so cannot coexist with a maximal N-cycle ψ. [Th 6.3 (iii).] Case (3) is similar to Case (2).

In Case (4) the falsity of (d) again implies that there exists an $h \in F^*$ which meets φ and ψ and is a meridian.

We infer the truth of (d).

The subset Ψ of Φ. An F^*-cycle which is N- or S-concave will be called *concave*. Let Ψ be the subset of concave F^*-cycles in Φ.

Lemma 8.2. *When there is no meridian the set Ψ is not empty and contains every cycle in Φ, excepting $\varphi_N = \beta(U_N \cup N) \mid (U_N \neq 0)$ when φ_N is a maximal N-cycle not S-concave, and excepting $\varphi_S = \beta(U_S \cup S) \mid (U_S \neq 0)$ when φ_S is a maximal S-cycle not N-concave.*

The set Ψ is not empty. For there is either exactly one cycle in Φ, or exactly two, or an inner cycle. In each of these cases L 8.1 implies the existence of a concave cycle in Φ and hence in Ψ. If $U_N \neq 0$ and φ_N is not in Ψ, φ_N is certainly not S-concave by definition of Ψ. It cannot be an inner cycle by L 8.1 (c). The case of an excepted φ_S is similar.

The critical elements in Φ. We aim at a finite decomposition of Σ using such basic regions as N-caps, S-caps, the Ûnion C of all concave annuli, and such other sets as may be necessary. No concave annulus A meets an N-cap or S-cap, since no N- or S-loop can enter A. Hence C cannot meet an N-cap U_N, or S-cap U_S. The question then is what is the nature of

$$\Sigma - U_N - U_S - C.$$

The problem is complicated by the fact that U_N, U_S or C may be empty. When $U_N \neq 0$ and $U_S \neq 0$, $U_N \cup N$ and $U_S \cup S$ have unique cycles, in Φ, φ_N and φ_S respectively, as boundaries [Th 6.3.] When $C \neq 0$ it will presently appear that it has unique elements ψ_N and ψ_S in Φ as boundaries, and apart from special cases one would expect that in Φ

$$\mathfrak{S} < \varphi_S < \psi_S < \psi_N < \varphi_N < \mathfrak{N}. \tag{8.1}$$

To simplify the problem, and to include all cases of the vanishing of U_N, U_S and C and the coalescence of their boundaries we define, de novo, four critical elements in Φ, namely,

$$\varphi_S,\ \psi_S,\ \psi_N,\ \varphi_N. \tag{8.2}$$

Let φ_N be $\mathfrak{N}$ if there are no N-loops and sup $\Phi \mid (\mathfrak{S} < \varphi < \mathfrak{N})$ otherwise.
Let φ_S be $\mathfrak{S}$ if there are no S-loops and inf $\Phi \mid (\mathfrak{S} < \varphi < \mathfrak{N})$ otherwise.
Let $\psi_N = \sup \Psi$, and $\psi_S = \inf \Psi$.

The elements so defined exist. For the set Ψ is not empty by L 8.2, nor is the set $\Phi \mid (\mathfrak{S} < \varphi < \mathfrak{N})$ since $\Phi \supset \Psi$. One makes use of the completeness of Φ. Moreover

$$\mathfrak{S} \leqq \varphi_S \leqq \psi_S \leqq \psi_N \leqq \varphi_N \leqq \mathfrak{N}. \tag{8.3'}$$

An alternative but equivalent definition of φ_N and φ_S follows.

Let φ_N be the closed curve in Φ with minimum carrier $\beta(U_N \cup N)$.

Let φ_S be the closed curve in Φ with minimum carrier $\beta(U_S \cup S)$.

We are assuming that there is no meridian, and in this case $\mathrm{Cl}\, U_N \cap \mathrm{Cl}\, U_S = 0$. It follows from the second definition of φ_N and φ_S that $\varphi_N > \varphi_S$. Moreover Ψ is not empty, so that for some cycle $\psi \in \Psi$, $\mathcal{S} < \psi \leqq \psi_N$, $\mathcal{N} > \psi \geqq \psi_S$. In summary,

$$\varphi_S < \varphi_N, \quad \mathcal{N} > \psi_S, \quad \mathcal{S} < \psi_N. \tag{8.3''}$$

At the end of this section we shall see that the conditions (8.3)′ and (8.3)″ are the only order conditions on the elements involved.

Lemma 8.3 (a). *If $\psi_N < \varphi_N$, ψ_N is N-concave. If $\psi_S > \varphi_S$, ψ_S is S-concave.*

(b). *If $\psi_S < \psi_N$, then ψ_S is N-concave or $\mathcal{S}$, and ψ_N is S-concave or $\mathcal{N}$.*

Proof of (a). If $\varphi_S = \psi_N < \varphi_N$, then ψ_N is the only cycle in Ψ. It cannot be S-concave since equal to φ_S, and so must be N-concave. If $\varphi_S < \psi_N < \varphi_N$ then ψ_N is an inner cycle in Ψ and must be N-concave by Th 6.3 (ii). Similarly ψ_S is S-concave if $\psi_S > \varphi_S$.

Proof of (b). If $\varphi_S = \psi_S$ and $\psi_S \neq \mathcal{S}$, then φ_S is in Ψ and hence N-concave, or not in Ψ and N-concave (a contradiction) because $\varphi_S = \beta$ Union R_n for a proper choice of R_n, as in the proof of L 7.1 [L 3.2]. If $\varphi_S < \psi_S < \psi_N$, ψ_S is an inner cycle, and N-concave since $< \varphi_N$. The case of ψ_N is similar.

The open sets $\{X, Y\}$. Let X and Y be any two successive elements in (8.3)′. If $X = Y$ let $\{X, Y\}$ be the empty set. If $X < Y$ let $\{X, Y\}$ be the open set bounded by X and Y. To describe F^* over Σ^* it is sufficient to describe F^* over the sets

$$\{\mathcal{S}, \varphi_S\} \quad \{\varphi_S, \psi_S\} \quad \{\psi_S, \psi_N\} \quad \{\psi_N, \varphi_N\} \quad \{\varphi_N, \mathcal{N}\}. \tag{8.4}$$

In the order written such sets are called, respectively, *S-caps*, *S-spiral annuli*, *central annuli*, *N-spiral annuli*, and *N-caps*, We have already characterized N- and S-caps. We shall characterize spiral and central annuli.

Improper annuli. A spiral annulus or a central annulus in which one at least of the the bounding elements is $\mathcal{N}$ or $\mathcal{S}$ will be called *improper*. Because of the conditions $\mathcal{S} < \psi_N$ and $\mathcal{N} > \psi_S$ improper spiral annuli take one of the forms

$$\{\mathcal{S}, \psi_S\} \quad \{\psi_N, \mathcal{N}\}.$$

Improper central annuli take one of the forms

$$\{\mathcal{S}, \psi_N\} \quad \{\psi_S, \mathcal{N}\} \quad \{\mathcal{S}, \mathcal{N}\}.$$

The last written annulus is realized in the case of a family F consisting of the parallels on Σ^*. If Σ^* is covered by just one of the open sets (8.4) this set must be a central annulus of the form $\{\mathcal{S}, \mathcal{N}\}$.

In § 5 we have analyzed concave annuli $A(\varphi_1, \varphi_2)$. Here φ_1 and φ_2 were cycles in Φ concave toward each other. The description of a central annulus follows.

Theorem 8.1. *A central annulus $\{\psi_S, \psi_N\} = C$ is the Ûnion of all concave annuli. Each element of F^* meeting C is in C, and is either a non-singular top circle or a non-singular element h which is an asymptote in each of its senses.*

We begin by establishing the following.

(a) *Each concave annulus $A(\varphi_1, \varphi_2)$ is in C.*

By definition of ψ_S and ψ_N, $\psi_S \leqq \varphi_1 < \varphi_2 \leqq \psi_N$, and (a) follows.

(b) *Each point of C is in a concave annulus.*

When C is non-empty and proper it is a concave annulus [L 8.3 (b)]. In this case (b) is trivial. When C is improper but not empty three cases are distinguished.

Case I. $C = \Sigma^*$. In this case

$$\psi_S = \mathcal{S} = \inf \Psi \qquad \psi_N = \mathcal{N} = \sup \Psi \tag{8.5}$$

and each cycle in Φ is an inner cycle. If φ_1 and φ_2 are cycles in Φ the annulus $A(\varphi_1, \varphi_2)$ is concave by L 5.2. It follows from (8.5) that every point of C is in some concave annulus $A(\varphi_1, \varphi_2)$.

Case II. $\psi_S = \mathcal{S}$, $\psi_N < \mathcal{N}$. Each cycle φ in C is an inner cycle in all cases. In Case II, ψ_N is S-concave by L 8.3 (b), so that $A(\varphi, \psi_N)$ is a concave annulus by L 5.2. Since $\mathcal{S}$ satisfies (8.5) it is clear that each point of C is in an annulus $A(\varphi, \psi_N)$.

Case III. Similar to Case II with S and N interchanged.

Thus (b) holds and it follows that C is the union of all concave annuli. Each element of F^* in C is in a concave annulus $A(\varphi_1, \varphi_2)$ and so has the nature stated in the theorem.

Theorem 8.2. *Let $\{\psi_N, \varphi_N\}$ be an N-spiral annulus $W \neq 0$.*

(i) *For η properly chosen as one of the cycles $\psi_N \pm$, each element in $F^* \mid W$ is non-singular and asymptotic to η with an initial limiting point in φ_N.*

(ii) *At most one such asymptote η has its initial point in a given N-loop ψ in φ_N.*

(iii) *When $W \neq 0$, φ_N is a maximal N-cycle or $\mathcal{N}$.*

Proof of (i). There is no cycle of Φ between φ_N and ψ_N when $\psi_N < \varphi_N$, since ψ_N is then the first cycle in Φ before φ_N. There are no N-loops meeting W, since φ_N is a maximal N-cycle or $\mathcal{N}$ by L 8.2. There are no elements in $F^* \mid W$ asymptotic to φ_N as a cycle, since the conditions $\psi_N < \varphi_N < \mathcal{N}$ imply that φ_N is not S-concave. Hence each $h \in F^* \mid W$ is asymptotic to $\eta = \psi_N \pm$ with a limiting initial point in φ_N. There is no singular point P in W; otherwise the elements in F^* meeting P would include at least two elements asymptotic to $\psi_N \pm$, contrary to Cor 5.1.

Proof of (ii). If (ii) were false there would exist points r and r' of ψ in Σ^* incident respectively with distinct elements h and h' in $F^* | W$, and asymptotic to η. There would then be two F^*-rays with initial point at r and asymptotic to η contrary to Cor 5.1.

Proof of (iii). This follows from L 8.2.

This completes the proof of the theorem. S-spiral annuli admit a similar description.

Covering by two sets $\{X, Y\}$. In any covering of Σ^* by two non-empty sets $\{X, Y\}$ and their common boundary β, the first set must be an S-cap, S-spiral or central annulus, and the second an N-cap, N-spiral or central annulus. There are thus nine a priori possibilities, but a central annulus cannot be combined with a central annulus, nor an N-cap with an S-cap, since $\varphi_S < \varphi_N$ when there is no meridian. There remain four combinations of two sets $\{X, Y\}$, namely

$$\{S\text{-cap},\ N\text{-spiral } A\}\ \{S\text{-cap},\ \text{central } A\} \tag{8.6}$$

$$\{S\text{-spiral } A,\ N\text{-spiral } A\}\ \{S\text{-spiral } A,\ \text{central } A\} \tag{8.7}$$

and three other combinations, obtained by interchanging N with S and inverting the order of the two sets.

It follows from L 8.3 that the common boundary β of the two sets is an N-concave φ_S in (8.6), and a non-singular inner cycle ψ_S in (8.7). The cycle $\beta = \varphi_S$ in (8.6) may be a maximal S-cycle or an inner cycle. Each of these possibilities is realizable.

In case meridians are absent Σ is decomposed as follows.

Theorem 8.3 (a). *The non-empty open sets* $\{X, Y\}$ *in* (8.4) *are disjoint, and, taken with their boundaries, cover* Σ.

(b). *A boundary cycle* φ *common to two of these open sets is singular at most if one of the sets is an* N*-cap or* S*-cap.*

(c). *A boundary of an* N*-cap* [S*-cap*] *is an inner cycle* φ *at most if* φ *is singular and* S*-concave* [N*-concave*], *and if any* N*-spiral* [S*-spiral*] *annulus is empty.*

(a) This needs no further proof.

(b) This follows from L 8.3, on recalling that φ is non-singular if both N- and S-concave.

(c) If φ_N [φ_S] is an inner cycle then it is in Ψ by L 8.2, and hence S-concave [N-concave], and any N-spiral [S-spiral] annulus is empty by Th 8.2 (iii).

Corollary 8.1. *When there is no meridian a necessary and sufficient condition that each element in F^* be non-singular is that there be no singular N- or S-loops.*

The corollary would follow immediately from the theorem, if the condition were that there exist no singular N- or S-loops or N- or S-circuits. However, it follows from Th 8.3 (c) that in the absence of meridians an N-circuit ψ (if it exists) is necessarily S-concave. Hence the elements in F^* meeting a singular point in ψ must carry an N-loop. The case of an S-circuit is similar. Hence the condition of the corollary is sufficient. It is trivial that it is necessary.

Construction of an F. Without going into details one can assert that any distribution of the signs $<$ and $=$ in (8.3)′ that is consistent with (8.3)″ is realizable in an example which is non-singular. One can, for example, make use of central annuli covered by a continuous 1-parameter family of closed curves. An N-cap can be defined which is bounded by N and a single non-singular N-loop One can use N-spiral annuli which are covered by aymptotes each of whose initial points are in N. S-caps and S-spiral annuli of similar character can be constructed when called for, and combined with these elementary N-caps, N-spiral annuli, and central annuli to form a family F as desired.

§ 9. Loop coverage

The case of loop coverage arises, by definition, when points on N- or S-loops are everywhere dense in Σ^*. Clearly a necessary and sufficient condition for loop coverage is that

$$\hat{\text{U}}\text{nion}\,[U_N, U_S] = \Sigma^*.$$

When there is loop coverage it will appear that there is at least one meridian in F^*. Cf. Cor 6.1. When there is at least one meridian the decomposition of Σ^* can be studied under the case of loop coverage and the case of no loop coverage. We here study loop coverage.

By a *maximal* N-loop is meant any N-loop φ such that $I\varphi \supset I\psi$ whenever ψ is an N-loop with $I\psi \cap I\varphi \neq 0$. A maximal S-loop is similarly defined.

We need further information regarding unbounded N-caps U_N.

Theorem 9.1. *If U_N is not bounded from S and $\neq \Sigma^*$, then each component R of U_N is either* i, *the interior of a maximal N-loop or,* ii, *an F-region bounded by $N \cup S$, by two disjoint meridians and at most countably many disjoint maximal S-loops,* iii, *a region bounded by a maximal S-cycle and by N.*

(a) *Two components of U_N of types* i *or* ii *have disjoint boundaries in Σ^*.*

(b) *There is at least one component of type* ii *or* iii. *Any component of U_N of type* iii *equals U_N. The number of components of type* ii *is finite.*

L 3.2 implies that each component R of U_N is an F-region. As given by Th 3.2 βR does not carry an N or S-circuit since U_N is not bounded from S. An R is then of type i if bounded from S, of type iii if N is isolated in βR, of type ii otherwise. The maximality of the loops follows from the definition of U_N.

Intersection of component boundaries. Two components of U_N cannot have an open boundary arc in common since U_N is an inner closure in Σ^*. If R_1 and R_2 are two components of U_N of types [i, ii], [i, i] or [ii, ii] then βR_1 and βR_2 cannot meet in a point in Σ^*; otherwise $\beta R_1 \cup \beta R_2$ would carry an N-loop with interior R_0 such that $\hat{\text{U}}$nion $[R_0, R_1, R_2]$ would be connected and in U_N, contrary to the nature of R_1 and R_2 as components of U_N.

Number of components. If there were no component of type ii or iii, U_N would be bounded from S, contrary to hypothesis; that any component of type iii is U_N itself follows from the relation

$$U_N \supset R \supset \Sigma^* - \bar{U}_S \supset U_N.$$

The number of components R of U_N of type ii is finite; for there exists in each such R an N-loop with diameter exceeding $\pi/2$ and the number of such N-loops in different components R is finite by L 3.1.

This completes the proof of the theorem.

By an argument similar to that used in the last paragraph of the proof one can show that the maximal number of meridians in any collection of disjoint meridians is finite in the case of loop coverage.

On setting

$$B = \beta U_N \cap \beta U_S \tag{9.1}$$

one obtains the following theorem.

Theorem 9.2. *In the case of loop coverage, the following is true.*

(a) *If U_N is bounded from S and U_S from N, $B = \beta U_N - N = \beta U_S - S$ is a top circle in F^*.*

(b) *If U_N is not empty and bounded from S, but U_S is not bounded from N, $B = \beta U_N$ is the carrier of a maximal N-cycle. A similar statement holds interchanging N and S.*

(c) *If U_N is not bounded from S nor U_S from N, the components of $B|\Sigma^*$ are simple and disjoint, and include a finite set (at least two) of meridians, and carry at most a countable set of maximal N- and S-loops.*

Statements (a) and (b) follow from Th 6.3 and statement (c) from Th 9.1.

Primitives. We shall give another decomposition of Σ^* in terms of a $\hat{\text{U}}$nion of certain elementary regions to be termed primitives. These primitives will enter into decompositions both in the case of loop coverage, and in the case where at least one meridian exists and there is no loop coverage.

Definition. Let $\varphi_1, \varphi_2, \ldots$ be a sequence of disjoint non-singular N-loops such that

$$I\varphi_1 \subset I\varphi_2 \subset I\varphi_3 \subset \ldots \quad . \tag{9.2}$$

Then Union $I\varphi_n$ will be called an *N-element* and denoted by $[\varphi]$. An N-element which is not a proper subset of any other N-element is called an *N-primitive*. *S-primitives* $[\varphi]$ are similarly defined.

We shall establish a number of propositions which lead up to a decomposition of U_N $[U_S]$ into a $\hat{\text{U}}$nion of disjoint N-primitives [S-primitives] countable in number. The essence of the analysis lies in the introduction of a partial order among N- or S-elements, and, for ordered subsets of N- or S-elements, in the reduction of this order to a numerical basis.

We say that two N-elements are *ordered* if one is included in the other. We similarly order interiors $I\psi$ of N-loops ψ. Strict inclusion of a set A in B will be denoted by the relation $A < B$ or $B > A$. If φ and ψ are non-singular N-loops and $I\varphi \cap I\psi \neq 0$ $I\varphi$ and $I\psi$ are ordered. We extend this fact as follows.

(α) *If N-elements $[\varphi]$ and $[\psi]$ intersect, $[\varphi]$ and $[\psi]$ are ordered.*

If $[\varphi] \cap [\psi] \neq 0$ then for suitable integers r and s, $I\varphi_r \cap I\psi_s \neq 0$, and hence $I\varphi_n \cap I\psi_m \neq 0$ for $n \geqq r$, $m \geqq s$. Hence the set of all loop interiors of the form $I\varphi_n$, $n \geqq r$, $I\psi_m$, $m \geqq s$, is ordered. From this set one can form an N-element $[\zeta]$ such that $[\zeta] \supset [\varphi]$ and $[\zeta] \supset [\psi]$. It is clear that either $[\zeta] = [\varphi]$, or $[\zeta] = [\psi]$, or that both of these equalities hold. Statement (α) follows.

(β) *If $[\varphi]$ and $[\psi]$ are ordered N-elements with diameters $D[\varphi]$ and $D[\psi]$ respectively, then $D[\varphi] > D[\psi]$ if and only if $[\varphi] > [\psi]$.*

Suppose that $I\varphi_n$ and $I\psi_m$ are ordered, and let $d(\varphi_n)$ and $d(\psi_n)$ be the diameters of φ_n and ψ_n respectively. It is clear that $d(\varphi_n) > d(\psi_m)$ if and only if $I\varphi_n > I\psi_m$.

(1) If $[\varphi] > [\psi]$, the above N-element $[\zeta]$ is such that for some integer t, and for $n \geqq t$, ζ_n is in the set $[\varphi_1, \varphi_2, \ldots]$ and $I\zeta_t \supset [\psi]$. Hence $D[\varphi] = D[\zeta] > d(\zeta_t) \geqq D[\psi]$.

(2) If $D[\varphi] > D[\psi]$, $[\zeta]$ has this same property so that $[\varphi] > [\psi]$.

(γ) *If $[\varphi]$ and $[\psi]$ are ordered N-elements, $D[\varphi] = D[\psi]$ if and only if $[\varphi] = [\psi]$.*

This is an immediate consequence of (β).

We state two basic lemmas independent of the hypothesis of loop coverage.

Lemma 9.1. *The Union V of N-elements [S-elements] in any ordered class K of N-elements [S-elements] is an N-element [S-element].*

It is sufficient to consider the case of N-elements. Set $\Delta = \sup D[\varphi] \,|\, ([\varphi] \in K)$ and distinguish two cases as follows.

Case I. For some $[\psi] \in K$, $D[\psi] = \Delta$. In this case it follows from (β) and (γ) that $[\psi]$ includes every $[\varphi] \in K$ and hence $[\psi] \supset V$. But $[\psi]$ is in K so that $V \supset [\psi]$. Hence $[\psi] = V$ and L 9.1 follows.

Case II. Not Case I. In Case II there is a sequence $[\varphi^r]$, $r = 0, 1, \ldots$ of N-elements in K, such that $D[\varphi^r]$ increases strictly as $r \uparrow \infty$ and tends to Δ as $r \uparrow \infty$. Then

$$[\varphi^0] < [\varphi^1] < [\varphi^2] < \ldots \qquad [\text{by } (\beta)]. \tag{9.3}$$

For n successively 1, 2, 3, ... one can choose a ψ_n in the set $[\varphi_1{}^n, \varphi_2{}^n, \ldots]$ such that

$$I\psi_n > [\varphi^{n-1}]$$

and hence

$$I\psi_1 < I\psi_2 < I\psi_3 < \ldots .$$

Thus

$$[\psi] > I\psi_n > [\varphi^{n-1}] \qquad (n = 1, 2, \ldots) \tag{9.4}$$

so that

$$D[\psi] \geqq D[\varphi^{n-1}] \qquad [n = 1, 2, \ldots]$$

and hence $D[\psi] \geqq \Delta$. But $D[\psi] \leqq \Delta$ by virtue of the definition of Δ. We conclude that $D[\psi] = \Delta$. But clearly $V = [\psi]$, and L 9.1 follows.

Corollary 9.1 *The union K of all N-elements [S-elements] which meet a given N-element [S-element] is an N-primitive [S-primitive].*

The elements in K each meet a given element $[\varphi]$, so that K is the union of an ordered set of N-elements [S-elements]. The corollary follows from the lemma.

Lemma 9.2. *A primitive R is an F-region. Each component of βR in Σ^* is concave toward R.*

The region R is an N-element and as such simply connected. It is an F-region and an F^*-set by L 3.2. Because R is an F^*-set the components of βR in Σ^* are concave toward R.

Lemma 9.3 (a). *Each point in an N-loop [S-loop] is in the closure of an N-primitive [S-primitive].*

(b) *No two primitives intersect.*

(c) *The number of disjoint primitives with diameters exceeding a positive constant is finite.*

Proof of (a). We treat the case of a point P in an N-loop φ. Let λ be a transversal ray in $I\varphi$ and incident with P. Let p_n, $n = 1, 2, \ldots$ be a sequence of points appearing on λ in the order $p_1, p_2, \ldots$ and tending to P as $n \uparrow \infty$. We can suppose that each point p_n is chosen so that the N-loop φ_n meeting p_n is non-singular. Then $\varphi_n \cap \lambda = p_n$ [L 3.0] and P is in $E\varphi_n$. It follows that

$$I\varphi_1 \subset I\varphi_2 \subset \ldots$$

so that $[\varphi]$ is an N-element and Cl $[\varphi]$ contains P. According to Cor 9.1 $[\varphi]$ is in an N-primitive.

The case of a point P in an S-loop is similar.

Proof of (b). An N-primitive $[\varphi]$ cannot meet an S-primitive $[\psi]$; otherwise some non-singular N-loop φ_n would be in the interior of some non-singular S-loop ψ_m. This is clearly impossible. Nor can an N-primitive $[\varphi]$ meet a different N-primitive $[\psi]$. For $[\varphi]$ and $[\psi]$ would then be ordered [cf. (α)], and be equal, since both are maximal N-elements. Statement (b) follows.

Proof of (c). In each primitive with diameter exceeding $c > 0$, there is a loop with diameter exceeding c and two such loops in disjoint primitives would have disjoint interiors. The number of such loops is, however, finite [L 3.1] and (c) follows.

L 9.3 yields the following theorem.

Theorem 9.3. *There is at most a countable number of N-primitives [S-primitives] which meet an N-cap* U_N, *[S-cap* U_S*], and* U_N $[U_S]$ *is the* $\hat{U}$*nion of these primitives.*

Corollary 9.2. *In the case of loop coverage there is at most a countable number of primitives in* Σ^*, *and* Σ^* *is the* $\hat{U}$*nion of these primitives.*

§ 10. *F*-guides

A pseudoharmonic function with the open arcs of F as level lines is strictly increasing or decreasing along a transversal. The existence of simple arcs on M which are finite sequences of transverse arcs will turn out to be of the greatest importance in the study of pseudoharmonic functions u on M, and in answering the question as to the nature of u as a function on Σ^*, in particular in finding pseudoharmonic functions which are single-valued on Σ^* and have the open arcs of F as level lines. F-guides, which we now define, are central in this study.

Definition. A non-singular arc on Σ^* is termed *m-transverse* if the union of m consequtive transverse arcs. A top circle on Σ^* is termed m-transverse ($m > 1$) if the union of m consecutive transverse arcs, and 1-transverse if every open subarc is a transversal. An m-transverse top circle separating N from S for which m is a minimum is called an *F-guide.*

The existence of an F-guide is most difficult to establish in the case in which there exists at least one meridian L, and this is the case where the F-guide is most useful. In case L exists an n-transverse top circle g in Σ^* which separates N from S, intersects L in a single point, and is such that n is a minimum subject to these conditions, is called an *FL-guide.* An FL-guide need not be an F-guide, but once the existence of an FL-guide is established the existence of an F-guide follows readily, even in the cases where there is no meridian.

Reversing points. A point of junction P of two successive transverse arcs whose union is an arc g, is called a reversing point of g if the sense of crossing of elements of F reverses at P. Recall that P is non-singular. It is clear that the junction point P of two successive transverse arcs in a finite minimal decomposition of an arc g into transverse arcs is a reversing point. Otherwise the two arcs would form a single transverse arc and g could not have been minimally decomposed.

The existence of an FL-guide. Except for one point in L, an FL-guide g, if it exists, will be in the region $\Lambda = \Sigma - L$. The region Λ is the homeomorph of a finite z-plane so that the results of MJ 2 can be applied to the family $F_0 = F|\Lambda$. In MJ 2 "bands" played a fundamental role. A band $R(N_p)$, relative to Λ, is defined as the union of all elements in F_0 which meet a right neighborhood N_p in Λ. As shown in MJ 2 a band $R(N_p)$ in Λ is an F_0-region, and has boundary components in Λ which are simple. If E is a set in Λ it will be necessary to distinguish between the boundary βE of E relative to Σ, and the boundary $\beta_0 E$ of E relative to Λ.

We begin with two lemmas.

Lemma 10.1. *Any two non-singular points p_1 and p_2 on the boundary βR of a band R in $\Lambda = \Sigma - L$ can be joined by an m-transverse arc g such that $g - p_1 - p_2$ is in R and $m \leq 3$.*

Any two points q_1 and q_2 in different elements of F_0 in R can clearly be joined by a transverse arc in R. But the given points p_1 and p_2 can be joined to points q_1 and q_2 in R and neighboring p_1 and p_2, respectively, by transverse arcs k_1, k_2, in R except for p_1 and p_2. One can suppose q_1 and q_2 so near p_1 and p_2, respectively, that k_1 does not meet k_2. Let k be a transverse arc joining q_1 to q_2 in R. If $k_1 \cap k = q_1$ and $k_2 \cap k = q_2$ the arc $g = k_1 k k_2$ satisfies the lemma. Otherwise let k_1' and k_2' be maximal initial subarcs of k_1 and k_2, respectively, intersecting k only in their endpoints q_1' and q_2', and let k' be the subarc $q_1' q_2'$ of k. Then the arc $g = k_1' k' k_2'$ satisfies the lemma.

Lemma 10.2. *If a meridian L exists an FL-guide g exists.*

Let p be an arbitrary non-singular point in L, and let λ and μ be sensed transverse arcs joining p to points P and Q respectively, on opposite sides of L. We suppose λ and μ so restricted that $\lambda \cap \mu = 0$, $\lambda \cap L = p$, $\mu \cap L = p$.

It follows from Th 9.1 of MJ 2 that there exists a finite set of disjoint bands

$$R_1, R_2, \ldots, R_m \qquad [m > 1] \tag{10.1}$$

of Λ whose $\hat{\text{U}}$nion is an F-region H which contains P and Q. Let P_1 and Q_1 be respectively the first intersection of λ and μ with βH. The point P_1 is not necessarily in βR_1, nor Q_1 in βR_m.

If R_i and R_j, $i \neq j$, are two bands in (10.1) whose $\hat{\text{U}}$nion is connected, $\beta_0 R_i \cap \beta_0 R_j$ includes at least one element $\alpha \in F_0$, so that one can connect R_i with R_j by an arc which crosses α at one point only. The points P_1 and Q_1 can accordingly be connected by a non-singular arc g, in H except for P_1 and Q_1, and meeting the respective boundaries $\beta_0 R_i$ in at most a finite set of s points. If then one chooses g so that s is minimal, it follows that $\beta R_i \cap g$, $i = 1, \ldots, m$, is either the empty set or two points p_i' and p_i'' appearing in this order on g. The points p_i' and p_i'' can be joined by a r-transverse arc g_i ($r \leq 3$) with $g_i - p_i' - p_i'' \subset R_i$. [L 10.1.]

If $p \neq P_1$ and $p \neq Q_1$, the subarcs pP_1 of λ and $Q_1 p$ of μ, united with the arcs g_i in proper order, give an n-transverse top circle, with $n \leq 3m + 2$, meeting L only at p. If $p = P_1$ the subarc of pP_1 of λ is not needed.

The case in which $p = Q_1$ is similar.

An n-transverse top circle meeting L only at p and for which n is minimal accordingly exists, and the lemma follows.

The principal theorem of this section follows. No hypothesis as to the existence of a meridian is made.

Theorem 10.1. *Corresponding to an arbitrary admissible family F defined on Σ^*, there always exists an F-guide g.*

Let h be a non-singular subarc of an element of F, with end points P_1 and P_2 in Σ^*. There exist top circles g_1 and g_2 in Σ^*, each separating N from S and with $g_1 \cap g_2 = 0$, $g_1 \cap h = P_1$, $g_2 \cap h = P_2$. Then g_1 and g_2 bound a doubly connected domain $X \subset \Sigma^*$. X is topologically equivalent to Σ^* under a mapping T of X onto Σ^*. Under T, $F|X$ goes into a family F' admissibly defined over Σ^*. In F', $T(h - P_1 - P_2)$ is a meridian L'. From L 10.2 we infer the existence of an $F'L'$-guide g'. For some finite m, $T^{-1}g'$ is m-transverse relative to F and the existence of an F-guide follows.

To apply this theorem certain definitions and lemmas are needed.

Let p be any non-singular point and N_p a right neighborhood of p with canonical coordinates u and v. Given $r \in N_p$ with $u \neq 0$ at r, a sensed transverse arc g meeting r will be said to be *sensed away from* p if $|u|$ is increasing on g as r is approached in g's positive sense. A similar definition is understood on M.

A construction for use in L 10.3. Given $h \in F^*_M$ let A and B be non-singular points in h. Let H be one of the two regions into which h divides M. In H suppose that there are $n \geqq 0$ F^*_M-rays $\pi_1, \ldots, \pi_n$ with end points in the arc AB of h. Suppose these rays written in the order in which they are met by an arc k joining A to B with $k - A - B \subset H$, and meeting each ray π_i in just one point. Let λ and μ be non-intersecting open transversals in H incident with A and B respectively. It follows from Th 2.3 that λ and μ meet none of the rays.

Lemma 10.3. *In the preceding configuration A can be joined to an arbitrary point $P \in \mu$, or to $P = B$, by an m-transverse arc g with $g - A - P \subset H - \lambda - \mu$ and such that g is sensed away from B when P is in μ. The minimum value of m is $n+1$ when P is in μ, and $n+2$ when $P = B$.*

The lemma is true when $n = 0$; in this case a minimum $m = 2$, when $P = B$. [Cf. Th 2.3.]

When $n > 0$ let P_i be a point in $\pi_i \cap H$, $i = 1, \ldots, n$, and set $P_0 = A$. It is clear that P_{i-1} can be joined to P_i by a 2-transverse arc k_i whose maximal open subarc is in the set

$$H^* = H - \text{Union}\ (\lambda, \mu, \pi_1, \ldots, \pi_n).$$

The junction point of the two transverse arcs composing k_i must be a reversing point [Th 2.3]. Moreover P_n can be joined to $P \in \mu$ by a 1-transverse arc sensed away from B at P. Let g be the arc joining A to P obtained by uniting these arcs. The points $P_1, \ldots, P_n$ in g are not reversing points. Thus g bears n reversing points. These reversing points divide g into $n+1$ transverse arcs.

Let g now be an arbitrary m-transverse arc satisfying the lemma. Suppose $P \in \mu$. Let Q_i be the first point of intersection of g with π_i and K_i the last point. Set $K_0 = A$. The subarcs $K_{m-1}Q_m$, $m = 1, \ldots, n$, intersect only when successive and then only in a common end point. They cannot be transverse arcs, by Th 2.3, and hence bear at least one reversing point. Thus g bears at least n reversing points so that $m \geqq n+1$.

The case in which $P = B$ is similar.

Corollary 10.1. *An F-guide g which meets the interior of a non-singular loop φ has precisely one reversing point in $I\varphi$.*

Let an N-loop g [S-loop g] into whose south side [north side] there enters just one F^*-ray with initial point in g, be termed *S-semi-concave* [N-semi-concave]. Cf. MJ 2 Th 8.1.

Lemma 10.4 (a). *A meridian h which is concave toward one of its sides is met by an F-guide in just one point.*

(b). *An F-guide g meets no S-concave or semi-concave N-loop, or N-concave or semi-concave S-loop each point of which is the limit point of a sequence of points on non-singular meridians.*

Proof of (a). The intersections of g with h are isolated on g and hence finite in number. If g meets h in more than one point it meets h in at least two points. One then uses L 10.3 to show that g can be modified so as to cross h just once and be an m-transverse arc with m smaller than previously. Since m is supposed to be a minimum for g this is impossible.

Proof of (b). We suppose (b) false in that g meets an S-semi-concave N-loop φ satisfying the conditions of the lemma. At the first point p of intersection of g with φ, g crosses $|\varphi|$. Otherwise g will have a reversing point at p and cross some non-singular meridian passing near p more than once, contrary to (a).

Let A and B then be two points, at which g enters $I\varphi$ and leaves $I\varphi$ respectively, bounding an open subarc $g(A, B)$ of g in $I\varphi$. Consider the case in which the F^*-ray π given in $E\varphi$ as incident with φ has its initial point r in the open subarc $\varphi(A, B)$ of φ. There will then be at least one F^*-ray in $I\varphi$ incident with r. Let P be a point in an open transversal in g just following $g(A, B)$. By L 10.3, $g(A, B)$ carries at least two reversing points. However, there exists a 2-transverse arc $g_1(A, P)$, in $E\varphi$ except for A in φ, which, substituted for $g(A, P)$, gives a simple closed curve g_1 in place of g, with a reversing point at A, and at only one other point of $g_1(A, P)$, but with no reversing point at P. Thus g_1 is an F-guide meeting φ without crossing φ. This we have seen is impossible.

The case in which no F^*-ray π is incident with $\varphi(A, B)$ is similar. This is the case which always occurs if φ is S-concave. The case of an S-loop is of like character. We infer then that g cannot cross loops conditioned as in the lemma.

The index ν (F) of F. The number of reversing points in an F-guide g is called the index ν (F) of F. It is independent of the choice of g as F-guide. The following theorem gives an evaluation of ν (F).

Theorem 10.2. *If there is at least one meridian in F^*, each reversing point of an F-guide g is in a primitive, while each primitive met by g contains just one reversing point of g. Thus the index ν (F) is the number of primitives met by g.*

Let P be a reversing point of g. A non-singular element $h \in F^*$ which meets g in a point $q \neq P$ sufficiently near P meets the two transversal subarcs of g incident with P. When a meridian exists, h as non-singular, must either be a meridian or a loop. But h

cannot be a meridian by L 10.4. Hence h carries a loop φ. Then P is in $I\varphi$. Otherwise g would enter $I\varphi$ at two points. This is impossible, for by L 10.3 an F-guide can meet a non-singular loop in at most two points. Similarly P is in the interior of each of a sequence $\varphi_1, \varphi_2, \ldots$ of disjoint loops whose carriers meet g in a sequence of pairs of points tending to P as a limit. If φ_n is properly chosen

$$I\varphi_1 \subset I\varphi_2 \subset \ldots$$

so that Union $I\varphi_n$ is an element containing P in its interior. By Cor 9.1 P is in a primitive containing this element. That each primitive met by g contains just one reversing point follows with the aid of Cor 10.1.

The theorem follows.

§ 11. No loop coverage, meridians present

To decompose Σ^* properly in this case a new type of covering region is needed to supplement N- and S-caps.

Meridional regions. A maximal connected open set $R \subset \Sigma^*$ in which the set of points on non-singular meridians is everywhere dense is called meridional. Equivalently a meridional region is a maximal connected open set $R \subset \Sigma^*$ which is the Union of non-singular meridians in R.

In the case at hand the open set

$$X = \Sigma^* - \mathrm{Cl}(U_N \cup U_S) \tag{11.0}$$

is not empty. We begin with a lemma.

Lemma 11.1. *Any element $h \in F^*$ which meets X is a meridian.*

Such an h cannot be a loop since h is not in the closure of U_N or U_S, nor a top circle, since it would then be carried by an N- or S-circuit and so bound U_N or U_S [Th 6.3 ii]. It cannot carry an asymptotic ray π since such a ray, from a certain point on is non-singular, contrary to the fact that π would meet any meridian in points following any prescribed point on π. Hence h is a meridian.

A "covering" in M of a meridian or N- or S-loop in Σ^* is called a meridian or N- or S-loop in M. Observing that no meridional region can intersect an N- or S-cap, the natural decomposition of Σ^* is here as follows.

Theorem 11.1. *In case loop coverage fails and a meridian exists, Σ^* is decomposed as follows.* Set $B = \beta(U_N \cup N) \cap \beta(U_S \cup S)$.

(a). *If $B = 0$, the set X in* (11.0) *is a doubly connected meridional region R bounded on the north by βU_N, or by N if $U_N = 0$, and on the south by βU_S, or by S if $U_S = 0$.*

(b). *If* $B \neq 0$ *each component* R *of* X *is an* F*-region* R *such that* R_M *in* M *is bounded by two disjoint meridians, whose projections in* Σ^* *intersect at most in a point, and by a set (possibly empty) of disjoint* N*- or* S*-loops.*

(c). *The number of components of* X *is finite.*

It follows from L 11.1 that any non-singular element in F^* which meets X is a meridian. Hence the components of X are meridional.

Proof of (a). It is clear that X is bounded as stated, and hence doubly connected.

Proof of (b). Here X is an F-set and satisfies Conditions Θ with U_N and U_S. Hence R does likewise [L 3.3]. Since there exists a non-singular meridian there is no N- or S-circuit. Hence the components of βR in Σ^* must consist of two meridians h and k, and a set (possibly empty) of disjoint N- and S-loops. R is in fact an F-region. The meridians h and k intersect at most in a point, since X and hence R is an inner closure. Statement (b) follows.

Proof of (c). If there were infinitely many components of X there would be infinitely many meridians in βX of which no two would intersect in more than a point. There would then be infinitely many components of $U_N \cup U_S$ whose closures would meet the equator of Σ, and by virtue of Th 9.3 infinitely many primitives with diameters at least $\frac{\pi}{2}$. This is impossible by L 9.3 (c).

This completes the proof of Th 11.1.

Lemma 11.2 (i). *If* R *is a meridional region each element* $k \in F^* | R$ *is carried by a meridian in* F^*.

(ii). *There is at most one element* $h \in F^* | R$ *incident with a given* N*- or* S*-loop in* βR*, and no such* h *is incident with a meridian in* βR.

(iii). *There is no singular point in* R.

Proof of (i). This follows from L 11.1.

Proof of (ii). Suppose two elements h and k in $F^* | R$ were incident with points p and q in an N- or S-loop φ in βR. Let $\varphi(p, q)$ be the arc of $|\varphi|$ between p and q in case $p \neq q$, and let $\varphi(p, q) = p$ in case $p = q$. Let h' and k' be meridians carrying h and k respectively. It is then clear that $h' \cup k' \cup \varphi(p, q)$ carries an N-loop and an S-loop intersecting in $\varphi(p, q)$. Since at least one of these loops meets R this is impossible. That no element $h \in F^* | R$ is incident with a meridian in βR is similarly proved.

Proof of (iii). The denial of (iii) implies the existence of a loop meeting R. Thus (iii) must be true.

Theorem 11.2 (a). *It R is a simply connected meridional region each F-guide g crosses R without reversing point in R and without meeting the loop boundaries of R. The union of the elements in F meeting $g \cap R$ is R.*

(b). *A doubly connected meridional region R exists if and only if there is an F-guide without reversing point, and in case R exists the union of all elements in F meeting an F-guide g is R.*

Proof of (a). By virtue of L 11.2 (ii) each N-loop [S-loop] in βR is S-concave or semi-concave [N-concave or semi-concave]. It follows from L 10.4 that an F-guide g meets no N- or S-loop in βR. Now g meets each non-singular meridian in precisely one point [L 10.4], and since non-singular meridians are everywhere dense in R there can be no reversing point in $g \cap R$. Each element $h \in F^* | R$ is carried by a meridian k in F^* [L 11.1], and k meets g. Since there is no singular point in R [L 11.2 (iii)], we conclude that $h \in F^* | R$ is an element in F meeting g. Hence R is the union of elements in F meeting g.

Proof of (b). Suppose an F-guide g exists without reversing points. Then B, in Th 11.1, $=0$. Otherwise g would enter U_N or U_S and hence meet a primitive [Th 9.3], and by Th 10.2 carry at least one reversing point, contrary to hypothesis. Hence $B = 0$ and we infer the existence of a doubly connected meridional region [Th 11.1 (a)].

Conversely the existence of a doubly connected meridional region R implies, as in the proof of (a), the existence of an F-guide without reversing point, and that R is the union of all elements in F meeting g.

The establishes Th 11.2.

There is no singular point in a meridional region [L 11.2 (iii)], and none in a central or spiral annulus [Ths 8.1 and 8.2] or in a boundary common to a central and a spiral annulus [Th 8.3 (b)]. Thus each singular point of F^* is in $\mathrm{Cl}\, U_N \cup \mathrm{Cl}\, U_S$. Hence the following theorem.

Theorem 11.3. *Regardless of loop coverage or the existence of meridians, a necessary and sufficient condition that F^* be non-singular is that there exist no N- or S-circuit or singular N- or S-loops.*

§ 12. Meridians present, no inner cycle

When there is at least one meridian we have distinguished the case of loop coverage from the case of no loop coverage. One can equally well make a different division into the cases in which an inner cycle exists and no inner cycle exists.

When there is both an inner cycle φ and a meridian, φ is the inner cycle both of an N- and an S-circuit. The cycle φ is the common curve boundary of $U_N \cup N$ and $U_S \cup S$.

Loop coverage thus occurs as in Th 9.2 (a). In this case Σ^* is the Ûnion of primitives as indicated in Cor 9.2. In this section we suppose that no inner cycle exists and divide Σ^* into canonical polar sectors.

Our decomposition of Σ^* into polar sectors is analogous to our decomposition of Σ^* into caps and annuli in § 8. We began there with a partial ordering of inner cycles, N- and S-cycles. We begin here with a partial ordering of meridians in M.

Order among meridians in M. Let θ represent the longitude of a point in Σ^*. On M we understand that the range at θ is the whole θ-axis. By a parallel in M is meant an unending open arc in M covering a parallel in Σ^*. By the *positive side* of a meridian x in M we understand that region in $M - x$ in which θ takes on arbitrarily large positive values on each parallel in M. The *negative side* of x is the complement in $M - x$ of the positive side of x.

Two meridians x and y in M which are not identical shall stand in the relation $x < y$ or $y > x$, if y meets the positive but not the negative side of x, or equivalently if x meets the negative but not the positive side of y. If $x < y$ the set $x \cap y$ may be empty, a point, an arc, or a half open arc whose projection in Σ has one end point in Σ^*, and a limiting end point either at N or at S.

A point p in M has coordinates $[\lambda, \theta]$ where θ and λ are respectively the longitude and latitude of p. There exists a top mapping T of M onto M such that the coordinates of Tp are $[\lambda, \theta + 2\pi]$. If E is an arbitrary set in M the set $T^n E$, $n = \pm 1, \pm 2, \ldots$ is termed *congruent* to E. We shall denote TE by $E^{(1)}$.

The covering in M of an N- or S-primitive, meridional region, N- or S-loop, etc. given in Σ^*, will be called by the same name as a subset of M. Conversely the projection into Σ^* of various sets first defined on M, such as polar sectors, cut sectors, etc. will be called by the same name as subsets of Σ^*.

Polar sectors. If x and y are meridians in M and if $x < y \leq x^{(1)}$, the intersection of the positive side of x with the negative side of y will be called a polar sector $\Pi = \Pi(x, y)$ in M. When no ambiguity can arise we speak of a polar sector as a sector. If $x \cap y = 0$, Π is connected. If $x \cap y$ is a point or arc, Π has two components, one an N-loop interior, the other an S-loop interior. If $x \cap y$ is a half open arc, Π has precisely one component of one of these types. If $y = x^{(1)}$ the projection of Π in Σ^* has just one boundary meridian and an inner closure in Σ^* which is Σ^*.

Cut sectors. Let $\Pi(x, y)$ be a sector in M such that $x \cap y \neq 0$, or such that $x \cap y = 0$ but there exists an open arc $c \in F_M^* | \Pi$ with end points in x and y respectively. We term Π a cut sector. When $x \cap y$ is a point we term Π *simply degenerate*; when $x \cap y$ is an arc or half open arc we term Π *doubly degenerate*. The open arc c in a non-degenerate cut sector is unique, and in such a sector $y \cap x^{(1)} = 0$; otherwise an inner cycle would exist in F^*, contrary to the hypothesis of this section. Hence c divides Π into an N-loop interior and an S-loop interior with c as common boundary.

The meridian class ξ. Let ξ be the class of meridians in Σ^* in the boundaries of meridional regions or of unbounded N- or S-primitives, that is primitives whose closures meet N and S. Cf. Th 6.1. The set ξ_M of meridians in M covering elements of ξ is ordered without exception. The set ξ may be empty. If ξ is empty there can be no unbounded primitives, and the only meridional region possible is one without meridional boundaries, that is, a doubly connected meridional region [Th 11.1]. The set ξ may contain only one meridian. There can then be no meridional region, since a meridional region, as an inner closure, cannot have a single meridian boundary. When ξ contains precisely one meridian and there is no inner cycle, this meridian must be the sole meridian boundary of an N- or S-primitive.

If y is the immediate successor of $x \in \xi_M$, x and y are termed adjacent in M. If ξ contains only one meridian h, h_M and $h_M^{(1)}$ are adjacent in M.

Lemma 12.1. *If x and y are adjacent meridians in ξ_M, then x and y are meridian boundaries in M either of a meridional region, an unbounded primitive, or of a maximal cut sector.*

The truth of this lemma will follow from (i) and (ii).

(i). *If for the given x and y, $\Pi(x, y)$ meets a meridional region H or unbounded primitive H in M, then x and y are meridian boundaries of H.*

Recall that H has unique meridian boundaries x_1 and y_1 with $x_1 < y_1$, and that x_1 and y_1 are adjacent in ξ_M. Hence $x \leq x_1 < y_1 \leq y$. But x and y are adjacent in ξ_M by hypothesis. Hence $x = x_1$, $y = y_1$ and (i) is proved.

(ii). *If for the given x and y, $\Pi(x, y)$ meets no meridional region or unbounded primitive, then $\Pi(x, y)$ is a cut sector which is maximal in M.*

If $x \cap y \neq 0$, Π is a cut sector by definition of a cut sector.

Suppose then that $x \cap y = 0$. Then Π is connected. Since Π contains no meridional region by hypothesis its homeomorphic projection $\Pi^* \subset \Sigma^*$ cannot meet the region X of (11.0). [L 11.1.] Hence

$$\Pi^* \subset \mathrm{Cl}(U_N \cup U_S).$$

The openness of Π^* and the disjointness of U_N and U_S then implies that

$$\Pi^* \cap \beta U_N = \Pi^* \cap \beta U_S. \tag{12.1}$$

The open F-set $\Pi^* \cap U_N$ is bounded from S. Otherwise there would be an unbounded N-element E in this set; the N-primitive which contains E is E, [cf. (γ) § 9] and in Π^*, contrary to hypothesis. Similarly $\Pi^* \cap U_S$ is bounded from N. Hence the set in (12.1) is bounded from N and from S. This set is an F-set separating N from S in the connected set Π^* and contains a finite collection of elements in F. There must then exist an open arc $c \in F_M^* | \Pi$ with end points in x and y. Thus Π is a cut sector. It remains to show that Π is maximal.

The meridian y is in the boundary of a meridional region or an unbounded primitive in the positive side of y. Hence there can be no cut sector which contains Π and meets the positive side of y. Similarly with the negative side of x. Hence Π is maximal and the lemma follows.

A polar sector whose meridian boundaries are those of a meridional region or of an unbounded primitive will be called a *meridional sector* or *primitive sector* respectively. In general a meridional sector is not a meridional region, nor a primitive sector a primitive. With this understood we state a theorem of paramount importance in the study of pseudo-harmonic functions on M.

Theorem 12.1 (a). *If there is at least one meridian in F^* and no inner cycle, then Σ^* is the $\hat{U}$nion of a finite non-empty set of disjoint polar sectors each of which is* (1) *a meridional sector,* (2) *a primitive sector, or* (3) *a maximal cut sector.*

(b). *The set of meridians bounding the canonical sectors in* (a) *is the set of meridians bounding the meridional and primitive sectors in* (a).

(c). *Any finite circular sequence of sectors of types* (1), (2), *or* (3) *is realizable subject to the following conditions. On M adjacent sectors of the same type must be primitive sectors. A doubly degenerate cut sector Π cannot be adjacent on M to two meridional sectors.*

Statement (a) of the theorem follows from L 12.1, Th 11.1 (c), and L 9.3 (c). Statement (b) follows from L 12.1 with particular reference to the definition of the set ξ_M. Turning to (c) we note that two maximal cut sectors Π_1 and Π_2 in M cannot be adjacent since $\hat{U}$nion (Π_1, Π_2) would then be a cut sector. Two meridional sectors cannot be adjacent to each other or to a doubly degenerate cut sector since the $\hat{U}$nion of these sectors would then be a meridional sector. That any finite circular sequence of sectors of types (1), (2), or (3) is realizable with the above exceptions is readily established by simple examples.

The non-singular case. When there are no singular points, $F^* = F$; there are then no N- or S-circuits so that φ_N [φ_S] is either N [S] or a maximal N-cycle [S-cycle]. Th 6.3 is accordingly simplified. If loop coverage occurs it is impossible that U_S be bounded from S and U_S bounded from N as well. Cf. Th 9.2 (a). In N-spiral [S-spiral] annuli all asymptotes have initial points at N [S]. Cf. Th 8.2; Th 8.3 reduces to Th 8.3 (a).

When a meridian exists there is no inner cycle, so that Th 12.1 covers the case where a meridian exists completely. In this theorem reference to cut sectors should be deleted. Th 12.1 (b) is trivial. When a meridian exists the index $\nu(F)$ of F is simply the number of disjoint unbounded primitives. An F-guide crosses each such primitive and has just one reversing point therein. Each open arc or top circle in the boundary of a region is concave toward that region.

Bibliography

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MJ 5. Conjugate nets on an open Riemann surface, *Report of the Conference at the University of Michigan*, June 1953, (to appear).

Other references:

[1]. W. M. Boothby, The topology of regular curve families with multiple saddle points, *American Journal of Mathematics*, vol. 73 (1951), pp. 405–438.

[2]. ——, The topology of level curves of harmonic functions with critical points, *American Journal of Mathematics*, vol. 73 (1951), pp. 512–538.

[3]. W. Kaplan, The structure of a curve-family on a surface in the neighborhood of an isolated singularity, *American Journal of Mathematics*, vol. 64 (1942), pp. 1–35.

[4]. M. Morse, Topological methods in the theory of functions of a complex variable, *Annals of Mathematics Studies*, Princeton, 1947.

[5]. Y. Tôki, A topological characterization of pseudoharmonic functions, *Osaka Mathematical Journal*, vol. 3 (1951), pp. 101–122.