THE INHOMOGENEOUS MINIMA OF BINARY QUADRATIC FORMS (IV)

BY

E. S. BARNES

1. The object of this paper is to show how the ideas of part III of this series may be applied to the problems considered in part I. No results from parts I and II are used, but a knowledge of sections 1 and 2 of part III is essential for an understanding of the method. For convenience of reference, the necessary definitions and theorems are repeated here.

Let $f(x, y) = a x^2 + b x y + c y^2$ be an indefinite binary quadratic form with real coefficients and discriminant $D = b^2 - 4ac > 0$. For any real numbers x_0, y_0 we define $M(f; x_0, y_0)$ to be the lower bound of $|f(x+x_0, y+y_0)|$ taken over all integer sets x, y. The inhomogeneous minimum $M(f)$ of $f(x, y)$ is now defined to be the upper bound of $M(f; x_0, y_0)$ over all sets x_0, y_0. It is convenient to identify pairs of real numbers with points of the Cartesian plane.

As in part III, we approach the problem of evaluating $M(f)$ geometrically, and consider an inhomogeneous lattice $\mathfrak{L}$ in the ξ, η-plane i.e. a set of points with coordinates

$$\begin{aligned} \xi &= \xi_0 + \alpha x + \beta y, \\ \eta &= \eta_0 + \gamma x + \delta y, \end{aligned} \tag{1.1}$$

where $\xi_0, \eta_0, \alpha, \beta, \gamma, \delta$ are real, $\alpha\delta - \beta\gamma \neq 0$, and x, y take all integral values. The determinant of $\mathfrak{L}$ is defined to be

$$\Delta = \Delta(\mathfrak{L}) = |\alpha\delta - \beta\gamma|.$$

If we suppose that $\mathfrak{L}$ has no point on either of the coordinate axes $\xi = 0$, $\eta = 0$, then $\mathfrak{L}$ has at least one divided cell: that is to say, there exist points A, B, C, D of $\mathfrak{L}$, one in each quadrant, such that $ABCD$ is a parallelogram of area Δ.

If we suppose further that there exists no lattice-vector of $\mathfrak{L}$ parallel to either of the coordinate axes (the condition for which is simply that the ratios α/β and γ/δ in (1.1) shall be irrational), then $\mathfrak{L}$ has an infinity of divided cells $A_n B_n C_n D_n$ $(-\infty < n < \infty)$. The relations between the vertices of successive cells are:

$$\left.\begin{aligned} A_{n+1} &= A_n - (h_n+1)\underset{\sim}{V}_n \\ B_{n+1} &= A_n - h_n \underset{\sim}{V}_n \\ C_{n+1} &= C_n + (k_n+1)\underset{\sim}{V}_n \\ D_{n+1} &= C_n + k_n \underset{\sim}{V}_n \end{aligned}\right\} \qquad (1.2)$$

where $\underset{\sim}{V}_n$ is the lattice-vector

$$\underset{\sim}{V}_n = A_n - D_n = B_n - C_n.$$

Here the convention is adopted that A_n, C_n lie one in each of the first and third quadrants, and B_n, D_n lie one in each of the second and fourth quadrants. The integers h_n, k_n are then uniquely determined by the cell $A_n B_n C_n D_n$; they are non-zero and have the same sign.

Let the ξ, η-coordinates of A_n, B_n, C_n, D_n be given by

$$C_n = (\xi_n, \eta_n), \quad B_n = (\xi_n + \alpha_n, \eta_n + \gamma_n), \quad D_n = (\xi_n + \beta_n, \eta_n + \delta_n),$$

$$A_n = (\xi_n + \alpha_n + \beta_n, \eta_n + \gamma_n + \delta_n),$$

so that

$$\underset{\sim}{V}_n = (\alpha_n, \gamma_n) = (-\beta_{n+1}, -\delta_{n+1}).$$

If we write

$$a_{n+1} = h_n + k_n, \qquad (1.3)$$

it follows that

$$\alpha_n = (-1)^n (\alpha_0 p_n - \beta_0 q_n),$$

$$\gamma_n = (-1)^n (\gamma_0 p_n - \delta_0 q_n),$$

where

$$p_{-1} = 0, \ q_{-1} = -1; \ p_0 = 1, \ q_0 = 0$$

and

$$\frac{p_n}{q_n} = a_1 - \frac{1}{a_2 -}\frac{1}{a_3 -}\cdots\frac{1}{a_n} = [a_1, a_2, \ldots, a_n] \qquad (n \geq 1),$$

$$\frac{(-q_{-n})}{(-p_{-n})} = a_0 - \frac{1}{a_{-1} -}\frac{1}{a_{-2} -}\cdots\frac{1}{a_{-n+2}} = [a_0, a_{-1}, \ldots, a_{-n+2}] \qquad (n \geq 2).$$

A passage to the limit gives

$$\frac{\alpha_0}{\beta_0} = \theta = [a_0, a_{-1}, a_{-2}, \ldots], \quad \frac{\delta_0}{\gamma_0} = \phi_0 = [a_1, a_2, a_3, \ldots],$$

(the continued fractions being necessarily convergent). If for each n we set

$$\varepsilon_n = h_n - k_n, \tag{1.4}$$

$$\theta_n = [a_n, a_{n-1}, a_{n-2}, \ldots], \quad \phi_n = [a_{n+1}, a_{n+2}, a_{n+3}, \ldots], \tag{1.5}$$

we have further

$$2\xi_0 + \alpha_0 + \gamma_0 = \sum_{n=1}^{\infty} (-1)^{n-1} \varepsilon_{-n} (\alpha_0 p_{-n} - \beta_0 q_{-n}) = \beta_0 \sum_{n=0}^{\infty} (-1)^n \frac{\varepsilon_{-n-1}}{\theta_{-1}\theta_{-2}\ldots\theta_{-n}}, \tag{1.6}$$

$$2\eta_0 + \gamma_0 + \delta_0 = \sum_{n=0}^{\infty} (-1)^n \varepsilon_n (\gamma_0 p_n - \delta_0 q_n) = \gamma_0 \sum_{n=0}^{\infty} (-1)^n \frac{\varepsilon_n}{\phi_1 \phi_2 \cdots \phi_n}. \tag{1.7}$$

To relate these ideas to that of the inhomogeneous minimum of a binary quadratic form $f(x, y)$, we observe that for points of $\mathfrak{L}$ we have

$$\xi\eta = (\xi_0 + \alpha_0 x + \beta_0 y)(\eta_0 + \gamma_0 x + \delta_0 y).$$

We call

$$f(x, y) = (\alpha_0 x + \beta_0 y)(\gamma_0 x + \delta_0 y)$$

a form associated with $\mathfrak{L}$; it has discriminant $D = (\alpha_0\delta_0 - \beta_0\gamma_0)^2 = \Delta^2$. If x_0, y_0 are any real numbers and we write

$$\xi_0 = \alpha_0 x_0 + \beta_0 y_0, \quad \eta_0 = \gamma_0 x_0 + \delta_0 y_0,$$

it is clear that $f(x + x_0, y + y_0)$ takes the same set of values for integral x, y as the product $\xi\eta$ for points of $\mathfrak{L}$. Thus

$$M(f; x_0, y_0) = \underset{(\xi,\eta)\in\mathfrak{L}}{\text{g.l.b.}} \; |\xi\eta|.$$

As was proved in part III, Theorem 5, this lower bound is also the lower bound of $|\xi\eta|$ taken over the vertices A_n, B_n, C_n, D_n of the chain of divided cells. It follows that

$$M(f; x_0, y_0) = \underset{n}{\text{g.l.b.}} \min \{|\xi_n \eta_n|, \; |(\xi_n + \alpha_n)(\eta_n + \gamma_n)|, \; |(\xi_n + \beta_n)(\eta_n + \delta_n)|, \; |(\xi_n + \alpha_n + \beta_n)(\eta_n + \gamma_n + \delta_n)|\}. \tag{1.8}$$

It follows also that $f(x + x_0, y + y_0)$ is equivalent to

$$f_n(x + x_n, y + y_n) = (\xi_n + \alpha_n x + \beta_n y)(\eta_n + \gamma_n x + \delta_n y) \tag{1.9}$$

for each n by a unimodular integral affine transformation, i.e. a transformation

$$x \to px + qy + a,$$
$$y \to rx + sy + b,$$

where $ps - qr = \pm 1$ and p, q, r, s, a, b are integral.

2. *I*-reduced forms

We shall say that an indefinite binary quadratic form $\phi(x, y)$, of determinant $D = \Delta^2 > 0$, is inhomogeneously reduced, or *I-reduced*, if it may be factorized in the form

$$\phi(x, y) = \lambda(\theta x + y)(x + \phi y),$$

where

$$|\theta| > 1, \quad |\phi| > 1.$$

By comparison of determinants it is clear that

$$\phi(x, y) = \pm \frac{\Delta}{|\theta\phi - 1|}(\theta x + y)(x + \phi y). \tag{2.1}$$

Lemma 2.1. *If $f(x, y)$ has integral coefficients and does not represent zero, there are only finitely many I-reduced forms equivalent to $f(x, y)$.*

Proof. Let $\phi(x, y) = ax^2 + bxy + cy^2$ be I-reduced and equivalent to $f(x, y)$. Then it is easily seen from the definition that each of $2|a|$, $2|c|$ lies strictly between $|b - \Delta|$ and $|b + \Delta|$, where Δ^2 is the discriminant of f and therefore of ϕ. Since the forms $ax^2 \pm bxy + cy^2$ are equivalent, it is sufficient to suppose that $b \geq 0$. Thus

$$|b - \Delta| < 2|a|, \qquad 2|c| < |b + \Delta|. \tag{2.2}$$

If now $0 \leq b < \Delta$, the relation

$$-4ac = \Delta^2 - b^2$$

shows that, for each b, there are only a finite number of possible values of each of a and c.

We cannot have $b = \Delta$, since f is not a zero form, so that it remains to consider $b > \Delta$. Since $4ac = b^2 - \Delta^2$, a and c have the same sign. Writing $k = c - a$, we have from (2.2)

$$|k| = |c - a| < \tfrac{1}{2}(b + \Delta) - \tfrac{1}{2}(b - \Delta) = \Delta;$$

also

$$(a + c)^2 - k^2 = 4ac = b^2 - \Delta^2,$$

whence

$$\Delta^2 - k^2 = (b - a - c)(b + a + c).$$

For each of the finite number of possible values of k, there are only a finite number of choices of $b - a - c$ and $b + a + c$. It follows at once that each of a, b and c can take only a finite number of values.

Lemma 2.2. *If $f(x, y)$ is an indefinite quadratic form which does not represent zero, there exists an I-reduced form equivalent to it.*

Proof. It is well known that $f(x, y)$ is equivalent to a form reduced in the sense of Gauss, i.e. a form (2.1) satisfying the more stringent inequalities

$$\theta < -1, \quad \phi > 1.$$

Now it was shown in part III, § 4, that any irrational α with $|\alpha| > 1$ may be expanded (in infinitely many ways) as a continued fraction

$$\alpha = a_1 - \frac{1}{a_2 -} \frac{1}{a_3 -} \cdots = [a_1, a_2, a_3, \ldots],$$

where the integers a_n satisfy the conditions: $|a_n| \geq 2$; a_n is not constantly equal to 2 or to -2 for large n. Conversely, if $\{a_n\}$ is any sequence of integers satisfying these conditions, the continued fraction $[a_1, a_2, a_3, \ldots]$ converges to a real number α with $|\alpha| > 1$.

From this point, the theory of chains of I-reduced forms and the associated continued fractions may be developed in a similar way to any of the classical theories of reduced forms. The important difference is that now there exist infinitely many chains of reduced forms equivalent to a given form; moreover, any single chain need not contain all the I-reduced forms equivalent to a given form.

Now let $\{a_n\}$ $(-\infty < n < \infty)$ be any chain of integers satisfying the conditions:

$$\text{(A)} \quad \begin{cases} |a_n| \geq 2; \\ a_n \text{ is not constantly equal to 2 or to } -2 \text{ for large } n \text{ of either sign.} \end{cases}$$

We can then associate with $\{a_n\}$ a chain of integer pairs $\{h_n, k_n\}$ satisfying the conditions

$$\text{(E)}: \begin{cases} \text{(i)} \quad h_n + k_n = a_{n+1}; \\ \text{(ii)} \quad h_n \text{ and } k_n \text{ are non-zero and have the same sign;} \\ \text{(iii)} \quad \text{neither } h_n \text{ nor } k_n \text{ is constantly equal to } -1 \text{ for large } n \text{ of either sign;} \\ \text{(iv)} \quad \text{the relation } h_{n+2r} = k_{n+2r+1} = 1 \text{ does not hold, for any } n, \text{ either for all} \\ \qquad r \geq 0 \text{ or for all } r \leq 0. \end{cases}$$

As was shown in part III, Lemma 1, the chain $\{h_n, k_n\}$ derived from the successive construction of divided cells of a lattice, according to the formula (1.2), satisfy the conditions (E) (ii), (iii), (iv); and then (E) (i) implies that the chain $\{a_n\}$ satisfies the condition (A). Thus (A) and (E) are *necessary* conditions for the constants h_n, k_n, a_n to correspond to a chain of divided cells of a lattice. We shall now show that they are *sufficient*.

We first note, as remarked above, that if $\{a_n\}$ satisfies (A), then the numbers θ_n, ϕ_n of (1.5) are well defined and satisfy $|\theta_n|>1$, $|\phi_n|>1$.

We next prove that the series (1.6) and (1.7), with $\varepsilon_n = h_n - k_n$, are absolutely convergent provided that $\{h_n, k_n\}$ satisfy E (i), (ii).

Lemma 2.3. *The series* $\sum_{n=1}^{\infty} \frac{|a_{n+1}|-2}{|\phi_1 \phi_2 \cdots \phi_n|}$ *is convergent and*

$$\sum_{n=0}^{\infty} \frac{|a_{n+1}|-2}{|\phi_1 \phi_2 \cdots \phi_n|} \leq |\phi_0| - 1. \tag{2.3}$$

Equality holds in (2.3) *if and only if* a_n *has constant sign.*

Proof. We have

$$\phi_{n-1} = [a_n, a_{n+1}, \ldots] = a_n - \frac{1}{\phi_n},$$

and so

$$|\phi_{n-1}| - 1 \geq |a_n| - \frac{1}{|\phi_n|} - 1 = |a_n| - 2 + \frac{|\phi_n| - 1}{|\phi_n|}. \tag{2.4}$$

Using this relation for $n = 1, 2, \ldots$, we have

$$|\phi_0| - 1 \geq |a_1| - 2 + \frac{|a_2| - 2}{|\phi_1|} + \cdots + \frac{|a_n| - 2}{|\phi_1 \phi_2 \cdots \phi_{n-1}|} + \frac{|\phi_n| - 1}{|\phi_1 \phi_2 \cdots \phi_n|}.$$

The first assertion of the lemma follows at once, since $|\phi_n| - 1 = 0$.

There is clearly inequality in (2.3) if there is inequality in (2.4) for any value of n; and equality holds in (2.4) if and only if a_n and ϕ_n have the same sign, i.e. if and only if a_n and a_{n+1} have the same sign, since always $|\phi_n - a_{n+1}| = \frac{1}{|\phi_{n+1}|} < 1$. Thus equality holds in (2.3) if and only if a_n has constant sign.

Lemma 2.4. *If* $\{h_n, k_n\}$ *satisfies the conditions* (E), *the series*

$$\sum_{n=0}^{\infty} (-1)^n \frac{\varepsilon_n}{\phi_1 \phi_2 \cdots \phi_n}$$

of (1.7) *is absolutely convergent, and its sum is numerically less than* $|\phi_0| - 1$.

Proof. Since $\varepsilon_n = h_n - k_n$ and h_n, k_n satisfy (E) (i) and (ii), we have

$$|\varepsilon_n| \leq |a_{n+1}| - 2$$

and so, using Lemma 2.3,

$$\left| \sum_{n=0}^{\infty} (-1)^n \frac{\varepsilon_n}{\phi_1 \phi_2 \cdots \phi_n} \right| \leq \sum_{n=0}^{\infty} \frac{|a_{n+1}| - 2}{|\phi_1 \phi_2 \cdots \phi_n|} \leq |\phi_0| - 1. \tag{2.5}$$

It remains to show that equality cannot hold throughout (2.5). By Lemma 2:3, there is certainly strict inequality unless all a_n have the same sign. Suppose then first that $a_n < 0$ for all n. Then $\phi_n < 0$ and so $\phi_1 \phi_2 \dots \phi_n = (-1)^n |\phi_1 \phi_2 \dots \phi_n|$; hence equality in (2.5) implies that $\pm \varepsilon_n = |a_{n+1}| - 2 = -a_{n+1} - 2$ for all n (with a fixed determination of $\pm$). But this gives

$$\textit{either} \quad h_n = \tfrac{1}{2}(\varepsilon_n + a_{n+1}) = -1 \quad \textit{or} \quad k_n = \tfrac{1}{2}(a_{n+1} - \varepsilon_n) = -1$$

for all n, contradicting E (iii).

Suppose next that $a_n > 0$ for all n, so that $\phi_n > 0$. Then equality in (2.5) implies that $\pm(-1)^n \varepsilon_n = a_{n+1} - 2$ for all n. But this gives

$$\textit{either} \quad h_{2n+1} = \tfrac{1}{2}(\varepsilon_{2n+1} + a_{2n+2}) = 1, \quad k_{2n} = \tfrac{1}{2}(a_{2n+1} - \varepsilon_{2n}) = 1,$$

$$\textit{or} \quad h_{2n} = \tfrac{1}{2}(a_{2n+1} + \varepsilon_{2n}) = 1, \quad k_{2n+1} = \tfrac{1}{2}(a_{2n+2} - \varepsilon_{2n+1}) = 1$$

for all n, contradicting E (iv). This completes the proof of the lemma.

Precisely similar results hold, by symmetry, for the series (1.6); and these results are clearly independent of the starting-point of the enumeration of the chains. Thus for each n we may define numbers ξ_n', η_n' by the formulae

$$2\xi_n' + \theta_n + 1 = \varepsilon_{n-1} - \frac{\varepsilon_{n-2}}{\theta_{n-1}} + \frac{\varepsilon_{n-3}}{\theta_{n-1}\theta_{n-2}} - \cdots, \tag{2.6}$$

$$2\eta_n' + 1 + \phi_n = \varepsilon_n - \frac{\varepsilon_{n+1}}{\phi_{n+1}} + \frac{\varepsilon_{n+2}}{\phi_{n+1}\phi_{n+2}} - \cdots, \tag{2.7}$$

provided that the chain $\{h_n, k_n\}$ satisfies (E); and then we have

$$|2\xi_n' + \theta_n + 1| < |\theta_n| - 1, \tag{2.8}$$

$$|2\eta_n' + 1 + \phi_n| < |\phi_n| - 1. \tag{2.9}$$

It may be immediately deduced from these inequalities that

$$\left.\begin{aligned} &\operatorname{sgn} \xi_n' = \operatorname{sgn} (\xi_n' + 1) = -\operatorname{sgn} \theta_n, \\ &\operatorname{sgn} (\xi_n' + \theta_n) = \operatorname{sgn} (\xi_n' + \theta_n + 1) = \operatorname{sgn} \theta_n; \end{aligned}\right\} \tag{2.10}$$

$$\left.\begin{aligned} &\operatorname{sgn} \eta_n' = \operatorname{sgn} (\eta_n' + 1) = -\operatorname{sgn} \phi_n, \\ &\operatorname{sgn} (\eta_n' + \phi_n) = \operatorname{sgn} (\eta_n' + \phi_n + 1) = \operatorname{sgn} \phi_n. \end{aligned}\right\} \tag{2.11}$$

The points

$$C_n = (\xi_n', \eta_n'), \quad B_n = (\xi_n' + \theta_n, \eta_n' + 1), \quad D_n = (\xi_n' + 1, \eta_n' + \phi_n),$$
$$A_n = (\xi_n' + \theta_n + 1, \eta_n' + 1 + \phi_n)$$

therefore lie one in each of the four quadrants and are the vertices of a divided cell of the lattice

$$\begin{cases} \xi = \xi_n' + \theta_n x + y \\ \eta = \eta_n' + x + \phi_n y. \end{cases}$$

Next, from the formulae (2.6), (2.7) we have

$$2\xi_{n+1}' + \theta_{n+1} + 1 = \varepsilon_n - \frac{\varepsilon_{n-1}}{\theta_n} + \cdots = \varepsilon_n - \frac{1}{\theta_n}(2\xi_n' + \theta_n + 1),$$

$$2\eta_n' + 1 + \phi_n = \varepsilon_n - \frac{1}{\phi_{n+1}}(2\eta_{n+1}' + 1 + \phi_{n+1}).$$

From these, using the identities

$$\theta_{n+1} = a_{n+1} - \frac{1}{\theta_n}, \quad \phi_n = a_{n+1} - \frac{1}{\phi_{n+1}}, \quad 2k_n = a_{n+1} - \varepsilon_n,$$

we deduce at once that

$$-\theta_n \xi_{n+1}' = \xi_n' + (k_n + 1)\theta_n, \tag{2.12}$$

$$-\frac{\eta_{n+1}'}{\phi_{n+1}} = \eta_n' + k_n + 1. \tag{2.13}$$

A simple calculation now shows that the four points

$$A_n - (h_n + 1)\underline{V}_n, \quad A_n - h_n \underline{V}_n, \quad C_n + (k_n + 1)\underline{V}_n, \quad C_n + k_n \underline{V}_n$$

(where $\underline{V}_n = A_n - D_n$) are

$$\{\beta(\xi_{n+1}' + \theta_{n+1}' + 1),\ \gamma(\eta_{n+1}' + 1 + \phi_{n+1}')\},\ \{\beta(\xi_{n+1}' + \theta_{n+1}),\ \gamma(\eta_{n+1}' + 1)\},$$
$$\{\beta\xi_{n+1}', \gamma\eta_{n+1}'\},\ \{\beta(\xi_{n+1}' + 1),\ \gamma(\eta_{n+1}' + \phi_{n+1})\},$$

where $\beta = -\theta_n$, $\delta = -\dfrac{1}{\phi_{n+1}}$, and so are again the vertices of a divided cell of the lattice.

Thus the divided cell $A_{n+1}B_{n+1}C_{n+1}D_{n+1}$ is obtained from the cell $A_nB_nC_nD_n$ by precisely the formulae (1.2).

It follows from the above results that, given any chain $\{h_n, k_n\}$ satisfying conditions (E), there exists a lattice $\mathfrak{L}$ whose chain of divided cells satisfies the recurrence relations (1.2). Moreover, $\mathfrak{L}$ is uniquely determined, apart from a constant multiple of each coordinate.

In particular, let $f(x, y)$ be a binary quadratic form of discriminant $\Delta^2 > 0$. Let $\{f_n\}$ be any chain of I-reduced forms equivalent to $f(x, y)$ and $\{a_n\}$ the associated chain of integers satisfying (A). Let $\{h_n, k_n\}$ be any chain satisfying (E). Then, for each n, the numbers ξ_n', η_n' given by (2.6), (2.7) define an inhomogeneous form

$$f_n(x+x_n, y+y_n) = \pm\frac{\Delta}{|\theta_n\phi_n - 1|}(\xi_n' + \theta_n x + y)(\eta_n' + x + \phi_n y) \tag{2.14}$$

equivalent to $f(x+x_0', y+y_0')$ for some x_0', y_0'. Conversely, corresponding to any real numbers x_0', y_0', there exist chains $\{f_n\}$, $\{a_n\}$, $\{h_n, k_n\}$ satisfying (A) and (E) such that $f(x+x_0', y+y_0')$ is equivalent to the form (2.14) for each n.

3. The determination of $M(f)$

It is more convenient for the applications to quadratic forms to work with the numbers ξ_n', η_n' of § 2 rather than the ξ_n, η_n of § 1; these are clearly connected by the relations

$$\xi_n = \beta_n\xi_n', \quad \eta_n = \gamma_n\eta_n', \quad \beta_n\gamma_n = \pm\frac{\Delta}{|\theta_n\phi_n - 1|}.$$

We therefore drop the prime from ξ_n', η_n' in all that follows.

For any fixed form $f(x, y)$ we write

$$M(P) = M(f; x_0, y_0),$$

where P is the point (x_0, y_0) of the x, y-plane. Then (1.8) gives

$$M(P) = \underset{n}{\text{g.l.b.}}\ M_n(P), \tag{3.1}$$

where

$$M_n(P) = \frac{\Delta}{|\theta_n\phi_n - 1|}\min\{|\xi_n\eta_n|,\ |(\xi_n+\theta_n)(\eta_n+1)|,$$
$$|(\xi_n+1)(\eta_n+\phi_n)|,\ |(\xi_n+\theta_n+1)(\eta_n+1+\phi_n)|. \tag{3.2}$$

The results of § 2 show that the set of values of $M(f; x_0, y_0)$ for real x_0, y_0 coincides with the set of values of $M(P)$ for all possible chains $\{h_n, k_n\}$ associated with $f(x, y)$. Hence

$$M(f) = \text{l.u.b.}\ M(P), \tag{3.3}$$

where the upper bound is taken over all chains associated with $f(x, y)$.

To determine the relations between the successive inhomogeneous forms

$$f_n(x+x_n, y+y_n),$$

arising from any chain $\{h_n, k_n\}$, we first note that, by (2.12) and (2.13),

$$-\theta_n\,\xi_{n+1}=\xi_n+\theta_n\,(k_n+1), \tag{3.4}$$

$$-\eta_{n+1}=\phi_{n+1}\,(\eta_n+k_n+1). \tag{3.5}$$

Hence, if we set

$$x'=y,$$

$$y'=-x-a_{n+1}\,y+k_n+1,$$

we have

$$\begin{aligned}\eta_{n+1}+x'+\phi_{n+1}y' &= -\phi_{n+1}(\eta_n+k_n+1)+y+\phi_{n+1}(-x-a_{n+1}y+k_n+1)\\ &= -\phi_{n+1}\left\{x+\left(a_{n+1}-\frac{1}{\phi_{n+1}}\right)y+\eta_n\right\}\\ &= -\phi_{n+1}\,(\eta_n+x+\phi_n\,y)\end{aligned}$$

and similarly

$$\xi_{n+1}+\theta_{n+1}x'+y'=-\frac{1}{\theta_n}\,(\xi_n+\theta_n\,x+y);$$

thus

$$\begin{aligned}f_{n+1}\,(x'+x_{n+1},\,y'+y_{n+1}) &= \pm\frac{\Delta}{|\,\theta_{n+1}\,\phi_{n+1}-1\,|}\,(\xi_{n+1}+\theta_{n+1}\,x'+y')\,(\eta_{n+1}+x'+\phi_{n+1}\,y')\\ &= \pm\frac{\Delta}{|\,\theta_n\,\phi_n-1\,|}\,(\xi_n+\theta_n\,x+y)\,(\eta_n+x+\phi_n\,y)\\ &= f_n\,(x+x_n,\,y+y_n).\end{aligned}$$

Also, using (3.4) and (3.5) again, we have

$$\theta_{n+1}\,x_{n+1}+y_{n+1}=\xi_{n+1}=-(x_n+a_{n+1}\,y_n+1+k_n)+\theta_{n+1}\,y_n,$$

$$x_{n+1}+\phi_{n+1}\,y_{n+1}=\eta_{n+1}=-(x_n+a_{n+1}\,y_n+1+k_n)\,\phi_{n+1}+y_n,$$

so that

$$\left.\begin{aligned}x_{n+1}&=y_n,\\ y_{n+1}&=-(x_n+a_{n+1}\,y_n+1+k_n)\end{aligned}\right\}. \tag{3.6}$$

In the practical problem of finding the numerical value of $M(f)$ for a given form $f(x, y)$, the success of the method depends upon the rapid convergence of the series (2.6), (2.7) defining ξ_n and η_n; the error made in replacing them by a partial sum is easily estimated from (2.8), (2.9). We have

$$\begin{aligned}2\,\eta_n+1+\phi_n=\varepsilon_n-\frac{\varepsilon_{n+1}}{\phi_{n+1}}+\dots+\frac{(-1)^r\,\varepsilon_{n+r}}{\phi_{n+1}\,\phi_{n+2}\cdots\phi_{n+r}}\\ +\frac{(-1)^{r+1}}{\phi_{n+1}\cdots\phi_{n+r+1}}\,(2\,\eta_{n+r+1}+\phi_{n+r+1}+1),\end{aligned}$$

and so, by (2.9),

$$2\eta_n + 1 + \phi_n = \varepsilon_n - \frac{\varepsilon_{n+1}}{\phi_{n+1}} + \dots + \frac{(-1)^r \varepsilon_{n+r}}{\phi_{n+1} \cdots \phi_{n+r}}$$

$$+ \left\| \frac{1}{\phi_{n+1} \cdots \phi_{n+r}} \left(1 - \frac{1}{|\phi_{n+r+1}|}\right) \right\|. \quad (3.7)$$

Here we use the (permanent) notation $\|x\|$ for a quantity whose modulus does not exceed $|x|$. In the same way we find

$$2\xi_n + \theta_n + 1 = \varepsilon_{n-1} - \frac{\varepsilon_{n-2}}{\phi_{n-1}} + \dots + \frac{(-1)^{r+1} \varepsilon_{n-r}}{\theta_{n-1} \dots \theta_{n-r+1}}$$

$$+ \left\| \frac{1}{\theta_{n-1} \dots \theta_{n-r+1}} \left(1 - \frac{1}{|\theta_{n-r}|}\right) \right\|. \quad (3.8)$$

In order to avoid excessive enumeration of cases, we now justify some formal operations on the chains $\{a_n\}$, $\{\varepsilon_n\}$.

Lemma 3.1. *The value of $M(P)$ is unaltered by any of the following operations:*

(i) *reversing the chains $\{a_{n+1}\}$, $\{\varepsilon_n\}$ about the same point;*

(ii) *changing the signs of all ε_n;*

(iii) *changing the signs of all a_n and of alternate ε_n.*

Proof. (i) Reversing the chains is equivalent to interchanging ξ_n, η_n and θ_n, ϕ_n; this does not affect the set of values of $M_n(P)$.

(ii) On replacing ε_n by $-\varepsilon_n$ for all n, we obtain the values $\bar{\xi}_n, \bar{\eta}_n$ in place of ξ_n, η_n, where

$$2\bar{\xi}_n + 1 + \theta_n = -(2\xi_n + 1 + \theta_n),$$

$$2\bar{\eta}_n + 1 + \phi_n = -(2\eta_n + 1 + \phi_n).$$

Hence $\bar{\xi}_n \bar{\eta}_n$, $(\bar{\xi}_n + \theta_n)(\bar{\eta}_n + 1)$, $(\bar{\xi}_n + 1)(\bar{\eta}_n + \phi_n)$, $(\bar{\xi}_n + \theta_n + 1)(\bar{\eta}_n + 1 + \phi_n)$ are respectively equal to $(\xi_n + \theta_n + 1)(\eta_n + 1 + \phi_n)$, $(\xi_n + 1)(\eta_n + \phi_n)$, $(\xi_n + \theta_n)(\eta_n + 1)$, $\xi_n \eta_n$. Thus $M_n(P)$ is unaltered.

(iii) On replacing a_n by $-a_n$ and ε_n by $(-1)^n \varepsilon_n$ for all n, we obtain values $\bar{\theta}_n, \bar{\phi}_n, \bar{\xi}_n, \bar{\eta}_n$ in place of $\theta_n, \phi_n, \xi_n, \eta_n$, where

$$\bar{\theta}_n = -\theta_n, \quad \bar{\phi}_n = -\phi_n,$$

$$2\bar{\eta}_n + 1 + \bar{\phi}_n = \sum_{n=1}^{\infty} (-1)^r \frac{(-1)^{n+r} \varepsilon_{n+r}}{\bar{\phi}_{n+1} \cdots \bar{\phi}_{n+r}}$$

$$= (-1)^n \sum_{r=0}^{\infty} (-1)^r \frac{\varepsilon_{n+r}}{\phi_{n+1} \cdots \phi_{n+r}} = (-1)^n (2\eta_n + 1 + \phi_n)$$

and similarly $2\bar{\xi}_n+\bar{\theta}_n+1=(-1)^{n-1}(2\xi_n+\theta_n+1)$. It is now easily verified that the four quantities $\bar{\xi}_n\bar{\eta}_n, \ldots$ are merely a permutation of $-\xi\eta, \ldots$, so that $M_n(P)$ is unaltered.

Using (ii), we see that the same result holds if we replace a_n by $-a_n$ and ε_n by $(-1)^{n+1}\varepsilon_n$.

A very useful result which enables us to eliminate by inspection most chains $\{f_n\}$ is given by:

Lemma 3.2. *For any chains $\{a_n\}$, $\{\varepsilon_n\}$, and for all n, we have*

$$M(P)\le M_n(P)\le\frac{\Delta}{4\,|\theta_n\phi_n-1|}\min\{|(\theta_n-1)(\phi_n-1)|,\ |(\theta_n+1)(\phi_n+1)|\}. \tag{3.9}$$

Proof. By (2.10), $\xi_n+\theta_n$ and ξ_n+1 have opposite signs, so that

$$2\,|(\xi_n+\theta_n)(\xi_n+1)|^{\frac{1}{2}}\le|\xi_n+\theta_n|+|\xi_n+1|=|(\xi_n+\theta_n)-(\xi_n+1)|=|\theta_n-1|;$$

similarly

$$2\,|(\eta_n+1)(\eta_n+\phi_n)|^{\frac{1}{2}}\le|\phi_n-1|,$$

whence

$$4\,|(\xi_n+\theta_n)(\eta_n+1)(\xi_n+1)(\eta_n+\phi_n)|^{\frac{1}{2}}\le|(\theta_n-1)(\phi_n-1)|.$$

In the same way, using the fact that ξ_n and $\xi_n+\theta_n+1$ have opposite signs, as do also η_n and $\eta_n+1+\phi_n$, we find

$$4\,|\xi_n\eta_n(\xi_n+\theta_n+1)(\eta_n+1+\phi_n)|^{\frac{1}{2}}\le|(\theta_n+1)(\phi_n+1)|.$$

From these results and the definition of $M_n(P)$, (3.9) follows immediately.

From Lemma 3.2 we can deduce a simple inequality for $M(P)$ or $M(f)$, which is in a sense best possible.[1]

Lemma 3.3. *Suppose that the chain $\{f_n\}$ contains the form $f(x,y)=ax^2+bxy+cy^2$. Then, for any choice of $\{\varepsilon_n\}$ we have for the corresponding point P,*

$$M(P)\le\min\tfrac{1}{4}\,|a\pm b+c|=\min|f(\tfrac{1}{2},\pm\tfrac{1}{2})|. \tag{3.10}$$

Equality can hold in (3.10) *only if $\varepsilon_n=0$ for all n, so that $P=(x_0,y_0)\equiv(\frac{1}{2},\frac{1}{2})$* (mod 1).

---

[1] This result is closely related to an estimate for $M(f)$ found in Barnes [1]. There it was shown that, if $f(x,y)=ax^2+bxy+cy^2$ (not necessarily reduced in any sense) and P is any point, then

$$M(P)\le\tfrac{1}{4}\max\{|a|,|c|,\min|a\pm b+c|\},$$

where equality is possible only if $P\equiv(\frac{1}{2},\frac{1}{2})$, $(0,\frac{1}{2})$ or $(\frac{1}{2},0)$ (mod 1).

Proof. Since

$$f_n(x, y) = \pm \frac{\Delta}{\theta_n \phi_n - 1} (\theta_n x + y)(x + \phi_n y),$$

(3.10) follows immediately from (3.9). Clearly we can have equality in (3.9) only if there is equality in the arithmetic-geometric mean; in either case this implies that

$$\xi_n + \theta_n = -(\xi_n + 1), \quad \eta_n + \phi_n = -(\eta_n + 1);$$

thus $2\xi_n + 1 + \theta_n = 2\eta_n + 1 + \phi_n = 0$, whence $\varepsilon_n = 0$ for all n. Finally, if this relation holds, we have

$$f_n(x + x_n, y + y_n) = \pm \frac{\Delta}{\theta_n \phi_n - 1} \{\theta_n (x - \tfrac{1}{2}) + (y - \tfrac{1}{2})\} \{(x - \tfrac{1}{2}) + \phi_n (y - \tfrac{1}{2})\},$$

so that $P \equiv (\frac{1}{2}, \frac{1}{2}) \pmod 1$.

4. In this and the following section we apply the methods established above to the determination of $M(f)$ for the particular forms $x^2 - 19y^2$ and $x^2 - 46y^2$. These two norm-forms had proved difficult to handle by the technique of Part I of this series, and so were examined to test the practical efficiency of the present methods.

In the evaluation of $M(f)$ for a given form $f(x, y)$, the first step is to find the I-reduced forms equivalent to it; for forms with integral coefficients, this is perhaps best carried out by the method of Lemma 2.1.

Now if $g(x, y)$ is I-reduced, so also are the equivalent forms $g(x, -y)$, $g(y, x)$, $g(y, -x)$; also, any chain containing one of these latter forms may be converted into a chain containing $g(x, y)$ by reversing the chain $\{a_n\}$, or replacing it by $\{-a_n\}$, or both [cf. Lemma 3.1]. It is therefore sufficient to list only those I-reduced forms $ax^2 + bxy + cy^2$ with

$$b \geq 0, \quad |a| \leq |c|. \tag{4.1}$$

For these we have the factorization

$$\left.\begin{aligned} ax^2 + bxy + cy^2 = \lambda(\alpha x + y)(x + \beta y), \quad |\alpha| > 1, \quad |\beta| > 1 \\ \alpha = \frac{b + \Delta}{2c}, \quad \beta = \frac{b + \Delta}{2a}, \quad \Delta^2 = b^2 - 4ac. \end{aligned}\right\} \tag{4.2}$$

with

For convenience, we shall write (a, b, c) for the form $ax^2 + bxy + cy^2$.

Theorem 1. *If* $f(x, y) = x^2 - 19y^2$, *then*

$$M(f) = \tfrac{170}{171}.$$

Note. The value of $M(f)$ is incorrectly given as 31/38 in Part I (Table, p. 315).

The proof of Theorem 1 will be given as a series of lemmas, representing the successive steps in the argument. For convenience we write

$$\delta=\sqrt{19}=\tfrac{1}{2}\Delta.$$

Lemma 4.1. *If $M(P)>\frac{3}{4}$, the chain $\{a_n\}$ is*

$$\{\overset{\times}{3}, 5, 3, -\overset{\times}{8}\} \tag{4.3}$$

or its negative (where the crosses denote infinite repetition of the period 3, 5, 3, −8).

Proof. If $M(P)>\frac{3}{4}$, Lemma 3.3 shows that only forms (a, b, c) with $|a+b+c|\geq 4$, $|a-b+c|\geq 4$ can occur in the chain $\{f_n\}$. With the restrictions and notation of (4.1), (4.2), all such forms are

$$g_1=(1, 8, -3), \qquad \alpha_1=-\frac{\delta+4}{3}, \quad \beta=\delta+4;$$

$$g_2=(-2, 10, -3), \quad \alpha_2=-\frac{\delta+5}{3}, \quad \beta_2=-\frac{\delta+5}{2}.$$

Also we have

$$\alpha_1=-\frac{\delta+4}{3}=-3+\frac{5-\delta}{3}=[-3, \beta_2],$$

$$\beta_1=\quad \delta+4=\quad 8+\delta-4=[8, \alpha_1],$$

$$\alpha_2=-\frac{\delta+5}{3}=-3-\frac{\delta-4}{3}=[-3, \beta_1],$$

$$\beta_2=-\frac{\delta+5}{2}=-5+\frac{5-\delta}{2}=[-5, \alpha_2],$$

where, in each case, the alternative expansion leads to a reduced form other than g_1 or g_2. The lemma now follows at once.

For the proof of Theorem 1 it now remains for us to consider the chains $\{\varepsilon_n\}$, which can be associated with the chain $\{a_n\}$ given by (4.3).

We number the chain $\{a_n\}$ so that

$$a_1=3, \quad a_2=5, \quad a_3=3, \quad a_4=-8;$$

then

$$\phi_0=\theta_3=\left[\overset{\times}{3}, 5, 3, -\overset{\times}{8}\right]=\frac{\delta+4}{3},$$

$$\phi_1=\theta_2=\left[\overset{\times}{5}, 3, -8, \overset{\times}{3}\right]=\frac{5+\delta}{2},$$

$$\phi_2=\theta_1=\left[\overset{\times}{3}, -8, 3, \overset{\times}{5}\right]=\frac{5+\delta}{3},$$

$$\phi_3=\theta_0=\left[-\overset{\times}{8}, 3, 5, \overset{\times}{3}\right]=-(\delta+4);$$

and each of a_n, θ_n, ϕ_n is periodic in n with period 4.

Since $|\varepsilon_n| \leq |a_{n+1}| - 2$ and ε_n has the same parity as a_{n+1}, the possible values of ε_n are given by:

$$\varepsilon_{4n} = \pm 1, \quad \varepsilon_{4n+1} = \pm 1, \pm 3, \quad \varepsilon_{4n+2} = \pm 1, \quad \varepsilon_{4n+3} = 0, \pm 2, \pm 4, \pm 6;$$

and it is easily verified that conditions (E) are satisfied for any choice of ε_n from these values.[1]

We now show that $M(P) < .982$ unless

$$\varepsilon_{4n} = \varepsilon_{4n+1} = \varepsilon_{4n+2} = 1, \quad \varepsilon_{4n+3} = -2 \tag{4.4}$$

for all n (or the negative of these values).

Lemma 4.2. *If* $M(P) \geq .93$, *then* $\varepsilon_1 = \pm 1$.

Proof. We have

$$2\xi_2 + 1 + \theta_2 = \varepsilon_1 + \left\| 1 - \frac{1}{\theta_1} \right\| = \varepsilon_1 + \left\| \frac{\delta - 3}{2} \right\|,$$

$$2\eta_2 + 1 + \phi_2 = \| \phi_2 - 1 \| = \left\| \frac{\delta + 2}{3} \right\|.$$

If now $\varepsilon_1 = 3$, we obtain

$$\xi_2 = -\frac{\delta + 1}{4} + \left\| \frac{\delta - 3}{4} \right\|, \quad \eta_2 = -\frac{8 + \delta}{6} + \left\| \frac{\delta + 2}{6} \right\|,$$

so that $0 < -(\xi_2 + 1) < \frac{\delta - 3}{2}, \quad 0 < \eta_2 + \phi_2 < \frac{\delta + 2}{3}$.

Since

$$\theta_2 \phi_2 - 1 = \delta \left(\frac{5 + \delta}{3} \right),$$

we have

$$M(P) \leq M_2(P) \leq \frac{2\delta}{\theta_2 \phi_2 - 1} |(\xi_2 + 1)(\eta_2 + \phi_2)|$$

$$< 2 \left(\frac{5 - \delta}{2} \right) \left(\frac{\delta - 3}{2} \right) \left(\frac{\delta + 2}{3} \right) = 14 - 3\delta < .93.$$

By Lemma 3.1, the same result holds if $\varepsilon_1 = -3$. This proves the lemma.

We now suppose therefore that

$$\varepsilon_{4n+1} = \pm 1 \quad \text{for all } n. \tag{4.5}$$

[1] It is clear that E (iii) and (iv) are always satisfied if each of the sequences $\{a_n\}$ and $\{a_{-n}\}$ ($n = 0, 1, 2, \ldots$) changes sign infinitely often.

Lemma 4.3. *Suppose that $M(P) \geq .83$. Then if $\varepsilon_2 = 1$ or if $\varepsilon_4 = 1$ we have $\varepsilon_3 \leq 0$; and if $\varepsilon_2 = -1$ or $\varepsilon_4 = -1$ we have $\varepsilon_3 \geq 0$.*

Proof. By symmetry, it suffices to show that $\varepsilon_3 \leq 0$ if $\varepsilon_2 = 1$. We have

$$2\xi_3 + 1 + \theta_3 = \varepsilon_2 - \frac{\varepsilon_1}{\theta_2} + \left\| \frac{1}{\theta_2}\left(1 - \frac{1}{\theta_1}\right) \right\|;$$

using (4.4) and the hypothesis $\varepsilon_2 = 1$, it follows that

$$2\xi_3 + 1 + \theta_3 > 1 - \frac{1}{\theta_2} - \frac{1}{\theta_2}\left(1 - \frac{1}{\theta_1}\right),$$

whence

$$2\xi_3 > -\frac{4\delta - 8}{3},$$

$$0 < -(\xi_3 + 1) < \frac{2\delta - 7}{3}.$$

Next,

$$2\eta_3 + 1 + \phi_3 = \varepsilon_3 + \left\| 1 - \frac{1}{\phi_4} \right\| = \varepsilon_3 + \| 5 - \delta \|.$$

If now we suppose, contrary to the assertion of the lemma, that $\varepsilon_3 \geq 2$, this gives

$$2\eta_3 + 1 + \phi_3 > 2 - (5 - \delta),$$

whence

$$0 < -(\eta_3 + \phi_3) < 4.$$

Since

$$|\theta_3 \phi_3 - 1| = (\delta + 4)\left(\frac{\delta + 4}{3}\right) + 1 = 2\delta\left(\frac{\delta + 4}{3}\right),$$

these inequalities give

$$M(P) \leq M_3(P) \leq \frac{2\delta}{|\theta_3 \phi_3 - 1|} |(\xi_3 + 1)(\eta_3 + \phi_3)|$$

$$< (\delta - 4)\left(\frac{2\delta - 7}{3}\right) 4 = 88 - 20\delta < .83.$$

Lemma 4.4. *Suppose that $M(P) \geq .89$ and that either $\varepsilon_2 = 1$ or $\varepsilon_4 = 1$. Then $\varepsilon_3 = 0$ or -2.*

Proof. By symmetry, it suffices to suppose that $\varepsilon_2 = 1$. After Lemma 4.3, we have $\varepsilon_3 \leq 0$, and so we have to show that ε_3 cannot be -4 or -6.

Suppose then that $\varepsilon_3 \leq -4$. We have

$$2\eta_3 + 1 + \phi_3 = \varepsilon_3 - \frac{\varepsilon_3}{\phi_4} + \frac{\varepsilon_5}{\phi_4 \phi_5} + \left\| \frac{1}{\phi_4 \phi_5}\left(1 - \frac{1}{\phi_6}\right) \right\|,$$

where $\varepsilon_3 \le -4$, $\varepsilon_4 = 1$ by Lemma 4.3, and, by (4.5), $\varepsilon_5 = \pm 1$. Hence

$$2\eta_3 + 1 + \phi_3 < -4 - \frac{1}{\phi_4} + \frac{1}{\phi_4\phi_5} + \frac{1}{\phi_4\phi_5}\left(1 - \frac{1}{\phi_6}\right)$$

$$0 < 2\eta_3 < -5 - \phi_3 - \frac{1}{\phi_4} + \frac{1}{\phi_4\phi_5} + \frac{1}{\phi_4\phi_5}\left(1 - \frac{1}{\phi_6}\right)$$

$$= -5 + (\delta + 4) - (\delta - 4) + (3\delta - 13) + (48 - 11\delta) = 38 - 8\delta,$$

$$0 < \eta_3 < 19 - 4\delta.$$

As in Lemma 4.3,

$$0 < -\xi_3 < \frac{2\delta - 4}{3}, \quad |\theta_3\phi_3 - 1| = 2\delta\left(\frac{\delta + 4}{3}\right).$$

Hence

$$M(P) \le M_3(P) \le \frac{2\delta}{|\theta_3\phi_3 - 1|}|\varepsilon_3\eta_3| < (\delta - 4)\left(\frac{2\delta - 4}{3}\right)(19 - 4\delta)$$

$$= 646 - 148\delta < .89,$$

contradicting our assumption.

Our next step is to eliminate the possibility that $\varepsilon_3 = 0$, which we do in two stages.

Lemma 4.5. *Suppose that* $\varepsilon_3 = 0$ *and that* $M(P) \ge .96$. *Then*

$$\varepsilon_1 = \varepsilon_2 = \varepsilon_4 = \varepsilon_5 = \pm 1.$$

Proof. By symmetry, we may take $\varepsilon_2 = +1$. Using (4.5), we certainly have $\varepsilon_1 = \pm 1$, $\varepsilon_4 = \pm 1$, $\varepsilon_5 = \pm 1$.

Suppose first that $\varepsilon_4 = -1$. Then since

$$2\xi_3 + 1 + \theta_3 = \varepsilon_2 - \frac{\varepsilon_1}{\theta_2} + \frac{\varepsilon_0}{\theta_2\theta_1} - \cdots = 1 - \frac{\varepsilon_1}{\theta_2} + \frac{\varepsilon_0}{\theta_2\theta_1} - \cdots$$

$$2\eta_3 + 1 + \phi_3 = \varepsilon_3 - \frac{\varepsilon_4}{\phi_4} + \frac{\varepsilon_5}{\phi_4\phi_5} - \cdots = -\frac{\varepsilon_4}{\phi_4} + \frac{\varepsilon_5}{\phi_4\phi_5} - \cdots,$$

we have

$$2\xi_3 + 1 + \theta_3 = 1 - \frac{\varepsilon_1}{\theta_2} + \left\|\frac{1}{\theta_2}\left(1 - \frac{1}{\theta_1}\right)\right\|$$

$$> 1 - \frac{1}{\theta_2} - \frac{1}{\theta_2}\left(1 - \frac{1}{\theta_1}\right),$$

$$-2(\xi_3 + 1) < -2 + \theta_3 + \frac{2}{\theta_2} - \frac{1}{\theta_1\theta_2} = \tfrac{2}{3}(2\delta - 7),$$

$$0 < -(\xi_3 + 1) < \frac{2\delta - 7}{3};$$

and, with $\varepsilon_4 = -1$,

$$2\eta_3+1+\phi_3=\frac{1}{\phi_4}+\left\|\frac{1}{\phi_4}\left(1-\frac{1}{\phi_5}\right)\right\|$$

$$>\frac{1}{\phi_4}-\frac{1}{\phi_4}\left(1-\frac{1}{\phi_5}\right)=\frac{1}{\phi_4\phi_5},$$

$$-2(\eta_3+\phi_3)<-\phi_3+1-\frac{1}{\phi_4\phi_5}=\tfrac{28}{3},$$

$$0<-(\eta_3+\phi_3)<\tfrac{14}{3}.$$

Hence

$$M(P)\le M_3(P)\le\frac{2\delta}{|\theta_3\phi_3-1|}|(\xi_3+1)(\eta_3+\phi_3)|$$

$$<(\delta-4)\left(\frac{2\delta-7}{3}\right)\tfrac{14}{3}$$

$$=\tfrac{14}{9}(22-5\delta)$$

$$<.96.$$

Thus we require $\varepsilon_4=+1$.

In order now to show that $\varepsilon_1=\varepsilon_5=1$, it suffices, by symmetry, to show that $\varepsilon_1=1$. Suppose to the contrary that $\varepsilon_1=-1$, so that now $\varepsilon_1=-1$, $\varepsilon_2=1$, $\varepsilon_3=0$, $\varepsilon_4=1$, $\varepsilon_5=\pm1$. Then

$$2\xi_3+1+\theta_3=1-\frac{\varepsilon_1}{\theta_2}+\left\|\frac{1}{\theta_2}\left(1-\frac{1}{\theta_1}\right)\right\|$$

$$>1+\frac{1}{\theta_2}-\frac{1}{\theta_2}\left(1-\frac{1}{\theta_1}\right)=1+\frac{1}{\theta_2\theta_1},$$

$$-2(\xi_3+1)<\theta_3-\frac{1}{\theta_2\theta_1}-2=2\delta-8,$$

$$0<-(\xi_3+1)<\delta-4;$$

also

$$2\eta_3+1+\phi_3=-\frac{\varepsilon_4}{\phi_4}+\cdots=\left\|1-\frac{1}{\phi_4}\right\|>-1+\frac{1}{\phi_4},$$

$$-2(\eta_3+\phi_3)<-\phi_3+2-\frac{1}{\phi_4}=10,$$

$$0<-(\eta_3+\phi_3)<5.$$

Hence

$$M(P)\le M_3(P)\le(\delta-4)(\delta-4)5=5(35-8\delta)<.65.$$

Lemma 4.6. *Suppose that*

$$\varepsilon_1=\varepsilon_2=\varepsilon_4=\varepsilon_5=\pm1,\quad \varepsilon_3=0.$$

Then $M(P)<.982$.

Proof. We have, taking the upper sign without loss of generality,

$$2\varepsilon_2+1+\theta_2=\varepsilon_1-\frac{\varepsilon_0}{\theta_1}+\left\|\frac{1}{\theta_1}\left(1-\frac{1}{|\theta_0|}\right)\right\|$$

$$<1+\frac{1}{\theta_1}+\frac{1}{\theta_1}\left(1-\frac{1}{|\theta_0|}\right),$$

$$2\xi_2+2\theta_2<\theta_2+\frac{2}{\theta_1}-\frac{1}{\theta_1|\theta_0|}=14-2\delta,$$

$$0<\xi_2+\theta_2<7-\delta;$$

also

$$2\eta_2+1+\phi_2=\varepsilon_2-\frac{\varepsilon_3}{\phi_3}+\frac{\varepsilon_4}{\phi_3\phi_4}-\frac{\varepsilon_5}{\phi_3\phi_4\phi_5}+\left\|\frac{1}{\phi_3\phi_4\phi_5}\left(1-\frac{1}{|\phi_6|}\right)\right\|$$

$$=1-\frac{1}{\phi_3}+\frac{1}{\phi_3\phi_4}-\frac{1}{\phi_3\phi_4\phi_5}+\left\|\frac{1}{\phi_3\phi_4\phi_5}\left(1-\frac{1}{|\phi_6|}\right)\right\|,$$

whence

$$-2\eta_2-2<\phi_2-2+\frac{1}{\phi_3}-\frac{1}{\phi_3\phi_4}+\frac{1}{\phi_3\phi_4\phi_5}+\frac{1}{\phi_3\phi_4\phi_5}\left(1-\frac{1}{|\phi_6|}\right)$$

$$=\frac{110\delta-476}{3},$$

$$0<-(\eta_2+1)<\frac{55\delta-238}{3}.$$

Thus, since $\theta_2\phi_2-1=\delta\left(\frac{5+\delta}{3}\right)$, we have

$$M(P)\le M_2(P)<(5-\delta)(7-\delta)\left(\frac{55\delta-238}{3}\right)=1942\delta-8464<.982,$$

as required.

It follows from Lemmas 4.5 and 4.6 that, for $M(P)\ge .982$, $\varepsilon_3\neq 0$, and so, by Lemma 4.4, $\varepsilon=\pm 2$.

Allowing for an eventual change of sign of all ε_n, we take $\varepsilon_3=-2$, so that, by Lemma 4.3, $\varepsilon_2=\varepsilon_4=1$.

Lemma 4.7. *If* $\varepsilon_2=\varepsilon_4=1$, $\varepsilon_3=-2$ *and* $M(P)\ge .79$, *then* $\varepsilon_1=\varepsilon_5=1$.

Proof. By symmetry, it suffices to show that $\varepsilon_1=1$. Suppose to the contrary that $\varepsilon_1=-1$. Then

$$2\xi_3+1+\theta_3=\varepsilon_2-\frac{\varepsilon_1}{\theta_2}+\left\|\frac{1}{\theta_2}\left(1-\frac{1}{\theta_1}\right)\right\|$$

$$>1+\frac{1}{\theta_2}-\frac{1}{\theta_2}\left(1-\frac{1}{\theta_1}\right)=1+\frac{1}{\theta_1\theta_2},$$

$$-2\xi_3-2<\theta_3-2-\frac{1}{\theta_1\theta_2}=2\delta-8,$$

$$0<-(\xi_3+1)<\delta-4;$$

also

$$2\eta_3+1+\phi_3=\varepsilon_3+\left\|1-\frac{1}{\phi_4}\right\|>-2-\left(1-\frac{1}{\phi_4}\right),$$

$$-2\eta_3-2\phi_3<-\phi_3+4-\frac{1}{\phi_4}=12,$$

$$0<-(\eta_3+\phi_3)<6;$$

hence,

$$M(P)\le M_3(P)\le 6(\delta-4)(\delta-4)=6(35-8\delta)<.79.$$

In Lemmas 4.2 to 4.6 we have now established that, if $M(P)\ge .982$, then $\varepsilon_1=\varepsilon_2=\varepsilon_4=\varepsilon_5=1$, $\varepsilon_3=-2$ (or the negative of these values). By the periodicity of the chain $\{a_n\}$ it follows that, for each n, either

$$\varepsilon_{4n+1}=\varepsilon_{4n+2}=\varepsilon_{4n+4}=\varepsilon_{4n+5}=1,\quad \varepsilon_{4n+3}=-2,$$

or

$$\varepsilon_{4n+1}=\varepsilon_{4n+2}=\varepsilon_{4n+4}=\varepsilon_{4n+5}=-1,\quad \varepsilon_{4n+3}=2.$$

Since $4n+5=4(n+1)+1$, it is immediately clear that if the first alternative holds for any one value of n, it holds for all n. Thus $\{\varepsilon_n\}$ is the periodic chain given by (4.4), or its negative.

We now complete the proof of Theorem 1 by proving

Lemma 4.8. *If the chains* $\{a_n\}$, $\{\varepsilon_n\}$ *are given by* (4.3) *and* (4.4),

$$M(P)=\tfrac{170}{171}.$$

Proof. The values of ξ_n and η_n may easily be calculated from the series for $2\xi_n+\theta_n+1$, $2\eta_n+1+\phi_n$, since each of these quantities is periodic in n with period 4. Thus

$$2\eta_0+1+\phi_0=\varepsilon_0-\frac{\varepsilon_1}{\phi_1}+\frac{\varepsilon_2}{\phi_1\phi_2}-\frac{\varepsilon_3}{\phi_1\phi_2\phi_3}+\frac{1}{\phi_1\phi_2\phi_3\phi_4}(2\eta_4+1+\phi_4),$$

whence

$$(2\eta_0+1+\phi_0)\left(1-\frac{1}{\phi_1\phi_2\phi_3\phi_4}\right)=1-\frac{1}{\phi_1}+\frac{1}{\phi_1\phi_2}+\frac{2}{\phi_1\phi_2\phi_3}.$$

Inserting the values of ϕ_n, we find

$$\eta_0 = -\frac{152+23\delta}{171}.$$

Similarly we may calculate

$$\xi_0 = \frac{57+20\delta}{57}.$$

Using the recursion formulae (3.4), (3.5), we then find

$$\xi_1 = -\frac{190+23\delta}{171}, \quad \eta_1 = -\frac{171+25\delta}{114}.$$

By the periodicity and symmetry of the chains, it is clear that

$$M(P) = \min\{M_0(P),\ M_1(P)\}.$$

A rough calculation easily decides the minimum of the four expressions defining $M_0(P)$, $M_1(P)$. Using the relations

$$|\theta_0\phi_0 - 1| = \frac{2\delta}{\delta-4}, \quad |\theta_1\phi_1 - 1| = \frac{2\delta}{5-\delta},$$

we have

$$\begin{aligned} M_0(P) &= \frac{2\delta}{|\theta_0\phi_0-1|}\,|(\xi_0+\theta_0)(\eta_0+1)| \\ &= (\delta-4)\left(\frac{37\delta+171}{57}\right)\left(\frac{23\delta-19}{171}\right) \\ &= \tfrac{170}{171}; \end{aligned}$$

$$\begin{aligned} M_1(P) &= \frac{2\delta}{|\theta_1\phi_1-1|}\,|(\xi_1+1)(\eta_1+\phi_1)| \\ &= (5-\delta)\left(\frac{57+16\delta}{57}\right)\left(\frac{19+23\delta}{171}\right) \\ &= \tfrac{170}{171}. \end{aligned}$$

Thus $M(P) = \frac{170}{171}$, as required. This completes the proof of Theorem 1.

Note. A simple calculation shows that the point P of Lemma 4.8 corresponds to taking

$$(x, y) \equiv (0, \pm\tfrac{20}{57}) \pmod 1$$

in $f(x, y) = x^2 - 19y^2$.

5. The form $x^2 - 46y^2$

Theorem 2. *If* $f(x, y) = x^2 - 46y^2$, *then*

$$M(f) = \tfrac{76877}{48668} = 1.5796\ldots. \tag{5.1}$$

The reduced forms $ax^2+bxy+cy^2$ equivalent to $f(x, y)$ with $|a \pm b + c| \geq 7$ are found to be $g_n(x, \pm y)$, $g_n(y, \pm x)$ $(n=1, 2, 3, 4)$ where

$$g_1=(-5, 8, 6), \quad \alpha_1=\frac{\delta+4}{6}, \qquad \beta_1=-\frac{\delta+4}{5},$$

$$g_2=(2, 12, -5), \quad \alpha_2=-\frac{\delta+6}{5}, \quad \beta_2=\frac{\delta+6}{2},$$

$$g_3=(1, 14, 3), \qquad \alpha_3=\frac{\delta+7}{3}, \qquad \beta_3=\delta+7,$$

$$g_3=(3, 16, 6), \qquad \alpha_4=\frac{\delta+8}{6}, \qquad \beta_4=\frac{\delta+8}{3},$$

where we have written

$$\delta=\sqrt{46}=\tfrac{1}{2}\Delta.$$

By Lemma 3.3 we see that, for any point P with $M(f; P) > \frac{3}{2}$, no other forms can occur in the corresponding chain of reduced forms.

Lemma 5.1. *If $M(P) > 1.5$, the chain $\{a_n\}$ is given by*

$$\{\overset{\times}{14}, 5, 2, -2, 6, -2, 2, \overset{\times}{5}\} \tag{5.2}$$

(or its negative).

Proof. It is sufficient to observe that, if the chain of reduced forms contains no other forms than those listed above, then we have the following unique expansions:

$$\alpha_1=[2, \beta_4], \qquad \beta_1=[-2, \beta_2],$$

$$\alpha_2=[-2, \alpha_1], \quad \beta_2=[6, \alpha_2],$$

$$\alpha_3=[5, \alpha_4], \qquad \beta_3=[14, \alpha_3],$$

$$\alpha_4=[2, \beta_1], \qquad \beta_4=]5, \beta_3],$$

thus

$$\alpha_1=\left[\overset{\times}{2}, 5, 14, 5, 2, -2, 6, -\overset{\times}{2}\right], \quad \beta_1=\left[-\overset{\times}{2}, 6, -2, 2, 5, 14, 5, \overset{\times}{2}\right],$$

$$\alpha_2=\left[-\overset{\times}{2}, 2, 5, 14, 5, 2, -2, \overset{\times}{6}\right], \quad \beta_2=\left[\overset{\times}{6}, -2, 2, 5, 14, 5, 2, -\overset{\times}{2}\right],$$

$$\alpha_3=\left[\overset{\times}{5}, 2, -2, 6, -2, 2, 5, \overset{\times}{14}\right], \quad \beta_3=\left[\overset{\times}{14}, 5, 2, -2, 6, -2, 2, \overset{\times}{5}\right],$$

$$\alpha_4=\left[\overset{\times}{2}, -2, 6, -2, 2, 5, 14, \overset{\times}{5}\right], \quad \beta_4=\left[\overset{\times}{5}, 14, 5, 2, -2, 6, -2, \overset{\times}{2}\right],$$

Hence all the above forms occur in the chain, and the chain $\{a_n\}$ is given by (5.2).

We number the chain (5.2) so that

$$a_{8n+1}=14$$

for all n. Then a_n, θ_n and ϕ_n are periodic with period 8 and

$$\begin{aligned}
\theta_1&=\phi_8=\delta+7,\\
\theta_2&=\phi_7=\frac{\delta+8}{3},\\
\theta_3&=\phi_6=\frac{\delta+4}{6},\\
\theta_4&=\phi_5=-\frac{\delta+6}{5},\\
\theta_5&=\phi_4=\frac{\delta+6}{2},\\
\theta_6&=\phi_3=-\frac{\delta+4}{5},\\
\theta_7&=\phi_2=\frac{\delta+8}{6},\\
\theta_8&=\phi_1=\frac{\delta+7}{3}.
\end{aligned}$$

We note that $\varepsilon_2=\varepsilon_3=\varepsilon_5=\varepsilon_6=0$, $|\varepsilon_0|\le 12$, $|\varepsilon_1|\le 3$, $|\varepsilon_4|\le 4$, $|\varepsilon_7|\le 3$.

Lemma 5.2. *If $M(P)>1.5$, then $\varepsilon_4=0$ or ± 2.*

Proof. It is sufficient to prove that $\varepsilon_4\neq 4$.

If $\varepsilon_3=4$, we have

$$\begin{aligned}
2\xi_5+1+\theta_5&=\varepsilon_4-\frac{\varepsilon_3}{\theta_4}+\frac{\varepsilon_2}{\theta_4\,\theta_3}+\left\|\frac{1}{|\theta_4|\,\theta_3}\left(1-\frac{1}{\theta_2}\right)\right\|\\
&=4+\left\|\frac{3\delta-20}{2}\right\|,
\end{aligned}$$

$$-2\xi_5<\theta_5-3+\frac{3\delta-20}{2}=2\delta-10,$$

$$0<-\xi_5<\delta-5;$$

$$\begin{aligned}
2\eta_5+1+\phi_5&=\varepsilon_5-\frac{\varepsilon_6}{\phi_6}+\left\|\frac{1}{\phi_6}\left(1-\frac{1}{\phi_7}\right)\right\|\\
&=\left\|\frac{9-\delta}{5}\right\|,
\end{aligned}$$

$$2\eta_5<-\phi_5-1+\frac{9-\delta}{5}=2,$$

$$0<\eta_5<1.$$

Since $|\theta_5 \phi_5 - 1| = \dfrac{2\delta}{\delta - 6}$, we therefore have

$$M(P) \le M_5(P) < (\delta - 6)(\delta - 5) = 76 - 11\delta = 1.39\cdots < 1.5,$$

contradicting our assumption that $M(P) > 1.5$.

Lemma 5.3. *If* $M(P) > 1.5$, *then* $\varepsilon_1 = \pm 1$, $\varepsilon_7 = \pm 1$.

Proof. By symmetry, it is sufficient to prove that $\varepsilon_1 \neq 3$. If $\varepsilon_1 = 3$ we have

$$2\xi_2 + 1 + \theta_2 = \varepsilon_1 + \left\| 1 - \frac{1}{\theta_1} \right\| = 3 + \left\| \frac{\delta - 4}{3} \right\|,$$

$$-2\xi_2 - 2 < \theta_2 - 4 + \frac{\delta - 4}{3} = \frac{2\delta - 8}{3},$$

$$0 < -(\xi_2 + 1) < \frac{\delta - 4}{3};$$

$$2\eta_2 + 1 + \phi_2 = \varepsilon_3 + \left\| 1 - \frac{1}{|\phi_3|} \right\| = \left\| \frac{10 - \delta}{6} \right\|,$$

$$2\eta_2 + 2\phi_2 < \phi_2 - 1 + \frac{10 - \delta}{6} = 2,$$

$$0 < \eta_2 + \phi_2 < 1.$$

Since $\theta_2 \phi_2 - 1 = \dfrac{2\delta}{8 - \delta}$, we therefore have

$$M(P) \le M_2(P) < (8 - \delta)\left(\frac{\delta - 4}{3}\right) = 4\delta - 26 = 1.12\cdots < 1.5,$$

contradicting our assumption that $M(P) > 1.5$.

Lemma 5.4. *If* $M(P) > 1.55$ *and* $\varepsilon_1 = 1$, *then* $\varepsilon_0 \ge 4$.

Proof. Suppose to the contrary that $\varepsilon_0 \le 2$. Then

$$\begin{aligned} 2\xi_2 + 1 + \theta_2 &= \varepsilon_1 - \frac{\varepsilon_0}{\theta_1} + \left\| \frac{1}{\theta_1}\left(1 - \frac{1}{\theta_2}\right) \right\| \\ &= 1 - \varepsilon_0\left(\frac{7 - \delta}{3}\right) + \left\| \frac{13\delta - 88}{3} \right\|, \end{aligned}$$

$$\begin{aligned} -2\xi_2 - 2 &= \theta_2 - 2 + \varepsilon_0\left(\frac{7 - \delta}{3}\right) + \left\| \frac{13\delta - 88}{3} \right\| \\ &< \theta_2 - 2 + 2\left(\frac{7 - \delta}{3}\right) + \frac{13\delta - 88}{3} \\ &= 4\delta - 24, \end{aligned}$$

$$0 < -(\xi_2 + 1) < 2\delta - 12.$$

Also

$$2\eta_2+1+\phi_2=\varepsilon_2-\frac{\varepsilon_3}{\phi_3}+\frac{\varepsilon_4}{\phi_3\phi_4}-\frac{\varepsilon_5}{\phi_3\phi_4\phi_5}+\left\|\frac{2}{|\phi_3\phi_4\phi_5|}\left(1-\frac{1}{\phi_6}\right)\right\|$$

$$=-\varepsilon_4\left(\frac{7-\delta}{3}\right)+\left\|\frac{41\delta-278}{6}\right\|,$$

$$2\eta_2+2\phi_2<\phi_2-1-\varepsilon_4\left(\frac{7-\delta}{3}\right)+\left\|\frac{41\delta-278}{6}\right\|;$$

since $|\varepsilon_4|\leq 2$, by Lemma 5.2, this gives

$$2\eta_2+2\phi_2<\phi_2-1+2\left(\frac{7-\delta}{3}\right)+\frac{41\delta-278}{6}=\frac{38\delta-248}{6},$$

$$0<\eta_2+\phi_2<\frac{19\delta-124}{6}.$$

Since $\theta_2\phi_2-1=\frac{2\delta}{8-\delta}$, we therefore have

$$M(P)\leq M_2(P)<(8-\delta)(2\delta-12)\left(\frac{19\delta-124}{6}\right)=7964-1174\delta=1.544\ldots,$$

contradicting our hypothesis that $M(P)>1.55$.

Corollary. *If* $M(P)>1.55$ *and* $\varepsilon_1=1$, *then* $\varepsilon_{-1}=1$.

For $\varepsilon_{-1}=\pm 1$, by Lemma 5.3, and $\varepsilon_{-1}=-1$ would imply $\varepsilon_0\leq -4$, by an application of Lemma 5.4 to the sequence $\{-a_{-n}\}$.

Lemma 5.5. *If* $M(P)>1.55$ and $\varepsilon_1=1$, *then* $\varepsilon_0=4$ *or* 6.

Proof. By Lemma 5.4 and its corollary, it is sufficient to show that if $\varepsilon_{-1}=1$, $\varepsilon_1=1$, $\varepsilon_0\geq 8$, then $M(P)\leq 1.55$.

Supposing then that $\varepsilon_{-1}=1$, $\varepsilon_1=1$, $\varepsilon_0\geq 8$, we have

$$2\xi_1+1+\theta_1=\varepsilon_0+\left\|1-\frac{1}{\theta_0}\right\|=\varepsilon_0+\|\delta-6\|,$$

$$-2\xi_1-2=\theta_1-1-\varepsilon_0+\|\delta-6\|,$$

$$<\theta_1-1-8+\delta-6=2\delta-8,$$

$$0<-(\xi_1+1)<\delta-4;$$

$$2\eta_1+1+\phi_1=\varepsilon_0-\frac{\varepsilon_2}{\phi_2}+\frac{\varepsilon_3}{\phi_2\phi_3}+\left\|\frac{1}{|\phi_2\phi_3|}\left(1-\frac{1}{\phi_4}\right)\right\|$$

$$=1+\left\|\frac{7\delta-47}{3}\right\|,$$

$$2\eta_1+2\phi_1<\phi_1+\frac{7\delta-47}{3}=\frac{8\delta-40}{3},$$

$$0<\eta_1+\phi_1<\frac{4\delta-20}{3}.$$

Since $\theta_1 \phi_1 - 1 = \frac{2\delta}{7-\delta}$, we therefore have

$$M(P) \le M_1(P) < (7-\delta)\left(\frac{4\delta-20}{3}\right)(\delta-4) = 1168 - 172\delta = 1.439\ldots,$$

so that certainly $M(P) \le 1.55$.

Lemma 5.6. *If* $M(P) > 1.55$ *and* $\varepsilon_1 = 1$, *then* $\varepsilon_4 = -2$.

Proof. By Lemma 5.2, it is enough to show that $\varepsilon_4 < 0$. Suppose to the contrary that $\varepsilon_4 \ge 0$. Then

$$2\xi_3 + 1 + \theta_3 = \varepsilon_2 - \frac{\varepsilon_1}{\theta_2} + \frac{\varepsilon_0}{\theta_2\theta_1} - \frac{\varepsilon_{-1}}{\theta_2\theta_1\theta_0} + \left\| \frac{1}{\theta_2\theta_1\theta_1}\left(1 - \frac{1}{\theta_{-1}}\right)\right\|$$

$$= -\frac{1}{\theta_2} + \frac{\varepsilon_0}{\theta_2\theta_1} - \frac{1}{\theta_2\theta_1\theta_0} + \left\| \frac{1}{\theta_2\theta_1\theta_0}\left(1 - \frac{1}{\theta_{-1}}\right)\right\|$$

(since $\varepsilon_{-1} = 1$ by Lemma 5.4, Corollary): hence, since $\varepsilon_0 \le 6$ by Lemma 5.5,

$$2\xi_3 + 2\theta_3 < \theta_3 - 1 - \frac{1}{\theta_2} + \frac{6}{\theta_2\theta_1} - \frac{1}{\theta_2\theta_1\theta_0} + \frac{1}{\theta_2\theta_1\theta_0}$$

$$= \frac{97-14\delta}{3},$$

$$0 < \xi_3 + \theta_3 < \frac{97-14\delta}{6}.$$

Also

$$2\eta_3 + 1 + \phi_3 = \varepsilon_3 - \frac{\varepsilon_4}{\phi_4} + \frac{\varepsilon_5}{\phi_4\phi_5} - \frac{\varepsilon_6}{\phi_4\phi_5\phi_6} + \left\| \frac{1}{|\phi_4\phi_5\phi_6|}\left(1 - \frac{1}{\phi_7}\right)\right\|$$

$$= -\frac{\varepsilon_4}{\phi_4} + \left\|\frac{129-19\delta}{5}\right\|,$$

$$2\eta_3 + 2 = -\phi_3 + 1 - \frac{\varepsilon_4}{\phi_4} + \left\|\frac{129-19\delta}{5}\right\|$$

$$< -\phi_3 + 1 + \frac{129-19\delta}{5} = \frac{138-18\delta}{5}$$

(since $\phi_4 > 0$, $\varepsilon_4 \ge 0$), whence

$$0 < \eta_3 + 1 < \frac{69-9\delta}{5}.$$

Since $|\theta_3\phi_3 - 1| = \frac{2\delta}{\delta-4}$, we therefore have

$$M(P) \le M_3(P) < (\delta-4)\left(\frac{97-14\delta}{6}\right)\left(\frac{69-9\delta}{5}\right)$$

$$= \tfrac{1}{2}(1323\delta - 8970) < 1.512,$$

contradicting our assumption that $M(P) > 1.55$.

Lemma 5.7. *If $M(P)>1.55$ and $\varepsilon_1>0$, then*

$$\varepsilon_{8n+1}=1,\ \varepsilon_{8n+2}=0,\ \varepsilon_{8n+3}=0,\ \varepsilon_{8n+4}=-2,\ \varepsilon_{8n+5}=0,\ \varepsilon_{8n+6},\ \varepsilon_{8n+7}=1$$

for all n, and $\varepsilon_{8n}=4$ or 6 for each n.

Proof. If, for any n, $\varepsilon_{8n+1}>0$, then preceding lemmas (with r replaced by $8n+r$) show that

$$\varepsilon_{8n-1}=1,\ \varepsilon_{8n}=4 \text{ or } 6,\ \varepsilon_{8n+2}=0,\ \varepsilon_{8n+3}=0,\ \varepsilon_{8n+4}=-2,\ \varepsilon_{8n+5}=0,\ \varepsilon_{8n+1}=0.$$

If now $\varepsilon_{8n-7}<0$, we have $\varepsilon_{8n-1}=-1$ (replacing n by $n-1$ and changing the sign of all ε_r); hence $\varepsilon_{8n-7}>0$. If $\varepsilon_{8n+7}<0$, we have $\varepsilon_{8n+4}=+2$ (considering the reversed sequence); hence $\varepsilon_{8n+7}>0$, and so $\varepsilon_{8n+9}>0$.

It now follows that if $\varepsilon_{8n+1}>0$ for any n, then $\varepsilon_{8n+1}>0$ for all n, and this now gives the result of the lemma.

Lemma 5.8. *If $\varepsilon_{8n}=\pm 4$ for any n, then $M(P)<1.577$.*

Proof. It suffices to show that $M(P)<1.577$ if $\varepsilon_0=4$. Using Lemma 7 we see that then $\varepsilon_{-1}=1$, $\varepsilon_0=4$, $\varepsilon_1=1$, $\varepsilon_2=0$, $\varepsilon_3=0$, $\varepsilon_4=-2$, $\varepsilon_5=0$, $\varepsilon_6=0$, $\varepsilon_7=1$, $\varepsilon_8=4$ or 6. Hence

$$\begin{aligned}2\xi_3+1+0_3&=-\frac{1}{\theta_2}+\frac{4}{\theta_2\theta_1}-\frac{1}{\theta_2\theta_1\theta_0}+\left\|\frac{1}{\theta_2\theta_1\theta_0}\left(1-\frac{1}{\theta_{-1}}\right)\right\|\\&<-\frac{1}{\theta_2}+\frac{4}{\theta_2\theta_1}\end{aligned}$$

(since $\theta_2\theta_1\theta_0>0$); thus

$$2\xi_3+2\theta_3+2<\theta_3+1-\frac{1}{\theta_2}+\frac{4}{\theta_2\theta_1}=23-3\delta,$$

$$0<\xi_3+\theta_3+1<\frac{23-3\delta}{2}.$$

Also

$$2\eta_3+1+\phi_3=\frac{2}{\phi_4}+\frac{1}{\phi_4\phi_5\phi_6\phi_7}-\frac{\varepsilon_8}{\phi_4\cdots\phi_8}+\left\|\frac{1}{|\phi_4\cdots\phi_8|}\left(1-\frac{1}{\phi_9}\right)\right\|$$

$$\begin{aligned}-2\eta_3-2-2\phi_3&=-\phi_3-1-\frac{2}{\phi_4}-\frac{1}{\phi_4\cdots\phi_7}-\frac{\varepsilon_8}{|\phi_4\cdots\phi_8|}+\left\|\frac{1}{|\phi_4\cdots\phi_8|}\right\|\\&<-\phi_3-1-\frac{2}{\phi_4}-\frac{1}{\phi_4\cdots\phi_7}=\frac{31\delta-206}{5}\end{aligned}$$

(since $\varepsilon_8\geq 4>1$),

$$0<-(\eta_3+1+\phi_3)<\frac{31\delta-206}{10}.$$

Since $|\theta_3\phi_3-1|=\frac{2\delta}{\delta-4}$, it follows that

$$M(P)\le M_3(P)<(\delta-4)\left(\frac{23-3\delta}{2}\right)\left(\frac{31\delta-206}{10}\right)$$
$$=4864.5-717\delta$$
$$<4864.5-717(6.78232)=1.57656.$$

Lemma 5.9. *If* $M(P)\ge 1.577$, *then* $\{\varepsilon_n\}$ *or* $\{-\varepsilon_n\}$ *is the periodic chain given by*

$$\{\overset{\times}{6},1,0,0,-2,0,0,\overset{\times}{1}\}, \tag{5.3}$$

where $\varepsilon_0=6$.

Proof. This follows at once from Lemmas 5.7 and 5.8.

Lemma 5.10. *If the chains* $\{a_n\}$, $\{\varepsilon_n\}$ *are given by* (5.2), (5.3), *then*

$$M(P)=\frac{76877}{48668}.$$

Proof. By (3.6) we have, in matrix notation,

$$\begin{pmatrix}x_{n+1}\\y_{n+1}\end{pmatrix}=\begin{pmatrix}0&1\\-1&-a_{n+1}\end{pmatrix}\begin{pmatrix}x_n\\y_n\end{pmatrix}-\begin{pmatrix}0\\k_n+1\end{pmatrix} \tag{5.4}$$

for all n, where $2k_n=a_{n+1}-\varepsilon_n$. Since x_n, y_n are periodic in n with period 8, we have

$$(x_8,y_8)=(x_0,y_0),$$

and so we find that

$$\begin{pmatrix}x_0\\y_0\end{pmatrix}=\begin{pmatrix}x_8\\y_8\end{pmatrix}=\begin{pmatrix}-781&-10764\\3588&49451\end{pmatrix}\begin{pmatrix}x_0\\y_0\end{pmatrix}-\begin{pmatrix}170&781\\-781&-3588\end{pmatrix}\begin{pmatrix}0\\5\end{pmatrix}$$
$$-\begin{pmatrix}-69&-170\\317&781\end{pmatrix}\begin{pmatrix}0\\3\end{pmatrix}-\begin{pmatrix}-32&69\\147&-317\end{pmatrix}\begin{pmatrix}0\\2\end{pmatrix}$$
$$-\begin{pmatrix}2&-5\\-9&23\end{pmatrix}\begin{pmatrix}0\\5\end{pmatrix}-\begin{pmatrix}0&1\\-1&-5\end{pmatrix}\begin{pmatrix}0\\2\end{pmatrix}-\begin{pmatrix}0\\3\end{pmatrix},$$
$$\begin{pmatrix}782&10764\\-3588&-49450\end{pmatrix}\begin{pmatrix}x_0\\y_0\end{pmatrix}=\begin{pmatrix}-3510\\16123\end{pmatrix},$$
$$\begin{pmatrix}x_0\\y_0\end{pmatrix}=-\frac{1}{48668}\begin{pmatrix}-49450&-10764\\3588&782\end{pmatrix}\begin{pmatrix}-3510\\16123\end{pmatrix}$$
$$=-\frac{1}{48668}\begin{pmatrix}21528\\14306\end{pmatrix},$$

whence

$$(x_0, y_0) = \left(-\frac{468}{1058}, \; -\frac{311}{1058}\right).$$

Using (5.4) successively, we now obtain

$$(x_1, y_1) = \left(-\frac{311}{1058}, \; -\frac{468}{1058}\right),$$

$$(x_2, y_2) = \left(-\frac{468}{1058}, \; -\frac{523}{1058}\right),$$

$$(x_3, y_3) = \left(-\frac{523}{1058}, \; -\frac{602}{1058}\right),$$

$$(x_4, y_4) = \left(-\frac{602}{1058}, \; -\frac{681}{1058}\right).$$

By symmetry and periodicity we clearly have

$$M(P) = \min \{M_1(P), \; M_2(P), \; M_3(P), \; M_4(P)\},$$

where, by (3.2),

$$M_n(P) = \min \{|f_n(x_n, y_n)|, \; |f_n(1+x_n, y_n)|, \; |f_n(x_n, 1+y_n)|, \; |f_n(1+x_n, 1+y_n)|\}.$$

Here we find easily that

$$f_1(x, y) = 3x^2 + 14xy + y^2$$

$$f_2(x, y) = 6x^2 + 16xy + 3y^2,$$

$$f_3(x, y) = -5x^2 + 8xy + 6y^2,$$

$$f_4(x, y) = 2x^2 + 12xy - 5y^2,$$

and a straightforward calculation gives

$$M_1(P) = -f_1\left(-\frac{311}{1058}, \; \frac{590}{1058}\right) = \frac{83939}{48668},$$

$$M_2(P) = -f_2\left(-\frac{468}{1058}, \; \frac{535}{1058}\right) = \frac{79707}{48668},$$

$$M_3(P) = f_3\left(\frac{535}{1058}, \; \frac{456}{1058}\right) = \frac{76877}{48668},$$

$$M_4(P) = f_4\left(\frac{456}{1058}, \; \frac{377}{1058}\right) = \frac{76877}{48668}.$$

It follows that

$$M(P) = \frac{76877}{48668},$$

as required.

Theorem 2 now follows immediately from Lemmas 5.9 and 5.10.

We note that, with $f(x, y) = x^2 - 46y^2$, we have

$$M(f; x_0, y_0) = M(f)$$

if and only if

$$x_0 \equiv \frac{1}{2}, \quad y_0 \equiv \pm \frac{311}{1058} \pmod{1}.$$

In conclusion I wish to express my gratitude to the University of Sydney for supplying me with a Brunsviga, on which the calculations of §§ 4 and 5 were carried out.

The University of Sydney, Australia

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