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Universal properties of $L(\mathbf{F}_\infty)$ in subfactor theory

by

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1. Introduction

Let $N \subset M$ be an inclusion of type II_1 von Neumann factors with finite Jones index. Let $N \subset M \subset M_1 \subset \ldots$ be the associated tower of factors that one gets by iterating the Jones basic construction [J1]. The lattice of inclusions of finite-dimensional algebras $M_i' \cap M_j$ obtained by considering the higher relative commutants of the factors in the Jones tower, endowed with the trace inherited from $\bigcup M_j$, is a natural invariant for the subfactor $N \subset M$.

A standard lattice $\mathcal{G}$ is an abstraction of such a system of higher relative commutants of a subfactor [P3]. That is to say, the relative commutants of an arbitrary finite index inclusion of II_1 factors satisfy the axioms of a standard lattice and, conversely, any standard lattice $\mathcal{G}$ can be realized as the system of higher relative commutants of some subfactor that can be constructed in a functorial way out of $\mathcal{G}$ (see [P3]).

The abstract objects $\mathcal{G}$ carry a very rich symmetry structure. They can be viewed as Jones' planar algebras [J2]. They can also be viewed as group-like objects, serving as generalizations of finitely generated discrete groups and large classes of Hopf algebras and quantum groups.

Along these lines, a subfactor $N \subset M$ can be viewed as encoding an "action" of the group-like object $\mathcal{G} = \mathcal{G}_{N \subset M}$. Given $\mathcal{G}$ it is thus important to understand whether or not it can "act" on a given II_1 factor M; i.e., whether $\mathcal{G}$ can be realized as $\mathcal{G}_{N \subset M}$ for some subfactor N of the given algebra M.

The functorial construction of a subfactor $N \subset M$ with a given standard lattice obtained in [P3], as well as the one preceding it [P1], used amalgamated free products and also depended on a choice of an algebra Q taken as "initial data". However, it remained an open problem whether one can construct a "universal" II_1 factor M that would contain subfactors with any given standard lattice as higher relative commutants, i.e., a factor M on which any $\mathcal{G}$ can "act". It also remained an open problem to identify the isomorphism class of the algebras in the inclusions realizing a given standard lattice as constructed in [P3].

We solve both of these problems in this paper. The following theorems summarize our results:

THEOREM 1.1. *Any standard lattice $\mathcal{G}$ can be realized as the system of higher relative commutants of a type* II_1 *subfactor* $P_{-1} \subset P_0$, *where both* P_{-1} *and* P_0 *are isomorphic to the free group factor* $L(\mathbf{F}_\infty)$.

Moreover, the construction of subfactors $P_{-1} \subset P_0$ *can be chosen to be a functor from the category of standard lattices (with commuting square inclusions as morphisms) to the category of subfactors (with commuting square inclusions as morphisms).*

THEOREM 1.2. *The type* II_1 *factors appearing in the inclusions constructed in* [P1], [P3], [P5], *for the initial data* $Q = L(\mathbf{F}_\infty)$, *are all isomorphic to the free group factor* $L(\mathbf{F}_\infty)$.

THEOREM 1.3. *Given an arbitrary inclusion of* II_1 *factors* $M_{-1} \subset M_0$, *there exists an inclusion* $\widehat{M}_{-1} \subset \widehat{M}_0$ *with the same standard lattice as* $M_{-1} \subset M_0$ *and so that* $\widehat{M}_i \cong M * L(\mathbf{F}_\infty)$.

In other words, $L(\mathbf{F}_\infty)$ is the desired universal type II_1 factor, whose subfactors realize all possible standard lattices; equivalently, any group-like $\mathcal{G}$ can "act" on $L(\mathbf{F}_\infty)$. Moreover, free products with $L(\mathbf{F}_\infty)$ do not "constrict" the set of allowable standard lattices of subfactors.

We note that these results are generalizations of earlier results about realization of finite-depth subfactors inside free group factors [R2], [D3], irreducible subfactors in $L(\mathbf{F}_\infty)$ [SU] and finite-depth subfactors of $M * L(\mathbf{F}_\infty)$, for M arbitrary [S3], as well as results on the fundamental group of $L(\mathbf{F}_\infty)$ [R1] and of arbitrary free products $M * L(\mathbf{F}_\infty)$ [S2].

It should be noted that free group factors $L(\mathbf{F}_n)$ cannot possess the universal property in Theorem 1.1 without being isomorphic to $L(\mathbf{F}_\infty)$. Indeed, if the property in Theorem 1.1 holds, and standard lattices coming from elements of the fundamental group of a II_1 factor can be realized as subfactors of $L(\mathbf{F}_n)$, $n < +\infty$, then the fundamental group of $L(\mathbf{F}_n)$ would be non-trivial, and hence $L(\mathbf{F}_n) \cong L(\mathbf{F}_\infty)$ (cf. [R2] and [D1]). Our constructions do not produce subfactors of $L(\mathbf{F}_n)$ for n finite.

We give two proofs of Theorem 1.1. The first proof consists in identifying the factors constructed in [P3] as being isomorphic to $L(\mathbf{F}_\infty)$, when the initial data involved in that construction is taken to be $L(\mathbf{F}_\infty)$ itself. This proves Theorem 1.2 as well. The second proof that we give to Theorem 1.1 also shows Theorem 1.3.

The principal technique underlying both proofs is a functorial construction associating to a given standard lattice $\mathcal{G}=(A_{ij})$ a pair of non-degenerate commuting squares

$$\begin{array}{ccc} \mathcal{B}_{-1} & \subset & \mathcal{B}_0 \\ \cup & & \cup \\ \mathcal{A}_{-1}^{-1} & \subset & \mathcal{A}_0^{-1}, \end{array} \qquad \begin{array}{ccc} \mathcal{A}_{-1}^{0} & \subset & \mathcal{A}_0^{0} \\ \cup & & \cup \\ \mathcal{A}_{-1}^{-1} & \subset & \mathcal{A}_0^{-1} \end{array} \tag{1.3.1}$$

in such a way that $\mathcal{B}_{-1}\subset\mathcal{B}_0$ is the infinite amplification of the standard model inclusion for $\mathcal{G}$, and $\mathcal{A}_i^j$ are type I von Neumann algebras with discrete centers and with the inclusion matrices between them given by the graphs of $\mathcal{G}$. Most importantly, the commuting squares in (1.3.1) satisfy $(\mathcal{A}_i^0)'\cap\mathcal{A}_j^{-1}=(\mathcal{B}_i)'\cap\mathcal{A}_j^{-1}=A_{ij}$. Thus, each one of them encodes the standard lattice $\mathcal{G}=(A_{ij})_{i,j}$. To construct such canonical commuting squares out of a given standard lattice or a subfactor, we use inductive limits of non-unital embeddings naturally associated to the duality isomorphisms in the Jones tower.

We then give the first proof of Theorem 1.1 by showing that the inclusion (compare [P1], [P3], [P5], [R2])

$$\mathcal{A}_{-1}^0 *_{\mathcal{A}_{-1}^{-1}} (Q\otimes\mathcal{A}_{-1}^{-1})\subset\mathcal{A}_0^0 *_{\mathcal{A}_0^{-1}} (Q\otimes\mathcal{A}_0^{-1}) \tag{1.3.2}$$

is isomorphic to (the infinite amplification of) the one constructed in [P3], for any arbitrary initial data Q. Then we prove that if $Q=L(\mathbf{F}_\infty)$ then both amalgamated free product algebras in (1.3.2) are isomorphic to $L(\mathbf{F}_\infty)\otimes B(H)$. This, of course, also proves Theorem 1.2.

The techniques needed for the identification of such amalgamated free products come from free probability theory pioneered by Voiculescu ([VDN]). The main observation is that the amalgamated free product algebra $\mathcal{A}_i^0 *_{\mathcal{A}_i^{-1}} (Q\otimes\mathcal{A}_i^{-1})$ is generated by $\mathcal{A}_i^0$ and $Q\cong L(\mathbf{F}_\infty)$; furthermore, Q has as generators an infinite semicircular system $X_1, X_2, \dots$ [V]. The position of this family relative to $\mathcal{A}_i^0$ is encoded in the statement that $\{X_n\}$ form an *operator-valued* semicircular system over $\mathcal{A}_i^0$ in the sense of [S2], [S3]. The rest of the proof involves manipulations with this semicircular system—in ways that parallel earlier random-matrix techniques of Voiculescu [V], [VDN], and developed in the context of amalgamated free products by F. Rădulescu [R1], [R2] (we mention also [D2], [D1], [D3], [DR]).

Our second proof considers the inclusion

$$\mathcal{B}_{-1} *_{\mathcal{A}_{-1}^{-1}} (Q\otimes\mathcal{A}_{-1}^{-1})\subset\mathcal{B}_0 *_{\mathcal{A}_0^{-1}} (Q\otimes\mathcal{A}_0^{-1}) \tag{1.3.3}$$

(notice that $\mathcal{B}_i$ are hyperfinite). Since the first commuting square in (1.3.1) encodes $\mathcal{G}$, this inclusion has $\mathcal{G}$ as its system of higher relative commutants. We then use free probability techniques to prove that each of the algebras in this inclusion is isomorphic to

$\widehat{\mathcal{B}} * L(\mathbf{F}_\infty) \otimes B(H)$ if $Q = L(\mathbf{F}_\infty)$, where $\widehat{\mathcal{B}}$ is hyperfinite. By the results of Ken Dykema, each of these algebras is isomorphic to $L(\mathbf{F}_\infty) \otimes B(H)$, giving another proof of Theorem 1.1.

More generally, if we are given an inclusion of II_1 factors $M_{-1} \subset M_0$ with standard lattice $\mathcal{G} = (M_i' \cap M_j)$, then the non-degenerate commuting square

$$\begin{array}{ccc} M_{-1} \otimes B(H) & \subset & M_0 \otimes B(H) \\ \cup & & \cup \\ \mathcal{B}_{-1} & \subset & \mathcal{B}_0 \end{array}$$

together with the first commuting square in (1.3.1) give rise to a non-degenerate commuting square

$$\begin{array}{ccc} M_{-1}^\infty = M_{-1} \otimes B(H) & \subset & M_0 \otimes B(H) = M_0^\infty \\ \cup & & \cup \\ \mathcal{A}_{-1}^{-1} & \subset & \mathcal{A}_0^{-1}. \end{array}$$

Once again this commuting square encodes $\mathcal{G}$, and the inclusion

$$\widehat{M}_{-1} = M_{-1}^\infty *_{\mathcal{A}_{-1}^{-1}} (Q \otimes \mathcal{A}_{-1}^{-1}) \subset M_0^\infty *_{\mathcal{A}_0^{-1}} (Q \otimes \mathcal{A}_0^{-1}) = \widehat{M}_0 \tag{1.3.4}$$

has the standard lattice $\mathcal{G}$. Using free probability again, we prove that

$$\widehat{M}_i \cong (M * L(\mathbf{F}_\infty)) \otimes B(H),$$

thus showing Theorem 1.3.

The rest of the paper is organized as follows. §2 describes the construction of the commuting squares (1.3.1). §3 deals with the necessary free probability techniques necessary in the identification of the various free product algebras. §4 presents the proofs of the main results of the paper. Thus Theorem 1.1 is proved in Theorem 4.3; Theorem 1.2 is proved in Theorems 4.2 and 4.3 (first proof); Theorem 1.3 is proved in Theorem 4.5.

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2. Some canonical commuting squares associated to a subfactor

Let $M_{-1} \subset M_0$ be an inclusion of type II_1 factors with finite Jones index. In this section we will associate to it a system of λ-Markov commuting squares of semifinite von Neumann algebras with trace-preserving expectations

$$\mathcal{C}^0 = \begin{array}{ccc} \mathcal{M}_{-1} & \subset & \mathcal{M}_0 \\ \cup & & \cup \\ \mathcal{B}_{-1} & \subset & \mathcal{B}_0 \\ \cup & & \cup \\ \mathcal{A}^0_{-1} & \subset & \mathcal{A}^0_0 \\ \cup & & \cup \\ \mathcal{A}^{-1}_{-1} & \subset & \mathcal{A}^{-1}_0 \end{array}$$

in which the upper commuting square is the ∞-amplification of

$$\begin{array}{ccc} M_{-1} & \subset & M_0 \\ \cup & & \cup \\ M^{\mathrm{st}}_{-1} & \subset & M^{\mathrm{st}}_0, \end{array}$$

$M^{\mathrm{st}}_{-1} \subset M^{\mathrm{st}}_0$ being the standard model associated with $M_{-1} \subset M_0$, and in which

$$\begin{array}{ccc} \mathcal{A}^0_{-1} & \subset & \mathcal{A}^0_0 \\ \cup & & \cup \\ \mathcal{A}^{-1}_{-1} & \subset & \mathcal{A}^{-1}_0 \end{array}$$

is a commuting square of inclusions of type I von Neumann algebras with atomic centers and inclusion matrices given by the graphs of $M_{-1} \subset M_0$. The construction of the commuting square

$$\begin{array}{ccc} \mathcal{B}_{-1} & \subset & \mathcal{B}_0 \\ \cup & & \cup \\ \mathcal{A}^0_{-1} & \subset & \mathcal{A}^0_0 \\ \cup & & \cup \\ \mathcal{A}^{-1}_{-1} & \subset & \mathcal{A}^{-1}_0 \end{array}$$

will in fact only depend on the standard invariant $\mathcal{G} = \mathcal{G}_{M_{-1}, M_0}$ of $M_{-1} \subset M_0$ and will be functorial in $\mathcal{G}$. Each one of the commuting squares

$$\begin{array}{ccc} \mathcal{B}_{-1} & \subset & \mathcal{B}_0 \\ \cup & & \cup \\ \mathcal{A}^{-1}_{-1} & \subset & \mathcal{A}^{-1}_0, \end{array} \qquad \begin{array}{ccc} \mathcal{A}^0_{-1} & \subset & \mathcal{A}^0_0 \\ \cup & & \cup \\ \mathcal{A}^{-1}_{-1} & \subset & \mathcal{A}^{-1}_0 \end{array}$$

will completely encode $\mathcal{G}$, as they will satisfy $(\mathcal{A}^0_i)' \cap \mathcal{A}^{-1}_j = (\mathcal{B}_i)' \cap \mathcal{A}^{-1}_j = \mathcal{M}'_i \cap \mathcal{A}^{-1}_j \simeq M'_i \cap M_j$ in the Jones towers for $\mathcal{C}^0$ and $M_{-1} \subset M_0$, respectively.

The commuting square $\mathcal{C}^0$ will be constructed as an inductive limit of non-unital trace-preserving embeddings of the commuting squares

$$\begin{array}{ccc} M_{2n-1} & \subset & M_{2n} \\ \cup & & \cup \\ A_{-\infty,2n-1} & \subset & A_{-\infty,2n} \\ \cup & & \cup \\ A_{-2,2n-1} & \subset & A_{-2,2n} \\ \cup & & \cup \\ A_{-1,2n-1} & \subset & A_{-1,2n} \end{array}$$

where $A_{ij}=M_i'\cap M_j$, $i,j\in\mathbf{Z}$, are the higher relative commutants in some tunnel-tower

$$\ldots\subset M_{-2}\overset{e_{-1}}{\subset}M_{-1}\overset{e_0}{\subset}M_0\overset{e_1}{\subset}M_1\overset{e_2}{\subset}M_2\subset\ldots$$

for $M_{-1}\subset M_0$ and $A_{-\infty,j}=\overline{\bigcup_{k\leqslant j}A_{kj}}$.

LEMMA 2.1. *For each $k\geqslant 0$, $n\geqslant 0$, let α_n^k be the map from M_{2n+k} into M_{2n+k+2} given by*

$$\alpha_n^k(x)=\lambda^{-k-1}e_{2n+1}e_{2n+2}\ldots e_{2n+k+1}e_{2n+k+2}xe_{2n+k+2}\ldots e_{2n+1},$$

$x\in M_{2n+k}$. Then α_n^k are non-unital $*$-isomorphisms and they satisfy:

(1) $\alpha_n^k(M_{2n+j-1})=e_{2n+1}M_{2n+j+1}e_{2n+1}$, $j=0,1,\ldots,k+1$, *with* $\alpha_n^{k+1}|_{M_{2n+j-1}}=\alpha_n^k$, *if* $j\leqslant k+1$.

(2) $\alpha_n^k(A_{i,2n+j-1})=e_{2n+1}A_{i,2n+j+1}e_{2n+1}$, $j=0,1,\ldots,k+1$, $-\infty\leqslant i\leqslant -1$.

(3) $\alpha_n^k(x)=\sigma'(x)e_{2n+1}$, $x\in M_{2n-1}'\cap M_{2n+k}$, *where σ' is the duality isomorphism on* $\bigcup_{i,j\in\mathbf{Z}}A_{ij}$ *(see e.g.* [P5]*).*

(4) *If* Tr_n *is the rescaled trace on* $\bigcup_k M_{2n+k}$ *given by* $\mathrm{Tr}_n=\lambda^{-n}\tau$ *then we have* $\mathrm{Tr}_{n+1}(\alpha_n^k(x))=\mathrm{Tr}_n(x)$, *for all* $x\in M_{2n+k}$, *for all* $k\geqslant 0$.

Proof. Since for all $x\in M_{2n+k}$ we have $[x,e_{2n+k+2}]=0$, and since the element $\lambda^{-k-1/2}e_{2n+1}\ldots e_{2n+k+2}$ is a partial isometry, it follows that α_n^k is a $*$-isomorphism. For the properties (1)–(4) we have:

(1) Since

$$e_{2n+j+1}M_{2n+j-1}e_{2n+j+1}=M_{2n+j-1}e_{2n+j+1}=e_{2n+j+1}M_{2n+j+1}e_{2n+j+1}$$

it follows that

$$\begin{aligned} e_{2n+1}\ldots e_{2n+k+2}M_{2n+j-1}e_{2n+k+2}\ldots e_{2n+1} &= e_{2n+1}\ldots e_{2n+j+1}M_{2n+j-1}e_{2n+j+1}\ldots e_{2n+1} \\ &= e_{2n+1}\ldots e_{2n+j+1}M_{2n+j+1}e_{2n+j+1}\ldots e_{2n+1} \\ &= e_{2n+1}M_{2n+j+1}e_{2n+1}. \end{aligned}$$

(2) Because $e_{2n+1}, e_{2n+2}, , ..., e_{2n+k+2} \in A_{-1,2n+k+2}$ and since $A_{i,2n+j-1}e_{2n+j+1} = e_{2n+j+1}A_{i,2n+j+1}e_{2n+j+1}$ for each $j=0,1,...,k+1$ and $-\infty \leqslant i \leqslant -1$, this part follows by (1).

(3) This is trivial by the definition of σ'.

(4) Since $\tau(xe_{2n+k+2}) = \lambda\tau(x)$ for $x \in M_{2n+k}$, one gets $\tau(\alpha_n^k(x)) = \lambda\tau(x)$ so that $\mathrm{Tr}_{n+1}(\alpha_n^k(x)) = \mathrm{Tr}_n(x)$. □

Notation 2.2. To simplify the notation we will denote by $\mathcal{C}_n$ the system of commuting squares

$$\begin{array}{ccccccccc}
M_{2n-1} & \subset & M_{2n} & \subset & \dots & \subset & M_{2n+k} & \subset & \dots \\
\cup & & \cup & & & & \cup & & \\
A_{-\infty,2n-1} & \subset & A_{-\infty,2n} & \subset & \dots & \subset & A_{-\infty,2n+k} & \subset & \dots \\
\cup & & \cup & & & & \cup & & \\
A_{-2,2n-1} & \subset & A_{-2,2n} & \subset & \dots & \subset & A_{-2,2n+k} & \subset & \dots \\
\cup & & \cup & & & & \cup & & \\
A_{-1,2n-1} & \subset & A_{-1,2n} & \subset & \dots & \subset & A_{-1,2n+k} & \subset & \dots
\end{array}$$

with $\mathcal{C}_n^k$ denoting its truncation up to k, $k=0,1,\dots$. Thus, with this notation Lemma 2.1 states that α_n^k identifies the commuting square $(\mathcal{C}_n^k, \mathrm{Tr}_n)$ with the "corner" $e_{2n+1}\mathcal{C}_{n+1}^k e_{2n+1}$ of the commuting square $(\mathcal{C}_{n+1}^k, \mathrm{Tr}_{n+1})$, endowed with the restriction of the trace Tr_{n+1} on it.

Moreover, since by Lemma 2.1 (1) we have $\alpha_n^{k+1}|_{M_{2n+j-1}} = \alpha_n^k$, for $0 \leqslant j \leqslant k+1$, with the sequence $\{\alpha_n^k(x)\}_k$ being constant from a certain point on, for each $x \in M_{2n+j}$, for all j, we immediately get the following:

COROLLARY 2.3. *For each $n \geqslant 0$ and $x \in \bigcup_{j \geqslant 0} M_{2n+j}$ let*

$$\alpha_n(x) \stackrel{\text{def}}{=} \lim_{k \to \infty} \alpha_n^k(x).$$

Then we have:

(1) $\alpha_n(\mathcal{C}_n) = e_{2n+1}\mathcal{C}_{n+1}e_{2n+1}$.

(2) $\alpha_n(x) = \sigma'(x)e_{2n+1}$, $x \in \bigcup_j A_{2n-1,2n+j} = \bigcup_j (M'_{2n-1} \cap M_{2n+j})$, *where σ' is the duality endomorphism on $\bigcup_j A_{0j}$ that sends A_{ij} onto $A_{i+2,j+2}$, for all $j \geqslant i \geqslant 0$ (as defined in* [P5]).

(3) $\mathrm{Tr}_{n+1} \circ \alpha_n = \mathrm{Tr}_n$ *and α_n takes the Tr_n-preserving expectations ($=\tau$-preserving expectations) in $\mathcal{C}_n$ into the restrictions to $e_{2n+1}\mathcal{C}_{n+1}e_{2n+1}$ of the Tr_{n+1}-preserving expectations in $\mathcal{C}_{n+1}$.*

(4) *The top row of commuting squares in $\mathcal{C}_n$ is a sequence of basic constructions of*

the initial homogeneous λ-Markov commuting square of inclusions

$$\mathcal{C}_n^0 = \begin{array}{ccc} M_{2n-1} & \subset & M_{2n} \\ \cup & & \cup \\ A_{-\infty,2n-1} & \subset & A_{-\infty,2n}. \end{array}$$

Moreover, $\alpha_n(1_{\mathcal{C}_n})=\alpha_n(1_{M_{2n}})=e_{2n+1}\in A_{-\infty,2n+1}$ *has scalar central trace in* $A_{-\infty,2n+1}$ (*which is regarded as an algebra in* $\mathcal{C}_{n+1}^0$), *so that* $\mathcal{C}_{n+1}^0$ *is the* λ^{-1}*-amplification of* $\mathcal{C}_n^0$.

Proof. α_n is well defined because for each x and k large enough one has $\alpha_n(x)=\alpha_n^k(x)$ (by Lemma 2.1 (1)). Then properties (1)–(3) are just reformulations of Lemma 2.1 (1)–(4). The last property (4) is well known (see e.g. [P4]). □

Definition 2.4. We define $\mathcal{C}$ to be the system of inclusions of von Neumann algebras

$$\begin{array}{ccccccccc} \mathcal{M}_{-1} & \subset & \mathcal{M}_0 & \subset & \dots & \subset & \mathcal{M}_k & \subset & \dots \\ \cup & & \cup & & & & \cup & & \\ \mathcal{B}_{-1} & \subset & \mathcal{B}_0 & \subset & \dots & \subset & \mathcal{B}_k & \subset & \dots \\ \cup & & \cup & & & & \cup & & \\ \mathcal{A}_{-1}^0 & \subset & \mathcal{A}_0^0 & \subset & \dots & \subset & \mathcal{A}_k^0 & \subset & \dots \\ \cup & & \cup & & & & \cup & & \\ \mathcal{A}_{-1}^{-1} & \subset & \mathcal{A}_0^{-1} & \subset & \dots & \subset & \mathcal{A}_k^{-1} & \subset & \dots \end{array}$$

obtained as the inductive limit of the sequence of non-unital trace-preserving embeddings of commuting squares

$$(\mathcal{C}_0;\mathrm{Tr}_0)\xhookrightarrow{\alpha_0}(\mathcal{C}_1;\mathrm{Tr}_1)\xhookrightarrow{\alpha_1}(\mathcal{C}_2;\mathrm{Tr}_2)\hookrightarrow \dots .$$

By this we mean the following:

(2.4.1) We first take the (non-unital!) algebraic inductive limit $\mathcal{M}_i^0$ of

$$M_i\xhookrightarrow{\alpha_0}M_{i+2}\xhookrightarrow{\alpha_1}M_{i+4}\hookrightarrow \dots .$$

We note that $\mathcal{M}_{-1}^0\subset\mathcal{M}_0^0\subset\dots$, in a natural way.

(2.4.2) For each $n\geqslant 0$, $j\geqslant -1$ and $x\in M_{i+2n}$ we denote by $\widetilde{\alpha}_n(x)=\dots\circ\alpha_{n+1}\circ\alpha_n(x)$ the image of x in $\mathcal{M}_i^0$. With this notation, we clearly have $\mathcal{M}_i^0=\bigcup_n\widetilde{\alpha}_n(M_{i+2n})$.

(2.4.3) On $\mathcal{M}_i^0$ we take the C^*-norm defined by $\|\widetilde{\alpha}_n(x)\|=\|x\|_{M_{i+2n}}$, if $x\in M_{i+2n}$.

(2.4.4) We define a positive tracial functional Tr on the algebras $\mathcal{M}_i^0$ by $\mathrm{Tr}(\widetilde{\alpha}_n(x))=\mathrm{Tr}_n(x)$, if $x\in M_{i+2n}$.

(2.4.5) We define $\mathcal{M}_i$ to be the completion of $\mathcal{M}_i^0$ in the topology of convergence in the norm $\|x\|_{2,\mathrm{Tr}}=\mathrm{Tr}(x^*x)^{1/2}$ on bounded sets (in C^*-norm) (note that $\mathcal{M}_i$ can also be defined through the GNS construction for $(\mathcal{M}_i^0,\mathrm{Tr})$).

(2.4.6) We note that Tr extends to a normal semifinite faithful trace on $\mathcal{M}_i$, still denoted Tr. Moreover, the algebras $\mathcal{M}_i$ defined in this way clearly satisfy $\mathcal{M}_i \subset \mathcal{M}_{i+1}$ with $\mathrm{Tr}_{\mathcal{M}_{i+1}}|_{\mathcal{M}_i} = \mathrm{Tr}_{\mathcal{M}_i}$ (the notation being self-explanatory).

(2.4.7) We define $\mathcal{B}_i$, $\mathcal{A}_i^{-1}$ and $\mathcal{A}_i^0$, $i \geqslant -1$, as the closure in the same topology of $\|\cdot\|_{2,\mathrm{Tr}}$-convergence on bounded sets of the $*$-subalgebras $\bigcup_n \tilde{\alpha}_n(A_{-\infty,i+2n})$ (for $\mathcal{B}_i$), $\bigcup_n \tilde{\alpha}_n(A_{-1,i+2n})$ (for $\mathcal{A}_i^{-1}$) and $\bigcup_n \tilde{\alpha}_n(A_{-2,i+2n})$ (for $\mathcal{A}_i^0$), respectively, all taken as subalgebras of $\mathcal{M}_i^0$.

(2.4.8) We note that the trace Tr on $\mathcal{M}_i$ restricts to semifinite traces on $\mathcal{A}_i^{-1}$ (and thus on $\mathcal{B}_i$ and $\mathcal{A}_i^0$ too), for each $i \geqslant -1$.

(2.4.9) If for each n we choose an inclusion $\mathcal{Q}_n \subset \mathcal{P}_n$ between two of the algebras in the commuting square $\mathcal{C}_n$, but so that for each n the algebras are chosen at the same "spot", and if we denote by $\mathcal{E}_{\mathcal{Q}}^{\mathcal{P}}$ the unique Tr-preserving expectation of the inductive limit $\mathcal{P} \stackrel{\text{def}}{=} \overline{\bigcup_n \tilde{\alpha}_n(P_n)}$ onto the inductive limit $\mathcal{Q} \stackrel{\text{def}}{=} \overline{\bigcup_n \tilde{\alpha}_n(Q_n)}$, then by Corollary 2.3 we have $\mathcal{E}_{\mathcal{Q}}^{\mathcal{P}}(\tilde{\alpha}_n(x)) = \tilde{\alpha}_n(E_{Q_n}^{P_n}(x))$, for $x \in P_n$.

In particular, by Corollary 2.3 (3), the properties (2.4.8) and (2.4.9) above show that the system of inclusions $\mathcal{C}$, endowed with the corresponding Tr-preserving expectations between its algebras, is a system of commuting squares.

We now examine more closely the main properties of $\mathcal{C}$.

LEMMA 2.5. *If for each $n \geqslant 0$ we let 1_n be the identity in $\mathcal{C}_n$, i.e. $1_n = 1_{M_{2n-1}} = 1_{A_{-1,2n-1}} = 1_{M_{2n+k}} = 1_{A_{-1,2n+k}}$, for all $k \geqslant 0$, and define $p_n = \tilde{\alpha}_n(1_n)$ then we have:*

(1) *p_n belong to $\mathcal{A}_{-1}^{-1}$, $\mathrm{Tr}\, p_n = \lambda^{-n}$ for all n, and $p_0 \leqslant p_1 \leqslant p_2 \leqslant \ldots$ with $p_n \nearrow 1_{\mathcal{A}_{-1}^{-1}}$ $(=1_{\mathcal{C}})$.*

(2) *For each n, $p_n \mathcal{C} p_n$ is naturally isomorphic to $\mathcal{C}_n$, via $\tilde{\alpha}_n$ (as commuting squares of trace-preserving expectations).*

(3) *p_n has scalar central trace in $p_{n+1}\mathcal{B}_{-1}p_{n+1}$, for all $n \geqslant 0$.*

(4) *For each $j \geqslant i \geqslant -1$ and $x \in A_{ij}$ there exists a unique element $\alpha(x)$ in $\bigcup_k \mathcal{M}_k$ such that $[\alpha(x), p_n] = 0$, for all n, $\alpha(x)p_n = \tilde{\alpha}_n(\sigma'^n(x))$, where σ' is the duality isomorphism as in Corollary 2.3 (3). Moreover, α is a $*$-isomorphism and $\alpha(A_{ij}) = \mathcal{M}_i' \cap \mathcal{M}_j = \mathcal{M}_i' \cap \mathcal{A}_j$, for all $j \geqslant i \geqslant -1$.*

(5) *$\alpha(e_j)$ belongs to $\mathcal{A}_j^{-1}$, for all $j \geqslant 1$, and $\alpha(e_0)$ belongs to $\mathcal{A}_0^0$. Also, $\alpha(e_{n+1})$ implements the Tr-preserving conditional expectation of $\mathcal{M}_n$ onto $\mathcal{M}_{n-1}$, for all $n \geqslant 0$.*

Proof. (1) is clear by the definitions, and so is the equality $p_n \mathcal{C} p_n = \tilde{\alpha}_n(\mathcal{C}_n)$ of condition (2). Then p_n have scalar central trace in $p_{n+1}\mathcal{B}_{-1}p_{n+1}$ because e_{2n+1} has scalar central trace in $A_{-\infty,2n+1}$ (see e.g. [P4]). This proves (3).

The first part in (4) follows by property (2) in Corollary 2.3. Then the equality $\alpha(A_{ij}) = \mathcal{M}_i' \cap \mathcal{M}_j$ is immediate by the definitions of $\alpha, \mathcal{M}_i, \mathcal{M}_j$.

Further on, by the way it is defined, $\alpha(A_{ij})$ is clearly contained in $\mathcal{A}_j^{-1}$, so that we have $\alpha(A_{ij})\subset\mathcal{M}_i'\cap\mathcal{A}_j^{-1}\subset(\mathcal{A}_i^0)'\cap\mathcal{A}_j^{-1}$.

To prove the opposite inclusion note that, since $A'_{-2,i}\cap A_{-1,j}=A_{ij}$, it follows that $A'_{-2,i+2n}\cap A_{-1,j+2n}=\sigma'^n(A_{ij})$ so that $\widetilde{\alpha}_n(A_{-2,i+2n})'\cap\widetilde{\alpha}_n(A_{-1,j+2n})=\widetilde{\alpha}_n(\sigma'^n(A_{ij}))$, which gives that $p_n((\mathcal{A}_i^0)'\cap\mathcal{A}_j^{-1})p_n=(p_n\mathcal{A}_i^0p_n)'\cap p_n\mathcal{A}_j^{-1}p_n=\alpha(A_{ij})p_n$. Since $p_n\nearrow 1$, this proves the last part of (4).

Since e_j lies in $A_{-1,j}$ for $j\geqslant 1$, it follows by (4) that $\alpha(e_j)$ lies in $\mathcal{A}_j^{-1}$, $j\geqslant 1$. Similarly, since e_0 lies in $A_{-2,j}$ for all $j\geqslant 0$, it follows that $\alpha(e_0)$ lies in $\mathcal{A}_0^0$.

Since e_{2k+1} implements the expectation of M_{2k} onto M_{2k-1}, it follows that $\widetilde{\alpha}_k(e_{2k+1})$ implements the conditional expectation of $p_k\mathcal{M}_0p_k$ onto $p_k\mathcal{M}_{-1}p_k$. Since $p_k\nearrow 1$ and $\alpha(e_1)p_k=\widetilde{\alpha}_k(e_{2k+1})$, we get the last part of (5) as well. □

The next lemma clarifies the structure of the inclusions $\mathcal{A}_{-1}^k\subset\mathcal{A}_0^k\subset\ldots$ for $k=-1,0$.

To state it, let us denote by $\Gamma=\Gamma_{M_{-1},M_0}=(a_{kl})_{k\in K,l\in L}$ the standard graph of $M_{-1}\subset M_0$ (or, equivalently, of $\mathcal{G}=\mathcal{G}_{M_{-1},M_0}$), which describes the sequence of inclusions $A_{-1,-1}\subset A_{-1,0}\subset A_{-1,1}\subset\ldots$. Thus, if $*\in K$ denotes the *initial vertex* of Γ and $K_n=(\Gamma\Gamma^t)^n(\{*\})$, $L_n=(\Gamma\Gamma^t)^n\Gamma(\{*\})$, then $K=\bigcup_n K_n$, $L_n=\bigcup_n L_n$, with the sets K_n, L_n having the following significance:

The set of simple summands of $\mathcal{Z}(A_{-1,2n-1})$ (resp. $\mathcal{Z}(A_{-1,2n})$) naturally identifies with the set K_n (resp. L_n), with the inclusion $K_n\subset K_{n+1}$ (resp. $L_n\subset L_{n+1}$) corresponding to the embedding of $\mathcal{Z}(A_{-1,2n-1})$ into $\mathcal{Z}(A_{-1,2n+1})$ (resp. of $\mathcal{Z}(A_{-1,2n})$ into $\mathcal{Z}(A_{-1,2n+2})$) given by the applications

$$\mathcal{Z}(A_{-1,j})\ni z\mapsto z'\in Z(A_{-1,j+2}),$$

with z' the unique element in $Z(A_{-1,j+2})$ such that $ze_{j+2}=z'e_{j+2}$.

Moreover, the inclusion graphs of $A_{-1,2n-1}\subset A_{-1,2n}$ (resp. $A_{-1,2n}\subset A_{-1,2n+1}$) are given by ${}_{K_n}\Gamma$ (resp. ${}_{L_n}\Gamma^t$).

Also, there exists a unique vector $\vec{s}=(s_k)_{k\in K}$ such that $s_*=1$, $\Gamma\Gamma^t\vec{s}=\lambda^{-1}\vec{s}$ and such that if $\vec{t}=(t_l)_{l\in L}=\lambda\Gamma^t\vec{s}$ then $(\lambda^n s_k)_{k\in K_n}$ (resp. $(\lambda^n t_l)_{l\in L_n}$) give the traces of the minimal projections in $A_{-1,2n-1}$ (resp. $A_{-1,2n}$).

Similarly, we denote by $\Gamma'=\Gamma_{M_{-2},M_{-1}}=(a'_{k'l'})_{k'\in K',l'\in L'}$ the standard graph of $M_{-2}\subset M_{-1}$ (or, equivalently, the "second" standard graph of $M_{-1}\subset M_0$; note that by duality $\Gamma'=\Gamma_{M_0,M_1}$ as well), with its standard vectors $\vec{s}'=(s_{k'})_{k'\in K'}$, $\vec{t}=(t_{l'})_{l'\in L'}$.

With this notation at hand we have:

LEMMA 2.6. *$\mathcal{A}_{-1}^k\subset\mathcal{A}_0^k\subset\ldots$ are inclusions of atomic von Neumann algebras, for each $k=-1,0$. More precisely, for each $n\geqslant 0$ the reduced sequence of inclusions $p_n(\mathcal{A}_{-1}^k\subset\mathcal{A}_0^k\subset\ldots)p_n$ is isomorphic via $\widetilde{\alpha}_n^{-1}$ to the sequence of inclusions $(A_{-1+k,2n-1}\subset$*

$A_{-1+k,2n}\subset\ldots$), with the trace Tr *on the former corresponding to the trace* Tr_n *on the latter.*

Moreover, if one identifies the set of factor summands of $\mathcal{A}_{-1}^{-1}$ (resp. $\mathcal{A}_{-1}^{0}$) which contain non-zero parts of the projection p_n with the set of factor summands of $A_{-1,2n-1}$ (resp. $A_{-2,2n-1}$), i.e., with K_n (resp. L_n'), via the identification of $p_n\mathcal{A}_{-1}^{-1}p_n$ with $A_{-1,2n-1}$ (resp. $A_{-2,2n-1}$), then the inclusion matrix for $\mathcal{A}_{-1}^{-1}\subset\mathcal{A}_0^{-1}$ (resp. $\mathcal{A}_{-1}^{0}\subset\mathcal{A}_0^0$) is given by Γ (resp. $(\Gamma')^t$), while the trace Tr *is given on the minimal projections of $\mathcal{A}_{-1}^{-1}$ (resp. $\mathcal{A}_{-1}^{0}$) by the eigenvector $\vec{s}=(s_k)_{k\in K}$ (resp. $\vec{t}'$) and on the minimal projections of $\mathcal{A}_0^{-1}$ (resp. $\mathcal{A}_0^0$) by the vector $\vec{t}$ (resp. $\lambda\vec{s}'$).*

Similarly, the inclusion graph for $\mathcal{A}_i^{-1}\subset\mathcal{A}_{i+1}^{-1}$ (resp. $\mathcal{A}_i^0\subset\mathcal{A}_{i+1}^0$) is given by Γ if i is odd and by Γ^t if i is even (resp. $(\Gamma')^t$ if i is odd and by Γ' if i is even), with the trace vector for the minimal projections of $\mathcal{A}_{2l-1}^{-1}$ and $\mathcal{A}_{2l}^{-1}$ (resp. $\mathcal{A}_{2l-1}^{0}$ and $\mathcal{A}_{2l}^{0}$) being given by $\lambda^k\vec{s}$ and $\lambda^k\vec{t}$ (resp. $\lambda'^k\vec{t}'$ and $\lambda'^{k+1}\vec{s}$).

Proof. We have already noted in Lemma 2.5 that the non-unital isomorphism $\widetilde{\alpha}_n$ takes the sequence of inclusions $(A_{-1,-1}\subset A_{-1,0}\subset A_{-1,1}\subset\ldots)$ onto the sequence of inclusions $p_n(\mathcal{A}_{-1}\subset\mathcal{A}_0\subset\ldots)p_n$, with $\mathrm{Tr}\circ\widetilde{\alpha}_n=\mathrm{Tr}_n$. Since A_{ij} are all atomic and $p_n\nearrow 1$, it follows that $\mathcal{A}_k$ are all atomic.

From the above and the discussion preceding Lemma 2.6, the last part now follows trivially. □

LEMMA 2.7. *The sequence of inclusions*

$$\mathcal{A}_{-1}^{-1}\subset\mathcal{A}_0^{-1}\overset{\alpha(e_1)}{\subset}\mathcal{A}_1^{-1}\overset{\alpha(e_2)}{\subset}\mathcal{A}_2^{-1}\ldots$$

is a Jones tower of λ-Markov inclusions.

Proof. By Lemma 2.5 (5), $\alpha(e_{n+1})$ belongs to $\mathcal{A}_{n+1}^{-1}$, and by commuting squares with $\mathcal{M}_{n-1}\subset\mathcal{M}_n$, it implements the Tr-preserving expectation of $\mathcal{A}_n^{-1}$ onto $\mathcal{A}_{n-1}^{-1}$.

By the definitions, we see that $p_n\mathcal{A}_1^{-1}p_n$ is contained in the linear span

$$\overline{\mathrm{sp}}(p_{n+1}\mathcal{A}_0^{-1}p_{n+1})\alpha(e_1)(p_{n+1}\mathcal{A}_0^{-1}p_{n+1}).$$

Since $p_n\nearrow 1$, this shows that $\overline{\mathrm{sp}}\,\mathcal{A}_0^{-1}\alpha(e_1)\mathcal{A}_0^{-1}=\mathcal{A}_1^{-1}$.

But by Lemma 2.6 the traces of the minimal projections in $\mathcal{A}_{-1}^{-1}\subset\mathcal{A}_0^{-1}$ satisfy the conditions in [J1]. Thus, the basic construction $\mathcal{A}_{-1}^{-1}\subset\mathcal{A}_0^{-1}\overset{e}{\subset}\langle\mathcal{A}_0^{-1},e\rangle$, where $e=e_{\mathcal{A}_{-1}^{-1}}$, has a λ-Markov trace that extends Tr.

Altogether, this shows that $\mathcal{A}_0^{-1}\ni x\mapsto x\in\mathcal{A}_0^{-1}$ and $e\mapsto\alpha(e_1)$ extends to a trace-preserving isomorphism of $\mathcal{A}_{-1}^{-1}\subset\mathcal{A}_0^{-1}\overset{\alpha(e_1)}{\subset}\mathcal{A}_1^{-1}$ onto $\mathcal{A}_{-1}^{-1}\subset\mathcal{A}_0^{-1}\overset{e}{\subset}\langle\mathcal{A}_0^{-1},e\rangle$. □

Let us summarize all the properties of the commuting square $\mathcal{C}$ emphasized thus far. To state it, recall from [P2], [P4] that an inclusion of von Neumann algebras $\mathcal{N}\subset\mathcal{M}$ with a conditional expectation $\mathcal{E}$ of finite index is called a λ-*Markov inclusion* if there exists an orthonormal basis (abbreviated hereafter as ONB) of $\mathcal{M}$ over $\mathcal{N}$ (with respect to $\mathcal{E}$), $\{m_j\}_j$, such that $\sum_j m_j m_j^* = \lambda^{-1}1$.

Also, recall from [P2] that in the case that $\mathcal{N}, \mathcal{M}$ are semifinite von Neumann algebras and the expectation $\mathcal{E}$ preserves a semifinite trace Tr on $\mathcal{M}$, then the above condition is equivalent to the existence of a semifinite trace $\mathrm{Tr}_{\mathcal{M}_1}$ on $\mathcal{M}_1 = \langle \mathcal{M}, e_{\mathcal{N}}\rangle$ that extends the trace Tr on $\mathcal{M}$ and satisfies $\mathrm{Tr}(xe_{\mathcal{N}}y) = \lambda \mathrm{Tr}(xy)$, for all $x, y\in\mathcal{M}$.

Definition 2.8. Let $\mathcal{Q}_i, \mathcal{P}_i$, $i=-1,0$, be arbitrary semifinite von Neumann algebras with inclusions

$$\begin{array}{ccc} \mathcal{P}_{-1} & \subset & \mathcal{P}_0 \\ \cup & & \cup \\ \mathcal{Q}_{-1} & \subset & \mathcal{Q}_0 \end{array}$$

with a normal semifinite faithful trace Tr on $\mathcal{P}_0$ which is semifinite on each of the smaller algebras and such that the corresponding Tr-preserving expectations make the above into a commuting square with both row inclusions of finite index. Then the commuting square is *non-degenerate* if any ONB of the bottom row is an ONB for the top row. The commuting square is λ-*Markov* if it is non-degenerate and the bottom (equivalently, the top) row inclusion is λ-Markov, in the sense explained above.

Note that if a commuting square is λ-Markov then both of its row inclusions must be λ-Markov. Conversely, if both row inclusions of a commuting square are λ-Markov, then the commuting square is automatically non-degenerate, hence λ-Markov itself. The same conclusion is true if only the bottom row is assumed to be λ-Markov, with the top one having index $\leqslant\lambda^{-1}$.

Note also that if one has a λ-Markov commuting square denoted as in Definition 2.8 then the projection $e = e^{\mathcal{P}_0}_{\mathcal{P}_{-1}}$ implements the basic construction for $\mathcal{Q}_{-1}\subset\mathcal{Q}_0$ as well. Moreover, the resulting system of inclusions

$$\begin{array}{ccc} \mathcal{P}_0 & \subset & \mathcal{P}_1 \\ \cup & & \cup \\ \mathcal{Q}_0 & \subset & \mathcal{Q}_1, \end{array}$$

where $\mathcal{Q}_1$ is the algebra generated by $\mathcal{Q}_0$ and e, is itself a λ-Markov commuting square (with respect to the Tr-preserving expectations). Thus, one can iterate the basic construction and obtain from the initial λ-Markov commuting square a whole Jones tower of λ-Markov commuting squares.

THEOREM 2.9. (1) *The commuting squares in the initial inclusion*

$$\mathcal{C}^0 = \begin{array}{ccc} \mathcal{M}_{-1} & \subset & \mathcal{M}_0 \\ \cup & & \cup \\ \mathcal{B}_{-1} & \subset & \mathcal{B}_0 \\ \cup & & \cup \\ \mathcal{A}^0_{-1} & \subset & \mathcal{A}^0_0 \\ \cup & & \cup \\ \mathcal{A}^{-1}_{-1} & \subset & \mathcal{A}^{-1}_0 \end{array}$$

of $\mathcal{C}$, *with its* Tr*-preserving expectations, are all* λ*-Markov.*

(2) $\mathcal{C}$ *is obtained by iterating the basic construction for* $\mathcal{C}^0$, *with* $\alpha(e_i)$, $i \geqslant 1$, *being the corresponding Jones projection.*

(3) *The commuting square*

$$\begin{array}{ccc} \mathcal{M}_{-1} & \subset & \mathcal{M}_0 \\ \cup & & \cup \\ \mathcal{B}_{-1} & \subset & \mathcal{B}_0 \end{array}$$

is isomorphic to the ∞*-amplification of the commuting square*

$$\begin{array}{ccc} M_{-1} & \subset & M_0 \\ \cup & & \cup \\ A_{-\infty,-1} & \subset & A_{-\infty,0}\,, \end{array}$$

i.e., it is obtained by tensoring the latter by $B(l^2(\mathbf{N}))$.

(4) *The commuting square*

$$\begin{array}{ccc} \mathcal{A}^0_{-1} & \subset & \mathcal{A}^0_0 \\ \cup & & \cup \\ \mathcal{A}^{-1}_{-1} & \subset & \mathcal{A}^{-1}_0 \end{array}$$

consists of infinite type I *von Neumann algebras with discrete centers. The bottom inclusion has graph given by* $\Gamma = \Gamma_{M_{-1},M_0}$, *and the top inclusion is given by the graph* $(\Gamma')^t$, *where* $\Gamma' = \Gamma'_{M_{-1},M_0} = \Gamma_{M_0,M_1}$. *The trace* Tr *is given on the minimal projections of* $\mathcal{A}^{-1}_{-1}$ *by* $\vec{s}$, *on* $\mathcal{A}^{-1}_0$ *by* $\vec{t} = \lambda\Gamma^t\vec{s}$, *on* $\mathcal{A}^0_{-1}$ *by* $\vec{t}'$, *and on* $\mathcal{A}^0_0$ *by* $\lambda\vec{s}'$.

(5) $\mathcal{M}'_i \cap \mathcal{M}_j = \mathcal{M}'_i \cap \mathcal{A}^{-1}_j = (\mathcal{B}_i)' \cap \mathcal{A}^{-1}_j = (\mathcal{A}^{-1}_i)' \cap \mathcal{A}^0_j$ *and* α *gives a natural isomorphism from*

$$\mathcal{G}_{M_{-1},M_0} = (M'_i \cap M_j)_{j \geqslant i \geqslant -1}$$

onto

$$(\mathcal{M}'_i \cap \mathcal{A}^{-1}_j)_{j \geqslant i \geqslant -1} = ((\mathcal{B}_i)' \cap \mathcal{A}^{-1}_j)_{j \geqslant i \geqslant -1} = ((\mathcal{A}^0_i)' \cap \mathcal{A}^{-1}_j)_{j \geqslant i \geqslant -1}.$$

The last result in this section describes the functoriality properties of the commuting squares appearing in $\mathcal{C}^0$. To state it, recall from [P3], [P5] that given two standard λ-lattices $\mathcal{G}^0=(A^0_{ij})_{j\geqslant i\geqslant -1}$, $\mathcal{G}=(A_{ij})_{j\geqslant i\geqslant -1}$, an *embedding* of $\mathcal{G}^0$ into $\mathcal{G}$ is a trace-preserving isomorphism $\imath$ from $\bigcup_n A^0_{-1,n}$ into $\bigcup_n A_{-1,n}$ such that $\imath(A^0_{ij})\subset A_{ij}$, for all $j\geqslant i\geqslant -1$, and such that $\imath$ takes the Jones λ-sequence of projections $\{e^0_n\}_{n\geqslant 1}$ of $\mathcal{G}^0$ into a Jones sequence of projections for $\mathcal{G}$, satisfying the smoothness condition

$$E_{A_{01}}(\imath(e^0_1))=\imath(E_{A^0_{01}}(e^0_1)). \tag{2.9.1}$$

Thus, one should keep in mind that a "morphism" between two standard lattices implicitly requires that both lattices have the same index (i.e., both be λ-lattices, with the same λ).

Note that by [P5], if $\imath$ is an embedding of a standard λ-lattice $\mathcal{G}_0$ into a standard lattice $\mathcal{G}$, then for any $-1\leqslant i\leqslant k\leqslant l\leqslant j$ one has commuting squares:

$$\begin{array}{ccc} A_{kl} & \subset & A_{ij} \\ \cup & & \cup \\ \imath(A^0_{kl}) & \subset & \imath(A^0_{ij}). \end{array}$$

THEOREM 2.10. (1) *The object $\mathcal{C}_{M_{-1},M_0}$ consisting of the commuting square*

$$\begin{array}{ccc} \mathcal{M}_{-1} & \subset & \mathcal{M}_0 \\ \cup & & \cup \\ \mathcal{A}^{-1}_{-1} & \subset & \mathcal{A}^{-1}_0 \end{array}$$

together with the fixed projection $p_0\in\mathcal{A}^{-1}_{-1}$ is canonically associated with $M_{-1}\subset M_0$.

(2) *The object $\mathcal{C}^{\mathrm{st}}_{\mathcal{G}}$ consisting of the commuting square*

$$\begin{array}{ccc} \mathcal{B}_{-1} & \subset & \mathcal{B}_0 \\ \cup & & \cup \\ \mathcal{A}^{-1}_{-1} & \subset & \mathcal{A}^{-1}_0 \end{array}$$

together with the fixed projection $p_0\in\mathcal{A}^{-1}_{-1}$ is canonically associated with the standard λ-lattice $\mathcal{G}$, and it is functorial in $\mathcal{G}$: If $\mathcal{G}_0\subset\mathcal{G}$ is a standard λ-lattice embedded in $\mathcal{G}$ then $\mathcal{C}^{\mathrm{st}}_{\mathcal{G}_0}$ is naturally non-degenerately embedded[1] *in $\mathcal{C}^{\mathrm{st}}_{\mathcal{G}}$ with commuting squares and with the corresponding projections p_0 coinciding.*

(3) *The object $\mathcal{C}_{\mathcal{G}}$ consisting of the commuting square*

$$\begin{array}{ccc} \mathcal{A}^0_{-1} & \subset & \mathcal{A}^0_0 \\ \cup & & \cup \\ \mathcal{A}^{-1}_{-1} & \subset & \mathcal{A}^{-1}_0 \end{array}$$

---

[1] This means that all the sides of the commuting "cube" arising from the inclusion of the two commuting squares are all non-degenerate commuting squares, with respect to the trace-preserving conditional expectations.

together with the fixed projection $p_0 \in \mathcal{A}_{-1}^{-1}$ *is canonically associated with the standard* λ*-lattice* $\mathcal{G}$*, and it is functorial in* $\mathcal{G}$*, in the same sense as in* (2).

Proof. (1) This part is clear by the construction of

$$\begin{array}{ccc} \mathcal{B}_{-1} & \subset & \mathcal{B}_0 \\ \cup & & \cup \\ \mathcal{A}_{-1} & \subset & \mathcal{A}_0 \end{array}$$

as the inductive limit of the canonical commuting squares

$$\begin{array}{ccc} M_{2n-1} & \subset & M_{2n} \\ \cup & & \cup \\ A_{-1,2n-1} & \subset & A_{-1,2n} \end{array}$$

via embeddings which are canonical as well (being defined by using only the Jones projections in the tower $e_1, e_2, \dots$). Also, $p_0 = \widetilde{\alpha}_0(1)$ so that the position of p_0 inside $\mathcal{A}_{-1}^{-1}$ is canonical as well.

(2) The fact that $\mathcal{C}_{\mathcal{G}}^{\mathrm{st}}$ is canonically associated with $\mathcal{G}$ follows by first noticing that the extended standard lattice $\widetilde{\mathcal{G}} = (A_{ij})_{i,j\in\mathbf{Z}}$, associated with $\mathcal{G}$ as in [P5], is canonically constructed from $\mathcal{G}$ by repeated basic constructions starting from the inclusion $A_{0,\infty} \subset A_{-1,\infty}$ (see the second paragraph in the proof of 2.2 in [P5]). In particular, the sequence of inclusions $A_{-\infty,-1} \subset A_{-\infty,0} \subset \dots$, with the whole system of inclusions of higher relative commutants into it, is therefore canonical. From this, an argument similar to the one in part (1) ends the proof.

If $\mathcal{G}_0 \subset \mathcal{G}$ in an embedding of standard λ-lattices with the same Jones projections then by the definition of the embeddings in the inductive limits of Definition 2.4, which only depends on the Jones projections, it follows that the inductive limit algebras involved in $\mathcal{C}_{\mathcal{G}_0}^{\mathrm{st}}$ are naturally embedded into the corresponding algebras of $\mathcal{C}_{\mathcal{G}}^{\mathrm{st}}$, with commuting squares. To see that the embedding of the two commuting squares is non-degenerate note that the embedding $\mathcal{G}^0 \subset \mathcal{G}$ implements a natural embedding between the corresponding extended standard lattices $\widetilde{\mathcal{G}}^0, \widetilde{\mathcal{G}}$ (thus, with commuting squares!). This fact in turn is an immediate consequence of the definitions, taking into account the smoothness condition (2.9.1).

(3) By the remarks following Definition 2.8, since the bottom row of $\mathcal{C}_{\mathcal{G}}$ is λ-Markov and the top row has index $\leqslant \lambda^{-1}$, $\mathcal{C}_{\mathcal{G}}$ is therefore λ-Markov as well.

The functoriality is trivial, by the definition of $\mathcal{C}_{\mathcal{G}}$, since the construction of $\mathcal{A}_{-1}^0 \subset \mathcal{A}_0^0$ only depends on the Jones projections in $\mathcal{G}$. Also, the commuting square conditions involved in the embedding $\mathcal{G}_0 \subset \mathcal{G}$ and the definition of the inductive limit, show that $\mathcal{C}_{\mathcal{G}_0}$ sits inside $\mathcal{C}_{\mathcal{G}}$ with non-degenerate commuting squares. $\square$

3. Amalgamated free products over type I algebras

We start with an easy lemma about compressions of amalgamated free products.

LEMMA 3.1. *Let $\mathcal{N}\subset\mathcal{M}^i$, $i=1,2$, be inclusions of von Neumann algebras with normal faithful conditional expectations $\mathcal{E}^i$. Assume that the projection $p\in\mathcal{N}$ has central support* 1 *in $\mathcal{N}$. Then*

$$p((\mathcal{M}^1,\mathcal{E}^1)*_{\mathcal{N}}(\mathcal{M}^2,\mathcal{E}^2))p=((p\mathcal{M}^1p,\mathcal{E}^1_p)*_{p\mathcal{N}p}(p\mathcal{M}^2p,\mathcal{E}^2_p)),$$

where $\mathcal{E}^i_p$ denotes conditional expectation of $p\mathcal{M}^ip$ onto $p\mathcal{N}p$ obtained by reducing $\mathcal{E}^i$ by p, $i=1,2$.

Proof. Since p has central support 1 in $\mathcal{N}$, there exists a family of partial isometries $v_i\in\mathcal{N}$ so that for all i, $v_i^*v_i\leqslant p$, and so that $\sum v_iv_i^*=1$ (in the sense of strong operator topology). Let $q_k=\sum_{i=1}^k v_iv_i^*$.

Let $w\in p((\mathcal{M}^1,\mathcal{E}^1)*_{\mathcal{N}}(\mathcal{M}^2,\mathcal{E}^2))p$ be an element. Then given a strong neighborhood U of w, one can find a k large enough so that a finite linear combination $w'=\sum w_i'$ of words w_i' each of the form

$$pq_km_1q_km_1'q_km_2q_km_2'\dots q_kp,\quad m_i\in\mathcal{M}^1,\ m_i'\in\mathcal{M}^2,$$

belongs to U. But such a word can be rewritten as

$$w_i=p\biggl(\sum_{i\leqslant k}v_iv_i^*\biggr)m_1\biggl(\sum_{i\leqslant k}v_iv_i^*\biggr)\dots\,.$$

Since each $v_i^*m_jv_j=pv_i^*m_jv_jp$ belongs either to $p\mathcal{M}^1p$ or $p\mathcal{M}^2p$, we deduce that

$$p((\mathcal{M}^1,\mathcal{E}^1)*_{\mathcal{N}}(\mathcal{M}^2,\mathcal{E}^2))p=W^*(p\mathcal{M}^1p,p\mathcal{M}^2p),$$

as subalgebras of $((\mathcal{M}^1,\mathcal{E}^1)*_{\mathcal{N}}(\mathcal{M}^2,\mathcal{E}^2))$.

We now note that the algebras $p\mathcal{M}^1p$ and $p\mathcal{M}^2p$ are free with amalgamation over $p\mathcal{N}p$ with respect to the reduced conditional expectation. This is immediate from the freeness condition. Since $\mathcal{E}^i_p$ are faithful, it follows that this von Neumann algebra is isomorphic to the free product $((p\mathcal{M}^1p,\mathcal{E}^1_p)*_{p\mathcal{N}p}(p\mathcal{M}^2p,\mathcal{E}^2_p))$, as claimed. □

COROLLARY 3.2. *If $\mathcal{E}^i\colon\mathcal{M}^i\to\mathcal{N}\subset\mathcal{M}^i$ are faithful conditional expectations, we have the isomorphism*

$$((\mathcal{M}^1,\mathcal{E}^1)*_{\mathcal{N}}(\mathcal{M}^2,\mathcal{E}^2))\otimes B(H)\cong(\mathcal{M}^2\otimes B(H),\mathcal{E}^1\otimes\mathrm{id})*_{\mathcal{N}\otimes B(H)}(\mathcal{M}^2\otimes B(H),\mathcal{E}^2\otimes\mathrm{id}).$$

We now turn to identification of amalgamated free products with the free group factor $L(\mathbf{F}_\infty)$.

THEOREM 3.3. *Let B be a von Neumann algebra, and $\mathcal{A}\subset B$ be a subalgebra. Let $E: B\to\mathcal{A}$ be a normal faithful conditional expectation. Assume that there exists a normal faithful semifinite trace* Tr *on $\mathcal{A}$, so that* $\mathrm{Tr}\circ E$ *is a trace on B. Assume lastly that $\mathcal{A}$ is of type* I *and has discrete center.*

Let

$$M=(B,E)*_{\mathcal{A}}(\mathcal{A}\otimes L(\mathbf{F}_\infty),\mathrm{id}\otimes\tau).$$

If B is of type II_∞ *and $p\in B$ is a projection,* $\mathrm{Tr}(p)=1$, *so that there is a system of matrix units $\{e_{ij}\}\subset B$ with $e_{11}=p$, $\sum e_{ii}=1$, then*

$$M\cong[(pBp,\mathrm{Tr}(p\cdot))*(L(\mathbf{F}_\infty),\tau)]\otimes B(H).$$

The proof of the theorem will consist of a sequence of lemmas. The notation and assumptions of the first paragraph of the theorem remain fixed throughout this section.

It is convenient to omit mentioning the specific conditional expectations in expressions for reduced amalgamated free products. It will always be clear from the context what conditional expectations are understood. Moreover, note that all of the conditional expectations in this paper are trace-preserving.

LEMMA 3.4. *M is a factor if and only if the centers $Z(\mathcal{A})\cap Z(B)$ have trivial intersection.*

Proof. By [P1], the relative commutant of $L(\mathbf{F}_\infty)$ inside M is equal to $\mathcal{A}$. It follows that $Z(M)\subset\mathcal{A}$, hence $Z(M)\subset Z(\mathcal{A})$. Since $\mathcal{A}\subset B$, also $Z(M)\subset Z(\mathcal{A})\cap Z(B)$. The other inclusion is trivial. □

Let Q be a von Neumann algebra with a semifinite normal trace Tr, and let $\eta_i: Q\to Q$ be normal completely positive maps. Assume that each η_i is self-adjoint, i.e., $\mathrm{Tr}(\eta_i(x)y)=\mathrm{Tr}(x\eta_i(y))$ for all x,y trace class in Q.

Define $\Phi(Q,\eta_1,\eta_2,...,\eta_n)$, where $n=1,2,...$ or $+\infty$, to be the von Neumann algebra generated by Q and the Q-semicircular family $X_1,X_2,...,X_n$, so that

(i) X_i are free with amalgamation over Q;

(ii) each X_i has covariance η_i.

Denote by E_Q the canonical conditional expectation from $\Phi(Q,\eta_1,\eta_2,...,\eta_n)$ onto Q. By [S3], $\mathrm{Tr}\circ E_Q$ is a trace on $\Phi(Q,\eta_1,...,\eta_n)$. Moreover, $E_Q(X_iqX_j)=\delta_{ij}\eta(q)$, for all $q\in Q$. Recall [S2] that X_i satisfy the inequality

$$\|X_i\|\leqslant 2\|\eta_i(1)\|^{1/2}.$$

Recall [S2] that if $q_i,r_i\in Q$ are elements, X is Q-semicircular of covariance η, then

$$Y_i=q_iXr_i+r_i^*Xq_i^*$$

is again Q-semicircular, of covariance

$$q \mapsto q_i \eta(r_i x r_i^*) q_i^* + r_i^* \eta(q_i^* x q_i) r_i + q_i \eta(r_i x q_i) r_i + r_i^* \eta(q_i^* x r_i^*) q_i^*.$$

In addition, $\{Y_i\}$ are free with amalgamation over Q if and only if $E_Q(Y_i q Y_j) = 0$ for all $q \in Q$ and $i \neq j$.

LEMMA 3.5. *$M \cong \Phi(B, E, E, E, ...)$ (infinite number of copies).*

Proof. By [S3],

$$\begin{aligned}\Phi(B, E, E, ...) &\cong (B, E) *_{\mathcal{A}} \Phi(\mathcal{A}, \mathrm{id}, \mathrm{id}, \mathrm{id}, ...)\\ &\cong (B, E) *_{\mathcal{A}} (\mathcal{A} \otimes \Phi(\mathbf{C}, \mathrm{id}, \mathrm{id}, ...))\\ &\cong (B, E) *_{\mathcal{A}} (\mathcal{A} \otimes L(\mathbf{F}_\infty)) = M.\end{aligned}$$

□

We need a slight modification of the construction Φ which works for semifinite completely positive maps, like $\mathrm{Tr}: B \to B$.

LEMMA 3.6. *Let $\eta_i: Q \to Q$, $\mu_i: Q \to Q$ be normal self-adjoint completely positive maps. Assume that for each i, there exist (possibly unbounded) operators x_i affiliated with Q, with (possibly unbounded) inverses, so that*

$$\mu_i(q) = x_i^* \eta_i(x_i q x_i^*) x_i \quad \text{for all } q \in Q.$$

Then $\Phi(Q, \eta_1, \eta_2, ...) \cong \Phi(Q, \mu_1, \mu_2, ...)$ in a way that preserves Q and E_Q. (The equation means that μ_i is the closure of the densely defined operator $q \mapsto x_i^* \eta_i(x_i q x_i^*) x_i$.)

Proof. By definition,

$$\Phi(Q, \eta_1, \eta_2, ...) = W^*(Q, X_1, X_2, ...),$$

where X_i are Q-semicircular, of covariance η_i. We claim that $x_i^* X_i x_i \in \Phi(Q, \eta_1, \eta_2, ...)$ (a priori, it may not be defined, since x_i may be unbounded). It is sufficient, by passing to the polar decomposition $x_i = u_i b_i$, $u_i \in Q$ unitary, to consider only the case that x_i are self-adjoint. Denote by x_i^t the value of the cut-off function $\{x \mapsto x\}|_{[-t,t]}$ applied to x_i. Let $Y_t = x_i^t X_i x_i^t$. Then Y_t is again Q-semicircular, of covariance

$$\eta_i^t(q) = x_i^t \eta_i(x_i^t q x_i^t) x_i^t.$$

In particular,

$$\|Y_t\| \leqslant 2 \|x_i^t \eta_i(x_i^t x_i^t) x_i^t\|^{1/2}.$$

Since $x_i^t x_i^t \leqslant x_i^2$ we get that

$$x_i^t \eta_i(x_i^t x_i^t) x_i^t \leqslant x_i^t \eta(x_i^2) x_i^t = \chi_{[-t,t]}(x_i) x_i \eta(x_i^2) x_i \chi_{[-t,t]}(x_i) \leqslant x_i \eta(x_i)^2 x_i = \mu(1).$$

Hence we have that

$$\|Y_t\| \leqslant 2\|\mu(1)\|^{1/2}.$$

Note that $Y_t = \chi_{[-t,t]}(x_i) x_i^s X_i x_i^s \chi_{[-t,t]}(x_i)$ if $t<s$. Hence Y_t are bounded, and moreover $\chi_{[-r,r]}(x_i) Y_t \chi_{[-r,r]}(x_i)$ does not depend on t once $t>r$. It follows that also the weak limit of Y_t exists and is bounded. We denote the limit by $x_i X_i x_i$. It is clear that $x_i X_i x_i$ is Q-semicircular of covariance $q \mapsto x_i \eta(x_i q x_i) x_i$. Note that $X_i \in W^*(Q, x_i X_i x_i)$ (one simply applies the same construction, starting with $x_i X_i x_i$ and using x_i^{-1} in the place of x_i).

Now,

$$\Phi(Q, \eta_1, \eta_2, \ldots) = W^*(Q, x_1 X_1 x_1, x_2 X_2 x_2, \ldots) \cong \Phi(Q, \mu_1, \mu_2, \ldots),$$

since $x_i X_i x_i$ has covariance $q \mapsto x_i \eta(x_i q x_i) x_i = \mu_i(x_i)$. □

Definition 3.7. $\Phi(Q, \mathrm{Tr}, \mathrm{Tr}, \ldots) = \Phi(Q, \eta, \eta, \ldots)$, where η is any normal completely positive map from Q to Q, so that $\eta(q) = x^* \mathrm{Tr}(xqx^*) x$ for some $x \in Q$, having a (possibly unbounded) inverse.

It is not hard to see, from Lemma 3.6, that this definition does not depend on the choice of η. Moreover, if the trace Tr is actually *finite*, then this coincides with the previous definition of $\Phi(Q, \mathrm{Tr})$.

Remark 3.8. The "unbounded semicircular element" of Rădulescu [R1] (see also [DR]) is precisely the "operator" one would get if in the construction of $\Phi(Q, \mathrm{Tr})$ one were to use a semifinite trace, but completely ignore the fact that Tr(1) is infinite. If $\eta(\cdot) = x \mathrm{Tr}(x \cdot x) x$ is as above, and X is Q-semicircular of covariance η, then Rădulescu's element would correspond to the operator $x^{-1} X x^{-1}$, which does not make sense as an operator, because Tr is not a normal self-adjoint map from Q to itself. Note that, as used in Rădulescu's work, the finite compressions $\chi_{[-t,t]}(x) x^{-1} X x^{-1} \chi_{[-t,t]}(x)$ do make sense as operators in $\Phi(Q, \mathrm{Tr})$. In particular, $\Phi(Q, \mathrm{Tr})$ is exactly the algebra $Q * \mathcal{S}X$ described in [DR].

PROPOSITION 3.9. *Let M be a von Neumann algebra with a semifinite faithful normal trace* Tr. *Then $\Phi(M, \mathrm{Tr}, \mathrm{Tr}, \ldots)$ is a factor of type* II_∞.

Proof. Choose $p_k \in M$ to be an increasing family of projections of finite trace, and so that $p_k \to 1$ strongly. Let $d = \sum (1/2^k) p_k$ and $\eta = d\,\mathrm{Tr}(d \cdot)$. Then $\Phi(M, \mathrm{Tr}, \mathrm{Tr}, \ldots) \cong$

$\Phi(M, \eta, \eta, \ldots)$, and is generated by M and M-semicircular elements $X_1, X_2, \ldots$ of covariance η. Consider the subalgebra $B_k \subset \Phi(M, \mathrm{Tr}, \mathrm{Tr}, \ldots)$, generated by $p_k M p_k$ and $p_k X_1 p_k, p_k X_2 p_k, \ldots$. Note that each $p_k X_i p_k$ is $p_k \eta(p_k \cdot p_k) p_k = p_k \mathrm{Tr}(dp_k \cdot)$-semicircular over $p_k M_k p_k$, and the restriction of the canonical semifinite trace on $\Phi(M, \eta, \eta, \ldots)$ to B_k is a finite trace (having value $\mathrm{Tr}(p_k)$ on the identity of B_k). Moreover,

$$B_k \cong \Phi(B_k, \mathrm{Tr}|_{B_k}, \mathrm{Tr}|_{B_k}, \ldots) \cong (B_k, 1/\mathrm{Tr}(p_k)) * L(\mathbf{F}_\infty),$$

and hence is a II_1 factor. Since $\Phi(M, \eta, \eta, \ldots)$ is the closure of $\bigcup_k B_k$, it follows that $\Phi(M, \eta, \eta, \ldots) \cong \Phi(M, \mathrm{Tr}, \mathrm{Tr}, \ldots)$ is a factor. Since it has a semifinite faithful normal trace, it must be a factor of type II_∞. $\square$

LEMMA 3.10. *Let $N = \Phi(Q, \eta_1, \eta_2, \ldots, \mu_1, \mu_2, \ldots)$. Denote by*

$$E_\eta : N \to \Phi(Q, \eta_1, \eta_2, \ldots) = N_\eta \quad \text{and} \quad E_\mu : N \to \Phi(Q, \mu_1, \mu_2, \ldots) = N_\mu$$

the canonical conditional expectations. Then

$$N \cong (N_\eta, E_Q) *_Q (N_\mu, E_Q) \cong \Phi(N_\eta, \mu_1 \circ E_\eta, \mu_2 \circ E_\eta, \ldots)$$

in a way that preserves N_η, Q and E_η, E_Q.

Proof. By definition,

$$N = W^*(Q, X_1, X_2, \ldots, Y_1, Y_2, \ldots),$$

where X_i and Y_i are free over Q, and X_i is η_i-semicircular over Q, Y_i is μ_i-semicircular over Q. The claimed decomposition as an amalgamated free product follows. The second isomorphism follows from the fact that Y_i, being free from $W^*(Q, X_1, X_2, \ldots) = N_\eta$ over Q, is $\mu_i \circ E_\eta$-semicircular over N_η (see [S2]). $\square$

LEMMA 3.11. *Assume that Q is a factor of type II_∞, and $\eta_1, \eta_2, \ldots$ are normal self-adjoint completely positive maps from Q to itself. Assume that $\eta_i \neq 0$ for all i, and that for each i, there exist subalgebras $\mathcal{A}_i$, each of type* I *with discrete center, so that*

$$\eta_i = E^Q_{\mathcal{A}_i}.$$

Then $\Phi(Q, \eta_1, \eta_1, \ldots, \eta_2, \eta_2, \ldots) \cong \Phi(Q, \mathrm{Tr}, \mathrm{Tr}, \mathrm{Tr}, \ldots)$ (each η_i is repeated an infinite number of times), in a way that maps Q to Q, and preserves E_Q.

Proof. Since

$$\Phi(Q, \eta_1, \ldots, \eta_2, \ldots) = \Phi(Q, \eta_1, \eta_1, \ldots) *_Q \Phi(Q, \eta_2, \eta_2, \ldots) *_Q \cdots,$$

it is sufficient to prove the result in the case that all η_i are the same. We can then clearly assume that all $\mathcal{A}_i=\mathcal{A}$, and $\eta_i=E_{\mathcal{A}}$.

Let $q_1, q_2, \ldots$ be the minimal central projections of $\mathcal{A}$, $\sum q_i=1$. Then $\mathcal{A}=\sum q_i\mathcal{A}q_i$, and each $q_i\mathcal{A}q_i$ is a type I factor; let $n_i\in\mathbf{N}\cup\{+\infty\}$ be the rank (square root of the dimension) of $q_i\mathcal{A}q_i$. Let e^i_{st}, $1\leqslant s,t\leqslant n_i$ be a system of matrix units for $q_i\mathcal{A}q_i$; that is,

$$e^i_{st}s^j_{t's'}=\delta_{ij}\delta_{tt'}e^i_{ss'},$$
$$e^i_{st}=(e^i_{ts})^*,$$
$$q_i=\sum_{1\leqslant s\leqslant n_i} e^i_{ss}.$$

CLAIM 3.12. *Let P be an algebra of type II_∞, $\widehat{P}\subset P$ be a unital subalgebra of type II_∞ so that $\widehat{P}$ is a factor, and $p\in\widehat{P}$ be a projection of finite trace. Let $\nu: Q\to Q$ be given by*

$$\nu(q)=p\operatorname{Tr}(pqp)p, \quad q\in P.$$

Then $\Phi(P,\operatorname{Tr},\operatorname{Tr},\ldots)\cong\Phi(P,\nu,\nu,\ldots)$ in a way that preserves P and E_P.

Proof. Choose matrix units $f_{ij}\in\widehat{P}$ so that $f_{11}=p$, $\sum f_{ii}=1$, $f_{ij}f_{j'i'}=\delta_{jj'}f_{ii'}$, $f^*_{ij}=f_{ji}$. Let $x=\sum_i f_{ii}/2^i$, and let $\mu(p)=x\operatorname{Tr}(xpx)x$, $p\in P$, be a completely positive map from P to itself. Then $\Phi(P,\operatorname{Tr},\operatorname{Tr},\ldots)\cong\Phi(P,\mu,\mu,\ldots)$. Let X_i be a P-semicircular family of covariance μ; thus $\Phi(P,\operatorname{Tr},\operatorname{Tr},\ldots)=W^*(P,X_1,X_2,\ldots)$. Let $X^k_{ij}=\operatorname{Re} f_{1i}X_kf_{j1}$, $i\leqslant j$, $Y^k_{ij}=\operatorname{Im} f_{1i}X_kf_{j1}$, $i<j$. Then $\Phi(P,\operatorname{Tr},\operatorname{Tr},\ldots)$ is generated by P and $\{X^k_{ij}\}_{k,i\leqslant j}\cup\{Y^k_{ij}\}_{k,i<j}$. A straightforward computation shows that $E(X^k_{ij}pX^{k'}_{i'j'})=\mathrm{const}\cdot\delta_{ii'}\delta_{jj'}\delta_{kk'}p\operatorname{Tr}(pqp)$, $E(Y^k_{ij}pY^{k'}_{i'j'})=\mathrm{const}\cdot\delta_{ii'}\delta_{jj'}\delta_{kk'}p\operatorname{Tr}(pqp)$ and $E(X^k_{ij}pY^{k'}_{i'j'})=0$. Hence, upon proper rescaling, $\{X^k_{ij}\}_{k,i\leqslant j}\cup\{Y^k_{ij}\}_{k,i<j}$ form a P-semicircular family of covariance ν. Hence $\Phi(P,\operatorname{Tr},\operatorname{Tr},\ldots)\cong\Phi(P,\nu,\nu,\ldots)$, as claimed. □

CLAIM 3.13. $\Phi(Q,\operatorname{Tr},\operatorname{Tr},\ldots)\cong\Phi(Q,\eta,\eta,\ldots,\operatorname{Tr},\operatorname{Tr},\operatorname{Tr},\ldots)$, *in a way that preserves Q and E_Q.*

Proof. Let $p_i=e^i_{11}\in Q$. Let $\nu_i(q)=p_i\operatorname{Tr}(p_iqp_i)p_i$. We first notice that, in view of Claim 3.12,

$$\begin{aligned}\Phi(Q,\operatorname{Tr},\operatorname{Tr},\ldots)&\cong(\Phi(Q,\operatorname{Tr},\operatorname{Tr},\ldots)*_Q(\Phi(Q,\operatorname{Tr},\operatorname{Tr},\ldots)*_Q\ldots))\\&\cong(\Phi(Q,\operatorname{Tr},\operatorname{Tr},\ldots,\nu_1,\nu_1,\ldots)*_Q(\Phi(Q,\operatorname{Tr},\ldots,\nu_2,\ldots)))*_Q\ldots\\&\cong\Phi(Q,\operatorname{Tr},\operatorname{Tr},\ldots,\nu_1,\nu_1,\ldots,\nu_2,\nu_2,\ldots)\\&\cong\Phi(Q,\nu_1,\nu_1,\ldots,\nu_2,\nu_2,\ldots)*_Q\Phi(Q,\operatorname{Tr},\operatorname{Tr},\ldots).\end{aligned}$$

Let X_j^i be Q-semicircular variables, free with amalgamation over Q, and so that the covariance of X_j^i is ν_i. Note that $X_j^i = e_{11}^i X_j^k e_{11}^i$. Let

$$Y_j = \sum_i \sum_{1 \leqslant s \leqslant n_i} e_{s1} X_j^i e_{1s} \cdot \frac{1}{\operatorname{Tr}(e_{11}^s)^{1/2}}.$$

This sum converges strongly, since X is diagonal relative to the orthogonal family of projections $\{e_{ss}^i\}$, and

$$\left\| e_{s1} X_j^i e_{1s} \cdot \frac{1}{\operatorname{Tr}(e_{11}^s)^{1/2}} \right\| \leqslant 2 \|\nu_i(1)\|^{1/2} \cdot \frac{1}{\operatorname{Tr}(e_{11}^s)^{1/2}} = 2.$$

It is not hard to see that $\{Y_j\}$ form a Q-semicircular family of covariance $E_{\mathcal{A}} = \eta$. Moreover, $\Phi(Q, \nu_1, \nu_1, ..., \nu_2, \nu_2, ...)$ is generated by Q and $\{Y_i\}_i$, since X_j^i is, up to a constant, $e_{11}^i Y_j e_{11}^i$. Hence $\Phi(Q, \nu_1, \nu_1, ..., \nu_2, \nu_2, ...) \cong \Phi(Q, \eta, \eta, ...)$. Thus

$$\begin{aligned} \Phi(Q, \operatorname{Tr}, \operatorname{Tr}, ...) &\cong \Phi(Q, \nu_1, \nu_1, ..., \nu_2, \nu_2, ...) *_Q \Phi(Q, \operatorname{Tr}, \operatorname{Tr}, ...) \\ &\cong \Phi(Q, \eta, \eta, ...) *_Q \Phi(Q, \operatorname{Tr}, \operatorname{Tr}, ...) \\ &\cong \Phi(Q, \eta, \eta, ..., \operatorname{Tr}, \operatorname{Tr}, ...). \end{aligned}$$

□

We now finish the proof of the lemma. By Lemma 3.10, we get that

$$\Phi(Q, \operatorname{Tr}, \operatorname{Tr}, ...) \cong \Phi(Q, \eta, \eta, ..., \operatorname{Tr}, \operatorname{Tr}, ...) \cong \Phi(\Phi(Q, \eta, \eta, ...), \operatorname{Tr}, \operatorname{Tr}, ...).$$

Noticing that $P = \Phi(Q, \eta, \eta, ...)$ contains a II_∞ factor $\widehat{P} = Q$, and setting $p = e_{11}^1 \in Q$, $\nu(x) = p \operatorname{Tr}(pxp) p$, $x \in \Phi(Q, \eta, \eta, ...)$, we get

$$\Phi(\Phi(Q, \eta, \eta), \operatorname{Tr}, \operatorname{Tr}, ...) \cong \Phi(\Phi(Q, \eta, \eta, ...), \nu, \nu, ...) \cong \Phi(Q, \eta, \eta, ..., \nu|_Q, \nu|_Q, ...),$$

the last isomorphism because

$$\nu = E_Q^{\Phi(Q,\eta,\eta,...)} \circ \nu \circ E_Q^{\Phi(Q,\eta,\eta,...)},$$

since $p \in Q$ (see Lemma 3.10).

Now, the algebra $\Phi(Q, \eta, \eta, ..., \nu|_Q, \nu|_Q, ...)$ is generated by Q and a Q-semicircular system $X_1, X_2, ..., Y_1, Y_2, ...$, where $\{X_i, Y_i\}_i$ are free with amalgamation over Q, X_i has covariance η and Y_i has covariance ν. Note that X_i commutes with $\mathcal{A}$ (containing $p = e_{11}^1$), and $Y_i = p Y_i p$, because of the form of ν. In particular, $X_i = \sum_{i, 1 \leqslant s \leqslant n_i} e_{ss}^i X_i e_{ss}^i$. It follows that $\Phi(Q, \eta, \eta, ..., \nu|_Q, \nu|_Q, ...)$ is generated by

$$\{q_1 X_i q_1\}_i, \quad \{(1-q_1) X_i (1-q_1)\}_i, \quad \{e_{11}^1 Y_i e_{11}^1\}_i, \quad Q.$$

Furthermore, $\Phi(Q,\eta,\eta,...)$ is generated by

$$\{q_1X_iq_1\}_i,\quad \{(1-q_1)X_i(1-q_1)\}_i,\quad Q.$$

Note that the three families $\{q_1X_iq_1\}_i$, $\{(1-q_1)X_i(1-q_1)\}_i$, $\{pY_ip\}_i$ are free with amalgamation over Q; this is because for all $q\in Q$,

$$E_Q(q_1X_iq_1q(1-q_1)X_j(1-q_1))=\delta_{ij}q_1E_{\mathcal{A}}(q_1q(1-q_1))(1-q_1)=0,$$

since q_1 is a central projection in $\mathcal{A}$.

Next, since X_i commutes with $\mathcal{A}$, we get that

$$q_1X_iq_1=\sum_{1\leqslant s\leqslant n_1}e^1_{ss}X_ie^1_{ss}=\sum_{1\leqslant s\leqslant n_1}e^1_{s1}e^1_{1s}X_ie^1_{s1}e^1_{1s}=\sum_{1\leqslant s\leqslant n_1}e^1_{s1}pX_ipe^1_{1s}.$$

It follows that $\Phi(Q,\eta,\eta,...,\nu|_Q,\nu|_Q,...)$ is generated by

$$\{pX_ip\}_i,\quad \{(1-q_1)X_i(1-q_1)\}_i,\quad \{pY_ip\}_i,\quad Q,$$

and the families

$$\{pX_ip\}_i,\quad \{(1-q_1)X_i(1-q_1)\}_i,\quad \{pY_ip\}$$

are free with amalgamation over Q. Moreover, $\Phi(Q,\eta,\eta,...)$ is generated by

$$\{pX_ip\}_i,\quad \{(1-q_1)X_i(1-q_1)\}_i,\quad Q.$$

Now, $\{pX_ip\}_i$ are free with amalgamation over Q, and pX_ip is Q-semicircular with covariance

$$q\mapsto E_Q(pX_ipqpX_ip)=pE_{\mathcal{A}}(pqp)p=\text{const}\cdot p\operatorname{Tr}(pqp)p=\text{const}\cdot\nu(q).$$

It follows that $\{pX_ip\}_i$ (upon rescaling by some non-zero constant) form an infinite Q-semicircular family of covariance $\nu|_Q$. Hence, by renumbering, we can join

$$\{pX_ip\}_i\cup\{pY_ip\}_i$$

into a single semicircular family of covariance ν. It follows that the algebras

$$W^*(\{pX_ip\}_i,\{(1-q_1)X_i(1-q_1)\}_i,\{pY_ip\}_i,Q)$$

and

$$W^*(\{pX_ip\}_i,\{(1-q_1)X_i(1-q_1)\}_i,Q)$$

are isomorphic to each other, in a way that maps Q to Q, and preserves E_Q. But we saw before that the first of these algebras is isomorphic to $\Phi(Q,\eta,\eta,\ldots,\nu|_Q,\nu|_Q,\ldots)$, while the second is isomorphic to $\Phi(Q,\eta,\eta,\ldots)$. $\square$

LEMMA 3.14. *If B is of type* II_∞ *and $p\in B$ is a projection,* $\mathrm{Tr}(p)=1$, *so that there is a system of matrix units $\{e_{ij}\}\subset B$ with $e_{11}=p$, $\sum e_{ii}=1$, then*

$$\Phi(B,\mathrm{Tr},\mathrm{Tr},\ldots)\cong[(pBp,\mathrm{Tr}(p\,\cdot))*(L(\mathbf{F}_\infty),\tau)]\otimes B(H).$$

Proof. Let $p_i=e_{ii}$ be a family of orthogonal projections in B, $\mathrm{Tr}(p_i)=1$, $\sum p_i=1$. Let $x=\sum p_n/2^n$, and let $\eta: B\to B$ be given by $\eta(b)=x\,\mathrm{Tr}(xbx)x$. Then $\Phi(B,\eta,\eta,\ldots)\cong\Phi(B,\mathrm{Tr},\mathrm{Tr},\ldots)$, by definition. Hence $\Phi(B,\mathrm{Tr},\mathrm{Tr},\ldots)\cong W^*(B,X_1,X_2,\ldots)$, where X_i are B-semicircular, each of covariance η. Then

$$p_1\Phi(B,\mathrm{Tr},\mathrm{Tr},\ldots)p_1\cong W^*(p_1Bp_1,\{X^r_{ij}\}_{r,i,j}),$$

where $X^r_{ij}=e_{1i}X_re_{j1}$. It is not hard to see that

$$\{X^r_{ii}\}\cup\{\mathrm{Re}\,X^r_{ij}:i>j\}\cup\{\mathrm{Im}\,X^r_{ij}:i>j\}$$

are free over p_1Bp_1 and are again a p_1Bp_1-semicircular family, each having covariance $2^{-i-j}\cdot\mathrm{Tr}(p_1\cdot p_1)$. Denoting $p=p_1$ and $\tau(\cdot)=\mathrm{Tr}(p\cdot p)$, we get (see [S3])

$$p\Phi(B,\mathrm{Tr},\mathrm{Tr},\ldots)p\cong\Phi(pBp,\tau,\tau,\ldots)\cong(B,\tau)*L(\mathbf{F}_\infty).\qquad\square$$

The following corollary, together with Lemma 3.14, implies Theorem 3.3.

COROLLARY 3.15. *Let B be a W^*-algebra with a semifinite normal faithful trace* Tr. *Let $\mathcal{A}\subset B$ be a type* I *subalgebra with discrete center. Set $M=\Phi(B,E,E,\ldots)$, where $E: B\to\mathcal{A}$ is the* Tr*-preserving conditional expectation. Then if M is a factor,*

$$M\cong\Phi(B,\mathrm{Tr},\mathrm{Tr},\mathrm{Tr},\ldots).$$

Proof. Let $F:\Phi(B,E,E,\ldots)\to\mathcal{A}$ denote the composition of

$$E: B\to\mathcal{A}\quad\text{and}\quad E_B:\Phi(B,E,E,\ldots)\to B.$$

Let $N=\Phi(B,\mathrm{Tr},\mathrm{Tr},\ldots)$, and denote by $G: N\to\mathcal{A}$ the composition of $E_B: N\to B$ and $E: B\to A$. Note that F, G and E all satisfy the hypothesis of Lemma 3.11; moreover, by

Proposition 3.9, N is a factor. We have

$$\begin{aligned} M &\cong \Phi(B,E,E,\ldots)\\ &\cong \Phi(\Phi(B,E,E,\ldots),F,F,F,\ldots)\\ &\cong \Phi(M,F,F,\ldots)\\ &\cong \Phi(M,\mathrm{Tr},\mathrm{Tr},\ldots)\\ &\cong \Phi(\Phi(B,E,E,\ldots),\mathrm{Tr},\mathrm{Tr},\ldots)\\ &\cong \Phi(B,E,E,E,\ldots,\mathrm{Tr},\mathrm{Tr},\ldots)\\ &\cong \Phi(B,\mathrm{Tr},\mathrm{Tr},\ldots,E,E,\ldots)\\ &\cong \Phi(\Phi(B,\mathrm{Tr},\mathrm{Tr},\ldots),G,G,\ldots)\\ &\cong \Phi(N,G,G,\ldots)\\ &\cong \Phi(N,\mathrm{Tr},\mathrm{Tr},\ldots)\\ &\cong \Phi(\Phi(B,\mathrm{Tr},\mathrm{Tr},\ldots),\mathrm{Tr},\mathrm{Tr},\ldots) \cong \Phi(B,\mathrm{Tr},\mathrm{Tr},\ldots). \end{aligned}$$

This completes the proof. □

We shall also need the following theorem:

THEOREM 3.16. *Let $\mathcal{A}\subset\mathcal{B}$ be an inclusion of type* I *von Neumann algebras with discrete centers. Let* Tr *be a semifinite normal trace on $\mathcal{B}$, and let $E:\mathcal{B}\to\mathcal{A}$ be the* Tr*-preserving conditional expectation. Let*

$$M=(\mathcal{B},E)*_{\mathcal{A}}(\mathcal{A}\otimes L(\mathbf{F}_\infty),\mathrm{id}\otimes\tau).$$

Then if M is a factor, $M\cong L(\mathbf{F}_\infty)\otimes B(H)$.

Proof. By tensoring B with $B(H)$, and noting that

$$\begin{aligned}(\mathcal{B}\otimes B(H),E\otimes\mathrm{id})*_{\mathcal{A}\otimes B(H)}&(\mathcal{A}\otimes L(\mathbf{F}_\infty)\otimes B(H),\mathrm{id}\otimes\tau\otimes\mathrm{id})\\ &\cong((\mathcal{B},E)*_{\mathcal{A}}(\mathcal{A}\otimes L(\mathbf{F}_\infty),\mathrm{id}\otimes\tau))\otimes B(H)\end{aligned}$$

(see Corollary 3.2), we may assume that $\mathcal{B}\cong\mathcal{B}\otimes B(H)$. Assume that M is a factor. By Corollary 3.15 we obtain the isomorphism

$$M\cong\Phi(\mathcal{B},\mathrm{Tr},\mathrm{Tr},\ldots);$$

it is therefore sufficient to prove that the latter algebra is isomorphic to $L(\mathbf{F}_\infty)\otimes B(H)$.

It is not hard to see that $\Phi(\mathcal{B},\mathrm{Tr},\mathrm{Tr},\ldots)\otimes B(H)\cong\Phi(\mathcal{B}\otimes B(H),\mathrm{Tr},\mathrm{Tr},\ldots)$; hence we may replace $\mathcal{B}=\bigoplus B(H)$ with $\bigoplus\mathbf{C}$, i.e., to assume that $\mathcal{B}$ is commutative.

We also have (arguing as before) that

$$\Phi(\mathcal{B}, \mathrm{Tr}, \mathrm{Tr}, \ldots) \cong \Phi(\mathcal{B}, \mathrm{id}, \mathrm{id}, \ldots, \mathrm{Tr}, \mathrm{Tr}, \ldots).$$

Setting $\widehat{B}=\Phi(\mathcal{B}, \mathrm{id}, \mathrm{id}, \ldots)=\mathcal{B}\otimes L(\mathbf{F}_\infty)$ gives that

$$\Phi(\mathcal{B}, \mathrm{Tr}, \mathrm{Tr}, \ldots) \cong \Phi(\widehat{B}, \mathrm{Tr}, \mathrm{Tr}, \ldots).$$

Note that $\widehat{B}=\bigoplus L(\mathbf{F}_\infty)$. Tensoring with $B(H)$ again allows us to replace $\widehat{B}$ with $B=\widehat{B}\otimes B(H)$. It thus remains to be proved that $\Phi(\widehat{B}, \mathrm{Tr}, \mathrm{Tr}, \ldots)\cong L(\mathbf{F}_\infty)\otimes B(H)$.

Denote by Ψ a choice of the semifinite trace on $L(\mathbf{F}_\infty)\otimes B(H)$. Then there exist numbers $\lambda_i>0$ so that $(B, \mathrm{Tr})\cong\bigoplus_i (L(\mathbf{F}_\infty)\otimes B(H), \lambda_i\Psi)$. Choose in each direct summand in B a projection p_i of trace 1. Let $p=\sum p_i$. Then B contains a set of matrix units e_{ij} with $e_{11}=p$ and $\sum e_{ii}=1$. Compressing to p gives that $\Phi(B, \mathrm{Tr}, \mathrm{Tr}, \ldots)\otimes B(H)\cong \Phi(A, \mathrm{Tr}', \mathrm{Tr}', \ldots)$, where $A=\bigoplus L(\mathbf{F}_\infty)$, and Tr' is the direct sum of the traces Ψ.

It follows that we may assume that the value of Tr on the minimal central projections of $\mathcal{B}$ is the same. It follows that the isomorphism class of $\Phi(\mathcal{B}, \mathrm{Tr}, \mathrm{Tr}, \ldots)$ does not depend on the choice of the normal faithful semifinite trace on $\mathcal{B}$; furthermore, it is sufficient to consider the case that $\mathcal{B}$ is commutative.

We now make a specific choice of $\mathcal{B}\cong l^\infty(\mathbf{Z})$ and the trace Tr:

$$\mathrm{Tr}(f) = \sum_{n\in\mathbf{Z}} 2^n f(n).$$

The translation action of $\mathbf{Z}$ on $\mathcal{B}$ gives rise to a trace-scaling action α of $\mathbf{Z}$ on $\Phi(\mathcal{B}, \mathrm{Tr}, \mathrm{Tr}, \ldots)$ (by naturality of the construction Φ and the fact that Tr is scaled by the action of $\mathbf{Z}$). Since $\Phi(\mathcal{B}, \mathrm{Tr}, \mathrm{Tr}, \ldots)$ is generated by a $\mathcal{B}$-semicircular family, it is easily seen that $N=\Phi(\mathcal{B}, \mathrm{Tr}, \mathrm{Tr}, \ldots)\rtimes_\alpha \mathbf{Z}$ is generated by a $B(H)\cong\mathcal{B}\rtimes\mathbf{Z}$-semicircular family, hence isomorphic to $\Phi(B(H), \eta, \eta, \eta, \ldots)$ for some $\eta: B(H)\to B(H)$. Note that N is a factor of type II_∞, since $\Phi(\mathcal{B}, \mathrm{Tr}, \mathrm{Tr}, \ldots)$ is a $\mathrm{III}_{1/2}$ factor. By Theorem 2.1 of [SU], $N\cong\Phi(\mathbf{C}, \mu, \mu, \mu)\otimes B(H)$ for some $\mu: \mathbf{C}\to B(H)$. Note that

$$\Phi(\mathbf{C}, \mu, \mu, \ldots) = \Phi(\mathbf{C}, \mu) * \Phi(\mathbf{C}, \mu) * \ldots$$

and is a a free Araki–Woods factor [S2], [S1]. Being type $\mathrm{III}_{1/2}$, it must be that $\Phi(\mathbf{C}, \mu)$ is isomorphic to the unique type $\mathrm{III}_{1/2}$ free Araki–Woods factor. Hence $\Phi(\mathcal{B}, \mathrm{Tr}, \mathrm{Tr}, \ldots)\cong L(\mathbf{F}_\infty)\otimes B(H)$, being the core of this factor. $\square$

4. Functorial constructions of subfactors via free products

Let us begin by recording the following general result (which is well known for semifinite inclusions with trace-preserving conditional expectations).

PROPOSITION 4.1. (a) *Let*

$$\begin{array}{ccc} \mathcal{P}_{-1} & \overset{\mathcal{F}_0}{\subset} & \mathcal{P}_0 \\ \cup \mathcal{E}_{-1} & & \cup \mathcal{E}_0 \\ \mathcal{Q}_{-1} & \overset{\mathcal{F}_{-1}}{\subset} & \mathcal{Q}_0 \end{array} \tag{4.1.1}$$

be a commuting square, and assume that $\mathcal{E}_i$ are faithful normal conditional expectations. Let Q be a diffuse finite von Neumann algebra with a normal finite faithful trace τ, and set

$$\mathcal{M}_i=((\mathcal{Q}_i \otimes Q), \mathrm{id}\otimes\tau) *_{\mathcal{Q}_i} (\mathcal{P}_i, \mathcal{E}_i).$$

Then

$$\begin{array}{ccc} \mathcal{M}_{-1} & \overset{\widehat{\mathcal{F}}}{\subset} & \mathcal{M}_0 \\ \cup & & \cup \\ \mathcal{P}_{-1} & \overset{\mathcal{F}_0}{\subset} & \mathcal{P}_0 \\ \cup \mathcal{E}_{-1} & & \cup \mathcal{E}_0 \\ \mathcal{Q}_{-1} & \overset{\mathcal{F}_{-1}}{\subset} & \mathcal{Q}_0 \end{array} \tag{4.1.2}$$

forms a commuting diagram of inclusions of von Neumann algebras. Moreover,

$$\mathcal{M}'_{-1} \cap \mathcal{M}_0 = \mathcal{P}'_{-1} \cap \mathcal{Q}_0.$$

(b) *Assume that* (4.1.1) *forms a commuting square, and $\mathcal{F}_i$ are finite-index conditional expectations. Assume also that* (4.1.1) *is non-degenerate, i.e., any ONB $\{m_i\}$ for the inclusion*

$$\mathcal{Q}_{-1} \overset{\mathcal{F}_1}{\subset} \mathcal{Q}_0$$

forms an ONB for the inclusion

$$\mathcal{P}_{-1} \overset{\mathcal{F}_1}{\subset} \mathcal{P}_0$$

(*equivalently,* $\overline{\mathrm{sp}}(\mathcal{Q}_0 \mathcal{P}_{-1}) = \mathcal{P}_0$).

Then all the commuting squares in (4.1.2) *are non-degenerate. In particular, the index of*

$$\mathcal{M}_{-1} \overset{\widehat{\mathcal{F}}}{\subset} \mathcal{M}_0$$

is given by $\sum m_j m_j^*$.

(c) *Assume that*

$$\begin{array}{ccccc} \mathcal{P}_{-1}^{-1} \subset \mathcal{P}_0^{-1} & & \mathcal{P}_{-1}^{0} \subset \mathcal{P}_0^{0} \\ \cup \qquad\quad \cup & \subset & \cup \qquad\quad \cup \\ \mathcal{Q}_{-1}^{-1} \subset \mathcal{Q}_0^{-1} & & \mathcal{Q}_{-1}^{0} \subset \mathcal{Q}_0^{0} \end{array}$$

is a non-degenerate inclusion of non-degenerate commuting squares (non-degeneracy here means that all of the 6 *commuting squares obtained by combining the inclusions of* $\mathcal{P}_i^j$ *and* $\mathcal{Q}_i^j$, *are non-degenerate). Set*

$$\mathcal{M}_i^j = \mathcal{P}_j^i *_{\mathcal{Q}_j^i} (\mathcal{Q}_j^i \otimes Q).$$

Then

$$\begin{array}{ccc} \mathcal{M}_{-1}^{0} & \subset & \mathcal{M}_0^{0} \\ \cup & & \cup \\ \mathcal{M}_{-1}^{-1} & \subset & \mathcal{M}_0^{-1} \end{array}$$

is again a non-degenerate commuting square.

Proof. (a) Note that the algebra generated by $\mathcal{P}_{-1}$ and Q inside $\mathcal{M}_0$ is isomorphic to $\mathcal{M}_1$; this is because Q and $\mathcal{P}_{-1}$ are free with amalgamation over $\mathcal{Q}_{-1}$, and the conditional expectations involved in the amalgamated free products are faithful.

(b) By the non-degeneracy and commuting square condition, an orthonormal basis $\{m_i\}$ for $\mathcal{Q}_{-1} \subset \mathcal{Q}_0$ "pulls out" to become an orthonormal basis for $\mathcal{M}_{-1} \subset \mathcal{M}_0$.

(c) By arguing as in part (a), we get the vertical inclusions in

$$\begin{array}{ccc} \mathcal{M}_{-1}^{0} & \subset & \mathcal{M}_0^{0} \\ \cup & & \cup \\ \mathcal{M}_{-1}^{-1} & \subset & \mathcal{M}_0^{-1}. \end{array}$$

Using the commuting square conditions and non-degeneracy, we see that an ONB for $\mathcal{M}_{-1}^{-1} \subset \mathcal{M}_0^{-1}$ (coming from an ONB for $\mathcal{Q}_{-1}^{-1} \subset \mathcal{Q}_0^{-1}$) is an ONB for $\mathcal{M}_{-1}^{0} \subset \mathcal{M}_0^{0}$. □

We now turn to the algebras constructed in [P3].

THEOREM 4.2. *Let* $\mathcal{G}$ *be a standard lattice, and let*

$$\mathcal{C}_{\mathcal{G}} = \begin{array}{ccc} \mathcal{A}_{-1}^{0} & \subset & \mathcal{A}_0^{0} \\ \cup & & \cup \\ \mathcal{A}_{-1}^{-1} & \subset & \mathcal{A}_0^{-1} \end{array}$$

be the commuting square associated to $\mathcal{G}$ *in Theorem* 2.10, *and let* p_0 *be the canonical projection in* $\mathcal{A}_{-1}^{-1} \subset \mathcal{P}_{-1}$. *Let* Q *be a tracial von Neumann algebra with diffuse center. Consider the inclusion of algebras*

$$\mathcal{P}_{-1} = \mathcal{A}_{-1}^{0} *_{\mathcal{A}_{-1}^{-1}} (Q \otimes \mathcal{A}_{-1}^{-1}) \subset \mathcal{P}_0 = \mathcal{A}_0^{0} *_{\mathcal{A}_0^{-1}} (Q \otimes \mathcal{A}_0^{-1}). \tag{4.2.1}$$

Then the inclusion

$$p_0\mathcal{P}_{-1}p_0 \subset p_0\mathcal{P}_0 p_0 \tag{4.2.2}$$

is isomorphic to the inclusion constructed in [P3].

Proof. Let us denote by $\mathcal{C}_0^I$ the bottom sequence of commuting squares in $\mathcal{C}_0$, i.e., corresponding to the case $n=0$ in Notation 2.2:

$$\begin{array}{ccccccccc} A_{-2,-1} & \subset & A_{-2,0} & \subset & \dots & \subset & A_{-2,k} & \subset & \dots \\ \cup & & \cup & & & & \cup & & \\ A_{-1,-1} & \subset & A_{-1,0} & \subset & \dots & \subset & A_{-1,k} & \subset & \dots . \end{array}$$

Note that this sequence of commuting squares coincides with the standard lattice associated to the subfactor $M_{-2}\subset M_{-1}$, which by duality is isomorphic to the standard lattice associated to $M_0\subset M_1$.

Also, denote by $\mathcal{C}^I$ the bottom sequence of commuting squares in $\mathcal{C}$:

$$\begin{array}{ccccccccc} \mathcal{A}_{-1}^0 & \subset & \mathcal{A}_0^0 & \subset & \dots & \subset & \mathcal{A}_k^0 & \subset & \dots \\ \cup & & \cup & & & & \cup & & \\ \mathcal{A}_{-1}^{-1} & \subset & \mathcal{A}_0^{-1} & \subset & \dots & \subset & \mathcal{A}_k^{-1} & \subset & \dots . \end{array}$$

Finally, denote by $\mathcal{A}_\infty^k=\overline{\bigcup_n \mathcal{A}_n^k}$, for $k=-1,0$, the closure being taken with respect to the strong topology implemented by the semifinite trace Tr on $\bigcup_n \mathcal{A}_n^0$.

Note that by Lemma 2.5, $\mathcal{C}_0^I$ is naturally isomorphic to $p_0\mathcal{C}^I p_0$, via $\widetilde{\alpha}_0$. Let us denote by $\widetilde{\alpha}$ this (trace-preserving) isomorphism. Thus we have

$$A_{-2,\infty}\subset A_{-1,\infty} \overset{\widetilde{\alpha}}{\simeq} (p_0\mathcal{A}_\infty^{-1}p_0\subset p_0\mathcal{A}_\infty^0 p_0)$$

as well. Also, by the irreducibility of the inclusion matrix for $\mathcal{A}^{-1}\subset\mathcal{A}_0^{-1}$ it follows that the central support of p_0 in $\mathcal{A}_\infty^{-1}$ is equal to 1. Thus, by Lemma 3.1, we have an isomorphism

$$(Q\otimes A_{-1,\infty} *_{A_{-1,\infty}} A_{-2,\infty}) \simeq p_0(Q\otimes\mathcal{A}_\infty^{-1} *_{\mathcal{A}_\infty^{-1}} \mathcal{A}_\infty^0)p_0,$$

that we still denote by $\widetilde{\alpha}$.

Moreover, inside of the algebra $p_0(Q\otimes\mathcal{A}_\infty^{-1} *_{\mathcal{A}_\infty^{-1}} \mathcal{A}_\infty^0)p_0$ we have the Jones tower of type II_1 factors

$$p_0(Q\otimes\mathcal{A}_{-1}^{-1} *_{\mathcal{A}_{-1}^{-1}} \mathcal{A}_{-1}^0)p_0 \subset p_0(Q\otimes\mathcal{A}_0^{-1} *_{\mathcal{A}_0^{-1}} \mathcal{A}_0^0)p_0 \subset \dots .$$

Denote by $M_1\subset M_0\subset\dots$ this Jones tower of type II_1 factors. Also, denote by $N_{-1}\subset N_0\subset\dots$ the Jones tower of factors constructed in [P3], [P5]. Thus, $N_\infty=Q\otimes A_{-1,\infty} *_{A_{-1,\infty}} A_{-2,\infty}$ and each of the factors N_k, $k\geqslant -1$, is defined as the smallest von Neumann subalgebra which contains Q as well as all the vector spaces $\Phi_k^n(Q \vee A_{-2,n})$, $n \geqslant k$, where Φ_k^n are the completely positive maps defined inside N_∞ out of the Jones projections, as in [P3], [P5].

Since $\widetilde{\alpha}(N_\infty) = M_\infty$ and $\widetilde{\alpha}$ takes Q onto Q, $A_{-2,n}$ onto $p_0 \mathcal{A}_n^0 p_0$, for all n, and Jones projections onto Jones projections, it follows that $\widetilde{\alpha}(N_k)$ is a subfactor inside M_k and that the system of inclusions

$$\begin{array}{ccccccccc} M_{-1} & \subset & M_0 & \subset & \dots & \subset & M_k & \subset & \dots \\ \cup & & \cup & & & & \cup & & \\ \widetilde{\alpha}(N_{-1}) & \subset & \widetilde{\alpha}(N_0) & \subset & \dots & \subset & \widetilde{\alpha}(N_k) & \subset & \dots \end{array}$$

has all squares commuting. Since $\widetilde{\alpha}(N_\infty) = M_\infty$, by commuting squares, the isomorphism $\widetilde{\alpha}$ takes N_k onto M_k, for all $k \geqslant -1$. □

We now have all the necessary ingredients to obtain the functorial constructions of subfactors in $L(\mathbf{F}_\infty)$. We denote by $\mathbf{G}$ the category whose objects are standard lattices. The morphisms in this category are by definition embeddings of standard lattices with the same index (i.e., embeddings of λ-lattices with the same λ), satisfying the smoothness condition (2.9.1).

THEOREM 4.3. *Let $\mathbf{G}$ be the category of standard lattices, with embeddings as morphisms. Let $\mathbf{S} = \mathbf{S}(L(\mathbf{F}_\infty))$ be the category of subfactors $(P_{-1} \subset P_0)$, $P_0 = L(\mathbf{F}_\infty)$, $P_{-1} \cong P_0$ of $L(\mathbf{F}_\infty)$ with morphisms $\imath: (P_{-1} \subset P_0) \to (Q_{-1} \subset Q_0)$ given by non-degenerate commuting square inclusions*

$$\begin{array}{ccc} Q_{-1} & \subset & Q_0 \\ \cup & & \cup \\ P_{-1} & \subset & P_0. \end{array}$$

Denote by $\mathcal{G}$ the functor $\mathcal{G}: \mathbf{S} \to \mathbf{G}$ assigning to an inclusion its standard lattice,

$$\mathcal{G}(P_{-1} \subset P_0) = \mathcal{G}_{\mathcal{P}_{-1} \subset P_0}.$$

Then there exists a functor $\mathcal{F}: \mathbf{G} \to \mathbf{S}$ which is a right inverse for $\mathcal{G}$:

$$\mathcal{G} \circ \mathcal{F} = \mathrm{id}.$$

Proof. We give two proofs to this theorem.

For the first proof, let $Q = L(\mathbf{F}_\infty)$ and define $\mathcal{F}(\mathcal{G})$ to be the inclusion (4.2.2) constructed in Theorem 4.2. By Theorem 4.2,

$$\mathcal{G} = \mathcal{G}_{\mathcal{P}_{-1} \subset \mathcal{P}_0} = \mathcal{G}(\mathcal{F}(\mathcal{G})),$$

so that $\mathcal{F}$ is the right inverse to $\mathcal{G}$. Moreover, by Proposition 4.1, $\mathcal{F}$ has the proper functorial properties. In fact, now that we have established that the construction of the subfactor $\mathcal{F}(\mathcal{G})$ coincides with the construction of subfactors in [P3], [P5], the functoriality of $\mathcal{F}$ also follows from the functoriality of the construction in those papers.

It remains to show that $p_0\mathcal{P}_i p_0 \cong L(\mathbf{F}_\infty)$, or, equivalently, that $\mathcal{P}_i \cong L(\mathbf{F}_\infty)\otimes B(H)$. Recall that $\mathcal{P}_i$ are given as amalgamated free products (4.2.1), with $Q = L(\mathbf{F}_\infty)$. Thus by Theorem 3.16, $\mathcal{P}_i \cong L(\mathbf{F}_\infty)\otimes B(H)$. This ends the first proof.

Now, for the second proof of the theorem, for a given standard λ-lattice $\mathcal{G}$ consider the λ-Markov commuting square

$$\mathcal{C}_{\mathcal{G}}^{\mathrm{st}} = \begin{array}{ccc} \mathcal{B}_{-1} & \subset & \mathcal{B}_0 \\ \cup & & \cup \\ \mathcal{A}_{-1}^{-1} & \subset & \mathcal{A}_0^{-1} \end{array}$$

as in Theorems 2.9 and 2.10. Recall that $\mathcal{A}_{-1}^{-1}$, $\mathcal{A}_0^{-1}$ are type I von Neumann algebras with discrete center. Moreover, each one of the algebras $\mathcal{B}_{-1}$, $\mathcal{B}_0$ is isomorphic to an algebra of the form $R_0\otimes B(H)$, where R_0 is hyperfinite of type II_1 (possibly with non-trivial center). Let $Q = L(\mathbf{F}_\infty)$. Denote by $\mathcal{F}(\mathcal{G})$ the compression of the inclusion

$$(Q\otimes\mathcal{A}_{-1}^{-1}) *_{\mathcal{A}_{-1}^{-1}} \mathcal{B}_{-1} \subset (Q\otimes\mathcal{A}_0^{-1}) *_{\mathcal{A}_0^{-1}} \mathcal{B}_0$$

to the canonical trace 1 projection in $\mathcal{A}_{-1}^{-1}$ (denoted by p_0 in Theorem 2.9). By Proposition 4.1, we get that the standard lattice of this inclusion is $\mathcal{G}$, i.e., $\mathcal{G}\circ\mathcal{F}(\mathcal{G}) = \mathcal{G}$. Proposition 4.1 implies that $\mathcal{F}$ is a functor between the categories **G** and **S**. Theorem 3.3 implies that each of the algebras involved is isomorphic to an algebra of the form $(R_0 * L(\mathbf{F}_\infty))\otimes B(H) \cong L(\mathbf{F}_\infty)\otimes B(H)$, where R is hyperfinite of type II_1 (the last isomorphism follows from the results of Dykema [D1]). It follows that the compressed inclusion $\mathcal{F}(\mathcal{G})$ consists of algebras isomorphic to $L(\mathbf{F}_\infty)$. □

COROLLARY 4.4. *Let $\mathcal{G}$ be any standard lattice. Then there exists an inclusion $P_{-1}\subset P_0$ having $\mathcal{G}$ as its system of higher relative commutants, and so that $P_{-1}\cong P_0\cong L(\mathbf{F}_\infty)$.*

We now describe some further universal properties of $L(\mathbf{F}_\infty)$.

THEOREM 4.5. *Let $M_{-1}\subset M_0$ be an inclusion of II_1 factors with finite index. Then there exists an inclusion $\widehat{M}_{-1}\subset\widehat{M}_0$ functorially associated to $M_{-1}\subset M_0$, with $\widehat{M}_{-1}\cong M_{-1} * L(\mathbf{F}_\infty)$, $\widehat{M}_0\cong M_0 * L(\mathbf{F}_\infty)$, so that $\widehat{M}_{-1}\subset\widehat{M}_0$ has the same index and the same standard lattice of higher relative commutants as $M_{-1}\subset M_0$.*

Proof. By Theorems 2.9 and 2.10, there exists a non-degenerate commuting square

$$\begin{array}{ccc} M_{-1}\otimes B(H) & \subset & M_0\otimes B(H) \\ \cup & & \cup \\ \mathcal{A}_{-1}^{-1} & \subset & \mathcal{A}_0^{-1} \end{array}$$

with $\mathcal{A}_{-1}^{-1}$ and $\mathcal{A}_0^{-1}$ type I with discrete centers. Letting

$$\mathcal{M}_{-1} = (M_{-1} \otimes B(H)) *_{\mathcal{A}_{-1}^{-1}} (L(\mathbf{F}_\infty \otimes \mathcal{A}_{-1}^{-1})),$$
$$\mathcal{M}_0 = (M_0 \otimes B(H)) *_{\mathcal{A}_0^{-1}} (L(\mathbf{F}_\infty \otimes \mathcal{A}_0^{-1})),$$

we see that $\mathcal{M}_{-1} \subset \mathcal{M}_0$ has the same higher relative commutants as $M_{-1} \subset M_0$. Compressing by a finite projection and noticing that in view of Theorem 3.3,

$$\mathcal{M}_{-1} \cong (M_{-1} * L(\mathbf{F}_\infty)) \otimes B(H),$$
$$\mathcal{M}_0 \cong (M_0 * L(\mathbf{F}_\infty)) \otimes B(H),$$

gives the result. □

References

[D1] Dykema, K., Free products of hyperfinite von Neumann algebras and free dimension. *Duke Math. J.*, 69 (1993), 97–119.

[D2] — Interpolated free group factors. *Pacific J. Math.*, 163 (1994), 123–135.

[D3] — Amalgamated free products of multi-matrix algebras and a construction of subfactors of a free group factor. *Amer. J. Math.*, 117 (1995), 1555–1602.

[DR] Dykema, K. & Rădulescu, F., Compressions of free products of von Neumann algebras. *Math. Ann.*, 316 (2000), 61–82.

[J1] Jones, V. F. R., Index for subfactors. *Invent. Math.*, 72 (1983), 1–25.

[J2] — Planar algebras. Preprint, Berkeley, 1999.

[P1] Popa, S., Markov traces on universal Jones algebras and subfactors of finite index. *Invent. Math.*, 111 (1993), 375–405.

[P2] — Classification of amenable subfactors of type II. *Acta Math.*, 172 (1994), 163–255.

[P3] — An axiomatization of the lattice of higher relative commutants of a subfactor. *Invent. Math.*, 120 (1995), 427–445.

[P4] — *Classification of Subfactors and Their Endomorphisms.* CBMS Regional Conf. Ser. in Math., 86. Amer. Math. Soc., Providence, RI, 1995.

[P5] — Universal construction of subfactors. *J. Reine Angew. Math.*, 543 (2002), 39–81.

[R1] Rădulescu, F., A one-parameter group of automorphisms of $\mathcal{L}(\mathbf{F}_\infty) \otimes B(H)$ scaling the trace. *C. R. Acad. Sci. Paris Sér. I Math.*, 314 (1992), 1027–1032.

[R2] — Random matrices, amalgamated free products and subfactors of the von Neumann algebra of a free group, of noninteger index. *Invent. Math.*, 115 (1994), 347–389.

[S1] Shlyakhtenko, D., Free quasi-free states. *Pacific J. Math.*, 177 (1997), 329–368.

[S2] — Some applications of freeness with amalgamation. *J. Reine Angew. Math.*, 500 (1998), 191–212.

[S3] — *A*-valued semicircular systems. *J. Funct. Anal.*, 166 (1999), 1–47.

[SU] Shlyakhtenko, D. & Ueda, Y., Irreducible subfactors of $L(\mathbf{F}_\infty)$ of index $\lambda > 4$. *J. Reine Angew. Math.*, 548 (2002), 149–166.

[V] Voiculescu, D., Circular and semicircular systems and free product factors, in *Operator Algebras, Unitary Representations, Enveloping Algebras, and Invariant Theory* (Paris, 1989), pp. 45–60. Progr. Math., 92. Birkhäuser Boston, Boston, MA, 1990.

[VDN] Voiculescu, D., Dykema, K. & Nica, A., *Free Random Variables.* CRM Monogr. Ser., 1. Amer. Math. Soc., Providence, RI, 1992.

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