A classification of Busemann G-surfaces which possess convex functions

by

NOBUHIRO INNAMI

University of Tsukuba, Sakura-mura Ibaraki, Japan

1. Introduction

A function φ defined on a complete Riemannian manifold M without boundary is said to be convex if φ is a one variable convex function on each arc-length parametrized geodesic. φ is locally Lipschitz continuous and hence continuous on M. It is a natural question to ask to what extent the existence of a convex function on M implies restrictions to the topology of M. In a recent work [4], the topology of M with locally nonconstant convex functions has been studied in detail. One of their results gives a classification theorem of 2-dimensional complete Riemannian manifolds which admit locally nonconstant convex functions: they are diffeomorphic to either a plane, a cylinder, or an open Möbius strip.

A classical result of Cohn-Vossen [3] states that a complete noncompact Riemannian 2-dimensional manifolds with nonnegative Gaussian curvature is homeomorphic to a plane, a cylinder, or an open Möbius strip. Moreover, Cheeger–Gromoll have proved in [2] that if a complete noncompact Riemannian manifold has nonnegative sectional curvature, then every Busemann function on it is convex (and locally nonconstant).

H. Busemann generalized Cohn-Vossen's result in [1] pp. 292–294, proving that a noncompact G-surface with finite connectivity and zero excess whose angular measure is uniform at π is topologically a plane, a cylinder, or a Möbius strip.

Now, the purpose of the present paper is to prove the following:

THEOREM 3.13. *Let R be a noncompact 2-dimensional G-space. If R admits a locally nonconstant convex function, then R is homeomorphic to either a plane, a cylinder $S^1 \times \mathbf{R}$, or an open Möbius strip.*

It should be noted that in the proof of the above result there is no analogy with the Riemannian case. This is because every point of a G-space R does not in general have convex balls around it. And, hence, for every closed convex set C of a G-space R and for every point $x \in R-C$ which is sufficiently close to C, we cannot conclude the uniqueness of a segment which connects x to a point on C, and whose length realizes the distance between x and C. It should also be noted that a convex function on a G-space R is in general not necessarily continuous. But in the case where $\dim R=2$, R is a topological manifold and every convex function on it is locally Lipschitz continuous.

In § 2 we shall give the definition and some basic notions for G-spaces which are used in this paper. They are found in the book of H. Busemann, [1]. In § 3 we shall discuss G-surfaces which possess locally nonconstant convex functions.

The author would like to express his hearty thanks to Professor K. Shiohama for his advice, encouragement and careful reading of the original manuscript.

2. G-spaces

Let R be a metric space and let pq denote the distance between points p and q on R. Let (pqr) denote that p, q and r are mutually distinct and $pq+qr=pr$; let $S(p, \varrho)$ denote the set $\{q; qp<\varrho\}$, which is called the (open) sphere with center p and radius ϱ. The axioms for a G-space R are:

(1) The space is a symmetric metric space with distance $pq=qp$.

(2) The space is finitely compact, i.e., a bounded infinite set has at least one accumulation point.

(3) The space is (Menger) convex, i.e., for given two distinct points p and r, a point q with (pqr) exists.

(4) To every point x of the space there corresponds $\varrho_x>0$ such that for any two distinct points p and q in $S(x, \varrho_x)$ a point r with (pqr) exists. (Axiom of local prolongation.)

(5) If (pqr_1), (pqr_2) and $qr_1=qr_2$, then $r_1=r_2$. (Axiom of uniqueness of prolongation.)

The axioms insure the existence of a (continuous) curve connecting two given points p and q whose length is equal to the distance between them, and this curve is called a segment and denoted by $T(p, q)$. If an r with (pqr) exists, then the segment $T(p, q)$ is unique. If $p, q \in S(r, \varrho)$ for some r in R, then $T(p, q) \subset S(r, 2\varrho)$. Let $\varrho(p)$ be the least upper bound of those ϱ_x which satisfy Axiom (4). Then either $\varrho(p)=\infty$ for all p or $0<\varrho(p)<\infty$ and $|\varrho(p)-\varrho(q)| \leq pq$, which implies the continuity of the function ϱ on R.

A geodesic g is a certain class of mappings of the real line into R which is locally a segment, i.e., g has a representation $x(\tau)$, $-\infty<\tau<\infty$, such that for every τ_0 there exists an $\varepsilon(\tau_0)>0$ such that $x(\tau_1)x(\tau_2)=|\tau_1-\tau_2|$ for $|\tau_0-\tau_i|\leq\varepsilon(\tau_0)$, $i=1$, 2, and for another representation $y(\tau)$, $-\infty<\tau<\infty$, there exist $\alpha=\pm1$, $\beta\in\mathbf{R}$ which satisfy that $x(\tau)=y(\alpha\tau+\beta)$ for all τ. If a representation of a geodesic is a globally isometric map of $\mathbf{R}$ into R, or of a plane circle into R, then we call it a straight line or a great circle, respectively. In the proof of our results the following properties of geodesics will be often used.

(2.1) If $y(\tau), \alpha\leq\tau\leq\beta, \alpha<\beta$, represents a segment in a G-space, then there is a unique representation $x(\tau)$, $-\infty<\tau<\infty$, of a geodesic such that $x(\tau)=y(\tau)$ for $\alpha\leq\tau\leq\beta$.

(2.2) If $x_\nu(\tau)$, $-\infty<\tau<\infty$, represents a geodesic, $\nu=1,2,\ldots$, and the sequence $\{x_\nu(\tau_0)\}$ is bounded, then $\{x_\nu(\tau)\}$ contains a subsequence $\{x_\lambda(\tau)\}$ which converges (uniformly in every bounded set of $\mathbf{R}$) to a representation $x(\tau)$, $-\infty<\tau<\infty$, of a geodesic.

(2.3) A class of homotopic closed curves through p which is not contractible contains a geodesic loop (a piece of a geodesic) with endpoint p.

We shall use the notion of dimension in the sense defined by Menger and Urysohn.

(2.4) A G-space of dimension 2 is a topological manifold. $S(p, \varrho(p))$ is homeomorphic to an open sphere in the plane $\mathbf{R}^2$.

(2.5) Every point of a 2-dimensional G-space is an interior point of a closed and of an open convex set whose boundary consists of three segments, i.e., a triangle, where a convex set C means that $p, q\in C$ implies that $T(p, q)$ exists uniquely and is contained in C.

3. *G*-surfaces which possess locally nonconstant convex functions

Let R be a 2-dimensional G-space and φ be a convex function on R. This means that for each geodesic with a representation $x(\tau)$, $-\infty<\tau<\infty, \varphi$ satisfies the inequality:

$$\varphi(x(\lambda\tau_1+(1-\lambda)\tau_2))\leq\lambda\varphi(x(\tau_1))+(1-\lambda)\,\varphi(x(\tau_2)),$$

for any $\lambda\in[0, 1]$, and for any τ_1, $\tau_2\in\mathbf{R}$.

The plane, cylinder and open Möbius strip with canonical metric evidently possess (locally nonconstant) convex functions. As we are interested in the topological structure of R, we may assume that a convex function φ is locally nonconstant, i.e., nonconstant on each open set of R. If a non-trivial convex function φ is constant on an open set $U\subset R$, then we can construct from R a topologically distinct R' on which a non-trivial convex function is defined. This is done as follows; There is a disk $D \subset U$. R' is obtained via the connected sum $(R-D)\# V$, where V is an arbitrary G-space with boundary S^1 which is identified with ∂D. The convex function on R' is equal to φ on $R'-V$ and is constant on V, and agree with φ on ∂D. Thus the existence of a non-trivial convex function does not imply a topological restriction on the G-space except a trivial one, namely, noncompactness (see [4]). Throughout this section, let φ be locally nonconstant on R. And let R^a and R_a^b denote the sets $\{q \in R; \varphi(q) \leq a\}$ and $\{q \in R; a \leq \varphi(q) \leq b\}$, respectively.

(3.1) φ is locally Lipschitz continuous on R.

Proof. We first show that φ is locally bounded above. Let q_1, q_2 and q_3 be the vertices of the convex triangle C mentioned in (2.5), and let $p' \in \operatorname{Int} C$. Choose q on $T(q_2, q_3)$ such that $p' \in T(q_1, q)$. Then by convexity of φ, we have

$$\varphi(p') \leq (p'q/q_1q)\,\varphi(q_1) + (q_1p'/q_1q)\,\varphi(q) \leq (p'q/q_1q)\,\varphi(q_1) + (q_1p'/q_1q)\,((qq_3/q_2q_3)\,\varphi(q_2) + (q_2q/q_2q_3)\,\varphi(q_3)).$$

Therefore $\varphi(p') \leq \max\{\varphi(q_1), \varphi(q_2), \varphi(q_3)\}$.

Secondly, we show that φ is locally bounded. Let $S(p, \varrho) \subset C$, $\varrho \leq \varrho(p)$, and let $q \in S(p, \varrho)$. Then by convexity of φ, we have

$$\varphi(p) \leq \tfrac{1}{2}(\varphi(q) + \varphi(q')),$$

where q' satisfies that (qpq') and $pq = pq'$. Hence $\varphi(q) \geq 2\varphi(p) - \max\{\varphi(q_1), \varphi(q_2), \varphi(q_3)\}$ for $q \in S(p, \varrho)$. Thus φ is locally bounded.

In order to prove the local Lipschitz continuity of φ, we work in the above $S(p, \varrho)$. Let $u, v \in S(p, \varrho/3)$. Extend $T(u, v)$ in both directions until its endpoints arrive at $\partial S(p, \varrho)$. Take points u_1, u_2, v_2 and v_1 in this extension of $T(u, v)$ such that u_1, u_2, u, v, v_2 and v_1 are in that order and $u_1u_2 = v_2v_1 = \varrho/3$ and $u_1, v_1 \in \partial S(p, \varrho)$. Then by convexity of φ,

$$(\varphi(u_2) - \varphi(u_1))/u_1u_2 \leq (\varphi(v) - \varphi(u))/uv \leq (\varphi(v_1) - \varphi(v_2))/v_2v_1.$$

Hence there is an $L > 0$ such that $|\varphi(v) - \varphi(u)| \leq Luv$ for u, $v \in S(p, \varrho/3)$. Thus φ is locally Lipschitz continuous.

(3.2) R_a^a, $a > \inf_R \varphi$, has the structure of an embedded 1-dimensional topological submanifold without boundary.

Proof. Let $p \in R_a^a$. There is q such that $q \in S(p, \varrho(p))$ and $\varphi(q) < \varphi(p)$. Take r on an extension of $T(p, q)$ such that $\varphi(r) > \varphi(p)$ and $r \in S(p, \varrho(p))$. Let T' be a segment through r and contained in $S(p, \varrho(p))$ and which intersects the extension of $T(p, q)$ at exactly r and on which $\varphi > \varphi(p)$. Then $T(q, q') \cap R_a^a$ is exactly one point for every $q' \in T'$, because φ is strictly monotone increasing along $T(q, q') \cap (R - R^{\varphi(q)})$, and the totality of those points is homeomorphic to T'. Hence this set is a neighborhood of p in R_a^a, and it has no selfintersections.

We conclude from (3.2) that R_a^a, $a > \inf_R \varphi$ is homeomorphic to either a real line $\mathbf{R}$ or a circle S^1.

(3.3) R is noncompact.

Concerning the number of components of a level set R_a^a, $a > \inf_R \varphi$, of φ, the following holds.

PROPOSITION 3.4. *Let p and q be distinct points of R_a^a, $a > \inf_R \varphi$. If there is a geodesic curve from p to q such that φ does not take $\inf_R \varphi$ on it, then p and q are contained in the same component of R_a^a.*

Proof. Let $x(\tau)$, $\alpha \leq \tau \leq \beta$, represent the geodesic curve in the assumption. If $\min \varphi(x(\tau)) = a$, then $\varphi(x(\tau)) = a$ for every τ, $\alpha \leq \tau \leq \beta$, by convexity of φ. So p and q are contained in the same component of R. Thus we may assume, without loss of generality, that there exists τ_0 such that $\varphi(x(\tau_0)) = \min \varphi(x(\tau)) < a$. Since $\varphi(x(\tau_0)) > \inf_R \varphi$, we can choose r such that $r \in S(x(\tau_0), \varrho(x(\tau_0))/3)$, $\varphi(r) < \varphi(x(\tau_0))$. Put $\alpha' := \max \{\alpha, \tau_0 - \varrho(x(\tau_0))/3\}$, $\beta' := \min \{\beta, \tau_0 + \varrho(x(\tau_0))/3\}$, and $m := \min \{(\varphi(x(\tau)) - \varphi(r))/rx(\tau);\ \alpha' \leq \tau \leq \beta'\}$. The choice of r implies $m > 0$. For each τ, $\alpha' \leq \tau \leq \beta'$, there is exactly one representation $y'_\tau(v)$, $-\infty < v < \infty$, of a geodesic by (2.1) which satisfies that $y'_\tau(0) = r$, and $y'_\tau(rx(\tau)) = x(\tau)$. Then we have; for every $v \geq rx(\tau)$,

$$m(v - rx(\tau)) + \varphi(x(\tau)) \leq ((\varphi(x(\tau)) - \varphi(r))/rx(\tau))\,(v - rx(\tau)) + \varphi(x(\tau)) \leq \varphi(y'_\tau(v)).$$

Since $y'_\tau(v)$, $rx(\tau) \leq v < \varrho(x(\tau_0))/3 + rx(\tau)$, is contained in $S(x(\tau_0), 2\varrho(x(\tau_0))/3)$ for each $\tau, \alpha' \leq \tau \leq \beta'$, there is a v such that $y'_\tau(v) \in S(x(\tau_0), \varrho(x(\tau_0)))$, and $m\varrho(x(\tau_0))/3 + \varphi(x(\tau)) \leq \varphi(y'_\tau(v))$.

Let $\varepsilon(\tau)$, $\alpha' \leq \tau \leq \beta'$, be a continuous function which satisfies that $\varepsilon(\alpha') = \varepsilon(\beta') = 0$, and $0 < \varepsilon(\tau) < m\varrho(x(\tau_0))/3$ for any τ, $\alpha' \leq \tau \leq \beta'$. Convexity of φ implies that the geodesic curve with a representation $y'_\tau(v)$, $v \geq 0$, intersects $R_{\varphi(x(\tau))+\varepsilon(\tau)}^{\varphi(x(\tau))+\varepsilon(\tau)}$ at exactly one point, which is denoted by $y(\tau)$. We are going to see that $y(\tau)$, $\alpha' \leq \tau \leq \beta'$, is a continuous curve such that

$y(\alpha')=x(\alpha')$ and $y(\beta')=x(\beta')$ and $y(\tau)\in S(x(\tau_0), \varrho(x(\tau_0)))$. Let a sequence $\{\tau_i\}$ tend to $\tau, \alpha'\leqslant\tau\leqslant\beta'$. Since $\{y(\tau_i)\}$ is bounded, the sequence $\{y(\tau_i)\}$ contains a subsequence $\{y(\tau_k)\}$ which converges to a point y_0. Then, since the equality $\varphi(y_0)=\varphi(x(\tau))+\varepsilon(\tau)$ follows from continuity of ε and φ, y_0 belongs to $R^{\varphi(x(\tau))+\varepsilon(\tau)}_{\varphi(x(\tau))+\varepsilon(\tau)}$. On the other hand, y_0 is on the geodesic curve with a representation $y'_\tau(\nu)$, $-\infty<\nu<\infty$, since the sequence $\{y'_{\tau_i}(\nu)\}$ of representations of geodesics converges to $y'_\tau(\nu)$, $-\infty<\nu<\infty$, by (2.2) when τ_i tends to τ. Therefore we have from definition of $y(\tau)$ that $y_0=y(\tau)$.

Next, for each $\tau, \alpha'\leqslant\tau\leqslant\beta'$, let $z'_\tau(\nu)$, $-\infty<\nu<\infty$, be a representation of a geodesic which satisfies that $z'(0)=x(\tau_0)$ and $z'_\tau(x(\tau_0)\,y(\tau))=y(\tau)$. That this is well-defined follows from the fact that $y(\tau)\in S(x(\tau_0)), \varrho(x(\tau_0)))$. From construction of $z'_\tau(\nu)$ we see that $z'_{\alpha'}(\nu)=x(\tau_0-\nu), z'_{\beta'}(\nu)=x(\tau_0+\nu)$, and hence each of them has a unique intersection with R^a_a. The desired curve from p to q in R^a_a is obtained as follows. From construction of $z'_\tau(\nu), \alpha'\leqslant\tau\leqslant\beta', \nu\geqslant 0$, we see that $\varphi(z'_{\alpha'}(\nu))$ and $\varphi(z'_{\beta'}(\nu))$ are monotone non-decreasing for $\nu\geqslant 0$, and moreover $\varphi(z'_\tau(\nu)), \alpha'<\tau<\beta'$, is strictly monotone increasing for $\nu\geqslant x(\tau_0)\,y(\tau)$. Thus for each $\tau, \alpha'\leqslant\tau\leqslant\beta', z'_\tau(\nu), \nu\geqslant 0$ has a unique intersection with R^a_a, which denoted by $z(\tau)$, and the intersection is continuous with τ. In fact, to prove it around $\tau=\alpha'$, fix $p'=z'_{\alpha'}(\tau_0-\alpha+1)$. Then convexity of φ along $z'_{\alpha'}(\nu), \nu\geqslant 0$, implies that $\varphi(p')>a$. There is a nieghborhood of p' on which $\varphi>a$. Therefore we find a $\delta_1>0$ such that $z'_\tau(\tau_0-\alpha+1)$ is in the neighborhood if $|\tau-\alpha'|<\delta_1$. Then continuity of $z(\tau), \alpha'\leqslant\tau<\alpha'+\delta_1$, is obvious. In the same way we find a $\delta_2>0$ such that $z(\tau), \beta'-\delta_2<\tau\leqslant\beta'$, is continuous. To prove continuity of $z(\tau), \alpha'+\delta_1\leqslant\tau\leqslant\beta'-\delta_2$, put $m_1:=\inf\{\varphi(y(\tau))-\varphi(x(\tau_0))/x(\tau_0)\,y(\tau);\ \alpha'+\delta_1\leqslant\tau\leqslant\beta'-\delta_2\}$. Then we can see that $m_1>0$ and that for each $\tau, \alpha'+\delta_1\leqslant\tau\leqslant\beta'-\delta_2$, for every $\nu\geqslant x(\tau_0)\,y(\tau)$,

$$m_1(\nu-x(\tau_0)\,y(\tau))+\varphi(y(\tau))\leqslant((\varphi(y(\tau))-\varphi(x(\tau_0)))/x(\tau_0)\,y(\tau))\ (\nu-x(\tau_0)\,y(\tau))+\varphi(y(\tau))\leqslant\varphi(z'_\tau(\nu)).$$

Thus the set $\{z(\tau);\ \alpha'+\delta_1\leqslant\tau\leqslant\beta'-\delta_2\}$ is bounded. Continuity of $z(\tau), \alpha'\leqslant\tau\leqslant\beta'$, holds by means of the same argument as in continuity of $y(\tau)$.

As a direct consequence of the proof of Proposition 3.4, we have:

(3.5) If p and q are taken from different components of R^a_a, then φ attains $\inf_R\varphi$ on every geodesic curve which joins p and q and $\inf_R\varphi$ is attained at exactly one point on it.

THEOREM 3.6. *If there is a value a such that R^a_a is not connected, then the following holds:*

(1) *φ attains* $\inf_R \varphi$.

(2) $R^{\min\varphi}_{\min\varphi}$ *is totally convex and it is either a straight line or a great circle.*

(3) $R-R^{\min\varphi}_{\min\varphi}$ *consists of two components. If* $b>\inf_R \varphi$, *then* R^b_b *has exactly two components.*

Proof. (1) has been proved. In (2), total convexity of $R^{\min\varphi}_{\min\varphi}$ is trivial. If $\partial R^{\min\varphi}_{\min\varphi}=\emptyset$ as a 1-dimensional manifold, then it follows from total convexity of $R^{\min\varphi}_{\min\varphi}$ and [1], p. 46 (9.6) that $R^{\min\varphi}_{\min\varphi}$ is either a straight line or a great circle. If $\partial R^{\min\varphi}_{\min\varphi}\neq\emptyset$ or $R^{\min\varphi}_{\min\varphi}$ has only one point then we can see that R^a_a is connected, a contradiction. In fact, we can prove this as follows. Take points p and q in R^a_a and r in $\partial R^{\min\varphi}_{\min\varphi}$, and join from r to p, and from r to q by segments $T(r, p)$ and $T(r, q)$. Since $S(r, \varrho(r))$ is not separated by $R^{\min\varphi}_{\min\varphi}$, we get a continuous curve $y(\tau), \alpha\leq\tau\leq\beta$, joining $\partial S(r, \varrho(r)/2)\cap T(r, p)$ and $\partial S(r, \varrho(r)/2)\cap T(r, q)$ which is contained in $S(r, \varrho(r))$ and does not intersect $R^{\min\varphi}_{\min\varphi}$. If for each τ, $\alpha\leq\tau\leq\beta$, $z(\tau)$ is defined by the intersection of a geodesic curve in the direction from r to $y(\tau)$ and R^a_a as in the proof of Proposition 3.4, then $z(\tau), \alpha\leq\tau\leq\beta$, is a continuous curve joining p and q.

To prove (3), fix a point p of $R^{\min\varphi}_{\min\varphi}$. Then by (2), $S(p, \varrho(p)/2)-R^{\min\varphi}_{\min\varphi}$ has exactly two components. We denote its components by V_1 and V_2. For each $q\in R-R^{\min\varphi}_{\min\varphi}$, let $x(\tau), 0\leq\tau\leq\alpha, x(0)=p, x(\alpha)=q$, be a representation of a geodesic curve from p to q. Then $x(\tau), 0<\tau<\varrho(p)/2$, is contained in only one of V_1 and V_2. Put $A:=\{q\in R-R^{\min\varphi}_{\min\varphi}$; all geodesic curves from p to q on which sufficiently small parts near p intersect $V_1\}$, $B:=\{q\in R-R^{\min\varphi}_{\min\varphi}$; all geodesic curves from p to q on which sufficiently small parts near p intersect $V_2\}$ and $C:=\{q\in R-R^{\min\varphi}_{\min\varphi}$; there are geodesic curves from p to q such that one of their representations, $y'(\tau), 0\leq\tau\leq\beta, y'(0)=p, y'(\beta)=q$, intersects V_1 and another $z'(\tau), 0\leq\tau\leq\gamma, z'(0)=p, z'(\gamma)=q$, intersects $V_2\}$.

Both A and B are open and connected in $R-R^{\min\varphi}_{\min\varphi}$ if they are nonempty. If we show that $A\cup B=R-R^{\min\varphi}_{\min\varphi}$, i.e., $C=\emptyset$, then the first part of (3) will be proved. In fact, if $C=\emptyset$, $V_1\subset A$ and $V_2\subset B$ will follow from the argument stated below.

Suppose $q\in C$ exists. Then we have a contradiction from the following considerations. Fix a point $q_0=y(\tau_0)$ such that $\varphi(q_0)=a$, where $y(\tau), -\infty<\tau<\infty$, is a representation of a geodesic determined by $y'(\tau), 0\leq\tau\leq\beta$, in the definition of C. And let $z''(\tau)$, $0\leq\tau\leq|\tau_0-\beta|+\gamma$, be a continuous curve from p to q_0 such that if $\tau_0\geq\beta$ then $z''(\tau)=z'(\tau)$ for $0\leq\tau\leq\gamma$ and $z''(\tau)=y(\tau+\beta-\gamma)$ for $\gamma\leq\tau\leq\tau_0-\beta+\gamma$ and if $\tau_0<\beta$ then $z''(\tau)=z'(\tau)$ for $0\leq\tau\leq\gamma$ and $z''(\tau)=y(\beta-\tau+\gamma)$ for $\gamma\leq\tau\leq\beta-\tau_0+\gamma$, where $z'(\tau), 0\leq\tau\leq\gamma$, is in the definition of C. We consider the class of all curves from p to q_0 whose interiors are in $R-R^{\min\varphi}_{\min\varphi}$

and such that sufficiently small parts near p meet V_2 but do not meet V_1. This class is nonempty. Since $R^{\min\varphi}_{\min\varphi}$ is totally convex and is either a great circle or a straight line, we find from (5.18), [1], p. 25, a geodesic curve from p to q_0, in the class, whose interior is contained in $R-R^{\min\varphi}_{\min\varphi}$ and which is different from a geodesic curve with a representation $y(\tau), 0\leqq\tau\leqq\beta$. Let $z(\tau), 0\leqq\tau\leqq\delta, z(0)=p, z(\delta)=q_0$, represent this geodesic curve. Using these representations $y(\tau), 0\leqq\tau\leqq\beta$, and $z(\tau), 0\leqq\tau\leqq\delta$, we connect any two points q_1 and q_2 in R^a_a by a continuous curve in R^a_a, a contradiction. This is done as follows: If $y_1(\tau), 0\leqq\tau\leqq\beta_1$, and $y_2(\tau), 0\leqq\tau\leqq\beta_2$, represents geodesic curves such that $y_1(0)=y_2(0)=p$ and $y_1(\beta_1)=q_1, y_2(\beta_2)=q_2$ and if $y_i(\tau), 0<\tau<\varrho(p)/2$, $i=1,2$, are contained in V_1, then we can connect $y_1(\varrho(p)/3)$ and $y_2(\varrho(p)/3)$ by a continuous curve in V_1 and, hence, as in Proposition 3.4, we find a curve in R^a_a which joins q_1 and q_2. Thus, without loss of generality, we may consider that $y_i(\tau), 0<\tau<\varrho(p)/2$, are contained in V_i for $i=1,2$. By the same idea as in Proposition 3.4, we can find two curves in R^a_a such that one of them joins q_1 to q_0 and the other joins q_2 to q_0. Thus R^a_a is connected, a contradiction. Hence $C=\emptyset$ is proved.

The above arguments show that $V_1\subset A$ and $V_2\subset B$, and hence they are not empty. Therefore $R-R^{\min\varphi}_{\min\varphi}$ is the disjoint union of A and B.

To prove the second part of (3) it is enough to see that both $R^b_b\cap A$ and $R^b_b\cap B$ are connected for any $b>\inf_R\varphi$. This is evident by the above argument.

To continue our investigation, we need the notion of an *end* ε which is, by definition, an assigment to each compact set K in R a component $\varepsilon(K)$ of $R-K$ in such a way that $\varepsilon(K_1)\supset\varepsilon(K_2)$ if $K_1\subset K_2$.

THEOREM 3.7. *If there is a compact component of a level set of φ, then all level sets are compact.*

Proof. Theorem 3.6 implies that every level set consists of one or two components. So we first consider the case where all level sets are connected.

Let R^a_a be compact. And suppose that R^b_b is noncompact for some b with $a<b$. We fix a point p in R^a_a and choose an unbounded sequence $\{q_i\}, q_i\in R^b_b$. Let $x_i(\tau), \tau\geqq0$, be a representation of a geodesic curve such that $x_i(0)=p$ and $x_i(pq_i)=q_i, i=1,2,\ldots$. Then we have a subsequence $\{x_k(\tau)\}$ of $\{x_i(\tau)\}$ which converges to a representation $x(\tau), \tau\geqq0$, of a geodesic curve. If we see that $\varphi(x(\tau))=a$ for any $\tau\geqq0$, then R^a_a is noncompact since $x(\tau)$, $\tau\geqq0$, represents a half-straight line. This is a contradiction. Thus, if R^a_a is compact then R^b_b is compact for all $b>a$. It remains to prove that $\varphi(x(\tau))=a$ for every $\tau\geqq0$. For each $\tau \geqslant$ the diameter of R_a^a, there is a number k_0 such that $pq_k > \tau$ for $k \geqslant k_0$. For such k_0, it follows from convexity of φ that $k \geqslant k_0$ implies $\varphi(x_k(\tau)) \leqslant b$. Therefore, we have:

$$\varphi(x(\tau)) = \varphi(\lim x_k(\tau)) = \lim \varphi(x_k(\tau)) \leqslant b.$$

On the other hand, since $\tau \geqslant$ the diameter of R_a^a, the following holds: $\varphi(x(\tau)) = \varphi(\lim x_k(\tau)) = \lim \varphi(x_k(\tau)) \geqslant a$. $\varphi(x(\tau))$, the diameter of $R_a^a \leqslant \tau < \infty$, is bounded and monotone non-decreasing, so φ is constant on it. Therefore, it follows from convexity of φ that $\varphi(x(\tau)) = a$ for $\tau \geqslant 0$.

Suppose that R_a^a is noncompact and R_b^b compact for some $a < b$. In this case, there are at least two ends of R because of the existence of a straight line intersecting R_b^b along which φ is nonconstant and monotone. In particular R is not simply connected. Under the assumption stated above, we claim that φ does not take $\inf_R \varphi$. In fact, suppose φ takes $\inf_R \varphi$. Then the minimum set is noncompact, otherwise all level sets are compact because of the above argument. Thus the minimum set consists of either a half-straight line or a straight line. Since R is not simply connected there is a non-null homotopy class of closed curves with any fixed point $p \in R_{\min \varphi}^{\min \varphi}$ as a base point. Then we get a geodesic loop at p by (2.3). Along this geodesic loop, φ is constant because $\varphi(x(\tau)) \leqslant \varphi(p)$, where $x(\tau), 0 \leqslant \tau \leqslant \alpha$, is a representation of this geodesic loop. Since this geodesic loop is contained neither in that half-straight line nor in that straight line, there is an open set U in the neighborhood of p such that φ is constant on U. This contradicts local nonconstancy of φ. Thus we can suppose that φ does not take $\inf_R \varphi$.

Now, we are going to obtain a final contradiction in this case. Put $t_0 := \inf\{t \in \mathbf{R};\ R_t^t$ is compact$\}$. If we prove that R^{t_0} is homeomorphic to the closed half-plane, which is proved in Proposition 3.8, then as R is not simply connected we have a geodesic loop with endpoint p where $\varphi(p) < t_0$. This geodesic loop must intersect $R_{t_0}^{t_0}$. Thus this contradicts convexity of φ.

Next we consider the case where there is a level set which is not connected. In this case it follows from Theorem 3.6 that φ takes $\inf_R \varphi$ and the minimum set is either a straight line or a great circle.

If the minimum set is a great circle, all level sets are compact by the same reason as we have already shown that R_b^b is compact if so is R_a^a for $a \leqslant b$.

In the case where the minimum set is a straight line, each component of any level set is noncompact. In fact, suppose that there is a compact component of some level. Then R has at least two ends and hence it is not simply connected. Then for every

$p\in R^{\min\varphi}_{\min\varphi}$ there is a goodesic loop at p which is not homotopic to a point curve. Thus the geodesic loop at p must lie in the minimum set which is a straight line, a contradiction. The proof is complete for all cases.

The following proposition is directly used in the proof of Theorem 3.7. Once we establish Theorem 3.7, we may ınd by the same reasoning as Proposition 3.8 that R^b_a, $b>a>\inf_R\varphi$, is topologically a part of a cylinder, $S^1\times[a,b]$, if R^b_b is compact.

PROPOSITION 3.8. *If there is a value* $b>\inf_R\varphi$ *such that* R^b_b *is noncompact then there exists a homeomorphism* h *of* $\mathbf{R}\times[a,b]$, $a>\inf_R\varphi$, *onto each component of* R^b_a *such that* $\varphi\circ h(u,v)=v$ *for every* $(u,v)\in\mathbf{R}\times[a,b]$.

It should be noted that each level set R^c_c, $a\leqslant c\leqslant b$, has a neighborhood which is homeomorphic to the union of triangles of $\mathbf{R}^2$ since, from the same idea as in (3.2), R^c_c is covered by triangles whose interiors are mutually disjoint. Hence our aim is to alter the above homeomorphism to globally satisfy the condition.

Proof. We know as in the first part of the proof of Theorem 3.7 that every c, $c\leqslant b$, R^c_c is noncompact and so homeomorphic to $\mathbf{R}$. Fix a value d, $\inf_R\varphi<d<a$, and choose an unbounded sequence $\{p_i\}_{-\infty<i<\infty}$ in R^b_b in that order of an orientation of R^b_b in both directions of R^b_b. For each i, $-\infty<i<\infty$, let f_i be a point in R^d_d such that $p_if_i=p_iR^d$. Then for each c, $a\leqslant c\leqslant b$, $T(p_i,f_i)\cap R^c_c$ is exactly one point which we denote by $p_i(c)$. Clearly $p_i(b)=p_i$ for every i, $-\infty<i<\infty$. And for every c, $a\leqslant c\leqslant b$, $\{p_i(c)\}_{-\infty<i<\infty}$ is unbounded in both directions of R^c_c. Otherwise, there exists a half-straight line starting at a point of R^d_d on which φ is bounded and which intersects R^c_c. Moreover $\{p_i(c)\}_{-\infty<i<\infty}$ is in this order. This is proved as follows. Let W be a nieghborhood of $T(p_0,f_0)$. $W\cap R^b_a$ is separated by $T(p_0,f_0)$ into two components W_{-1} and W_1. Let F be a function of R^b_a to $\{-1,0,1\}$ which satisfies that if $q\in T(p_0(b),p_0(a))$ then $F(q)=0$, if $q\in R^b_a-T(p_0(b),p_0(a))$, $\varphi(q)=c$ and the subarc of R^c_c from $p_0(c)$ to q meet W_{-1} then $F(q)=-1$, and otherwise $F(q)=1$. By the remark above the proof, F is continuous except on $T(p_0(b),p_0(a))$, and hence for every c, $a\leqslant c\leqslant b$, and for every integer $k\geqslant 1$, $p_k(c)$ is in only one of two sides of $T(p_0(b),p_0(a))$. Also for $k\leqslant -1$, $p_k(c)$ is in the other side. Since this fact is true for each i, $-\infty<i<\infty$, $\{p_i(c)\}_{-\infty<i<\infty}$ is in this order on R^c_c for every c, $a\leqslant c\leqslant b$.

Let M_i denote the set which is surrounded by R^b_b, R^a_a, $T(p_i(b),p_i(a))$ and $T(p_{i+1}(b)$,

$p_{i+1}(a)$), more precisely, the totality of the subarcs of R_c^c's from $p_i(c)$ to $p_{i+1}(c)$, for all c, $a \leq c \leq b$.

It suffices to construct a homeomorphism h_i of the domain $\{(u, v) \in \mathbf{R}^2;\ i \leq u \leq i+1, a \leq v \leq b\}$ onto M_i such that $\varphi \circ h_i(u, v) = v$. Because if we connect h_i's we get h. This is done as follows. Put $\varrho := \min\{\varrho(q)/2;\ q \in M_i \cap R_b^b\}$. If we consider a neighborhood $U := \{q;\ qR_b^b < \varrho\}$ of $M_i \cap R_b^b$, then there is an $\varepsilon > 0$ and $b' < b$ such that $M_i \cap R_{b'-\varepsilon}^b \subset U$. Choose $s_j \in R_b^b \cap U$ and $r_j \in R_{b'-\varepsilon}^{b'-\varepsilon} \cap U$, $j = 0, 1, \ldots, n$, in this order, such that $s_0 = p_i(b)$, $s_n = p_{i+1}(b)$, $r_0 = p_i(b'-\varepsilon)$ and $r_n = p_{i+1}(b'-\varepsilon)$ and s_j, s_{j+1}, r_j, $r_{j+1} \in S(t_j, \varrho/2)$ for some $t_j \in R_b^b, j = 0, 1, \ldots, n-1$. For each j, $0 \leq j \leq n-1$, let M_{ij} be the domain which is surrounded by $R_b^b, R_{b'}^{b'}$, $T(s_j, r_j)$ and $T(s_{j+1}, r_j)$ and let M'_{ij} be the domain which is surrounded by $T(s_{j+1}, r_j)$, $T(s_{j+1}, r_{j+1})$ and $R_{b'}^{b'}$. Then, we can construct (by the same techniques as (3.2) and Proposition 3.4) a homeomorphism of M_{ij} onto the trapezoid in $\mathbf{R}^2$ whose vertices are $(i+j/n, b)$, $(i+j/n, b')$, $(i+(j+1)/n, b)$ and $(i+j/n+\varepsilon/n(b-b'+\varepsilon), b')$, and a homeomorphism of M'_{ij} onto the triangle in $\mathbf{R}^2$ whose vertices are $(i+(j+1)/n, b)$, $(i+j/n+\varepsilon/n(b-b'+\varepsilon), b')$ and $(i+(j+1)/n, b')$. The homeomorphisms agree on the segment $T(r_j, s_{j+1}) \cap R_{b'}^b$. If we connect these homeomorphisms we get a homeomorphism $(h'_i)^{-1}$ of $M_i \cap R_{b'}^b$ to $\{(u, v) \in \mathbf{R}^2;\ i \leq u \leq i+1,\ b' \leq v \leq b\}$ which satisfies $\varphi \circ h'_i(u, v) = v$ for all $u \in [i, i+1]$. We do not know whether M_i is compact, so the desired homeomorphism is obtained as follows. Let b_0 be the most lower bound of $\{b';\ R_{b'}^b \cap M_i$ has a homeomorphism $(h'_i)^{-1}$ which satisfies the condition$\}$. Then $b_0 = a$. Otherwise we can construct by the same way as above a homeomorphism $(h'_i)^{-1}$ of $R_{b'}^b \cap M_i$, for some $b' < b_0$, which satisfies the condition, a contradiction.

Clearly h_i^{-1} and h_{i+1}^{-1} agree on the segment $T(p_{i+1}(b), p_{i+1}(a))$, so we obtain the desired homeomorphism h^{-1} of R_a^b onto $\{(u, v) \in \mathbf{R}^2;\ -\infty < u < \infty, a \leq v \leq b\}$ after connecting h_i^{-1} and h_{i+1}^{-1} for all i, $-\infty < i < \infty$.

We shall observe how the existence of a locally nonconstant convex function on R will restrict the number of ends of R.

LEMMA 3.9. *If there is a compact level set of φ, then R has at most two ends.*

Proof. Suppose that R has more than two ends. Then there is a compact set K such that $R-K$ contains exactly three unbounded connected components U_1, U_2 and U_3. We will prove that φ is bounded above on two of the U_1, U_2 and U_3. Therefore φ is bounded above on exactly two of them since φ is not bounded above. In order to see this, we may suppose that $\sup \varphi(U_1) = \sup \varphi(U_2) = \infty$. Then we can find such a high level set that it does not intersect K but intersects U_1 and U_2. This implies that this level set is not connected. Therefore φ is bounded above on U_3 since Theorem 3.6 says that all level sets except the minimum set consist of exactly two components. Choose a point p in the minimum set and an unbounded sequence $\{q_i\}$ in U_3. Let $x_i(\tau), \tau \geqslant 0$, represents a geodesic curve, $i=1,2,\ldots$, which satisfies that $x_i(0)=p$ and $x_i(pq_i)=q_i$. Because of (2.2) there is a subsequence $\{x_k(\tau)\}$ of $\{x_i(\tau)\}$ which converges to a representation $x(\tau), \tau \geqslant 0$, of a geodesic curve. In the same way as in the proof of Theorem 3.7, we see that $\{x(\tau); \tau \geqslant 0\}$ belongs to the minimum set. Since $x(\tau), \tau \geqslant 0$, represents a half-straight line, this contradicts the compactness of level sets.

Thus we may suppose, without loss of generality, that φ is bounded above on U_2 and U_3. If we put $m := \min \varphi(K)$, then $m = \inf_R \varphi$. In fact, if we suppose that there exists a point q with $\varphi(q) < m$, then we can find a half-straight line on which φ is constant equal to $\varphi(q)$ or non-increasing in the same way as above argument after taking an unbounded sequence $\{q_i\}$ contained in U_j, $j=2$ or 3, which does not contain q. However, since this half-straight line intersects K, this is impossible.

Let $p \in K$ satisfy that $\varphi(p)=m$. Then there is a half-straight line emanating from p, and an unbounded subarc of which lies in $K \cup U_2$. But φ is constant, equal to m, on the half-straight line. Thus R_m^m is noncompact, contradicting Theorem 3.7.

LEMMA 3.10. *Suppose there exists a noncompact level set. If R has more than one end, then φ attains* $\inf_R \varphi$ *and the minimum set intersects every $\varepsilon(K)$, where ε is an arbitrary end and K is any compact set of R.*

It turns out from Proposition 3.12 that if φ has a noncompact level then R has exactly one end. But this lemma gives a step of the proof of Proposition 3.12.

Proof. From assumption there exist two ends ε_1 and ε_2 and a compact set K which satisfies that $\varepsilon_1(K)$ and $\varepsilon_2(K)$ are distinct components of $R-K$. Put $a := \min \varphi(K)$, $b := \max \varphi(K)$. We will find that $a := \inf_R \varphi$. Otherwise there is a point p such that $\varphi(p) < a$. We may suppose, without loss of generality, that $p \notin \varepsilon_1(K)$. If φ is bounded on $\varepsilon_1(K)$, then making use of p and an unbounded sequence in $\varepsilon_1(K)$, in the same way as before, we obtain a representation $x(\tau)$, $\tau \geqslant 0$, of a geodesic curve intersecting K and on which φ is constant. This contradicts the choice of p and a.

The above argument also shows that every $\varepsilon(K)$ intersects the minimum set if $R-K$ has at least two unbounded components.

Let K_1 be any compact set. Then there is a compact set $K \supset K_1$ such that every

$\varepsilon(K)$ intersects the minimum set, and hence so does $\varepsilon(K_1)$. Thus the proof of the final statement is complete.

THEOREM 3.11. *If R is a noncompact G-surface which admits a locally nonconstant convex function, then the number of ends is at most two.*

Proof. Suppose that the number of ends of R is not less than three. Then Lemma 3.9 says that all level sets are noncompact and Lemma 3.10 concludes that φ takes $\inf_R \varphi$ and the minimum set intersects every $\varepsilon(K)$. Since the noncompact minimum set is a half-straight line or a straight line, it cannot intersect more than two $\varepsilon(K)$'s. This is a contradiction.

Now we will classify G-surfaces which admit locally nonconstant convex functions. First we consider the case where R has two ends.

PROPOSITION 3.12. *If R has two ends, then each component of each level set is homeomorphic to a circle S^1 and R is homeomorphic to a cylinder $S^1 \times \mathbf{R}$.*

Proof. In the first step we will see that all level sets are compact. Suppose that there is a noncompact level set. Lemma 3.10 and two ends of R imply that φ takes $\inf_R \varphi$ and the minimum set is a straight line. Since R has two ends there is a geodesic loop whose endpoint is contained in the minimum set and which is not homotopic to a point curve, a contradiction. We know the existence of a homeomorphism of R to a cylinder from the remark above Proposition 3.8.

In the case that φ does not take $\inf_R \varphi$, then all level sets are connected. Therefore each level set intersects a straight line at one point which connects two ends. Thus R is topologically a cylinder by the remark above Proposition 3.8.

Next we claim that if φ takes $\inf_R \varphi$, then the minimum set is a great circle. Otherwise, since it is a point or a segment, all level sets are connected. The existence of two ends implies that there is a compact set K such that $R-K$ consists of exactly two unbounded components, so φ is bounded above on one of the components of $R-K$. Thus, in the same way as in the proof of Theorem 3.7, we can derive a contradiction, namely the minimum set contains a half-straight line. It turns out at the same time from this consideration that there exist level sets which are not connected. Therefore each component of a level set intersects a straight line at one point which passes through the minimum set and connect two ends.

Now we can conclude the following:

THEOREM 3.13. *If R is a noncompact G-surface which admits a locally nonconstant convex function, then R is homeomorphic to either a plane, a cylinder $S^1 \times \mathbf{R}$, or an open Möbius strip.*

Proof. The case where R has two ends has already been treated in Proposition 3.12. We may suppose by Theorem 3.11 that R has one end. First we will prove that if there is a compact level set, φ takes $\inf_R \varphi$. In fact, suppose that φ does not take $\inf_R \varphi$. Then we can produce a straight line through a certain compact level set by choosing two sequences $\{q_i\}$ and $\{q_i'\}$ which satisfy that $\lim \varphi(q_i) = \inf_R \varphi$, and $\lim \varphi(q_i') = \infty$, and connecting q_i and q_i' by a segment. Hence R has at least two ends, a contradiction. Therefore, the minimum set of φ is either a point, a segment, or a great circle. Since the detailed construction of homeomorphisms is the same as Proposition 3.8, we need only to see how to map a level set.

When the minimum set is a point, we map it to the origin of canonical plane $\mathbf{R}^2$ and R_a^a, $a > \min \varphi$, onto a circle in $\mathbf{R}^2$ with center $(0, 0)$ and radius $a - \min \varphi$.

When the minimum set is a segment, we map it to a segment T in canonical plane $\mathbf{R}^2$ and R_a^a, $a > \min \varphi$, onto the set $\{w \in \mathbf{R}^2;\ wT = a - \min \varphi\}$ in $\mathbf{R}^2$.

If the minimum set is a great circle, we map it to the shortest great circle T in canonical open Möbius strip M and R_a^a, $a > \min \varphi$, onto the set $\{w \in M;\ wT = a - \min \varphi\}$ in M.

Next we consider the case where all level sets are noncompact. If φ does not take $\inf_R \varphi$, Proposition 3.8 implies that R_a^a, $a > \inf_R \varphi$, is mapped onto the set $\{(u, v) \in \mathbf{R}^2;\ -\infty < u < \infty,\ v = a\}$ in canonical plane $\mathbf{R}^2$ and R is topologically a plane. Hence we examine the case where φ takes $\inf_R \varphi$.

When the minimum set is a half-straight line (a straight line), we map it to a half-straight line T (a straight line T') in canonical plane $\mathbf{R}^2$ and R_a^a, $a > \min \varphi$, onto the set $\{w \in \mathbf{R}^2;\ wT = a - \min \varphi\}$ ($\{w \in \mathbf{R}^2;\ wT' = a - \min \varphi\}$) in $\mathbf{R}^2$.

Finally, if the minimum set is a great circle, then R is topologically an open Möbius strip. Otherwise R is topologically a plane $\mathbf{R}^2$.

References

[1] BUSEMAN, H., *The geometry of geodosics*. Academic Press, New York, 1955.

[2] CHEEGER, J. & GROMOLL, D., On the structure of complete manifolds of nonnegative curvature. *Ann. of Math.*, 96 (1972), 415–443.

[3] COHN-VOSSEN, S., Kürzeste Wege und Totalkrummung auf Flächen. *Compositio Math.*, 2 (1935), 63–133.

[4] GREENE, R. E. & SHIOHAMA, K., Convex functions on complete noncompact manifolds: Topological structure. *Inventiones Math.*, 63 (1981), 129–157.
[5] ROBERTS-VERBERG, *Convex functions*. Academic Press, New York, 1973.
[6] THORBERGSSON, G., Closed geodesics on non-compact Riemannian manifolds. *Math. Z.*, 159 (1978), 245–258.

Received February 2, 1981.