Acta Math., 179 (1997), 1–39

Hausdorff dimension and Kleinian groups

by

CHRISTOPHER J. BISHOP and PETER W. JONES

SUNY at Stony Brook
Stony Brook, NY, U.S.A.

Yale University
New Haven, CT, U.S.A.

1. Statement of results

Consider a group G of Möbius transformations acting on the 2-sphere S^2. Such a group G also acts as isometries on the hyperbolic 3-ball $\mathbf{B}$. The limit set, $\Lambda(G)$, is the accumulation set (on S^2) of the orbit of the origin in $\mathbf{B}$. We say the group is discrete if it is discrete as a subgroup of $\mathrm{PSL}(2, \mathbf{C})$ (i.e., if the identity element is isolated). The ordinary set of G, $\Omega(G)$, is the subset of S^2 where G acts discontinuously, i.e., $\Omega(G)$ is the set of points z such that there exists a disk around z which hits itself only finitely often under the action of G. If G is discrete, then $\Omega(G)=S^2\setminus\Lambda(G)$. G is called a Kleinian group if it is discrete and $\Omega(G)$ is non-empty (some sources permit $\Lambda=S^2$ in the definition of Kleinian group, but our results are easier to state by omitting it). The limit set $\Lambda(G)$ has either 0, 1, 2 or infinitely many points and G is called elementary if $\Lambda(G)$ is finite.

The *Poincaré exponent* (or *critical exponent*) of the group is

$$\delta(G)=\inf\left\{s:\sum_G \exp(-s\varrho(0,g(0)))<\infty\right\},$$

where ϱ is the hyperbolic metric in $\mathbf{B}^3$. A point $x\in\Lambda(G)$ is called a *conical limit* point if there is a sequence of orbit points which converges to x inside a (Euclidean) non-tangential cone with vertex at x (such points are sometimes called radial limit points or points of approximation). The set of such points is denoted $\Lambda_c(G)$. G is called *geometrically finite* if there is a finite-sided fundamental polyhedron for G's action on $\mathbf{B}$ and *geometrically infinite* otherwise. A result of Beardon and Maskit [6] says that G is geometrically finite if and only if $\Lambda(G)$ is the union of $\Lambda_c(G)$, the rank 2 parabolic fixed points and doubly cusped rank 1 parabolic fixed points of G. This makes it clear that $\dim(\Lambda_c)=\dim(\Lambda)$ and $\mathrm{area}(\Lambda)=0$ in the geometrically finite case.

The first author is partially supported by NSF Grant DMS-92-04092 and an Alfred P. Sloan research fellowship. The second author is partially supported by NSF Grant DMS-92-13595.

For any Kleinian group $R=\Omega(G)/G$ is a (branched) Riemann surface. If R is of finite type (i.e, a finite union of compact surfaces with at most finitely many punctures and branch points), then G is called *analytically finite*. These are the groups with which we will work. By the Ahlfors finiteness theorem ([2], [8]), any finitely generated group is analytically finite, so our results hold for finitely generated groups. This case is slightly easier because by Selberg's lemma any finitely generated discrete matrix group contains a finite index subgroup without torsion. This subgroup must have the same limit set as the original group, so for finitely generated groups it will always be sufficient to assume that G has no torsion.

In this paper "circle" will always refer to Euclidean circles or lines (e.g., circles on the sphere). Similarly the terms "disk" or "ball" will always denote spherical balls. The main results of this paper are the following.

THEOREM 1.1. *If G is a non-elementary, discrete Möbius group on* $\mathbf{B}$ *then* $\delta(G)=\dim(\Lambda_c(G))$.

THEOREM 1.2. *If G is an analytically finite Kleinian group which is geometrically infinite then* $\dim(\Lambda(G))=2$.

Theorem 1.1 uses nothing but the definitions and a few simple properties of Möbius transformations. The direction $\dim(\Lambda_c(G))\leqslant\delta(G)$ is easy and well known. We have only been able to locate the opposite inequality in the literature under the additional assumption that G is geometrically finite or Fuchsian (e.g., see [57], [60], [49] and [34]). More detailed information about the conical limit points can be found in [64] and [65]. Theorem 1.2 can be sharpened in some cases, e.g., in [12] we show that the limit set has positive Hausdorff measure with respect to the function $t^2\sqrt{\log 1/t \log\log\log 1/t}$ if we assume that the injectivity radius of $M=\mathbf{B}/G$ is bounded away from zero.

We should also note that our proof of Theorem 1.1 works for any discrete group of Möbius transformations acting on the hyperbolic ball in any dimension and in the rank 1 symmetric space case in general. In [25] Corlette proved the rank 1 case of Theorem 1.1 for geometrically finite groups without cusps and in [26] Corlette and Iozzi prove it for rank 1 geometrically finite groups with cusps. The case of finite-volume, rank 1 spaces is considered in [3], [4], [27].

It is well known that $x\in\Lambda_c$ if and only if the geodesic ray from $0\in\mathbf{B}$ to x corresponds to a geodesic on $M=\mathbf{B}/G$ which returns to some compact subset of M infinitely often. Our proof shows that for any $\varepsilon>0$ there is a subset of $\Lambda_c(G)$ of dimension $\geqslant\delta(G)-\varepsilon$ which corresponds to geodesics which never leave the ball of radius $R(\varepsilon)$ around x. Thus if M is a hyperbolic manifold with finitely generated fundamental group and $x\in M$, then the set of directions corresponding to geodesic rays starting at x which have compact closure has dimension $\delta(G)$. This had been proven for geometrically finite Kleinian groups by Fernández and Melián [34] and by Stratmann [54]. Fernández and Melián also show in their paper that this holds for all Fuchsian groups.

Theorem 1.2 was previously known in special cases. Examples of groups with $\dim(\Lambda(G))=2$ were constructed by Sullivan in [58], and Canary [20] proved that Theorem 1.2 holds if $M=\mathbf{B}/G$ is a "topologically tame" manifold such that the thin parts have bounded type (in particular, if the injectivity radius is bounded away from zero). Our result shows these extra hypotheses are unnecessary.

Sullivan [60] and Tukia [63] independently showed that if G is a geometrically finite group then $\dim(\Lambda(G))<2$. Thus Theorem 1.2 implies

COROLLARY 1.3. *An analytically finite Kleinian group is geometrically finite if and only if* $\dim(\Lambda(G))<2$.

The proof of Theorem 1.2 divides into two cases. First, if $\delta(G)=2$, it follows immediately from Theorem 2.1. Secondly, if $\delta(G)<2$ and G is geometrically infinite we will show that $\operatorname{area}(\Lambda(G))>0$. Since it is known that $\delta(G)=\dim(\Lambda(G))$ for geometrically finite groups, we also obtain

COROLLARY 1.4. *Suppose that G is a non-elementary, analytically finite Kleinian group and that* $\operatorname{area}(\Lambda(G))=0$. *Then* $\delta(G)=\dim(\Lambda)$.

For geometrically finite groups Stratmann and Urbanski [55] proved that the Hausdorff and Minkowski dimensions of Λ agree. Since Theorem 1.2 clearly implies this for geometrically infinite groups we get

COROLLARY 1.5. *If G is an analytically finite Kleinian group then the Minkowski dimension of Λ exists and equals the Hausdorff dimension.*

For alternate proofs of this result which do not need Theorem 1.2, see [10] and [11]. Another corollary of Theorem 1.2 is the following.

THEOREM 1.6. *If $\{G_n\}$ is a sequence of N-generated Kleinian groups which converges algebraically to G then* $\dim(\Lambda(G))\leqslant\liminf_n \dim(\Lambda(G_n))$.

Using the proof of Theorem 2.1 it is easy to see that $\delta(G)$ is lower semi-continuous under algebraic convergence, so using Corollary 1.4, the only case that causes problems is when the $\{G_n\}$ are geometrically finite and G is geometrically infinite with positive area limit set.

As final applications of Theorem 1.2 we have:

COROLLARY 1.7. *Suppose that G is a finitely generated Kleinian group and Ω is a simply-connected, invariant component of $\Omega(G)$. Then* $\dim(\partial\Omega)=1$ *if and only if $\partial\Omega$ is a circle.*

COROLLARY 1.8. *If G is a finitely generated Kleinian group then its limit set is either totally disconnected, a circle or has Hausdorff dimension >1.*

These two results were previously known for geometrically finite groups (see [21]), and are clear for geometrically infinite groups since the limit set has dimension 2. This type of result was first formulated by Bowen [15] for quasi-Fuchsian groups with no parabolics. The geometrically finite, cocompact Kleinian group (also called "convex cocompact") case was proven by Sullivan [57] and by Braam [16]. See also [59], [49]. There are more elementary proofs of these results which apply to all analytically finite groups at once, and these are given in [12]. The proof given there also shows that in Corollary 1.7, $\delta(G)=1$ if and only if Λ is a circle. In particular, this implies that if G is analytically finite, but geometrically infinite, then $\delta(G)>1$.

The Ahlfors conjecture states that the limit set of a finitely generated discrete group of Möbius transformations is either the whole sphere or has zero area. Thus by Corollary 1.4 the Ahlfors conjecture implies $\delta(G)=2$ for any geometrically infinite group. We do not know an argument for the converse direction, but both results are known to be true for topologically tame groups, [20].

The elementary groups have to be excluded in Theorem 1.1 and Corollary 1.4 because a cyclic group consisting of parabolics has a one-point limit set, but $\delta(G)=\frac{1}{2}$. Also the invariance of Ω is necessary in Corollary 1.7; the boundary of a general component can have dimension strictly less than $\delta(G)$.

Very interesting pictures of limit sets can be found in several sources such as [17], [46], [50].

The remaining sections of this paper are organized as follows:

§2: We prove Theorem 1.1 and deduce from the proof that $\delta(G)$ is lower semi-continuous with respect to algebraic convergence.

§3: We collect some facts about the convex core of a hyperbolic 3-manifold that we will need in later sections.

§4: We record some facts about the heat kernel and prove an estimate on Green's function which we need to prove Theorem 1.2.

§5: We prove Theorem 1.2 (geometrically infinite implies that Λ has dimension 2).

§6: We prove Theorem 1.6 ($\dim(\Lambda)$ is lower semi-continuous).

§7: We deduce some corollaries of our results in the special case when the groups belong to $\overline{T(S)}$, the closure of the Teichmüller space of a finite-type surface S. For example,

$\dim(\Lambda(G))$ is a lower semi-continuous function on $\overline{T(S)}$ and is continuous everywhere except at the geometrically finite cusps, where it is discontinuous.

Many of the auxiliary results we use could probably be found in the literature (or easily deduced from the literature). We have included proofs of most of these results to try to make the paper as accessible to non-experts as possible and to emphasize the elementary nature of our approach.

We thank many people for helpful conversations and suggestions: Michael Anderson, Itai Benjamini, Mladen Bestvina, Dick Canary, Jay Jorgenson, Irwin Kra, Curt McMullen, Bernie Maskit, Yair Minsky, Yuval Peres and Bernd Stratmann among others. The first author particularly thanks Ed Taylor for explaining the results of [21] and for suggesting the problem of proving that the Hausdorff dimension of degenerate limit sets is strictly greater than 1. It was by considering this problem that we were led to the other results in this paper.

Originally this paper was circulated as a preprint which also contained the results of [12]. To improve the readability, the preprint was cut into two shorter papers for publication.

2. The conical limit set and critical exponent

First we recall the definition of Hausdorff dimension. Given an increasing function φ on $[0,\infty)$, we define

$$H^\delta_\varphi(E)=\inf\Big\{\sum\varphi(r_j):E\subset\bigcup_j D(x_j,r_j),\, r_j<\delta\Big\}$$

and

$$H_\varphi(E)=\lim_{\delta\to 0}H^\delta_\varphi(E).$$

This is the Hausdorff measure associated to φ. H^∞_φ is called the Hausdorff content. It is not a measure, but has the same null sets as H_α. When $\varphi(t)=t^\alpha$ we denote the measure H_φ by H_α and we define

$$\dim(E)=\inf\{\alpha:H_\alpha(E)=0\}.$$

For $\alpha=1$ we sometimes denote H_1 by l (for "length"). An upper bound for $\dim(E)$ can be produced by finding appropriate coverings of the set. We will be more interested in finding lower bounds. The usual idea is the mass distribution principle: construct a positive measure μ on E which satisfies $\mu(D(x,r))\leqslant Cr^\alpha$. This implies $\dim(E)\geqslant\alpha$ since for any covering of E we have

$$\sum_j r_j^\alpha\geqslant C^{-1}\sum_j\mu(D(x_j,r_j)\geqslant C^{-1}\mu(E)>0.$$

Taking the infimum over all covers gives

$$H_\alpha^\infty(E) > C^{-1}\mu(E) > 0,$$

which implies $\dim(E) \geqslant \alpha$.

Next we will prove that $\delta(G) = \dim(\Lambda_c)$ for any non-elementary, discrete Möbius group. This was previously known for geometrically finite Kleinian groups and all Fuchsian groups, and can be proven in many cases by considering the Patterson measure on the limit set (see [51], [52], [48], [56] and [60]). We shall also build a measure on the conical limit set, but ours is not a group-invariant construction. We do not assume that G is finitely generated (much less geometrically finite) and G is allowed to have torsion.

THEOREM 2.1. *Suppose that G is a discrete group of Möbius transformations with more than one limit point. Then $\delta(G) = \dim(\Lambda_c)$.*

Proof. If G has only two limit points, then it is easy to check that it is generated by a loxodromic (with the given fixed points) and a finite group of elliptics. Thus $\delta(G) = 0 = \dim(\Lambda_c)$. Therefore we may assume that G has more than two limit points (and hence infinitely many).

Let $\delta = \delta(G)$ be the critical exponent for the Poincaré series of G. The direction $\delta \geqslant \dim(\Lambda_c)$ is easy and well known (e.g., Corollary 4.4.3 of [49]). Briefly it goes as follows. Let $\mathcal{G} = G(0)$ denote the orbit of $0 \in \mathbf{B}$ under G. Fix a large number M and for each $g \in G$ let B_g be the Euclidean ball centered at $g(0)/|g(0)|$ (the radial projection of the orbit point onto the sphere) and radius $M(1-|g(0)|)$. Let E_M be the set of points which are in infinitely many of the balls B_g. Since

$$\sum_g \operatorname{diam}(B_g)^{\delta+\varepsilon} < \infty,$$

for any $\varepsilon > 0$, we see that $\dim(E_M) \leqslant \delta(G)$ for any M. On the other hand, any point of $\Lambda_c(G)$ is in E_M for some M. Thus $\dim(\Lambda_c(G)) \leqslant \delta(G)$, as desired.

To prove the other direction, $\delta \leqslant \dim(\Lambda_c)$, we will construct a subset $\mathcal{C} \subset \mathcal{G}$ of the orbit of 0. We will give $\mathcal{C}$ the structure of a tree with root at 0, i.e., we have $\mathcal{C} = \bigcup_{n=0}^\infty \mathcal{C}_n$ and each $z \in \mathcal{C}_n$, $n \geqslant 1$, will have a unique "parent" in $\mathcal{C}_{n-1}$ and a collection of "children" $\mathcal{C}(z) \subset \mathcal{C}_{n+1}$. We will show that we can construct $\mathcal{C}$ so that its boundary $E \subset S^2$ (i.e., the limit points of all sequences $\{z_n\} \subset \mathcal{C}$ where $z_{n+1} \in \mathcal{C}(z_n)$ for all n) is a subset of Λ_c and has dimension as close to δ as we wish. See Figure 2.1.

Let $B(z,r)$ denote the Euclidean ball of radius r around z. We will show that for any $\varepsilon > 0$ there are numbers $1 \leqslant C_0, N < \infty$ and points $\mathcal{C} \subset \mathcal{G}$ (as above) so that:

(1) If $w \in \mathcal{C}(z)$ then $w \in B(z, N(1-|z|))$.

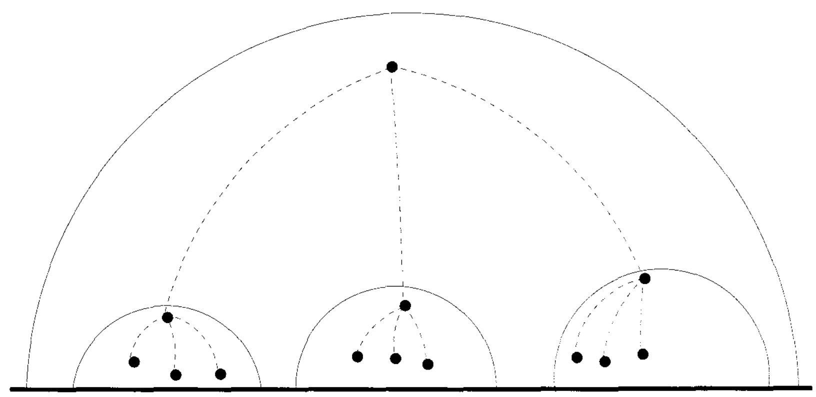

Fig. 2.1. The tree of orbit points

(2) If $w \in \mathcal{C}(z)$, then $C_0^{-1} \leqslant (1-|w|)/(1-|z|) \leqslant \frac{1}{2}$.

(3) For $w_1, w_2 \in \mathcal{C}(z)$ distinct, $B(w_1, 2N(1-|w_1|))$ and $B(w_2, 2N(1-|w_2|))$ are disjoint.

(4) $\sum_{w\in\mathcal{C}(z)}(1-|w|)^{\delta-2\varepsilon} \geqslant C_0^2(1-|z|)^{\delta-2\varepsilon}$.

First we will show how to construct such a collection of points. We will then show that the boundary of the tree on S^2 has dimension at least $\delta-2\varepsilon$ and lies inside Λ_c.

Let $\{z_n\}$ be an enumeration of the orbit of 0. Choose a point $x_0 \in S^2 = \partial\mathbf{B}$ so that

$$\sum_{j:|z_j-x|<r} (1-|z_j|)^{\delta-\varepsilon} = \infty,$$

for every $r>0$ (here $|z-x|$ denotes the Euclidean metric on $\overline{\mathbf{B}}$). We can do this by a simple compactness argument. Since G is non-elementary, x_0 is not fixed by every element of G. Therefore we can choose an element $g \in G$ so that $x_1 = g(x_0) \neq x_0$. Fix $r>0$ to be so small that the balls B_0, B_1 on S^2 (in the Euclidean metric) of radius r around the points x_0, x_1 have disjoint doubles.

Since B_0 and B_1 have disjoint doubles they are separated by some positive angle θ when viewed from the origin. In other words, two geodesic rays from 0 landing in B_0 and B_1 respectively must make an angle of at least θ at the origin. Using simple properties of hyperbolic geodesics one can check the following:

Property A. There is an $N<\infty$ (depending only on θ) such that if $z \neq 0$ then any geodesic ray starting at z and making angle at least $\frac{1}{2}\theta$ with the radial segment $[0, z]$ stays inside the (Euclidean) ball $B(z, N(1-|z|))$. This is the N which will work in conditions (1)–(4) above.

Property B. There is a K, $1 \leqslant K < \infty$ (also depending only on θ), so that if the geodesic segment from z to w makes angle at least $\frac{1}{2}\theta$ with $[0, z]$ and has hyperbolic length L then

$$1 \leqslant \frac{(1-|w|)e^L}{1-|z|} \leqslant K. \tag{2.1}$$

Note also that if $L > 2\log K$ then we have $1-|w| \leqslant \frac{1}{2}(1-|z|)$. Thus $C_0 = e^{-L}$ will work as the constant in (2).

Let $A_n = \{z \in \mathbf{B} : 2^{-n-1} \leqslant 1-|z| < 2^{-n}\}$. We claim that for $i=1,2$,

$$\limsup_{n\to\infty} \sum_{j:z_j\in B_i\cap A_n} (1-|z_j|)^{\delta-2\varepsilon} = \infty. \tag{2.2}$$

If not there would be an $M < \infty$ so that

$$\sum_{j:z_j\in B_i\cap A_n} (1-|z_j|)^{\delta-2\varepsilon} \leqslant M,$$

for all n, and hence we obtain the contradiction,

$$\sum_{j:z_j\in B_i} (1-|z_j|)^{\delta-\varepsilon} \leqslant C \sum_n 2^{-n\varepsilon} \sum_{j:z_j\in B_i\cap A_n} (1-|z_j|)^{\delta-2\varepsilon} \leqslant CM \sum_n 2^{-n\varepsilon} < \infty.$$

We can easily restate (2.2) as

$$\limsup_{n\to\infty} \frac{\#(z_j \in B_i \cap A_n)}{2^{n(\delta-2\varepsilon)}} = \infty. \tag{2.3}$$

Since the z_j's make up the orbit of a single point, they are uniformly separated in the hyperbolic metric of $\mathbf{B}$ (i.e., if $j \neq k$ then $\varrho(z_j, z_k) > \varepsilon_0$ for some $\varepsilon_0 > 0$ independent of j and k). Thus for any $A < \infty$ we may split the sequence into a finite number B of sequences (depending on A) each of which is separated by at least A in the hyperbolic metric. Moreover,

Property C. Given the N above there is an A so that for $w_1, w_2 \in A_n$, $\varrho(w_1, w_2) \geqslant A$ implies

$$B(w_1, 3N(1-|w_1|) \cap B(w_2, 3N(1-|w_2|)) = \varnothing.$$

By the preceding remark and (2.3), to each of the points x_0, x_1, we may associate a collection of points $V_0, V_1 \subset \{z_j\}$ such that $V_i \subset B_i \cap A_{n_i}$, $i=0,1$, and which satisfy the following two conditions:

$$B(w_1, 3N(1-|w_1|) \cap B(w_2, 3N(1-|w_2|)) = \varnothing, \tag{2.4}$$

for any distinct $w_1, w_2 \in V_i$, and

$$\sum_{j:z_j\in V_i} (1-|z_j|)^{\delta-2\varepsilon} \geqslant e^{-(\delta-2\varepsilon)L_i} \#(V_i) \geqslant C_0^4, \tag{2.5}$$

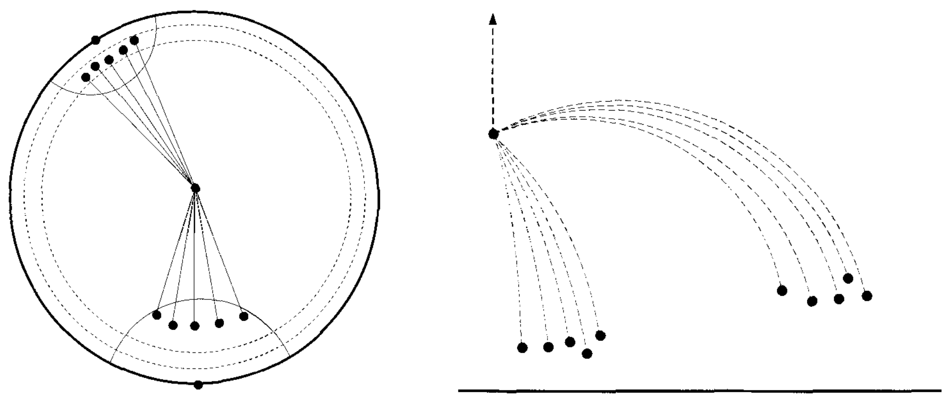

Fig. 2.2. V_0, V_1 and $g(V_0), g(V_1)$

where L_i is chosen so that

$$V_i \subset \{z \in \mathbf{B} : L_i \geqslant \varrho(0,z) > L_i - 1\}$$

and C_0 is as above. Since we may take n_0, n_1 as large as we wish, we may assume that $L_i \geqslant 2\log C_0$ for $i=0,1$. See Figure 2.2.

We now define the collection $\mathcal{C} = \bigcup_n \mathcal{C}_n$ by induction. Start with $\mathcal{C}_0 = \{0\}$. Let $\mathcal{C}_1 = V_0$ (V_1 would work just as well). In general, suppose that $z = g(0) \in \mathcal{C}_n$. Consider the cone of geodesic rays from z to $g(B_0)$ and $g(B_1)$ respectively. These cones are separated by angle at least θ from each other, so at least one of them is separated by angle at least $\frac{1}{2}\theta$ from the radial segment $[0,z]$. Let $i \in \{0,1\}$ be an index for which this happens and set $\mathcal{C}(z) = g(V_i)$. (This is the step of the proof which is not group invariant and the reason why the measure we construct is not group invariant.)

We can now easily verify each of the four desired conditions. Condition (2.4) and the fact that the geodesic segments from z to its children have hyperbolic length at least $2\log C$ prove (1), (2) and (3). Equations (2.1) and (2.5) imply

$$\sum_{w \in \mathcal{C}(z)} (1-|w|)^{\delta-2\varepsilon} \geqslant C_0^{-\delta+2\varepsilon} e^{(\delta-2\varepsilon)L_i} (1-|z|)^{\delta-2\varepsilon} \#(V_i) \geqslant C_0^2 (1-|z|)^{\delta-2\varepsilon}.$$

This is (4).

We now use conditions (1)–(4) to finish the proof of the theorem. Let E be the boundary of the tree on S^2, i.e., E is the set of all limit points of sequences $z_1, z_2, \ldots$ where z_{n+1} is a child of z_n. Note

Property D. If $z \in S^2$ is in $B(z_n, 2N(1-|z_n|))$, then z_n lies in a cone with vertex z and angle depending only on N.

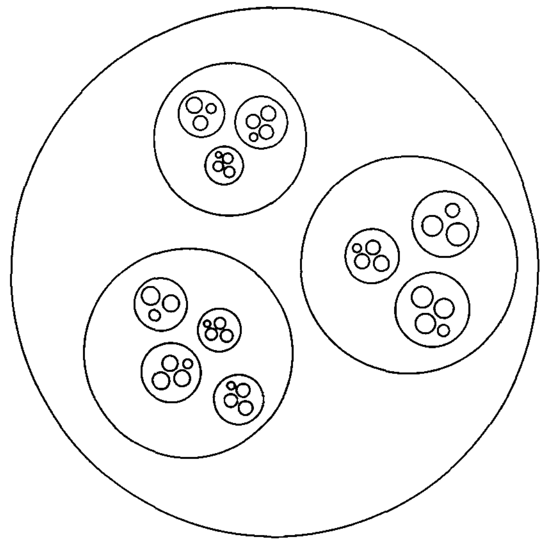

Fig. 2.3. The Cantor set formed by the disks

Since points in the boundary of tree are in infinitely many such balls, they are vertices of cones containing infinitely many orbit points, i.e., $E \subset \Lambda_c$.

Thus all we have left to do is to prove $\dim(E) \geqslant \delta - 2\varepsilon$. This is a standard argument which goes as follows.

Let $D_z = B(z, 2N(1-|z|)) \cap S^2$. For $z \in \mathcal{C}_n$ these are disjoint disks by (2) and (3). Define $E_n = \bigcup_{z \in \mathcal{C}_n} D_z$. Thus $E = \bigcap_n E_n$. See Figure 2.3. Define a probability measure μ on E by setting $\mu(E_0) = 1$, and for $z \in \mathcal{C}_n$ with "parent" $z' \in \mathcal{C}_{n-1}$, set

$$\mu(D_z) = \frac{(1-|z|)^{\delta-2\varepsilon}}{\sum_{w \in \mathcal{C}(z')} (1-|w|)^{\delta-2\varepsilon}} \mu(B_{z'}).$$

It is easy to see by induction that

$$\mu(D_z) \leqslant (1-|z|)^{\delta-2\varepsilon} \leqslant C \operatorname{diam}(D_z)^{\delta-2\varepsilon},$$

for each z in $\mathcal{C}$. We want to show that this inequality is true for any disk D on S^2. Let D be any disk and let $D_0 = D_z$ be the lowest generation disk in our construction so that $D_0 \cap D \neq \varnothing$ but $D \not\subset 2D_0$. Let D_1 be the parent of D_0. By the maximality of D_0 we have $D \subset 2D_1$. Since $2D_1$ is disjoint from any other balls of the same generation,

$$\begin{aligned} \mu(D) \leqslant \mu(D_1) \leqslant C \operatorname{diam}(D_1)^{\delta-2\varepsilon} &\leqslant C(NC)^{\delta-2\varepsilon} \operatorname{diam}(D_0)^{\delta-2\varepsilon} \\ &\leqslant C(2NC)^{\delta-2\varepsilon} \operatorname{diam}(D)^{\delta-2\varepsilon}. \end{aligned}$$

This is the desired inequality (the constant in front is larger, but is uniform over all disks; the power is the same). We now have

$$\dim(\Lambda_c(G)) \geqslant \dim(E) \geqslant \delta - 2\varepsilon,$$

by the mass distribution principle. Since ε is arbitrary, we get Theorem 2.1. □

The proof of Theorem 2.1 has the following corollaries.

COROLLARY 2.2. *Suppose $r>0$. There are constants $C=C(r)$ and $N=N(r)$ so that the following holds. Suppose that G is a non-elementary group of Möbius transformations on $\mathbf{B}$ and suppose that there are integers n_0, n_1 and balls B_0, B_1 with disjoint doubles of (Euclidean) radius r and a collection of points $F_i \subset G(0) \cap A_{n_i}$, $i=0,1$, which satisfy both*

$$z, w \in F_i,\ z \neq w,\ \textit{implies}\ |z-w| \geqslant N2^{-n_i},\ i=0,1,$$

and

$$\sum_{z \in F_i \cap B_i} (1-|z|)^{\alpha} \geqslant C.$$

Then $\delta(G)=\dim(\Lambda_c)\geqslant\alpha$.

Once we have the conditions in the hypothesis, the proof of Theorem 2.1 proves the corollary. The corollary is technical looking, but it shows that getting a lower bound for $\delta(G)$ only requires information about a finite number of orbit points. For example, one obvious corollary is that

$$\delta(G) = \sup\{\delta(G') : G' \subset G,\ G' \text{ finitely generated}\}. \tag{2.6}$$

This had been proven earlier by Sullivan [58]. If $\widetilde{G}$ is another group which is very close to G then $\widetilde{G}$ will also satisfy these conditions with only slightly worse constants (since they only involve a finite number of elements in the group). Moreover, our proof of Theorem 2.1 shows that any discrete G with more than one limit point satisfies the conditions of Corollary 2.2 with $\alpha=\delta(G)-\varepsilon$ for any $\varepsilon>0$. Thus

COROLLARY 2.3. *Suppose that G is a discrete Möbius group with more than one limit point generated by $\{g_1, ..., g_n\}$. Given any $\delta_0>0$ there is an $\varepsilon_0>0$ (depending only on δ and G) such that if $\widetilde{G}$ is a group containing elements $\{\tilde{g}_1, ..., \tilde{g}_n\}$ with $\|g_i-\tilde{g}_i\|<\varepsilon_0$ (as elements of $\mathrm{PSL}(2,\mathbf{C})$) then*

$$\delta(\widetilde{G}) \geqslant \delta(G) - \delta_0.$$

Suppose that $\{G_n\}$ is a sequence of m-generated Möbius groups each with a specific listing of its generators $G_n=\{g_{1n}, ..., g_{mn}\}$. We say that G_n converges *algebraically* to a Kleinian group G with generators $\{g_1, ..., g_m\}$ if $g_{jn}\to g_j$ for each $1\leqslant j\leqslant m$, as elements of $\mathrm{PSL}(2,\mathbf{C})$. See [40]. If we identify groups with points in $\mathrm{PSL}(2,\mathbf{C})^m$, this is just convergence in the product topology.

COROLLARY 2.4. *Suppose that G is discrete and has more than one limit point. If $\{G_n\}$ is a sequence of discrete Möbius groups converging algebraically to G, then*

$$\delta(G) \leqslant \liminf_{n\to\infty} \delta(G_n).$$

This says that $\delta(G)$ is lower semi-continuous with respect to algebraic convergence. Strict inequality is possible even for sequences of Kleinian groups (e.g., we will see later that one can choose a sequence $\{G_n\}$ of geometrically infinite groups (with $\delta(G_n)=2$) in the boundary of Teichmüller space converging to a geometrically finite cusp group G (with $\delta(G)<2$). The hypothesis that G has 2 or more limit points is also necessary, since hyperbolic cyclic groups (with $\delta=0$) can converge to a parabolic cyclic group (with $\delta=\frac{1}{2}$).

Let $M=\mathbf{B}/G$. If G has no torsion then M is a manifold and otherwise it is an orbifold. Suppose that $x\in M$ is the point which projects to $0\in\mathbf{B}$. It is well known that geodesic rays from 0 to points of Λ_c lift exactly to the geodesic rays starting at x which return to some compact set of M infinitely often (the compact set depends on the ray). Thus the unit tangent directions at x corresponding to such geodesic rays has Hausdorff dimension exactly $\delta(G)$.

In the proof of Theorem 2.1, the set $E\subset\Lambda_c$ which is constructed has an additional property. The geodesic ray from $0\in\mathbf{B}$ to each point of the set E corresponds to a geodesic ray in $M=\mathbf{B}/G$ which remains in a bounded part of M. To see this, note that each z in the set we construct is contained in a nest sequence of balls of the form $\bigcap_{n=1}^{\infty} B(z_i^*, A(1-|z_i|))=\bigcap B_i$, where z_i is some sequence of orbit points. Now if γ is the geodesic from 0 to z, then let $\gamma_n=\gamma\cap(B_n\setminus B_{n+1})$. Since $1-|z_{i+1}|\geqslant C^{-1}(1-|z_i|)$, we see that $z\in\gamma_n$ implies $\varrho(z,z_n)\leqslant C'$ for some uniform $C'<\infty$, i.e., the projection of γ to a geodesic ray in M never leaves a C'-ball around the base point. Therefore we have the following.

COROLLARY 2.5. *If G is any non-elementary, discrete Möbius group, $\varepsilon>0$ and $x\in M=\mathbf{B}/G$, then there is an $R=R(\varepsilon,x)$ such that the set of directions (i.e., unit tangents at x) which correspond to geodesic rays starting at x which never leave the ball of radius R around x has dimension $\geqslant\delta(G)-\varepsilon=\dim(\Lambda_c(G))-\varepsilon$. In particular, the set of directions at x of bounded geodesic rays has dimension exactly $\delta(G)$.*

As mentioned in the introduction this was proven for geometrically finite groups by Fernández and Melián [34] and by Stratmann [54].

The only facts about hyperbolic geometry we used in Theorem 2.1 were Properties A–D. Versions of these facts hold for rank 1 symmetric spaces. Such spaces consist of the usual hyperbolic upper half-spaces over the reals, and the analogous spaces over the complex numbers and quaternions plus one exceptional case corresponding to the Cayley numbers (see [25], [38], [47]). They all have negative curvature bounded and bounded away from zero, so comparison theorems such as Toponogov's comparison theorem (e.g., [24]) can be used to deduce these properties. Also see Theorem 2.2 of [25]. Thus Theorem 1.1 also holds in the rank 1 case (the Euclidean metric on the boundary is replaced by a sub-Riemannian metric and Hausdorff dimension on the boundary is computed with respect to this metric).

COROLLARY 2.6. *If G is a non-elementary, discrete group of isometries of a rank* 1 *symmetric space then* $\delta(G)=\dim(\Lambda_c(G))$.

3. The convex hull of the limit set

A discrete group of Möbius transformations is called geometrically finite if there is a finite-sided fundamental polyhedron for its action on $\mathbf{B}$ and otherwise it is called geometrically infinite. We will make use of several different characterizations of geometrically finite groups. Recall that a rank 1 parabolic fixed point p is called doubly cusped if there are two disjoint balls in $\Omega(G)$, tangent at p, and both invariant under the parabolic subgroup fixing p. Beardon and Maskit [6] proved that if G is a finitely generated Kleinian group then G is geometrically finite if and only if every point of $\Lambda(G)$ is either a conical limit point, a rank 2 parabolic fixed point or a doubly cusped rank 1 parabolic fixed point. Geometrical finiteness can also be characterized in terms of the convex hull of the limit set. If K is a compact set on $S^2=\partial\mathbf{B}$ we will let $C(K)\subset\mathbf{B}$ denote its convex hull with respect to the hyperbolic metric on $\mathbf{B}$. If G is a Kleinian group, we let $M=\mathbf{B}/G$ be the hyperbolic 3-manifold (or orbifold) associated to G. Then $C(M)=C(\Lambda(G))/G\subset M$ is called the convex core of M.

For $r>0$ we define the radius r-neighborhood of $C(M)$ as

$$C_r(M)=\{z\in\mathbf{B}:\operatorname{dist}(z,C(M))<r\},$$

where distance is measure in the hyperbolic metric. Although we do not need it here, we should point out that Thurston has shown that G is geometrically finite if and only if $C_r(G)$ has finite volume for some (all) $r>0$. We cannot take $r=0$ because if G is any Fuchsian group, then $\Lambda(G)$ is contained in a circle, so the convex hull of $\Lambda(G)$ is contained in a hyperplane and hence has zero volume. Thus $C(M)$ has finite volume even for infinitely generated Fuchsian groups. However, for finitely generated groups this is not a problem. If G is a finitely generated Kleinian group and $C(M)$ has finite volume then G is geometrically finite. We define

$$M_{\mathrm{thin}(\varepsilon)}=\{x\in M:\operatorname{inj}(x)\leqslant\varepsilon\},$$

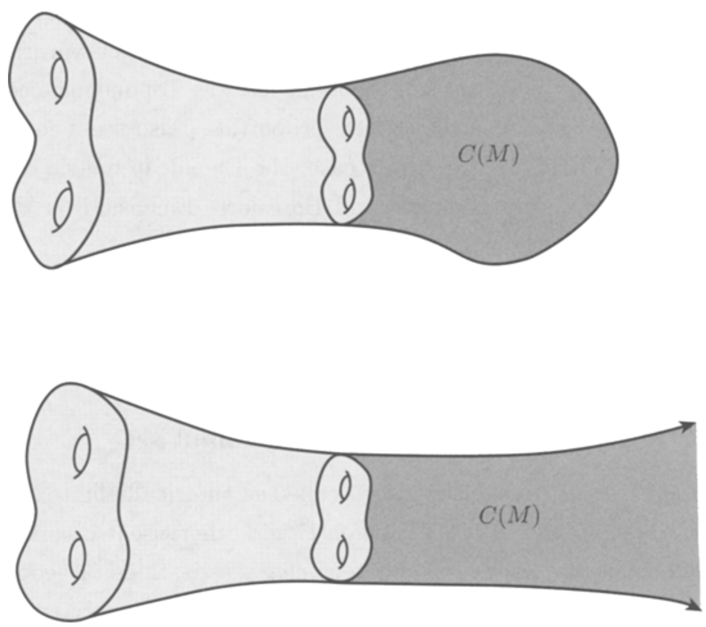

Fig. 3.1. Geometrically finite and infinite manifolds

and denote the complement in M by $M_{\text{thick}(\varepsilon)}$. Another equivalent definition of geometrically finite is that $M_{\text{thick}(\varepsilon)} \cap C(M)$ is compact for all $\varepsilon > 0$. The equivalence of the many formulations of geometric finiteness is proven in [14]. See Figure 3.1.

Suppose $E \subset S^2$ and let u be the solution of the Dirichlet problem on $\mathbf{B}$ with boundary values χ_E, i.e., u is the hyperbolic harmonic function with boundary values 1 on E and 0 on $S^2 \setminus E$. This function can also be defined by integrating χ_E by the appropriate version of the Poisson kernel on S^2, as described in Chapter 5 of [49]. For $z \in \mathbf{B}$, we define the *harmonic measure* of E with respect to z as

$$\omega(z, E, \mathbf{B}) = u(z).$$

This also represents the probability that a (hyperbolic) Brownian motion started at z will hit the sphere at infinity at a point of E.

For $z \in \mathbf{B}$ define

$$w(z) = \max_{D \subset \Omega(G)} \omega(z, D, \mathbf{B}),$$

where the max is over all round disks in $\Omega(G)$. Then $C(\Lambda) = \{z \in \mathbf{B} : w(z) \leqslant \frac{1}{2}\}$. Note that ω is G-invariant and so is well defined on M.

Next we want to show that the convex core can be separated from the geometrically finite ends by a "nice" surface of finite hyperbolic area. To prove this we need to recall the notion of uniformly perfect sets. For $z \in \Omega(G)$ define

$$d(z) = \operatorname{dist}(z, \partial\Omega(G)) = \operatorname{dist}(z, \Lambda),$$

where "distance" means spherical distance.

A compact set K is called *uniformly perfect* if there is an $\varepsilon > 0$ such that for any $x \in K$ and $r < \operatorname{diam}(K)$ there exists $y \in K$ such that

$$\varepsilon r \leqslant |x-y| \leqslant r.$$

There are several well-known equivalent formulations of this condition (e.g., [33], [35]). Suppose that K is compact and Ω is its complement. Then the following are known to be equivalent:

(1) K is uniformly perfect.

(2) There is a positive lower bound for the length of the shortest closed hyperbolic geodesic in Ω.

(3) There is a constant $C < \infty$ so that

$$\frac{1}{C} \cdot \frac{|dz|}{d(z)} \leqslant |d\varrho(z)| \leqslant C \frac{|dz|}{d(z)}.$$

The following was proved in [53]. Also see [18] and [39].

LEMMA 3.1. *If G is an analytically finite, non-elementary Kleinian group then Λ is uniformly perfect. In particular, if ϱ is the hyperbolic metric on a component Ω of $\Omega(G)$, then*

$$|d\varrho(z)| \sim \frac{|dz|}{d(z)}.$$

We will also use two elementary facts about Lipschitz graphs which we state as lemmas for the convenience of the reader. A Lipschitz graph in $\mathbf{B}$ over an open set $\Omega \subset S^2$ is a set of the form $S = \{f(x) \cdot x : x \in \Omega\}$, where $f: \Omega \to \left[\frac{1}{2}, 1\right]$ is a Lipschitz function, i.e., there is an $M < \infty$ so that

$$\frac{|f(x)-f(y)|}{|x-y|} \leqslant M,$$

for all $x, y \in \Omega$.

LEMMA 3.2. *Suppose that $\Omega \subset S^2$ is open, $c_1 > 0$ and consider a family of disks D_α with diameters satisfying*

$$c_1 \leqslant \frac{\operatorname{diam}(D_\alpha)}{\operatorname{dist}(D_\alpha, \partial\Omega)} \leqslant \frac{1}{2},$$

and such that each point $z \in \Omega$ is the center of at least one of the disks. Let $H_\alpha \subset \mathbf{B}$ be the hyperbolic half-space bounded by D_α. Then $\bigcup_\alpha H_\alpha$ is bounded by Ω and a Lipschitz *graph* $\mathcal{L}=\{f(x)\cdot x : x\in\Omega\}$ *with constant* M *depending only on* c_1. *Moreover, we have a* $c_2>0$ *(again depending only on* c_1*) so that*

$$c_2 \leqslant \frac{f(x)}{\operatorname{dist}(x,\partial\Omega)} \leqslant \frac{1}{c_2}. \tag{3.1}$$

LEMMA 3.3. *Suppose that* Ω *is an open set on* S^2 *with uniformly perfect boundary and that* $\mathcal{L}$ *is an* M*-Lipschitz graph supported on* Ω *which satisfies* (3.1). *Then the radial projection* $f(x)\cdot x\to x$ *is bi-Lipschitz between the hyperbolic metric restricted to* $\mathcal{L}$ *and the hyperbolic metric on* Ω. *The bi-Lipschitz constant depends only on the uniform perfectness constant of* $\partial\Omega$, *and the constants* c_2 *and* M.

The proofs are easy and left to the reader.

Our next goal is to see that $C(M)$ can be separated from $\Omega(G)$ by a finite-area surface. It is true that $\partial C(M)$ itself has finite area and is a Lipschitz surface in M ([32]), but it is more convenient to replace $\partial C(M)$ by a surface at bounded distance from $C(M)$ which corresponds to a Lipschitz graph in $\mathbf{B}$. The following is easy to prove and is sufficient for our purposes.

LEMMA 3.4. *Suppose that* G *is an analytically finite Kleinian group and* $M=\mathbf{B}/G$. *Let* $\{\Omega_j\}_1^N$ *be the conjugacy classes of components of* $\Omega(G)$ *(i.e., the geometrically finite ends of* M*). For each* Ω_j *there is a locally Lipschitz surface* S_j *in* M *so that the following holds.*

(1) $\operatorname{dist}(S_j, C(M))>2$.

(2) *Each* S_j *has hyperbolic finite area.*

(3) *The function* $f(y)=(\operatorname{vol}(B(y,1))^{-1/2}$ *is integrable over* S_j, *i.e.,*

$$\int_{S_j} (\operatorname{vol}(B(y,1))^{-1/2}\, dA(y) < \infty$$

($B(y,1)$ *is the ball in* M *of hyperbolic radius* 1 *around* y, vol *denotes hyperbolic volume in* M *and* dA *is hyperbolic area on* S_j*).*

(4) *The surfaces* S_j *separate* $C(M)$ *from the geometrically finite ends of* M, *i.e., there is an* $\varepsilon_0>0$ *so that* $w(z)\leqslant 1-\varepsilon_0$ *on the component* M_1 *of* $M\setminus\bigcup_j S_j$ *containing* $C(M)$.

(5) *There is a constant* $C<\infty$ *so that the Hausdorff distance between* S *and* $\partial C(M)$ *is less than* C. *In other words,*

$$S\subset\{x\in M : \operatorname{dist}(x,\partial C(M))<C\}$$

and

$$\partial C(M)\subset\{x\in M : \operatorname{dist}(x,S)<C\}.$$

Proof. Fix a fundamental domain $\mathcal{F}\subset\Omega(G)$ for G. Since $\partial\Omega=\Lambda$ is uniformly perfect, we know that $d\varrho\sim d(z)^{-1}|dz|$, where $d(z)$ and $|dz|$ denote the spherical metric. Take $\varepsilon\ll\frac{1}{10}$ small and let $D_z=\{w\in S^2:|z-w|\leqslant\varepsilon d(z)\}$. Let $\{D_\alpha\}$ be the collection of all such disks and all their images under elements of G. By the Koebe distortion theorem all these disks satisfy the estimate in Lemma 3.2, so the union of the corresponding hyperbolic half-spaces defines a Lipschitz graph $\mathcal{L}$ above Ω. By Lemma 3.3 the radial projection from $\mathcal{L}$ to Ω is bi-Lipschitz between the hyperbolic metric on $\mathbf{B}$ restricted to $\mathcal{L}$ and the hyperbolic metric on Ω. Also $\mathcal{L}$ is clearly G-invariant by definition, so its quotient by G is a surface S in M.

Simple geometric arguments and Lemma 3.1 show that if ε is small enough then S is at least hyperbolic distance 2 from $C(\Lambda)$. For each component Ω_j we simply take S_j to be the intersection of a fundamental polygon in $\mathbf{B}$ for G with S. Thus the hyperbolic area of S is bounded by at most a constant times the area of a fundamental region in Ω. This is (2). To prove (3) we split the integral over S_j into pieces: one for each part of S_j lying over a horoball in Ω and the remaining compact part of S_j. The integral over the compact part is clearly bounded. The integral over each of the cusps is easily bounded using the fact that the injectivity radius decreases exponentially in the cusps (in terms of hyperbolic distance) and the observation that for $y\in S$, $\mathrm{vol}(B(y,1))\sim\mathrm{inj}(y)$. Here $\mathrm{inj}(y)$ denotes the injectivity radius of M at y. Details are left to the reader. Conditions (1) and (4) are easy to check if ε_0 is small enough.

To check (5), we note that any point $x\in\partial C(\Lambda)$ is on the boundary of some hyperbolic half-space which meets S^2 in a disk contained in Ω, and the spherical radius of this disk is comparable to the spherical distance from its center to Λ. Thus the point of $\mathcal{L}$ over the center of the disk is a bounded spherical distance from x (depending only on ε in the definition of $\mathcal{L}$). Conversely, given any point z on $\mathcal{L}$ we can find a point $w_1\in\Lambda$ with $\mathrm{dist}(z,w_1)\sim\mathrm{dist}(z,\Lambda)$ (distances are spherical, as above). By uniform perfectness of Λ, we can find a second point $w_2\in\Lambda$ with $\mathrm{dist}(w_1,w_2)\sim\mathrm{dist}(z,w_1)$. Thus the top of the geodesic from w_1 to w_2 is a point of $C(\Lambda)$ which is a bounded distance from z. This completes the proof. □

We want to use the fact that if G is geometrically infinite then we can find a sequence of points $x\in C(M)$ tending to infinity and with $\mathrm{inj}(x)$ bounded away from zero. This follows from the Margulis lemma (see, e.g., [7]) and an alternate definition of geometric finiteness. According to the Margulis lemma (e.g., p. 134 of [7]) there is an $\varepsilon>0$ (independent of G) so that if $y\in\mathbf{B}$, then the group $G_\varepsilon(x)$ generated by

$$\{g\in G:\varrho(x,g(x))\leqslant\varepsilon\}$$

is almost-nilpotent and hence elementary. Let

$$T_\varepsilon(G)=\{x\in\mathbf{B}:G_\varepsilon(x)\text{ is infinite}\}.$$

According to Definition GF4 in [14], G is geometrically finite if and only if $C(M)\backslash T_\varepsilon(G)$ is compact for some ε less than the Margulis constant. If G has no elliptics this is the same as saying that $C(M)\backslash M_{\text{thin}}(\varepsilon)$ is compact, where

$$M_{\text{thin}(\varepsilon)}=\{x\in M:\text{inj}(x)\leqslant\varepsilon\}.$$

Thus if G is geometrically infinite and has no elliptics (which we may assume if G is finitely generated by Selberg's lemma), we obtain the desired sequence.

If G has elliptics then the only difference between the conditions is the possibility of points with small injectivity radius due to the action of finite subgroups of G. By considering the possible finite elementary groups (as listed in [41] or §5 of [5] for example), we see that for groups of bounded order we can always move unit hyperbolic distance and reach a point with injectivity radius bounded uniformly away from zero. The only finite Kleinian groups of high order are the cyclic groups $\mathbf{Z}_n$ and the corresponding dihedral groups.

Thus if G is geometrically infinite we can find points $x_n\in C(M)$ tending to infinity with either the injectivity radius bounded away from zero or the points are in the thin parts of finite cyclic groups of arbitrarily high order. If the latter case occurs then choose lifts of the points in a convex fundamental domain for G in $\mathbf{B}$ and assume that the points accumulate to a single point of S^2 (which we may assume by passing to a subsequence if necessary). Then the geodesic connecting two of the points x_n, x_m lies in the fundamental domain and $C(M)$, and has large distance from 0 (depending on n, m). Moreover, if x_n, x_m are in the thin parts of different finite groups, there is a point on the geodesic which is on the boundary of the thin part of one of the groups and not in any other thin part (otherwise the group associated to that point would not be elementary). This point has injectivity radius bounded uniformly away from zero, and so we can construct the desired sequence.

Note that this argument also implies that for any geodesic ray in $\mathbf{B}$ which lands in the limit set, either the landing point is fixed by some element of G or the ray leaves every thin part (and hence the injectivity radius is bigger than some $\varepsilon>0$ on the ray arbitrarily close to the landing point).

We will need the following lemma in the proof of Theorem 1.6. Note that if we replaced $C(M)$ by an r-neighborhood of itself, we would simply be stating one of the equivalent definitions of geometrically finite.

LEMMA 3.5. *If G is a Kleinian group (not necessarily finitely generated) and $\Lambda(G)$ has positive area then $C(M)$ has infinite volume.*

Note that we need the hypothesis that G is Kleinian (i.e., $\Lambda \neq S^2$), since it is possible for G to be a discrete group with $\Lambda(G)=S^2$, but $C(M)=M$ to have finite volume (e.g., if G is co-compact).

Proof. We will show that there is a sequence $\{x_n\} \in C(M)$ with $\mathrm{dist}(x_n, \partial C(M)) \to \infty$, and $\mathrm{inj}(x_n) > \varepsilon$ for all n. Thus $C(M)$ contains infinitely many disjoint balls of fixed volume, proving the lemma.

As above, for $z \in \mathbf{B}$ define

$$w(z) = \max_{D \subset \Omega(G)} \omega(z, D, \mathbf{B}),$$

where the max is over all round disks in $\Omega(G)$. It is easy to see that for any $R>0$ there is an ε so that $w(x)<\varepsilon$ implies $\mathrm{dist}(x, \partial C(M))>R$.

Since $\Lambda(G)$ has positive area the Lebesgue density theorem gives us a point of density $z_0 \in \Lambda(G)$. We may also assume that z_0 is not one of the (countably many) points fixed by some element of G. Let γ be the hyperbolic geodesic connecting the origin to z_0 (i.e., a radius of $\mathbf{B}$) and consider points $x \in \gamma$ converging to z_0. Clearly $w(x) \to 0$, so $\mathrm{dist}(x, \partial C(M)) \to \infty$ as $x \to z_0$. Therefore we only have to show that the injectivity radius of x is $\geqslant \varepsilon$ along some subsequence converging to z_0.

If this were false then eventually γ would remain in one of the thin components for all points close enough to the boundary. This implies that z_0 is fixed by some element of G, and this contradiction completes the proof. □

We will use the following result in the proof of Theorem 1.2 (but it is not essential; we will also sketch a proof which does not require it).

LEMMA 3.6. *Suppose that G is analytically finite and geometrically infinite. Then there is a sequence $\{x_n\} \in C(M)$ with $\mathrm{dist}(x_n, \partial C(M)) \to \infty$ and $\mathrm{inj}(x_n) > \varepsilon$ for all n.*

Proof. By our earlier remarks, we can find $\varepsilon>0$ and $\{x_n\} \subset C(M)$ tending to infinity with $\mathrm{inj}(x) > \varepsilon$. All we need to do is to check that there is a subsequence with $\mathrm{dist}(x_n, \partial C(M)) \to \infty$.

If not, there is some $A<\infty$ such that $\mathrm{dist}(x_n, \partial C(M))<A$ for all n. By Lemma 3.4 these points are also a bounded distance A' from the surface S we constructed. Therefore we can construct a sequence $\{y_n\} \subset S$ such that $\varrho(x_n, y_n)<A'$. For each of the finitely many cusps on S, there is a neighborhood of the cusp so that any point in M within hyperbolic distance A' of the cusp has injectivity radius $<\frac{1}{2}\varepsilon$ (just consider the parabolic fixing the cusp point on S^2). Thus the $\{y_n\}$ must all lie in a compact region of S, for otherwise there would be an infinite sequence of disjoint disks in S all with a given area, contradicting the fact that S has finite area. This is clearly a contradiction since $x_n \to \infty$ implies $y_n \to \infty$, proving the lemma. □

4. The heat kernel

In this section we recall some facts about the heat kernel $K(x,y,t)$ on $M=\mathbf{B}/G$ and prove a simple estimate which we will need in the proof of Theorem 1.2. The heat kernel is the fundamental solution for the heat equation on M, i.e.,

$$u(x,t) = \int_M u_0(y) K(x,y,t)\, dy,$$

is the solution of the heat equation $u_t = \Delta u$ with initial value u_0. It can also be interpreted in terms of Brownian motion: $K(x,y,t)\,dy$ is the distribution at time t of a Brownian motion started at the point x.

We will also need the fact that the Green function $G(x,y)$ for M may be obtained by integrating the heat kernel, i.e.,

$$G(x,y) = \int_0^\infty K(x,y,t)\, dt.$$

We are mainly interested in the Green function, but it is easier to deal with the heat kernel because of the semi-group associated to K and the fact that this kernel can be easily expressed in terms of the eigenfunctions of the Laplacian on M.

The estimate we need is known. For example, in [28] E. B. Davies shows

THEOREM 4.1. *If $\delta > 0$ then there is a constant $C = C(\delta)$ such that*

$$0 \leqslant K(x,y,t) \leqslant C \operatorname{vol}(B(x,t^{1/2}))^{-1/2} \operatorname{vol}(B(y,t^{1/2}))^{-1/2} e^{-\varrho(x,y)^2 (4+\delta)^{-1} t^{-1}},$$

for $0 < t < 1$, where $\varrho(x,y)$ denotes the hyperbolic distance between x and y in M, and $\operatorname{vol}(B(x,t))$ *denotes the volume (in M) of the ball of radius t around x. Furthermore,*

$$0 \leqslant K(x,y,t) \leqslant C \operatorname{vol}(B(x,1))^{-1/2} \operatorname{vol}(B(y,1))^{-1/2} e^{(\delta - \lambda_0)t} e^{-\varrho(x,y)^2 (4+\delta)^{-1} t^{-1}},$$

for $1 \leqslant t < \infty$, where λ_0 is the base eigenvalue for the Laplacian on M.

See also [28], [31] and [37] and their references. For completeness, we include the proof of the following estimate which is weaker than Davies', but which is sufficient for our purposes. We state our result in three dimensions only, but a similar estimate holds in any dimension.

LEMMA 4.2. *Let G be any discrete Möbius group and let $M=\mathbf{B}/G$. Then there is an absolute $C<\infty$ so that for any $x,y\in M$ and $t>0$ we have*

$$K(x,y,t)\leqslant \frac{C(1+\varrho(x,y))\,t^{-3/2}e^{-t}e^{-t^{-1}\varrho(x,y)^2/8}}{\operatorname{vol}(B(x,1))^{1/2}\operatorname{vol}(B(y,1))^{1/2}}, \tag{4.1}$$

for $0<t\leqslant\frac{1}{8}\varrho(x,y)$, and

$$K(x,y,t)\leqslant \frac{Ce^{-\lambda_0 t}}{\operatorname{vol}(B(x,1))^{1/2}\operatorname{vol}(B(y,1))^{1/2}}, \tag{4.2}$$

for $t\geqslant 1$. Here λ_0 denotes the base eigenvalue of M.

Using these estimates and integrating over t we obtain the estimate on Green's function which we will use in the next section.

COROLLARY 4.3. *Let G be any discrete Möbius group and let $M=\mathbf{B}/G$. Suppose that the base eigenvalue λ_0 is non-zero. Then there are constants $C<\infty$ and $c>0$ (depending only on λ_0) so that for any x,y with $\varrho(x,y)\geqslant 8$, we have*

$$G(x,y)\leqslant C\operatorname{vol}(B(x,1))^{-1/2}\operatorname{vol}(B(y,1))^{-1/2}e^{-c\varrho(x,y)}.$$

Proof. We set $R=\frac{1}{8}\varrho(x,y)$ and split the integral

$$G(x,y)=\int_0^\infty K(x,y,t)\,dt=\int_0^R K(x,y,t)\,dt+\int_R^\infty K(x,y,t)\,dt.$$

Now use the estimates from the lemma (we drop the volume terms since they do not involve t) to get

$$\begin{aligned}
&\leqslant C\left[\int_0^R (1+\varrho(x,y))\,t^{-3/2}e^{-t}e^{-t^{-1}\varrho(x,y)^2/8}+\int_R^\infty e^{-\lambda_0 t}\right]\\
&\leqslant C\left[(1+\varrho(x,y))\int_0^R t^{-3/2}e^{-t}e^{-t^{-1}\varrho(x,y)^2/16}e^{-t^{-1}\varrho(x,y)^2/16}+\int_R^\infty e^{-\lambda_0 t}\right]\\
&\leqslant C\left[(1+\varrho(x,y))\,e^{-\varrho(x,y)/2}\int_0^R t^{-3/2}e^{-t^{-1}/16}+\frac{1}{\lambda_0}e^{-\lambda_0 R}\right]\\
&\leqslant Ce^{-\varrho(x,y)/4}+\frac{C}{\lambda_0}e^{-\lambda_0\varrho(x,y)/8}\leqslant\frac{C}{\lambda_0}e^{-c\varrho(x,y)},
\end{aligned}$$

where $c=\min\left(\frac{1}{8}\lambda_0,\frac{1}{4}\right)$. $\square$

This estimate on Green's function is not as strong as possible, but is more than we need. All we will use is that

$$G(x,y)\leqslant C\operatorname{vol}(B(x,1))^{-1/2}\operatorname{vol}(B(y,1))^{-1/2}\eta(\varrho(x,y)),$$

where $\eta(t)\to 0$ as $t\to\infty$.

Proof of Lemma 4.2. We begin with the proof of estimate (4.1). We recall that if $k(w,z,t)$ is the heat kernel on the hyperbolic ball $\mathbf{B}$, then ([29, p. 178])

$$k(w,z,t)=(4\pi t)^{-3/2}\frac{\varrho(w,z)}{\sinh(\varrho(w,z))}e^{-t-\varrho(w,z)^2/4t}.$$

Using the observation that

$$\frac{r}{\sinh(r)}\sim(1+r)e^{-r},$$

for $r>0$ gives,

$$k(w,z,t)\sim t^{-3/2}(1+\varrho(w,z))\exp\left(-t-\varrho(w,z)-\frac{\varrho(w,z)^2}{4t}\right). \tag{4.3}$$

Fix $x,y\in M$ and let $w\in\mathbf{B}$ correspond to x via the covering map and let $\{z_j\}\subset\mathbf{B}$ be all the points corresponding to y. Then

$$K(x,y,t)=\sum_j k(w,z_j,t).$$

Let

$$S_n=\{z_j: n\leqslant\varrho(w,z_j)-\varrho(x,y)<n+1\},$$

so that $\{z_j\}=\bigcup_n S_n$. Notice that each S_n can be covered by at most $Ce^{2(n+\varrho(x,y))}$ hyperbolic balls of radius 1. (This is a very weak estimate; with more work the 2 can be replaced by $\delta(G)+\varepsilon$ for any $\varepsilon>0$.)

We claim that if $B\subset\mathbf{B}$ is some hyperbolic ball of radius 1, then the number points of $\{z_j\}$ which can lie in B is at most

$$C\operatorname{vol}(B(y,1))^{-1}. \tag{4.4}$$

To prove this let U be a connected, simply-connected neighborhood of $y\in M$ which lies in the ball of radius 1 around y and has the same volume as this ball. We associate to each z_j the component U_j of the lift of U which contains z_j. If $z_j\in B$ then U_j is in the double of B (same center, radius 2). Since different U_j are disjoint and each has volume $\operatorname{vol}(B(y,1))$, we see that there are at most $C\operatorname{vol}(B(y,1))^{-1}$ of them, where C is the volume of a ball of radius 2 in $\mathbf{B}$.

Combining estimates (4.4) and (4.3) we obtain for $0<t<\frac{1}{8}\varrho(x,y)$,

$$\begin{aligned} K(x,y,t) &= \sum_{n=0}^{\infty} \sum_{z_j \in S_n} k(w,z_j,t) \\ &\leqslant C \operatorname{vol}(B(y,1))^{-1} \sum_{n=0}^{\infty} e^{2n+2\varrho(x,y)} t^{-3/2}(1+\varrho(x,y)) \\ &\quad \times \exp\left(-t-\varrho(x,y)-n-\frac{(\varrho(x,y)+n)^2}{4t}\right) \\ &\leqslant C \operatorname{vol}(B(y,1))^{-1} t^{-3/2} e^{-t}(1+\varrho(x,y)) \\ &\quad \times \sum_{n=0}^{\infty} \exp\left(\varrho(x,y)+n-\frac{\varrho(x,y)^2+2n\varrho(x,y)+n^2}{4t}\right). \end{aligned}$$

The sum is (recall $t\leqslant\frac{1}{8}\varrho(x,y)$)

$$\begin{aligned} &\leqslant e^{-\varrho(x,y)^2/8t} \sum_{n=0}^{\infty} \exp\left(\varrho(x,y)+n-\frac{1}{8t}\varrho(x,y)^2-\frac{1}{2t}n\varrho(x,y)-\frac{n^2}{4t}\right) \\ &\leqslant e^{-\varrho(x,y)^2/8t} \sum_{n=0}^{\infty} \exp(\varrho(x,y)+n-\varrho(x,y)-4n-0) \leqslant Ce^{-\varrho(x,y)^2/8t}. \end{aligned}$$

Thus

$$K(x,y,t) \leqslant C \operatorname{vol}(B(y,1))^{-1}(1+\varrho(x,y))t^{-3/2}e^{-t}e^{-\varrho(x,y)^2/8t}. \tag{4.5}$$

A standard fact about the heat kernel is its symmetry, i.e.,

$$K(x,y,t)=K(y,x,t).$$

Thus

$$K(x,y,t)=(K(x,y,t)\cdot K(y,x,t))^{1/2}.$$

Estimate (4.1) now follows from this symmetry and by applying the estimate above twice.

Now we turn to the proof of (4.2). We start by setting $x=y$ and $t=1$ in (4.5) to get

$$K(x,y,1) \leqslant C \operatorname{vol}(B(x,1))^{-1}. \tag{4.6}$$

We now use an argument shown to us by Jay Jorgenson to handle the case $t\geqslant 1$. First, let us assume that M is a manifold (so that the group has no elliptic elements). Let $\{\Omega_n\}$ be an exhaustion of $M=\mathbf{B}/G$ by compact submanifolds (or suborbifolds) with smooth boundaries and let $x,y\in\Omega_n$ for all n. Denote by $K_n(x,y,t)$ the heat kernel on Ω_n. Then we have

$$K_n(x,y,t) \to K(x,y,t),$$

as $n\to\infty$. Fix some n and recall that $L^2(\Omega_n)$ has a complete orthonormal basis of eigenfunctions for the Laplacian, $\{\varphi_j^n\}_{j=0}^{\infty}$, with eigenvalues $\{\lambda_j^n\}$ chosen so that $\lambda_j^n\leqslant\lambda_{j+1}^n$ (see [23]). Then

$$K_n(x,y,t)=\sum_{j=0}^{\infty}\varphi_j^n(x)\varphi_j^n(y)\,e^{-\lambda_j^n t}=e^{-\lambda_0^n t}\sum_{j=0}^{\infty}\varphi_j^n(x)\varphi_j(y)\,e^{-(\lambda_j^n-\lambda_0^n)t}.$$

Now if $y=x$ we notice that each term of the series, $\varphi_j^n(x)^2e^{-(\lambda_j^n-\lambda_0^n)t}$ is a decreasing function of t, and hence taking $n\to\infty$, we get for $t\geqslant 1$,

$$K(x,x,t)\leqslant e^{-\lambda_0 t}\lim_{n\to\infty}\sum_{j=0}^{\infty}\varphi_j^n(x)^2e^{-\lambda_j+\lambda_0}=e^{\lambda_0-\lambda_0 t}K(x,x,1). \tag{4.7}$$

The Cauchy–Schwarz inequality implies

$$K(x,y,t)\leqslant(K(x,x,t)\cdot K(y,y,t))^{1/2},$$

so combining this with the estimates (4.6) and (4.7) gives for $t\geqslant 1$,

$$K(x,y,t)\leqslant C\,\mathrm{vol}(B(x,1))^{-1/2}\,\mathrm{vol}(B(y,1))^{-1/2}e^{-\lambda_0 t},$$

as desired.

We now outline the proof for the case when M is an orbifold (i.e., the group G has elliptic elements). We first look at the hyperbolic ball $\mathbf{B}$ and let X denote the (countable) collection of all elliptic axes in $\mathbf{B}$. We claim that X is a closed set. To see this, first find for each elliptic axis X_j a group element g_j fixing the axis and so that $\varrho(x,X_j)=1$ implies $\varrho(x,g_j(x))>\frac{1}{2}$ (we can do this by taking some power of the elliptic fixing X_j). Let $x_0\in\mathbf{B}$ and suppose that there are an infinite number of distinct X_j coming within distance 1 of x_0. It is an elementary compactness argument to find a sequence of distinct axes which converge in the Hausdorff metric in the ball of radius 2 around x_0 so that the corresponding elements g_j also converge (to a non-identity element). Thus $g_jg_{j+1}^{-1}$ is a sequence of non-identity elements of the group which converge to the identity, contradicting the discreteness of G. Hence X is closed.

Let $\widetilde{M}=(\mathbf{B}\setminus X)/G$ be the manifold obtained by removing from M the elliptic axes. Then the previous argument for manifolds shows that (4.2) holds for $\widetilde{K}(x,y,t)$, the heat kernel for $\widetilde{M}$. Therefore $\widetilde{G}$, the Green function for $\widetilde{M}$, obeys the estimate of Corollary 4.3 (the only estimate we really require). The proof is concluded by noting that, from elementary results on Newtonian capacity (see e.g., [22]), the Newtonian capacity of X is zero. Thus the Green capacity (i.e., the capacity with respect to the Green kernel on $\mathbf{B}$)

of X is zero. Lifting $\widetilde{G}$ to $\mathbf{B}\backslash X$, we obtain a function harmonic and bounded away from the poles of the lift of $\widetilde{G}$. Therefore (since the capacity of X is zero) the lift of $\widetilde{G}$ is harmonic on $\mathbf{B}$ minus the poles. Thus $\widetilde{G}$ agrees with the usual Green function for M. $\square$

It may be worth pointing out that Davies' estimate gives an interesting bound on the counting function of a Kleinian group. Given $x, y \in M = \mathbf{B}/G$, assume that x corresponds to an orbit containing 0 via the covering map and y to an orbit $\{y_j\}$. Let $\varrho = \varrho(x, y)$ and

$$N_k = \#(\{y_j : \varrho + k \leqslant \varrho(0, y_j) < \varrho + k + 1\}).$$

Then Davies' estimate implies that for any $\varepsilon > 0$,

$$N_k \leqslant C_\varepsilon \frac{\exp((1+\varepsilon)\delta(\varrho+k)[1-((1-\lambda_0)^{1/2}/\delta)(1-((2k\varrho+k^2)/(\varrho+k)^2)^{1/2})])}{\mathrm{vol}(B(x,1))^{1/2}\,\mathrm{vol}(B(y,1))^{1/2}},$$

where $\lambda_0 = \delta(2-\delta)$. See [11] for details.

5. Geometrically infinite groups have dimension 2

In this section we will prove

THEOREM 5.1. *If G is analytically finite and geometrically infinite then*

$$\dim(\Lambda(G)) = 2.$$

If $\delta(G) = 2$ then this follows from Theorem 1.1. Therefore we may assume $\delta(G) < 2$. In this case Theorem 5.1 follows from

THEOREM 5.2. *If G is an analytically finite, geometrically infinite group and $\delta(G) < 2$ then $\Lambda(G)$ has positive area.*

Proof of Theorem 5.2. Let $\Lambda = \Lambda(G)$ be the limit set of G. Let $\delta = \delta(G)$ be the critical index for the Poincaré series and λ_0 the base eigenvalue for the Laplacian on $M = \mathbf{B}/G$. The Elstrodt–Patterson formula relates δ and the base-eigenvalue by

$$\lambda_0 = \begin{cases} 1, & \text{if } \delta < 1, \\ \delta(2-\delta), & \text{if } \delta \geqslant 1 \end{cases}$$

(e.g., Theorem 2.18 of [61]). In particular, $\delta(G) < 2$ implies $\lambda_0 > 0$.

Let $C(\Lambda)$ be the convex hull of the limit set in $\mathbf{B}$ and $C(M) = C(\Lambda)/G \subset M$ be the convex core of M (see §3). Let $S = \bigcup_j S_j$ be the surface given by Lemma 3.4 which separates $C(M)$ from the geometrically finite ends, $\Omega(G)$. Let $\omega(x, \Omega(G))$ denote the harmonic measure of the geometrically finite ends with respect to the point $x\in M$. More precisely, let u be the solution of the Dirichlet problem on $\mathbf{B}$ with boundary values 1 on Ω and 0 on Λ (e.g., see Chapter 5 of [49] for a description of u in terms of the Poisson kernel of χ_Ω). Then u is a G-invariant harmonic function on $\mathbf{B}$ and so defines a harmonic function on M which we denote $\omega(x,\Omega(G))$. One easily sees that

$$u(0)=\frac{\operatorname{area}(\Omega)}{\operatorname{area}(S^2)}.$$

If $x\in C(M)$ then the surface S separates x from $\Omega(G)$, so the Gauss theorem yields

$$\omega(x,\Omega(G))=C\int_S \frac{\partial G}{\partial n}(x,y)\,dA(y), \tag{5.1}$$

where $\partial/\partial n$ is the normal derivative to S and dA is area measure on S. This equality is easier to see when lifted to $\mathbf{B}$. Let $\widetilde{S}$ denote the lift of S to $\mathbf{B}$ and let B_N denote the hyperbolic ball of radius N in $\mathbf{B}$ centered at 0. Let $\widetilde{B}_N$ be the subdomain of B_N consisting of all points separated by $\widetilde{S}$ from 0, and let $R_N=\partial\widetilde{B}_N\cap\partial B_N$. Then $\partial\widetilde{B}_n=R_N\cup\widetilde{S}_N$, where $\widetilde{S}_N=\widetilde{S}\cap B_N$. Now setting g to be the Green function for $\mathbf{B}$ with pole at 0,

$$\omega(0,\Omega(G))=\operatorname{area}(\Omega)=\lim_{N\to\infty}\frac{\operatorname{area}(R_N)}{\operatorname{area}(\partial B_N)}=\lim_{N\to\infty} c\int_{\widetilde{S}\cap B_n}\frac{\partial g}{\partial n}\,dA,$$

where the last equality uses Green's theorem. Also note that although the hyperbolic area of ∂B_N grows to infinity as $N\to\infty$, the measure $(\partial g/\partial n)\,dA$ is simply (normalized) Lebesgue measure on the sphere. Similarly, because $\widetilde{S}$ is a Lipschitz graph we have

$$\int_{\widetilde{S}}\left|\frac{\partial g}{\partial n}\right| dA<\infty.$$

Write $\widetilde{S}=\bigcup\widetilde{S}_j$ where $S_j=\gamma_j(S_0)$ is the image of some fundamental region for $\widetilde{S}$. Then letting g_j denote the Green function on $\mathbf{B}$ with pole at $z_j=\gamma_j(0)$,

$$\int_{\partial S}\frac{\partial g}{\partial n}\,dA=\sum_j\int_{\widetilde{S}_j}\frac{\partial g}{\partial n}\,dA=\sum_j\int_{\widetilde{S}_0}\frac{\partial g_j}{\partial n}\,dA=\int_{\widetilde{S}_0}\sum_j\frac{\partial g_j}{\partial n}\,dA=\int_{\widetilde{S}_0}\frac{\partial G}{\partial n}\,dA=\int_S\frac{\partial G}{\partial n}\,dA,$$

where the sum and integral can be interchanged using the Lebesgue dominated convergence theorem. This completes the verification of (5.1).

Suppose $\varrho(x,y)\geqslant 1$. By lifting from M to the hyperbolic ball and using Harnack's inequality we obtain the estimate $G(x,z)\leqslant CG(x,y)$ for all $z\in B_\varrho(y,\frac{1}{2})$. To see this, let Π denote the projection from $\mathbf{B}$ to M and let $u(w)=G(x,\Pi(w))$. Then if $\Pi(0)=y$, we see that u is a positive harmonic function on $\{w\in\mathbf{B}:\varrho(0,w)<1\}$, so we can apply Harnack's inequality to deduce $u(w) \leqslant Cu(0)$ on $\{w \in \mathbf{B} : \varrho(0,w) < \frac{1}{2}\}$. Projecting back to M gives the desired estimate.

Next we want to estimate the gradient

$$|\nabla_y G(x,y)| \leqslant CG(x,y), \tag{5.2}$$

as long as $\varrho(x,y) \geqslant 1$. This follows because harmonic functions satisfy the mean value property on balls, i.e.,

$$u(x) = \int_{B(x,1)} u(w)\, dw.$$

Thus

$$|u(x)-u(z)| \leqslant \int_{B(x,1)\Delta B(z,1)} |u(w)|\, dw \leqslant \|u\|_\infty \operatorname{vol}(B(x,1)\Delta B(z,1)) \leqslant C\|u\|_\infty |x-z|,$$

where $E\Delta F = (E \setminus F) \cup (F \setminus E)$. This proves the estimate if we take $u(x) = G(x,y)$.

Thus by (5.1) and (5.2),

$$\omega(x,\Omega(G)) = C\int_S \frac{\partial G}{\partial n}(x,y)\, dA(y) \leqslant C\int_S G(x,y)\, dA(y).$$

By Corollary 4.3 this gives

$$\omega(x,\Omega(G)) \leqslant C\int_S \operatorname{vol}(B(x,1))^{-1/2} \operatorname{vol}(B(y,1))^{-1/2} e^{-c\varrho(x,y)}\, dA(y).$$

By Lemma 3.6 we can choose a point $x \in C(M)$ where the injectivity radius $\operatorname{inj}(x) > \varepsilon > 0$ is uniformly bounded below and

$$\operatorname{dist}(x,S) \geqslant \operatorname{dist}(x,\partial C(M)) \equiv R,$$

is as large as we wish. For such a point we get

$$\omega(x,\Omega(G)) \leqslant C\varepsilon^{-1/2} e^{-cR} \int_S \operatorname{vol}(B(y,1))^{-1/2}\, dA(y).$$

By Lemma 3.4, the integral is bounded, so

$$\omega(x,\Omega(G)) \leqslant C\varepsilon^{-1/2} e^{-cR} \leqslant \tfrac{1}{2},$$

if R is large enough. Thus we can find a point $x \in \mathbf{B}$ at which the harmonic measure of $\Omega(G)$ is strictly less than 1. This means that $\Omega(G)$ has less than full area measure on the sphere, so the limit set must have positive area. This proves the theorem. □

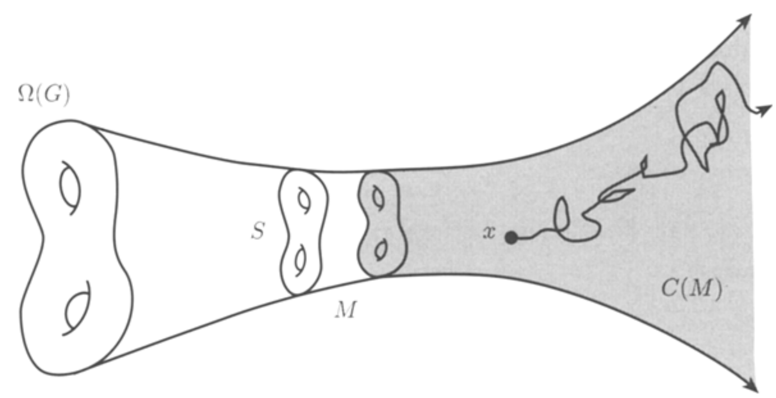

Fig. 5.1. Brownian motion escapes in the convex core

Sketch of proof using Brownian motion. The preceding argument can be rewritten in terms of Brownian motion on the manifold M as follows. To show that $\Lambda(G)$ has positive area, it suffices to show that a Brownian motion in the ball has a positive probability of first hitting the boundary in $\Lambda(G)$. Projecting to $M=\mathbf{B}/G$, it suffices to find a point $x \in C(M)$ so that a Brownian motion started at x has a positive probability of going to infinity in M without ever crossing the surface S. See Figure 5.1.

Let U be a unit neighborhood of S. The expected time a Brownian motion started at x spends in U is expressed in terms of the heat kernel by

$$\int_0^\infty \left[\int_U K(x,y,t)\,dy\right] dt.$$

The heat kernel estimates of the previous section imply that if we choose $x \in C(M)$ with injectivity radius bounded away from zero and with $\operatorname{dist}(x,U)$ very large, then the expected time a Brownian motion started at x spends in U is as small as we wish.

However, the expected time it takes a Brownian motion started at a point y of S to first leave U (i.e., to travel unit distance from S) is bounded away from zero independent of the starting point y. This is because the expected time to travel distance 1 in M is greater than or equal to the expected time to travel this distance in the covering space $\mathbf{B}$, and this is bounded away from zero. Let $t_0>0$ be a lower bound for the expected time to travel distance 1. Thus the expected time a Brownian motion started at x spends in U is at least the probability that it ever hits S multiplied by the bound t_0. From this it is easy to see that the chance that a Brownian motion started at x ever hits S is small if $\operatorname{dist}(x,S)$ is large. Thus with positive probability the Brownian path never crosses S, as desired. $\square$

Sketch of proof using doubling. It actually suffices to use a weaker estimate on the heat kernel and less information on the convex hull. All we need to know is that Brownian motion is transient on a complete, connected, infinite-volume Riemannian manifold with lowest eigenvalue bounded away from zero. For example, the estimate

$$K(x,y,t) \leqslant C_1 e^{-C_2 t},$$

for some fixed x and y and all $t > T_0$ would be sufficient for this.

We pay for the less precise estimate by a more involved construction on the orbifold. Let S be the surfaces described by Lemma 3.4 which separate the convex core from the geometrically finite ends. Cut M along S and let M_1 be the component containing $C(M)$. Glue two copies of M_1 along S. We claim that the resulting manifold N (the double of M_1) has lowest eigenvalue bounded away from zero. If so then the heat kernel estimates apply to the new manifold N and we deduce that the expected time a Brownian motion spends in U (the unit radius neighborhood of the S) is finite. By the Borel–Cantelli lemma this says that there are points x in M from which the probability of ever hitting S is strictly less than 1 (in fact, it is as small as we wish). Thus there is a point $x \in N \backslash U$ from which the chance of ever hitting U is less than $\frac{1}{2}$. But the two components of $N \backslash S$ are both exactly M_1. Thus Brownian motion in M_1 must have a positive probability of tending to infinity without ever hitting S.

This proves the result, except for verifying that N has first eigenvalue bounded away from 0. We will not verify this in detail, but simply note that since M has constant negative curvature and lowest eigenvalue >0, Buser's inequality (e.g. [19]) implies that the Cheeger constant for M is bounded away from zero. From this one proves that the Cheeger constant for the manifold with boundary M_1 is non-zero, and hence that the Cheeger constant for the doubled manifold N is non-zero. Then Cheeger's estimate says that the first eigenvalue for N is non-zero, as desired. □

6. Lower semi-continuity of Hausdorff dimension

In this section we will prove

THEOREM 6.1. *If G is a finitely generated Kleinian group and $\{G_n\}$ is a sequence of Kleinian groups converging algebraically to G then*

$$\dim(\Lambda(G)) \leqslant \liminf_n \dim(\Lambda(G_n)).$$

In particular, if G is geometrically infinite then this result and Theorem 1.2 imply that $\lim_n \dim(\Lambda(G_n)) = 2$.

If G is elementary then $\dim(G)=0$, and the result is trivial so we may assume that G is non-elementary. Similarly, if $\liminf \dim(\Lambda(G_n))=2$, there is nothing to do, so assume (after passing to a subsequence) that $\lim \dim(\Lambda(G_n))$ exists and is strictly less than 2. In particular, we may assume that all the G_n's are geometrically finite and hence $\delta(G_n)=\dim(\Lambda(G_n))$. If G is geometrically finite then $\delta(G)=\dim(\Lambda(G))$, so Theorem 6.1 follows from Corollary 2.4. Therefore we may also assume that G is geometrically infinite. The following result shows that this is impossible and completes the proof of Theorem 6.1.

THEOREM 6.2. *If $\{G_n\}$ is a sequence of geometrically finite Kleinian groups which converges algebraically to a finitely generated, geometrically infinite discrete group G then $\delta(G_n)\to 2$.*

This result follows from two known results:

THEOREM 6.3 (Canary). *If G is an n-generated, geometrically finite group then*

$$\lambda_0 \leqslant \frac{A_n}{\operatorname{vol}(C(M))},$$

where A_n is a constant that only depends on the number of generators of G.

Proof. This is essentially Theorem A of [19] except that there Canary proves

$$\lambda_0 \leqslant A\frac{\chi(\partial C(M))}{\operatorname{vol}(C(M))},$$

where A is an absolute constant and where χ denotes the Euler characteristic. However, the Euler characteristic of $\partial C(M)$ is the same as that of $\Omega(G)/G$, because there is always a homeomorphism between the two (e.g., see Epstein and Marden's paper [32]). By the Bers inequality [8] (a quantitative version of the Ahlfors finiteness theorem) the area, and hence the Euler characteristic, of $\Omega(G)/G$ can be bounded in terms of n, the number of generators of the group G. Thus Canary's result says that

$$\lambda_0 \leqslant \frac{A_n}{\operatorname{vol}(C(M))},$$

where A_n depends only on the number of generators. $\square$

THEOREM 6.4. *Suppose that $\{G_n\}$ are geometrically finite Kleinian groups such that*

$$\sup_n \operatorname{vol}(C(M)) < \infty.$$

If the sequence $\{G_n\}$ converges algebraically to a finitely generated, geometrically infinite, discrete group G then $\Lambda(G)$ has zero area.

Proof. This is an easy case of a result obtained by E. Taylor in [62] and is probably well known. Here we will sketch a proof which follows an argument given by Jørgensen and Marden in [40].

Suppose that $\Lambda(G)$ has positive area. We will derive a contradiction. Suppose that $\{G_n\}$ is a sequence of Kleinian groups converging algebraically to the Kleinian group G. The set of compact subsets of a compact metric space is itself compact with the Hausdorff metric

$$d(E,F)=\max_{z\in E}\operatorname{dist}(z,F)+\max_{w\in F}\operatorname{dist}(w,E),$$

so by passing to a subsequence (which we also denote $\{G_n\}$) we may assume that the sets $\Lambda_n=\Lambda(G_n)$ converge in the Hausdorff metric to a compact set Λ_∞.

We say that a sequence $\{G_n\}$ converges *polyhedrally* to a group H if H is discrete and for some $x_0\in\mathbf{B}$ the fundamental polyhedra (the Dirichlet polyhedron)

$$P(G_n)=\{z\in\mathbf{B}:\varrho(z,x_0)\leqslant\varrho(z,g(x_0))\text{ for all }g\in G_n\}$$

converge to $P(H)$ uniformly on compact subsets of $\mathbf{B}$. By Proposition 3.8 of [40] any algebraically converging subsequence has a polyhedrally convergent subsequence and the polyhedral limit group contains the algebraic limit group (but they need not be equal).

A third notion of convergence of groups is geometric convergence. Given a sequence of groups $\{G_n\}$ we define

$$\operatorname{Env}\{G_n\}=\{g\in\operatorname{PSL}(2,\mathbf{C}):g=\lim_n g_n,\ g_n\in G_n\},$$

and we say that $G_n\to H=\operatorname{Env}\{G_n\}$ *geometrically* if for every subsequence $\{G_{n_j}\}$, $\operatorname{Env}\{G_{n_j}\}=\operatorname{Env}\{G_n\}$. Proposition 3.10 of [40] says that G_n converges geometrically to H if and only if it converges polyhedrally to H.

Thus we may assume that we have groups $G\subset H$ such that

(1) $G_n\to G$ algebraically,

(2) $G_n\to H$ polyhedrally and geometrically,

(3) $\Lambda_n\to\Lambda_\infty$ in the Hausdorff metric.

If we can show that the convex core of $\mathbf{B}/H$ has finite volume, then we can obtain a contradiction as follows. Since $\Lambda(G)\subset\Lambda(H)$, the latter set also has positive area. If $\Lambda(H)\neq S^2$ then this contradicts Lemma 3.5. On the other hand, if $\Lambda(H)=S^2$ then we get $\operatorname{vol}(\mathbf{B}/H)<\infty$, which implies that the thick part of $\mathbf{B}/H$ is compact, which implies that the thick part of $\mathbf{B}/G_n$ is eventually compact, another contradiction.

Thus it suffices to prove that the convex core of $\mathbf{B}/H$ has finite volume.

We first claim that $\Lambda(H)\subset\Lambda_\infty$. If $\Lambda_\infty=S^2$ there is nothing to do, so we may assume that Λ_∞ is not the whole sphere. In this case we follow the proof of Proposition 4.2 of [40]. Let $\Omega=S^2\backslash\Lambda_\infty$ and suppose that K,K' are compact sets such that

$$K\subset\operatorname{int}(K')\subset K'\subset\Omega.$$

Suppose $h \in H$. Because $\{G_n\}$ converges geometrically to H we can write $h = \lim_n g_n$ with $g_n \in G_n$.

We claim that $h(K) \subset \Omega$. If not then $h(K)$ intersects Λ_∞, so $\mathrm{int}(h(K'))$ also hits Λ_∞. This implies that $\mathrm{int}(h(K')$ intersects Λ_n for all large enough n, say $n \geqslant N_1$. Therefore $\mathrm{int}(g_n(K'))$ hits Λ_m for all $m \geqslant N_1$, for all sufficiently large n, say $n \geqslant N_2$. So if $N_3 = \max(N_1, N_2)$, then $n \geqslant N_3$ implies

$$\mathrm{int}(g_n(K')) \cap \Lambda_n \neq \varnothing.$$

Therefore

$$\mathrm{int}(K') \cap g_n^{-1}(\Lambda_n) = \mathrm{int}(K') \cap \Lambda_n \neq \varnothing.$$

This is a contradiction, so we must have $h(K) \subset \Omega$. This implies $h(\Omega) \subset \Omega$. Since the same argument applies to h^{-1}, we see that $h(\Omega) = \Omega$, or equivalently, $h(\Lambda_\infty) = \Lambda_\infty$. Since Λ_∞ is a closed set which is invariant under the group H we must have $\Lambda(H) \subset \Lambda_\infty$ as desired (recall that the limit set is the smallest closed H-invariant set if H is non-elementary).

Since $\Lambda(H) \subset \Lambda_\infty$ the convex hull $C(\Lambda(H))$ of $\Lambda(H)$ in $\mathbf{B}$ is contained in the convex hull $C(\Lambda_\infty)$ of Λ_∞. The convex hulls $C(\Lambda_n)$ of the sets Λ_n converge, uniformly on compacta, to $C(\Lambda_\infty)$. Thus for any $R < \infty$,

$$\mathrm{vol}(C(\Lambda(H)) \cap P(H) \cap B(x_0, R)) \leqslant \liminf_n \mathrm{vol}(C(\Lambda_n) \cap P(G_n) \cap B(x_0, R)) \leqslant M,$$

by the lemma below. Thus $C(\Lambda(H)) \cap P(H)$ has finite volume, as desired. $\square$

The above inequality uses the the following lemma about the convergence of convex hulls.

LEMMA 6.5. *Suppose that $\{K_n\} \subset S^2$ are compact sets which converge in the Hausdorff metric to K. Then for any $R < \infty$,*

$$\lim_{n \to \infty} \mathrm{vol}(C(K_n) \cap B(0, R)) = \mathrm{vol}(C(K) \cap B(0, r)).$$

Proof. Let χ_n denote the characteristic function of $C(K_n)$ and χ_K the characteristic function of K. We first claim that

$$\lim_{n \to \infty} \chi_n(z) = \chi_K(z),$$

for all $z \notin \partial C(K)$. First suppose $z \in \mathrm{int}(C(K))$. Then there are four points $z_1, ..., z_4 \in \mathrm{int}(C(K))$, so that the convex hull of these points contains a neighborhood of z. Since $K_n \to K$ in the Hausdorff metric it is easy to see that for any $\varepsilon > 0$, and all large enough n,

$C(K_n)$ will contain points within ε of each of the four points, and thus will contain z (if ε is small enough).

On the other hand, suppose $z \notin C(K)$. Then there is a closed hyperbolic half-space which contains z but does not hit $C(K)$. This half-space hits S^2 in a closed disk which is a positive distance from K. Thus for all large enough N the disk does not hit K_n and so $z \notin C(K_n)$. This proves the claimed convergence.

Finally, since each point of $\partial C(K)$ is on the boundary of an open half-space which misses $\partial C(K)$, the Lebesgue density theorem implies that $\partial C(K)$ has zero volume. Thus the Lebesgue dominated convergence theorem implies the lemma. $\square$

Now that we have the two results, we can can finish the proof of Theorem 6.2. Suppose that G is a finitely generated, geometrically infinite discrete group and that $\{G_n\}$ are geometrically finite groups converging to G algebraically. If $\Lambda(G)$ has zero area then $\delta(G)=\dim(\Lambda(G))=2$ by Theorem 1.4 and $\delta(G_n)\to 2$ by Corollary 2.4. Thus we may assume that $\Lambda(G)$ has positive area. By Theorem 6.4 we must have $\mathrm{vol}(C(G_n))\to\infty$, so by Theorem 6.3 we get $\lambda_0(G_n)\to 0$. Thus by the Elstrodt–Patterson formula $\delta(G_n)\to 2$, as $n\to\infty$. This completes the proof of Theorem 6.2.

7. Teichmüller spaces

In this section we shall consider $\dim(\Lambda(G))$ as a function on the closure of the Teichmüller space $T(S)$ of a finite-type hyperbolic surface S.

Given a finite-type surface S (compact with a finite number of punctures, possibly none), the Teichmüller space $T(S)$ is the set of equivalence classes of quasiconformal mappings of S to itself. Each such is represented by a Beltrami differential $\tilde{\mu}$ which may be lifted to a Beltrami differential μ on the the upper half-plane $\mathbf{H}$. Let Γ be a Fuchsian group acting on $\mathbf{H}$ such that $\mathbf{H}/\Gamma=S$. There is a quasiconformal mapping F of the plane which fixes $0, 1, \infty$ and such that $\bar{\partial}F/\partial F=\mu$ on $\mathbf{H}$ and so that F is conformal on the lower half-plane. On the lower half-plane the Schwarzian derivative $S(F)$ satisfies

$$\|S(F)\| = \sup_z |\mathrm{Im}(z)|^2 |S(F)(z)| \leqslant 6 < \infty.$$

This realizes $T(S)$ as a bounded subset of a Banach space and gives a metric on $T(S)$. The closure of $T(S)$ with respect to this metric is denoted $\overline{T(S)}$ and the boundary $\partial T(S)$. Points of $\overline{T(S)}$ may be identified with certain Kleinian groups which are isomorphic to Γ. Moreover, convergence in the Teichmüller metric implies algebraic convergence of the groups. A group is called degenerate if $\Omega(G)$ has exactly one component and this component is simply-connected. Such groups must be geometrically infinite by a result of Greenberg [36]. $G \in \partial T(S)$ is called a cusp if there is a hyperbolic element in Γ which becomes parabolic in G. Bers showed that $\partial T(S)$ consists entirely of degenerate groups and cusps, and that degenerate groups form a dense G_δ-set in $\partial T(S)$ in [9] (in fact, the cusps lie on a countable union of real codimension 2 surfaces). McMullen [45] proved that there is a dense set of geometrically finite cusps in $\partial T(S)$.

Recall that Theorem 1.6 says that if $\{g_n\}$ converges algebraically to G then

$$\dim(\Lambda(G)) \leqslant \liminf_{n\to\infty} \dim(\Lambda(G_n)).$$

One special case where this holds is for $G \in \overline{T(S)}$, the closure of the Teichmüller space of a finite-type hyperbolic Riemann surface S. Since $\Lambda(G)$ is at most 2 and is lower semi-continuous, it is continuous whenever it takes the value 2 (i.e., at the geometrically infinite groups). Since these points are dense on the boundary of Teichmüller space, this function must be discontinuous at the geometrically finite cusps on the boundary. Thus,

COROLLARY 7.1. *Suppose that S is a hyperbolic Riemann surface of finite type. Then* $\dim(\Lambda(G))$ *is lower semi-continuous on* $\overline{T(S)}$ *and continuous everywhere except at the geometrically finite cusps in* $\partial T(S)$ *(where it must be discontinuous).*

This also shows that equality in Corollary 2.4 and Theorem 1.6 need not occur, because a geometrically finite cusp ($\delta(G)<2$) can be approximated by degenerate groups (so $\delta(G_n)\to 2$). The discontinuity at the geometrically finite cusps had been proved earlier by Taylor in [62]; he showed that for each geometrically finite cusp G there is a sequence $G_n \to G$ algebraically, but $G_n \to H$ geometrically where H is a geometrically finite group containing G and

$$\dim(\Lambda(G_n)) \to \dim(\Lambda(H)) > \dim(\Lambda(G)).$$

If f is lower semi-continuous then $\{f \leqslant \alpha\}$ is closed. Thus,

COROLLARY 7.2. *The set* $E_\alpha = \{G \in \overline{T(S)} : \dim(\Lambda(G)) \leqslant \alpha\}$ *is closed in* $\overline{T(S)}$. *The set* $F_\alpha = \{G \in \partial T(S) : \dim(\Lambda(G)) \leqslant \alpha < 2\}$ *is a closed, nowhere dense subset of* $\partial T(S)$.

Since a lower semi-continuous function takes a minimum on a compact set, $\dim(\Lambda)$ attains a minimum value on $\partial T(S)$. We prove in [12] that Λ is either totally disconnected, a circle or has dimension >1. Limit sets corresponding to points of $\partial T(S)$ cannot be circles or totally disconnected, so the third option holds for them. Thus,

COROLLARY 7.3. $\dim(\Lambda(G))$ *takes a minimum value on* $\partial T(S)$ *and this minimum is strictly larger than* 1.

It is not clear where the minimum occurs. Canary has suggested that it might occur at the cusp group corresponding to shrinking a minimum length geodesic on S to a parabolic, since this requires the "least" deformation of the Fuchsian group (in some sense). Since $\dim(\Lambda(G))$ takes a minimum on $\partial T(S)$ which is >1, any group in $\overline{T(S)}$ with small enough dimension must be quasi-Fuchsian. Thus,

COROLLARY 7.4. *Suppose that $\{G_n\}$ is a sequence of quasiconformal deformations of a Fuchsian group G (i.e., $\{G_n\}$ is a sequence in $T(S)$, $S=\mathbf{D}/G$). If $\dim(\Lambda(G_n))\to 1$, then $G_n \to G$.*

Proof. This is immediate from Theorem 6.1 and the fact (deduced from Theorem 1.7) that G is the only point in $\overline{T(S)}$ where $\dim(\Lambda)=1$. (Actually, since the groups involved are all geometrically finite we could use Corollary 2.4 and the geometrically finite case of Theorem 1.4 instead of the more difficult result Theorem 6.1.) □

COROLLARY 7.5. *Suppose that G is finitely generated and has a simply-connected invariant component Ω_0 (possibly not unique). Let $\Omega_0/G=S$. Then for any $\varepsilon>0$ there is a δ, depending only on S and ε, such that $\dim(\Lambda(G))<1+\delta$ implies that G is an ε-quasiconformal deformation of a Fuchsian group.*

Proof. If G is a quasi-Fuchsian group this follows from the previous result. Maskit [43] proved that a finitely generated Kleinian group with two invariant components is quasi-Fuchsian, so we may now assume that G is a b-group (i.e., has exactly one simply-connected, invariant component). If G is geometrically infinite then $\dim(\Lambda(G))=2$, which contradicts our assumption. Therefore, G must be geometrically finite. Abikoff [1] proved that every geometrically finite b-group covering S is on the boundary of the Teichmüller space $T(S)$, and so its dimension is bounded away from 1 by Corollary 7.3. □

It is not true that the ε in Corollary 7.5 can be taken to depend only on the topological type of S (e.g., the number of generators of G). For example, given a surface with punctures S, it is possible to use the combination theorems to construct a b-group G with $\dim(\Lambda(G))$ as close to one as we wish and so that Ω_0/G is homeomorphic to (though not conformally equivalent to) S.

Larman showed that there is an ε_0, such that if $\{D_j\}$ is a collection of three or more disjoint open disks then the dimension of $\overline{\mathbf{C}}\setminus\bigcup_j D_j$ is larger than $1+\varepsilon_0$. A careful reading of Larman's paper [42] shows that his proof gives

THEOREM 7.6. *There is an $\varepsilon_0>0$ such that if $\{D_j\}$ is any collection of three or more disjoint open ε_0-quasidisks, then $\dim(\mathbf{C}\setminus\bigcup_j D_j)>1+\varepsilon_0$.*

Recall that a *web group* is a finitely generated Kleinian group each of whose component subgroups is quasi-Fuchsian. Suppose that G is a web group. If $\Omega(G)$ has only two components then G is quasi-Fuchsian ([44]). So suppose that G has three or more components and let $\{G_1, ..., G_n\}$ be representatives of each conjugacy class of component subgroups. By the last corollary either one of these subgroups has a limit set with dimension $>1+\delta$ (where $\delta=\delta(G_n)$) or all the limit sets are ε_0-quasicircles. In the latter case, Larman's theorem implies that $\Lambda(G)$ has dimension bigger than $1+\varepsilon_0$. In either case the dimension is bounded away from 1 by a number which only depends on the conformal structure of $\Omega(G)/G$.

COROLLARY 7.7. *Suppose that G is a finitely generated web group which is not quasi-Fuchsian. Then* $\dim(\Lambda)>1+\varepsilon$ *where* $\varepsilon>0$ *depends only on the conformal types of the components of* $\Omega(G)/G$.

It is not true that the dimension of limit sets of proper web groups (i.e., not quasi-Fuchsian) is bounded uniformly away from 1. Canary, Minsky and Taylor have constructed examples of proper web groups (with a fixed number of generators) whose limit sets have dimension arbitrarily close to 1 (personal communication).

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CHRISTOPHER J. BISHOP
Department of Mathematics
SUNY at Stony Brook
Stony Brook, NY 11794-3651
U.S.A.
bishop@math.sunysb.edu

PETER W. JONES
Department of Mathematics
Yale University
New Haven, CT 06520
U.S.A.

Received November 29, 1994
Received in revised form January 28, 1997