LINEAR RELATIONS BETWEEN E-FUNCTIONS AND BESSEL FUNCTIONS

BY

F. M. RAGAB

Abstract

In this paper, some new linear relations for MacRobert's E-Functions are established. They are formulae (1), (10), (15), (20), (22) and (28) below. For the definitions and properties of these functions, the reader is referred to MacRobert, "Functions of a Complex Variable" (3rd ed., London 1946), p. 348. This work will be denoted by the letters C.V. Also some expansions of Bessel Functions are deduced.

§ 1. The first formula to be proved is

$$\sum_{r=0}^{n} \frac{(-1)^r\, {}^n c_r\,(\alpha;\, r)\,(1+\tfrac{1}{2}\alpha;\, r)}{(\tfrac{1}{2}\alpha;\, r)\,(1+\alpha+n;\, r)}\, x^{-2r} \times$$

$$\times E\left\{\begin{matrix} \tfrac{1}{2}+\alpha+2r,\ 1+2\alpha+2n+2r,\ \alpha_1+2r, \ldots,\ \alpha_p+2r;\ x \\ \tfrac{1}{2}+\alpha+n+2r,\ 1+2\alpha+4r,\ \varrho_1+2r, \ldots,\ \varrho_q+2r \end{matrix}\right\} =$$

$$= 2^{2n}\,(1+\alpha;\, n)\, E(p;\, \alpha_r : q;\, \varrho_s;\, x). \tag{1}$$

To prove (1), consider the special case $p=q=0$; then the coefficient of $\left(-\dfrac{1}{x}\right)^s$ on the L.H.S. is equal to

$$\frac{\Gamma(\tfrac{1}{2}+\alpha+s)\,\Gamma(1+2\alpha+2n+s)}{\Gamma(\tfrac{1}{2}+\alpha+n+s)\,\Gamma(1+2\alpha+s)} \cdot \frac{1}{\lfloor s} +$$

$$+ \frac{(-n)\ (\alpha;\ 1)\left(1+\dfrac{\alpha}{2};\ 1\right)}{\lfloor 1 \left(\dfrac{\alpha}{2};\ 1\right)(1+\alpha+n;\ 1)} \cdot \frac{\Gamma(\alpha+\tfrac{1}{2}+2+s-2)\,\Gamma(1+2\alpha+2n+2+s-2)}{\Gamma(\tfrac{1}{2}+\alpha+n+2+s-2)\,\Gamma(1+2\alpha+4+s-2)\lfloor s-2} +$$

$+\cdots\cdots$

$+\cdots\cdots$

$$=\frac{\Gamma(\tfrac{1}{2}+\alpha+s)\,\Gamma(1+2\alpha+2n+s)}{\Gamma(\tfrac{1}{2}+\alpha+n+s)\,\Gamma(1+2\alpha+s)\lfloor s}\times F\left(\begin{matrix}\alpha,\,1+\frac{\alpha}{2},\,-\frac{s}{2},\,\frac{1}{2}-\frac{s}{2},\,-n;\,1\\ \frac{\alpha}{2},\,1+\alpha+\frac{s}{2},\,\frac{1}{2}+\alpha+\frac{s}{2},\,1+\alpha+n\end{matrix}\right).$$

Now sum the generalized hypergeometric function by means of Dougall's second theorem (Proc. Edinb. Math. Soc., XXV, 1906, p. 10), namely

$$F\left(\begin{matrix}\alpha,\,1+\frac{\alpha}{2},\,\beta,\,\gamma,\,\delta;\,1\\ \frac{\alpha}{2},\,1+\alpha-\beta,\,1+\alpha-\gamma,\,1+\alpha-\delta\end{matrix}\right)=$$

$$=\frac{\Gamma(1+\alpha-\beta)\,\Gamma(1+\alpha-\gamma)\,\Gamma(1+\alpha-\delta)\,\Gamma(1+\alpha-\beta-\gamma-\delta)}{\Gamma(1+\alpha)\,\Gamma(1+\alpha-\beta-\gamma)\,\Gamma(1+\alpha-\beta-\delta)\,\Gamma(1+\alpha-\gamma-\delta)},\qquad (2)$$

where one of the parameters β,γ,δ is a negative integer and $R(\alpha-\beta-\gamma-\delta)>-1$; thus if $R(\alpha+n+s-\frac{1}{2})>-1$, the last coefficient is equal to

$$\frac{\Gamma(\tfrac{1}{2}+\alpha+s)\,\Gamma(1+2\alpha+2n+s)}{\Gamma(\tfrac{1}{2}+\alpha+n+s)\,\Gamma(1+2\alpha+s)\lfloor s}\times$$

$$\times\frac{\Gamma(\alpha+\tfrac{1}{2}s+1)\,\Gamma(\alpha+\tfrac{1}{2}s+\tfrac{1}{2})\,\Gamma(\alpha+n+1)\,\Gamma(\alpha+s+n+\tfrac{1}{2})}{\Gamma(1+\alpha)\,\Gamma(\alpha+s+\tfrac{1}{2})\,\Gamma(\alpha+\tfrac{1}{2}s+n+\tfrac{1}{2})\,\Gamma(\alpha+\tfrac{1}{2}s+n+1)}=$$

$$=\frac{\Gamma(\tfrac{1}{2}+\alpha+s)\,\Gamma(1+2\alpha+2n+s)}{\Gamma(\tfrac{1}{2}+\alpha+n+s)\,\Gamma(1+2\alpha+s)\lfloor s}\cdot\frac{\Gamma(2\alpha+s+1)\,\Gamma(\alpha+n+1)\,\Gamma(\alpha+s+n+\tfrac{1}{2})}{\Gamma(\alpha+1)\,\Gamma(\alpha+s+\tfrac{1}{2})\,\Gamma(2\alpha+s+2n+1)}\cdot 2^{2n}=$$

$=(1+\alpha;n)\dfrac{1}{\lfloor s}\cdot 2^{2n}=$ coefficient of $\left(-\dfrac{1}{x}\right)^{s}$ on the R.H.S. of (1). Thus (1) is proved for the special case $p=q=0$.

The general case can be deduced in the in the usual way (Ragab, F. M., Proc. Glasgow Math. Assoc., I, p. 192). Also the restriction $R(\alpha+n+s-\frac{1}{2})>-1$ can be removed by analytical continuation.

Particular cases: In (1) take $p=2$, $q=1$, $x=-1$ and get

$$\sum_{r=0}^{n}\frac{(-1)^{r}\,{}^{n}c_{r}\,(\alpha;r)\left(1+\frac{\alpha}{2};r\right)\Gamma(\tfrac{1}{2}+\alpha+2r)\,\Gamma(1+2\alpha+2n+2r)\,\Gamma(\delta+2r)\,\Gamma(\beta+2r)}{(\tfrac{1}{2}\alpha;r)\,(1+\alpha+n;r)\,\Gamma(\tfrac{1}{2}+\alpha+n+2r)\,\Gamma(1+2\alpha+4r)\,\Gamma(\gamma+2r)}\times$$

$$\times {}_4F_3\left(\begin{matrix}\tfrac12+\alpha+2r,\ 1+2\alpha+2n+2r,\ \delta+2r,\ \beta+2r;\ 1\\ \tfrac12+\alpha+n+2r,\ 1+2\alpha+4r,\ \gamma+2r\end{matrix}\right)=$$

$$=2^{2n}(1+\alpha;\,n)\frac{\Gamma(\delta)\,\Gamma(\beta)\,\Gamma(\gamma-\delta-\beta)}{\Gamma(\gamma-\delta)\,\Gamma(\gamma-\beta)} \tag{3}$$

where $R(\gamma)>0,\ R(\gamma-\delta-\beta)>0.$

In (1) take $p=2,\ q=1,\ x=-2$ and apply the formula due to Gauss, namely

$$F(2\beta,2\gamma;\,\beta+\gamma+\tfrac12;\,\tfrac12)=\frac{\Gamma(\beta+\gamma+\tfrac12)\,\Gamma(\tfrac12)}{\Gamma(\beta+\tfrac12)\,\Gamma(\gamma+\tfrac12)}, \tag{4}$$

so getting

$$\sum_{r=0}^{n}(-2)^{-2r}\times$$

$$\times\frac{(-1)^r\,{}^nc_r(\alpha;\,r)(1+\tfrac12\alpha;\,r)\,\Gamma(\tfrac12+\alpha+2r)\,\Gamma(1+2\alpha+2n+2r)\,\Gamma(2\beta+2r)\,\Gamma(2\gamma+2r)}{(\tfrac12\alpha;\,r)(1+\alpha+n;\,r)\,\Gamma(\tfrac12+\alpha+n+2r)\,\Gamma(1+2\alpha+4r)\,\Gamma(\beta+\gamma+\tfrac12+2r)}\times$$

$$\times{}_4F_3\left(\begin{matrix}\tfrac12+\alpha+2r,\ 1+2\alpha+2n+2r,\ 2\beta+2r,\ 2\gamma+2r;\ \tfrac12\\ \tfrac12+\alpha+n+2r,\ 1+2\alpha+4r,\ \beta+\gamma+\tfrac12+2r\end{matrix}\right)=$$

$$=2^{2n+\beta+\gamma-2}\pi^{-\frac12}(1+\alpha;\,n)\,\Gamma(\beta)\,\Gamma(\gamma). \tag{5}$$

Also in (1), write $1/x^2$ for x, take $p=1,\ q=2$ and apply the formula

$$\{J_\nu(x)\}^2=\frac{1}{\{\Gamma(\nu+1)\}^2}(x/2)^{2\nu}\,{}_1F_2(\nu+\tfrac12;\,\nu+1,\,2\nu+1;\,-x^2), \tag{6}$$

so getting

$$\{J_\nu(x)\}^2=\frac{2^{-2n}x^{2\nu}}{\sqrt{\pi}\,(1+\alpha;\,n)}\times$$

$$\times\sum_{r=0}^{n}\frac{(-1)^r\,{}^nc_r(\alpha;\,r)\left(1+\dfrac{\alpha}{2};\,r\right)\Gamma(\tfrac12+\alpha+2r)\,\Gamma(1+2\alpha+2n+2r)\,\Gamma(\tfrac12+\nu+2r)}{\left(\dfrac{\alpha}{2};\,r\right)(1+\alpha+n;\,r)\,\Gamma(\tfrac12+\alpha+n+2r)\,\Gamma(1+2\alpha+4r)\,\Gamma(\nu+1+2r)\,\Gamma(2\nu+1+2r)}\times$$

$$\times x^{4r}\,{}_3F_4\left(\begin{matrix}\tfrac12+\alpha+2r,\ 1+2\alpha+2n+2r,\ \nu+\tfrac12+2r;\ -x^2\\ \tfrac12+\alpha+n+2r,\ 1+2\alpha+4r,\ \nu+1+2r,\ 2\nu+1+2r\end{matrix}\right). \tag{7}$$

Again in (1) write $(2x)$ for x, take $p=2,\ q=0$ with $\alpha_1=\tfrac12+\nu,\ \alpha_2=\tfrac12-\nu$, apply the formula [C.V., p. 351], namely

$$\cos n\pi\,E(\tfrac12+n,\ \tfrac12-n::2z)=\sqrt{(2\pi z)}\,e^z K_n(z), \tag{8}$$

so getting

$$K_\nu(x) = 2^{-2n} \frac{\cos \nu\pi}{\sqrt{(2\pi x)}\,(1+\alpha;n)} e^{-x} \sum_{r=0}^{n} \frac{(-)^r\, {}^n c_r(\alpha;r)\left(1+\frac{\alpha}{2};r\right)}{\left(\frac{\alpha}{2};r\right)(1+\alpha+n;r)} (2x)^{-2r} \times$$

$$\times E\left\{\begin{matrix} \frac{1}{2}+\alpha+2r,\ 1+2\alpha+2n+2r,\ \frac{1}{2}+\nu+2r,\ \frac{1}{2}-\nu+2r : 2x \\ \frac{1}{2}+\alpha+n+2r,\ 1+2\alpha+4r \end{matrix}\right\}. \qquad (9)$$

§ 2. The second formula to be proved is

$$\sum_{r=0}^{n} \frac{(-1)^r\, {}^n c_r(\alpha;r)\left(1+\frac{\alpha}{2};r\right)}{(\frac{1}{2}\alpha;r)} E\left\{\begin{matrix} \varrho-\alpha,\ \varrho+n,\ \alpha_1,\ldots,\ \alpha_r : x \\ \varrho+r,\ \varrho-\alpha+n-1,\ \varrho_1,\ldots,\ \varrho_q \end{matrix}\right\} =$$

$$= (\varrho-\alpha-1-n)\, E(p;\alpha_r:q,\varrho_s:x) - \frac{1}{x} E(p;\alpha_r+1:q;\varrho_s+1:x). \qquad (10)$$

To prove (10), consider the special case with $p=q=0$. Then the coefficient of $\left(-\frac{1}{x}\right)^s$ on the L.H.S. is

$$\sum_{r=0}^{n} \frac{(-1)^r\, {}^n c_r(\alpha;r)(1+\frac{1}{2}\alpha;r)}{(\frac{1}{2}\alpha;r)\lfloor s} \cdot \frac{\Gamma(\varrho-\alpha+s)\,\Gamma(\varrho+n+s)}{\Gamma(\varrho+r+s)\,\Gamma(\varrho-\alpha+n-1+s)} =$$

$$= \frac{\Gamma(\varrho-\alpha+s)\,\Gamma(\varrho+n+s)}{\lfloor s\, \Gamma(\varrho+s)\,\Gamma(r-\alpha+n-1+s)} F\left(\begin{matrix} -n,\ \alpha,\ 1+\frac{1}{2}\alpha;\ 1 \\ \frac{1}{2}\alpha,\ \varrho+s \end{matrix}\right).$$

But since

$$\frac{\left(1+\frac{\alpha}{2};r\right)}{\left(\frac{\alpha}{2};r\right)} = 1+\frac{2r}{\alpha},$$

it follows

$$F\left(\begin{matrix} -n,\ \alpha,\ 1+\frac{\alpha}{2};\ 1 \\ \frac{1}{2}\alpha,\ \varrho+s \end{matrix}\right) = F\left(\begin{matrix} -n,\ \alpha;\ 1 \\ \varrho+s \end{matrix}\right) + \frac{(-n)(\alpha)}{\left(\frac{\alpha}{2}\right)(\varrho+s)} F\left(\begin{matrix} -n+1,\ \alpha+1;\ 1 \\ \varrho+s+1 \end{matrix}\right) =$$

$$= \frac{\Gamma(\varrho+s)\,\Gamma(\varrho+s+n-\alpha)}{\Gamma(\varrho+s+n)\,\Gamma(\varrho+s-\alpha)} + \frac{(-2n)}{(\varrho+s)} \cdot \frac{\Gamma(\varrho+s+1)\,\Gamma(\varrho-\alpha+s+n-1)}{\Gamma(\varrho+s+n)\,\Gamma(\varrho+s-\alpha)},$$

so that the last coefficient is equal to

$$\frac{1}{\lfloor s}\{(\varrho+s+n-\alpha-1)-2n\} = \frac{1}{\lfloor s}(\varrho+s-n-\alpha-1).$$

Also the coefficient of $\left(-\frac{1}{x}\right)^s$ on the R.H.S. of (10) is equal to when $p=q=0$

$$\frac{1}{\lfloor s}(\varrho-\alpha-n-1)+\frac{1}{\lfloor s-1}=\frac{1}{\lfloor s}(\varrho-\alpha-1-n+s).$$

Hence (10) is proved for the special case $p=q=0$. The general formula can be deduced in the usual way.

Particular cases: In (10) take $x=-1$, $p=2$, $q=1$, so getting

$$\sum_{r=0}^{n}\frac{(-1)^r\, {}^nc_r(\alpha;r)\left(1+\frac{\alpha}{2};r\right)\Gamma(\varrho-\alpha)\,\Gamma(\varrho+n)\,\Gamma(\beta)\,\Gamma(\delta)}{(\frac{1}{2}\alpha;r)\,\Gamma(\varrho+r)\,\Gamma(\varrho-\alpha+n-1)\,\Gamma(\gamma)}\times$$
$$\times{}_4F_3\left(\begin{matrix}\varrho-\alpha,\ \varrho+n,\ \beta,\ \delta;\ 1\\ \varrho+r,\ \varrho-\alpha+n-1,\ \gamma\end{matrix}\right)=$$
$$=(\varrho-\alpha-n-1)\frac{\Gamma(\beta)\,\Gamma(\delta)\,\Gamma(\gamma-\beta-\delta)}{\Gamma(\gamma-\beta)\,\Gamma(\gamma-\delta)}-\frac{\Gamma(\beta+1)\,\Gamma(\delta+1)\,\Gamma(\gamma-\beta-\delta-1)}{\Gamma(\gamma-\beta)\,\Gamma(\gamma-\delta)},\quad(11)$$

where

$$R(\gamma)>0,\ R(\gamma-\beta-\delta)>0.$$

Also in (10) write $(4/x^2)$ for x, take $p=0$, $q=1$ and apply the formula

$$E(:\nu+1:z)=z^{\frac{1}{2}\nu}J_\nu(2/\sqrt{z}),\quad(12)$$

so getting

$$\sum_{r=0}^{n}\frac{(-1)^r\, {}^nc_r(\alpha;r)\left(1+\frac{\alpha}{2};r\right)\Gamma(\varrho-\alpha)\,\Gamma(\varrho+n)}{(\frac{1}{2}\alpha;r)\,\Gamma(\varrho+r)\,\Gamma(\varrho-\alpha+n-1)\,\Gamma(\mu+1)}\times$$
$$\times{}_2F_3\left(\begin{matrix}\varrho-\alpha,\ \varrho+n;\ -x^2/4\\ \varrho+r,\ \varrho-\alpha+n-1,\ \mu+1\end{matrix}\right)=$$
$$=(4/x^2)^{\frac{1}{2}\mu}\left[(\varrho-\alpha-n-1)\,J_\mu(x)-\frac{x}{2}J_{\mu+1}(x)\right].\quad(13)$$

Again, (10) in combination with (6) gives

$$\sum_{r=0}^{n}\frac{(-1)^r\, {}^nc_r(\alpha;r)\left(1+\frac{\alpha}{2};r\right)\Gamma(\varrho-\alpha)\,\Gamma(\varrho+n)\,\Gamma(\nu+\frac{1}{2})}{(\frac{1}{2}\alpha;r)\,\Gamma(\nu+1)\,\Gamma(2\nu+1)\,\Gamma(\varrho+r)\,\Gamma(\varrho-\alpha+n-1)}\times$$
$$\times{}_3F_4\left(\begin{matrix}\varrho-\alpha,\ \varrho+n,\ \nu+\frac{1}{2};\ -x^2\\ \varrho+r,\ \varrho-\alpha+n-1,\ \nu+1,\ 2\nu+1\end{matrix}\right)+$$
$$+\frac{\Gamma(\nu+\frac{3}{2})}{\Gamma(\nu+2)\,\Gamma(2\nu+2)}x^2\,{}_1F_2(\nu+\tfrac{3}{2};\ \nu+2,\ 2\nu+2;\ -x^2)=$$
$$=\sqrt{\pi}\,x^{-2\nu}(\varrho-\alpha-n-1)\,\{J_\nu(x)\}^2.\quad(14)$$

§ 3. The third formula to be proved is

$$\sum_{r=0}^{n} \frac{{}^{n}c_r\,(\alpha\,;\,r)\,(1+\tfrac{1}{2}\,\alpha\,;\,r)}{(\tfrac{1}{2}\,\alpha\,;\,r)}\,(2x)^{-r}\times$$

$$\times E\left\{\begin{matrix}\frac{\alpha}{2}+r,\ \frac{\alpha+1}{2}+r, \frac{1}{2}+\frac{n}{2}+\frac{r}{2}, \frac{n}{2}+\frac{r}{2}, 1+\alpha+n+r,\ \alpha_1+r,\ldots,\ \alpha_p+r:x \\ \frac{\alpha}{2}+\frac{n}{2}+r, \frac{\alpha+1}{2}+\frac{n}{2}+r,\ n+r,\ \frac{1}{2}+r,\ 1+\alpha+2r,\ \varrho_1+r,\ldots,\ \varrho_q+r\end{matrix}\right\}=$$

$$=E\,(p\,;\,\alpha_r:q\,;\,\varrho_s:x). \tag{15}$$

The following formula is required in the prof of (15);

$${}_4F_3\left(\begin{matrix}\alpha,\ 1+\tfrac{1}{2}\,\alpha,\ \beta,\ -n\,;\ 1 \\ \tfrac{1}{2}\,\alpha,\ 1+\alpha-\beta,\ 1+2\beta-n\end{matrix}\right)=\frac{(\alpha-2\beta\,;\,n)\,(-\beta\,;\,n)}{(1+\alpha-\beta\,;\,n)\,(-2\beta\,;\,n)}. \tag{16}$$

(Bailey, W. N., Cambr. Tracts in Math., 32, p. 30, eq. 1.3).

To prove (15), consider the special case with $p=q=0$; then the coefficient of $\left(-\frac{1}{x}\right)^s$ on the L.H.S. is equal to

$$\frac{\Gamma\left(\frac{\alpha}{2}+s\right)\Gamma\left(\frac{\alpha+1}{2}+s\right)\Gamma\left(\frac{1}{2}+\frac{n}{2}+s\right)\Gamma\left(\frac{n}{2}+s\right)\Gamma(1+\alpha+n+s)}{\Gamma\left(\frac{\alpha}{2}+\frac{n}{2}+s\right)\Gamma\left(\frac{\alpha+1}{2}+\frac{n}{2}+s\right)\Gamma(n+s)\,\Gamma\left(\frac{1}{2}+s\right)\Gamma(1+\alpha+s)}\cdot\frac{1}{\lfloor s}+$$

$$+\frac{(-1)^1(-n\,;\,1)\,(\alpha\,;\,1)\left(1+\frac{\alpha}{2}\,;\,1\right)}{\left(\frac{\alpha}{2}\,;\,1\right)\lfloor 1\,(-1)^1}\,2^{-1}\times$$

$$\times\frac{\Gamma\left(\frac{\alpha}{2}+1+s-1\right)\Gamma\left(\frac{\alpha+1}{2}+1+s-1\right)\Gamma\left(\frac{n+1}{2}+\frac{1}{2}+s-1\right)\cdot}{\Gamma\left(\frac{\alpha+n}{2}+1+s-1\right)\Gamma\left(\frac{\alpha+1}{2}+\frac{n}{2}+1+s-1\right)\cdot}$$

$$\frac{\cdot\,\Gamma\left(\frac{n+1}{1}+s-1\right)\Gamma(1+\alpha+n+1+s-1)}{\cdot\,\Gamma(n+1+s-1)\,\Gamma\left(\frac{i}{2}+1+s-1\right)\Gamma(1+\alpha+2+s-1)\lfloor s-1}+$$

$+\cdots\cdots$

$+\cdots\cdots$

$$=\frac{\Gamma\left(\frac{\alpha}{2}+s\right)\Gamma\left(\frac{\alpha+1}{2}+s\right)\Gamma(1+\alpha+n+s)\,\Gamma\left(\frac{1}{2}+\frac{n}{2}+s\right)\Gamma\left(\frac{n}{2}+s\right)}{\Gamma\left(\frac{\alpha}{2}+\frac{n}{2}+s\right)\Gamma\left(\frac{\alpha+1}{2}+\frac{n}{2}+s\right)\Gamma(n+s)\,\Gamma\left(\frac{1}{2}+s\right)\Gamma(1+\alpha+s)}\cdot\frac{1}{\lfloor s}\times$$

$$\times\frac{2^{-n-2s+r-r}\,\Gamma\left(1-\frac{n}{2}-s\right)\Gamma\left(\frac{1}{2}-\frac{n}{2}-s\right)}{\sqrt{\pi}\,\Gamma(1-n-2s)}\times{}_4F_3\left(\begin{matrix}\alpha,\ 1+\frac{\alpha}{2},\ -s,\ -n;\ 1\\ \frac{\alpha}{2},\ 1+\alpha+s,\ 1-2s-n\end{matrix}\right),$$

by using the relation

$$(\alpha;\ -r)=(-1)^r/(1-\alpha;\ r),$$

where r is any positive integer.

Now substitute for the generalized hypergeometric function from (16) with $\beta=-s$ and get coefficient of $\left(-\frac{1}{x}\right)^s$ on the L.H.S. of (15) $=\frac{1}{\lfloor s}=$ coefficient of $\left(-\frac{1}{x}\right)^s$ on the R.H.S. of (15) with $p=q=0$. Thus (15) is proved for this special case and the general case can then be deduced in the usual way.

Particular cases: (15) in combination with (8) gives

$$K_\mu(x)=\frac{\cos\mu\pi}{\sqrt{(2\pi x)}}e^{-x}\sum_{r=0}^{n}\frac{{}^nc_r(\alpha;\ r)\left(1+\frac{\alpha}{2};\ r\right)}{\left(\frac{\alpha}{2};\ r\right)}(4x)^{-r}\times$$

$$\times E\left\{\begin{matrix}\frac{\alpha}{2}+r,\ \frac{\alpha+1}{2}+r,\ \frac{1}{2}+\frac{n+r}{2},\ \frac{n+r}{2},\ 1+\alpha+n+r,\ \frac{1}{2}+\mu+r,\ \frac{1}{2}-\mu+r:2x\\ \frac{\alpha+n}{2}+r,\ \frac{\alpha+n+1}{2}+r,\ n+r,\ \frac{1}{2}+r,\ 1+\alpha+2r\end{matrix}\right\}. \tag{17}$$

Also (15) in combination with (12) gives

$$J_\mu(x)=\left(\frac{x}{2}\right)^\mu\sum_{r=0}^{n}\frac{{}^nc_r(\alpha;\ r)\left(1+\frac{\alpha}{2};\ r\right)}{\left(\frac{\alpha}{2};\ r\right)}(8/x^2)^{-r}\times$$

$$\times\frac{\Gamma\left(\frac{\alpha}{2}+r\right)\Gamma\left(\frac{\alpha+1}{2}+r\right)\Gamma\left(\frac{1+n+r}{2}\right)\Gamma\left(\frac{n+r}{2}\right)\Gamma(1+\alpha+n+r)}{\Gamma\left(\frac{\alpha+n}{2}+r\right)\Gamma\left(\frac{\alpha+n+1}{2}+r\right)\Gamma(n+r)\,\Gamma\left(\frac{1}{2}+r\right)\Gamma(1+\alpha+2r)\,\Gamma(1+\mu+r)}\times$$

$$\times{}_5F_6\left(\begin{matrix}\frac{\alpha}{2}+r,\ \frac{\alpha+1}{2}+r,\ \frac{1+n+r}{2},\ \frac{n+r}{2},\ 1+\alpha+n+r;\ -x^2/4\\ \frac{\alpha+n}{2}+r,\ \frac{\alpha+n+1}{2}+r,\ n+r,\ \frac{1}{2}+r,\ 1+\alpha+2r,\ 1+\mu+r\end{matrix}\right). \tag{18}$$

Again in (15) take $x=-1,\ p=2,\ q=1$ and get

$$\sum_{r=0}^{n} \frac{(-2)^{-r}\,{}^{n}c_r(\alpha;r)\left(1+\frac{\alpha}{2};r\right)\Gamma\left(\frac{\alpha}{2}+r\right)\Gamma\left(\frac{\alpha+1}{2}+r\right)\Gamma\left(\frac{1+n+r}{2}\right)\cdot}{\left(\frac{\alpha}{2};r\right)\Gamma\left(\frac{\alpha+n}{2}+r\right)\Gamma\left(\frac{\alpha+n+1}{2}+r\right)\cdot}$$

$$\frac{\cdot\,\Gamma\left(\frac{n+r}{2}\right)\Gamma(1+\alpha+n+r)\,\Gamma(\beta+r)\,\Gamma(\delta+r)}{\cdot\,\Gamma(n+r)\,\Gamma\left(\frac{1}{2}+r\right)\Gamma(1+\alpha+2r)\,\Gamma(\gamma+r)}\times$$

$$\times\,{}_7F_6\left(\begin{matrix}\frac{\alpha}{2}+r,\ \frac{\alpha+1}{2}+r,\ \frac{1+n+r}{2},\ \frac{n+r}{2},\ 1+\alpha+n+r,\ \beta+r,\ \delta+r;\ 1\\ \frac{\alpha+n}{2}+r,\ \frac{\alpha+n+1}{2}+r,\ n+r,\ \frac{1}{2}+r,\ 1+\alpha+2r,\ \gamma+r\end{matrix}\right)=$$

$$=\frac{\Gamma(\beta)\,\Gamma(\delta)\,\Gamma(\gamma-\beta-\delta)}{\Gamma(\gamma-\beta)\,\Gamma(\gamma-\delta)}, \tag{19}$$

where

$$R(\gamma)>0,\ \ R(\gamma-\beta-\delta)>0.$$

§ 4. The fourth formula to be proved is

$$\sum_{r=0}^{n} {}^{n}c_r(\alpha;r)(2x)^{-r}\times$$

$$\times E\left\{\begin{matrix}\frac{\alpha}{2}+r,\frac{\alpha+1}{2}+r,\ \frac{\alpha}{2}+n+r,\ \frac{1+n+r}{2},\ \frac{n+r}{2},\ 1+\alpha+n+r,\ 1+\frac{\alpha}{2}+r,\ \alpha_1+r,\ \ldots,\ \ldots,\ \alpha_p+r:x\\ \frac{\alpha+n}{2}+r,\ \frac{\alpha+n+1}{2}+r,\ 1+\frac{\alpha}{2}+n+r,\ \frac{\alpha}{2}+r,\ n+r,\ 1+\alpha+2r,\ \frac{1}{2}+r,\ \varrho_1+r,\ \ldots,\ \ldots,\ \varrho_q+r\end{matrix}\right\}=$$

$$=E(p;\alpha_r:q;\varrho_s:x). \tag{20}$$

(20) can be proved in the same manner as formula (15) except instead of applying (16), the following formula (Bailey, W. N., Cambr. Tracts in Math., 32, p. 30, eq. 1.2), namely

$${}_3F_2\left(\begin{matrix}\alpha,\ \beta,\ -n;\ 1\\ 1+\alpha-\beta,\ 1+2\beta-n\end{matrix}\right)=\frac{(\alpha-2\beta;n)\left(1+\frac{\alpha}{2}-\beta;n\right)(-\beta;n)}{(1+\alpha-\beta;n)\left(\frac{\alpha}{2}-\beta;n\right)(-2\beta;n)} \tag{21}$$

is applied.

§ 5. The fifth formula to be proved is

$$\Gamma(\varrho_1)\sum_{r=0}^{n}(-1)^r\,{}^nc_r\,\frac{1}{\Gamma(\varrho_1-n+r)}\,x^{-r}\,E\,(\alpha_1+r,\ldots,\alpha_p+r:\varrho_1+r,\ldots,\varrho_q+r:x)=$$
$$=E\,(p\,;\,\alpha_r:\varrho_1-n,\,\varrho_2,\ldots,\varrho_q:x). \qquad (22)$$

Since the E-functions is symmetrical in the ϱ's formula (22) is equivalent to q similar relations. (22) generalizes the formula [C.V., p. 356, ex. 2, (iii)], namely

$$E\,(p\,;\,\alpha_r:\varrho_1-1,\,\varrho_2,\ldots,\varrho_q:x)=(\varrho_1-1)\,E\,(p\,;\,\alpha_r:q\,;\,\varrho_s:x)-\frac{1}{x}E\,(p\,;\,\alpha_r+1\,;\,q\,;\,\varrho_s+1;\,x), \quad (23)$$

which is (22) with $n=1$.

To prove (22), suppose it is true for a particular value of n, thus (22) with (ϱ_1-1) instead of ϱ_1 becomes

$$\Gamma(\varrho_1-1)\sum_{r=0}^{n}(-1)^r\,{}^nc_r\,\frac{x^{-r}}{\Gamma(\varrho_1-n+r-1)}\,E\,(\alpha_1+r,\ldots,\alpha_p+r:\varrho_1+r-1,\,\varrho_2+r,\ldots,\varrho_q+r:x)=$$
$$=E\,(p\,;\,\alpha_r:\varrho_1-n-1,\,\varrho_2,\ldots,\varrho_q:x).$$

Now apply (23) to each term on the left, so getting

$$E\,(p\,;\,\alpha_r:\varrho_1-n-1,\,\varrho_2.\,\varrho_3,\ldots,\varrho_q:x)=$$
$$=\Gamma(\varrho_1-1)\sum_{r=0}^{n}(-1)^r\,{}^nc_r\,\frac{(\varrho_1+r-1)}{\Gamma(\varrho_1-n+r-1)}\,x^{-r}\,E\,(\alpha_1+r,\ldots,\alpha_p+r:\varrho_1+r,\ldots,\varrho_q+r:x)+$$
$$+\Gamma(\varrho_1-1)\sum_{r=0}^{n}(-1)^{r+1}\,{}^nc_r\,\frac{x^{-r-1}}{\Gamma(\varrho_1-n+r-1)}\,E\,(\alpha_1+r+1,\ldots,\alpha_p+r+1\,;\,\varrho_1+r+1,\ldots,\varrho_q+r+1:x).$$

In the second of these series write $(r-1)$ for r, add the two series applying the identity

$$(\varrho_1+r-1)\,{}^nc_r+(\varrho_1-n+r-2)\,{}^nc_{r-1}=(\varrho_1-1)\,{}^{n+1}c_r\,;$$

then (22) is proved with $(n+1)$ in place of n. But it is true when $n=1$; hence it is true for all positive integral values of n.

Particular cases: In (22) take $\varrho_1=\alpha_1$; then it gives

$$ {}_1F_1\,(\varrho\,;\,\varrho-n\,;\,x)=e^x\,{}_1F_1\,(-n\,;\,\varrho-n\,;\,-x), \qquad (24)$$

which is a particular case of the known transformation

$$ {}_1F_1\,(\alpha\,;\,\beta\,;\,x)=e^x\,{}_1F_1\,(\beta-\alpha\,;\,\beta\,;\,-x). \qquad (25)$$

Again in (1) take $p=2$, $q=1$, $x=-1$ and it becomes

$$
{}_3F_2\left(\begin{matrix}\alpha_1, \alpha_2, -n; 1\\ \varrho_1-n, 1+\alpha_1+\alpha_2-\varrho_1\end{matrix}\right)=
$$
$$
=\frac{\Gamma(1+\alpha_1+\alpha_2-\varrho_1)\,\Gamma(1+\alpha_1-\varrho_1+n)\,\Gamma(1+\alpha_2-\varrho_1+n)\,\Gamma(1-\varrho_1)}{\Gamma(1+n-\varrho_1)\,\Gamma(1+\alpha_2-\varrho_1)\,\Gamma(1+\alpha_1-\varrho_1)\,\Gamma(1+\alpha_1+\alpha_2+n-\varrho_1)}, \qquad (26)
$$

which is Saalchûtz's theorem.

Also when $p=3$, $q=2$ (22) gives, if $R(\varrho_2-\alpha_2-\alpha_3-n)>0$, $\varrho_1=\alpha_1$,

$$
{}_3F_2\left(\begin{matrix}\alpha_1, \alpha_2, \alpha_3; 1\\ \alpha_1-n, \varrho_2\end{matrix}\right)=\frac{\Gamma(\varrho_2)\,\Gamma(\varrho_2-\alpha_2-\alpha_3)}{\Gamma(\varrho_2-\alpha_2)\,\Gamma(\varrho_2-\alpha_3)}\,{}_3F_2\left(\begin{matrix}-n, \alpha_2, \alpha_3; 1\\ 1+\alpha_2+\alpha_3-\varrho_2, \alpha_1-n\end{matrix}\right), \qquad (27)
$$

which was given by G. H. Hardy ("A chapter from Ramanujan's notebook", Proc. Cambr. Phil. Soc., 21, 1923, p. 498, eq. 5.2).

§ 6. The last formula to be proved is

$$
\sum_{r=0}^{n}(-1)^r\,{}^nc_r\,\frac{(\alpha_1;n)}{(\alpha_1;r)}\,x^{-r}\,E(\alpha_1+r,\ldots,\alpha_p+r:q;\varrho_s+r:x)=
$$
$$
=E(\alpha_1+n,\alpha_2,\ldots,\alpha_p:q;\varrho_s:x). \qquad (28)
$$

(28) generalizes the formula [C.V., p. 356, ex. 2, (i)], namely

$$
\alpha_1\,E(\alpha_1,\alpha_2,\ldots,\alpha_p:q;\varrho_s:x)-\frac{1}{x}\,E(\alpha_1+1,\alpha_2+1,\ldots,\alpha_p+1:q;\varrho_s+1:x)=
$$
$$
=E(\alpha_1+1,\alpha_2,\ldots,\alpha_p:q;\varrho_s:x). \qquad (29)
$$

Also formula (28) is equivalent to p similar relations.

To prove (28), assume that it is true for a particular value of n; thus (28) with (α_1+1) instead of α_1 becomes

$$
\sum_{r=0}^{n}(-1)^r\,{}^nc_r\,\frac{(\alpha_1+1;n)}{(\alpha_1+1;r)}\,x^{-r}\,E(\alpha_1+r+1,\alpha_2+r,\ldots,\alpha_p+r:q;\varrho_s+r:x)=
$$
$$
=E(\alpha_1+n+1,\alpha_2,\ldots,\alpha_p:q;\varrho_s:x).
$$

Here apply (29) to each term on the left, so getting

$$
E(\alpha_1+n+1,\alpha_2,\ldots,\alpha_p:q;\varrho_s:x)=
$$
$$
=\sum_{r=0}^{n}(-1)^r\,{}^nc_r\,\frac{(\alpha_1;n+1)}{(\alpha_1;r)}\,x^{-r}\,E(\alpha_1+r,\ldots,\alpha_p+r:q;\varrho_s+r:x)+
$$
$$
+\sum_{r=0}^{n}(-1)^{r+1}\,{}^nc_r\,\frac{(\alpha_1;n+1)}{(\alpha_1;r+1)}\,x^{-r-1}\,E(\alpha_1+r+1,\ldots,\alpha_p+r+1:q;\varrho_s+r+1:x).
$$

In the second of these two series, write $(r-1)$ for r, then add the two series applying the identity

$$^{n}c_r + {^{n}c_{r-1}} = {^{n+1}c_r},$$

then the last expression becomes (28) with $(n+1)$ in place of n. But (29) is (28) with $n=1$; therefore (28) is tiue for all positive integral values of n.

Particalar cases: In (28) write $-1/x$ for x and take $p=2$, $q=1$, $\alpha_1=\varrho_1=\varrho$, so getting

$$F(\beta, \alpha; \varrho; x) = (1-x)^{-\alpha} F\left(\alpha, \varrho-\beta; \varrho; \frac{x}{x-1}\right), \tag{30}$$

which is the known Euler Transformation with $\beta=-n$.

Finally, if in (28) $x=-1$, $p-2$, $q=1$; then it becomes Gauss's theorem.