ON THE DIFFERENTIAL EQUATIONS OF HILL IN THE THEORY OF THE MOTION OF THE MOON

BY

J. F. STEFFENSEN

in Copenhagen

1. G. W. Hill's equations define the movement of a body with infinitely small mass, attracted by Newton's law of gravitation towards two bodies moving in circles by that same law. Of these two bodies the mass of one is negligible in comparison with that of the other, and the distance of the body with infinitely small mass from the smaller of the two other bodies is assumed to be negligible in comparison with the distance between these. The whole movement takes place in a plane and is referred to the uniformly rotating axes. It is, thus, a degenerate case of the problem of three bodies.

Hill employed rectangular coordinates with the time as independent variable. Let p and q be the coordinates, t the time, then Hill's equations are[1]

$$\left\{\begin{aligned} \frac{d^2 p}{d t^2} - 2\frac{d q}{d t} &= 3p - \frac{p}{r^3} \\ \frac{d^2 q}{d t^2} + 2\frac{d p}{d t} &= -\frac{q}{r^3} \end{aligned}\right. \tag{1}$$

with Jacobi's integral

$$\left(\frac{d p}{d t}\right)^2 + \left(\frac{d q}{d t}\right)^2 = 3p^2 + \frac{2}{r} - C. \tag{2}$$

It has been shown[2] that, introducing polar coordinates $p = r\cos l$, $q = r\sin l$, putting at the same time

[1] See, for instance, H. C. Plummer: *An introductory treatise on dynamical astronomy* (1918), 265–6.

[2] J. F. Steffensen: Les orbites périodiques dans le problème de Hill. *Académie royale de Danemark, Bulletin* 1909 n⁰ 3, 320–3.

12 – 543809. *Acta Mathematica*. 93. Imprimé le 13 août 1955.

(3) $$\varepsilon=\frac{1}{r^3},\quad \varrho=\frac{1}{r}\frac{dr}{dt},\quad \omega=1+\frac{dl}{dt}$$

and eliminating t, the following system of three equations is obtained

(4) $$\begin{cases} (\omega-1)\dfrac{d\varepsilon}{dl}=-3\varrho\varepsilon \\ (\omega-1)\dfrac{d\varrho}{dl}=\omega^2-\varrho^2+\frac{3}{2}\cos 2l+\frac{1}{2}-\varepsilon \\ (\omega-1)\dfrac{d\omega}{dl}=-2\varrho\omega-\frac{3}{2}\sin 2l \end{cases}$$

with Jacobi's integral

(5) $$\tfrac{1}{2}\varrho^2+(\tfrac{1}{2}\omega^2-\omega)-\tfrac{1}{4}(1+3\cos 2l)+(\tfrac{1}{2}C\varepsilon^{\frac{2}{3}}-\varepsilon)=0.$$

We proceed to remove the trigonometric functions from this system. Putting

(6) $$\cos 2l=x,\quad \omega=1+\eta,\quad \varrho=\zeta\sin 2l$$

we have $dx=-2\sin 2l\,dl$ and find

(7) $$\begin{cases} 2\eta\dfrac{d\varepsilon}{dx}=3\zeta\varepsilon \\ 2(1-x^2)\eta\dfrac{d\zeta}{dx}=(1-x^2)\zeta^2-\eta^2+\varepsilon+2\eta(\zeta x-1)-\frac{3}{2}(1+x) \\ \eta\dfrac{d\eta}{dx}=\zeta(\eta+1)+\frac{3}{4} \end{cases}$$

with Jacobi's integral

(8) $$C=[2\varepsilon+\tfrac{3}{2}(1+x)-(1-x^2)\zeta^2-\eta^2]\,\varepsilon^{-\frac{2}{3}}.$$

2. We now try to satisfy (7) with the power series

(9) $$\begin{cases} \zeta=\sum\limits_{\nu=0}^{\infty}A_\nu x^\nu \\ \eta=\sum\limits_{\nu=0}^{\infty}B_\nu x^\nu \\ \varepsilon=\sum\limits_{\nu=0}^{\infty}C_\nu x^\nu. \end{cases}$$

It follows from general considerations that a region exists in ζ, η and ε where this solution converges for sufficiently small $|x|$.[1] We may therefore insert the series

[1] See, for instance, ÉMILE PICARD: *Traité d'analyse* II, Chap. XI.

(9) in (7) and demand that the coefficients of x^n shall vanish. In this way we obtain recurrence formulas for the determination of A_ν, B_ν and C_ν. These formulas we arrange in such a way that on the left we have A_{n+1}, B_{n+1} and C_{n+1}, while on the right only coefficients with lower indices enter. In stating the results, the cases $n=0$ and $n=1$ must be stated separately. We find:

$$n=0$$

$$(10)\qquad \begin{cases} 2\,A_1\,B_0 = A_0^2 - (B_0+1)^2 + C_0 - \frac{1}{2} \\ B_1\,B_0 = A_0 + A_0\,B_0 + \frac{3}{4} \\ 2\,C_1\,B_0 = 3\,A_0\,C_0. \end{cases}$$

$$n=1$$

$$(11)\qquad \begin{cases} 2\,A_2\,B_0 = (A_1+B_0)\,(A_0-B_1) + \frac{1}{2}C_1 - B_1 - \frac{3}{4} \\ 2\,B_2\,B_0 = A_1\,(B_0+1) + B_1\,(A_0-B_1) \\ 4\,C_2\,B_0 = 3\,A_0\,C_1 + 3\,A_1\,C_0 - 2\,B_1\,C_1. \end{cases}$$

$$n \geq 2$$

$$(12)\qquad \begin{aligned} 2\,(n+1)\,A_{n+1}\,B_0 = & \sum_{\nu=0}^{n} (A_\nu\,A_{n-\nu} - B_\nu\,B_{n-\nu}) - \sum_{\nu=2}^{n} A_{\nu-2}\,A_{n-\nu} + \\ & + 2\sum_{\nu=1}^{n} \nu\,(A_{\nu-1}\,B_{n-\nu} - A_\nu\,B_{n-\nu+1}) - 2\,B_n + C_n. \end{aligned}$$

$$(13)\qquad (n+1)\,B_{n+1}\,B_0 = A_n + \sum_{\nu=0}^{n} A_\nu\,B_{n-\nu} - \sum_{\nu=1}^{n} \nu\,B_\nu\,B_{n-\nu+1}.$$

$$(14)\qquad 2\,(n+1)\,C_{n+1}\,B_0 = 3\sum_{\nu=0}^{n} C_\nu\,A_{n-\nu} - 2\sum_{\nu=1}^{n} \nu\,C_\nu\,B_{n-\nu+1}.$$

3. For $x=0$ we obtain from (9) $\zeta = A_0$, $\eta = B_0$, $\varepsilon = C_0$, which we will take as constants of integration, corresponding to $l = \frac{\pi}{4}$, since $x = \cos 2\,l$. In order to compare with the orbit calculated by me in the paper quoted above I therefore put $l = \frac{\pi}{4}$ in formula (41) l.c. The result is

$$A_0 = 177649$$

$$B_0 = 12{\cdot}370483$$

$$C_0 = 179{\cdot}22909.$$

These figures are, of course, approximations but must, in calculating the following coefficients, be treated as arbitrarily chosen exact starting values. Only on this assumption can it be defended to retain so many figures in the coefficients as I have done in the table below.

ν	A_ν	B_ν	C_ν
0	$\cdot 177649$	$12\cdot 370483$	$179\cdot 22909$
1	$-\cdot 0^3 3704984$	$\cdot 2526379$	$3\cdot 860788$
2	$\cdot 0^5 5457$	$-\cdot 0^3 96596$	$-\cdot 00186694$
3	$\cdot 0^7 426$	$\cdot 0^4 14547$	$\cdot 0^4 9422$

As a check on these calculations I have calculated Jacobi's constant C by (8) for $x=0$, $x=0\cdot 6$ and $x=1$ and found the following values:

$$x=0, \quad C=6\cdot 5085385$$

$$x=0\cdot 6, \quad C=6\cdot 5085385$$

$$x=1, \quad C=6\cdot 5085384.$$

The agreement between these figures is as good as can be desired.

The control by Jacobi's integral of which no other use has been made is, of course, particularly valuable in a case like the present one where the remainder-terms of the expansions are unknown.

The chief advantage of the present method is that the calculation of the coefficients A_ν, B_ν and C_ν is much easier than the calculation of the corresponding coefficients in the trigonometrical series employed in the earlier paper, where it was necessary to proceed by successive approximations. On the other hand the convergence in the numerical example is nearly as rapid, as appears from a comparison of the coefficients given above with those of formula (41) in the earlier paper.

4. In order to examine the convergence from a purely theoretical point of view we begin by writing (12), (13) and (14) in the following form, where we have isolated the constants of integration A_0, B_0 and C_0. When the upper limit of summation is <1, the sum in question is simply left out.

$$(15) \quad \begin{aligned} 2(n+1) A_{n+1} B_0 &= 2 A_0 (A_n - A_{n-2} + B_{n-1}) + 2 B_0 (n A_{n-1} - B_n) + C_n - 2 B_n + \\ &+ \sum_{\nu=1}^{n-1} (A_\nu A_{n-\nu} - B_\nu B_{n-\nu}) + 2 \sum_{\nu=1}^{n-2} (\nu+1) A_\nu B_{n-\nu-1} - 2 \sum_{\nu=1}^{n} \nu A_\nu B_{n-\nu+1} - \sum_{\nu=1}^{n-3} A_\nu A_{n-\nu-2}. \end{aligned}$$

$$(16) \quad (n+1) B_{n+1} B_0 = A_0 B_n + B_0 A_n + A_n + \sum_{\nu=1}^{n-1} A_\nu B_{n-\nu} - \sum_{\nu=1}^{n} \nu B_\nu B_{n-\nu+1}.$$

$$(17)\qquad 2(n+1)\,C_{n+1}B_0 = 3A_0C_n + 3C_0A_n + 3\sum_{\nu=1}^{n-1} C_\nu A_{n-\nu} - 2\sum_{\nu=1}^{n} \nu\, C_\nu B_{n-\nu+1}.$$

We now write for $\nu \geq 1$

$$(18)\qquad K_\nu = \frac{\lambda^\nu}{\nu(\nu+1)} \quad (\lambda > 0)$$

and will assume that for $1 \leq \nu \leq n$ we have proved that

$$(19)\qquad |A_\nu| \leq \alpha K_\nu,\quad |B_\nu| \leq \beta K_\nu,\quad |C_\nu| \leq \gamma K_\nu.$$

We propose to find conditions which are sufficient to ensure that these inequalities are then also valid for $\nu = n+1$, that is, always.

In this investigation certain sums occur which are obtained by the identity

$$(20)\qquad K_\nu K_{m-\nu} = \lambda^m \left[\left(\frac{1}{\nu} + \frac{1}{m-\nu}\right)\frac{1}{m(m+1)} - \left(\frac{1}{\nu+1} + \frac{1}{m-\nu+1}\right)\frac{1}{(m+1)(m+2)}\right].$$

Putting for abbreviation

$$(21)\qquad s_n = \sum_{\nu=1}^{n} \frac{1}{\nu} \leq n$$

we find

$$(22)\qquad \sum_{\nu=1}^{n-1} K_\nu K_{n-\nu} = 2\,\frac{n-1+2s_{n-1}}{n(n+1)(n+2)}\,\lambda^n.$$

$$(23)\qquad \sum_{\nu=1}^{n-3} K_\nu K_{n-\nu-2} = 2\,\frac{n-3+2s_{n-3}}{n(n-1)(n-2)}\,\lambda^{n-2}.$$

$$(24)\qquad \sum_{\nu=1}^{n-2} (\nu+1) K_\nu K_{n-\nu-1} = \frac{n-2+2s_{n-2}}{n(n-1)}\,\lambda^{n-1}.$$

$$(25)\qquad \sum_{\nu=1}^{n} \nu K_\nu K_{n-\nu+1} = \frac{n+2s_n}{(n+2)(n+3)}\,\lambda^{n+1}.$$

5. We first obtain from (15) by (18) and (19)

$$(26)\qquad \begin{aligned} 2(n+1)|B_0|\cdot|A_{n+1}| &\leq 2|A_0|(\alpha K_n + \alpha K_{n-2} + \beta K_{n-1}) + 2|B_0|(n\alpha K_{n-1} + \beta K_n) + \\ &+ (\gamma + 2\beta)K_n + (\alpha^2+\beta^2)\sum_{\nu=1}^{n-1} K_\nu K_{n-\nu} + 2\alpha\beta\sum_{\nu=1}^{n-2}(\nu+1)K_\nu K_{n-\nu-1} + \\ &+ 2\alpha\beta\sum_{\nu=1}^{n} \nu K_\nu K_{n-\nu+1} + \alpha^2\sum_{\nu=1}^{n-3} K_\nu K_{n-\nu-2}. \end{aligned}$$

If, now, the right-hand side of this inequality is $\leq 2(n+1)|B_0|\cdot\alpha K_{n+1}$, then $|A_{n+1}| \leq \alpha K_{n+1}$, so that the inequality $|A_\nu| \leq \alpha K_\nu$ is valid for all ν.

By (18) and (22)–(25) this condition may, after multiplication by $(n+2)\,\lambda^{2-n}$, be written in the form

$$(27)\qquad \begin{aligned} &2\,|A_0|\left(\frac{n+2}{n(n+1)}\alpha\lambda^2+\frac{n+2}{(n-1)(n-2)}\alpha+\frac{n+2}{n(n-1)}\beta\lambda\right)+\\ &+2\,|B_0|\left(\frac{n+2}{n-1}\alpha\lambda+\frac{n+2}{n(n+1)}\beta\lambda^2\right)+(\gamma+2\beta)\frac{n+2}{n(n+1)}\lambda^2+\\ &+2(\alpha^2+\beta^2)\frac{n-1+2s_{n-1}}{n(n+1)}\lambda^2+2\alpha\beta\frac{(n-2+2s_{n-2})(n+2)}{n(n-1)}\lambda+\\ &+2\alpha\beta\frac{n+2s_n}{n+3}\lambda^3+2\alpha^2\frac{(n-3+2s_{n-3})(n+2)}{n(n-1)(n-2)}\leq 2\alpha\lambda^3|B_0|. \end{aligned}$$

This condition we will replace by a simpler though more rigid sufficient condition, obtained by replacing the fractions depending on n by numbers independent of n but at least as large as the original fractions.

As regards the fractions where s_ν does not enter simple considerations show that for $n\geq 3$

$$(28)\qquad \frac{n+2}{n(n+1)}\leq\tfrac{5}{12},\quad \frac{n+2}{(n-1)(n-2)}\leq\tfrac{5}{2},\quad \frac{n+2}{n(n-1)}\leq\tfrac{5}{6},\quad \frac{n+2}{n-1}\leq\tfrac{5}{2}.$$

In dealing with the fractions in which s_ν figures we employ the notations[1]

$$(29)\qquad \begin{cases} T_n=\dfrac{n-1+2s_{n-1}}{n(n+1)}, & U_n=\dfrac{(n-2+2s_{n-2})(n+2)}{n(n-1)},\\[2ex] V_n=\dfrac{(n-3+2s_{n-3})(n+2)}{n(n-1)(n-2)}, & W_n=\dfrac{n+2s_n}{n+3}, \end{cases}$$

T_n will decrease for increasing n if $T_n>T_{n+1}$, and, since $s_n=s_{n-1}+\dfrac{1}{n}$, this condition may be written in the form

$$n+4s_{n-1}>4$$

which is satisfied for $n\geq 2$, so that, in particular,

$$(30)\qquad T_n\leq\tfrac{5}{12}\quad(n\geq 3).$$

In the corresponding way is shown that $U_n\geq U_{n+1}$ if

$$2(n+5)s_{n-2}\geq n+13$$

which is satisfied for $n\geq 3$. Here we have $U_3=U_4=\frac{5}{2}$, so that

$$(31)\qquad U_n\leq\tfrac{5}{2}\quad(n\geq 3).$$

[1] It follows from (22)–(24) that T_n only occurs for $n\geq 2$, U_n for $n\geq 3$ and V_n for $n\geq 4$.

The condition for $V_n > V_{n+1}$ is

$$n(n-1)+4(n+4)s_{n-3} > 24$$

which is satisfied for $n \geq 4$. Hence

$$V_n \leq \tfrac{3}{4} \qquad (n \geq 4). \tag{32}$$

We finally have $W_n > W_{n+1}$, if

$$s_n > \tfrac{5}{2} + \frac{2}{n+1}.$$

This condition is satisfied for $n=9$, since $s_9 = 2\cdot82 \cdots > 2\cdot7$ and is therefore satisfied for all $n \geq 9$, because s_n increases but $\dfrac{2}{n+1}$ decreases for increasing n. For $n \geq 9$ we therefore have $W_n \leq W_9$ and in particular, for simplicity's sake,

$$W_n < \tfrac{5}{4}. \tag{33}$$

If this inequality is written in the form

$$s_n < \frac{n+15}{8},$$

a table of s_n shows that it is also valid for $n < 9$, hence for all n.

By (28) and (30)–(33) we now obtain from (27), when the terms depending on $|B_0|$ are collected on the right, the sufficient condition

$$\begin{aligned} 5\,|A_0|\,(\tfrac{1}{6}\alpha\lambda^2 + \tfrac{1}{3}\beta\lambda + \alpha) + \tfrac{5}{2}\alpha\beta\lambda^3 + \tfrac{5}{6}(\alpha^2+\beta^2+\beta+\tfrac{1}{2}\gamma)\lambda^2 + \\ + 5\alpha\beta\lambda + \tfrac{3}{2}\alpha^2 \leq 2\,|B_0|\,\lambda\,(\alpha\lambda^2 - \tfrac{5}{12}\beta\lambda - \tfrac{5}{2}\alpha). \end{aligned} \tag{34}$$

It follows from (19) and (18) that the three power series (9) will be convergent if $\Sigma K_\nu x^\nu$ or $\Sigma \dfrac{(\lambda x)^\nu}{\nu(\nu+1)}$ converges, that is, if $|x| \leq \dfrac{1}{\lambda}$.

If, as an example, we choose the simple figures

$$\lambda = 2, \quad \alpha = \tfrac{1}{2}, \quad \beta = \tfrac{1}{3}, \quad \gamma = 4, \tag{35}$$

the conditions (19) are satisfied in our numerical example for $1 \leq \nu \leq 3$, and (34) is reduced to

$$95\,|A_0| + 258 + \tfrac{5}{12} \leq 34\,|B_0| \tag{36}$$

which is satisfied in the example.

6. As regards (16), we find, proceeding in the same way,

$$(37)\qquad (n+1)\,|B_0|\cdot|B_{n+1}|\le(\beta\,|A_0|+\alpha\,|B_0|+\alpha)\,\frac{\lambda^n}{n\,(n+1)}+ \\ +2\,\alpha\,\beta\,\lambda^n\,\frac{n-1+2\,s_{n-1}}{n\,(n+1)\,(n+2)}+\beta^2\,\lambda^{n+1}\,\frac{n+2\,s_n}{(n+2)\,(n+3)}$$

whence, if we demand that the right-hand side shall be $\le(n+1)\,|B_0|\,\frac{\beta\,\lambda^{n+1}}{(n+1)\,(n+2)}$, and multiply by $(n+2)\,\lambda^{-n}$

$$(38)\qquad (\beta\,|A_0|+\alpha\,|B_0|+\alpha)\,\frac{n+2}{n\,(n+1)}+2\,\alpha\,\beta\,T_n+\beta^2\,\lambda\,W_n\le\beta\,\lambda\,|B_0|,$$

a sufficient condition corresponding to (27).

Inserting finally the limits given by (28), (30) and (33), collecting the terms depending on $|B_0|$ on the right, we find as a sufficient condition

$$(39)\qquad 5\,\beta\,|A_0|+5\,\alpha+10\,\alpha\,\beta+15\,\beta^2\,\lambda\le|B_0|\,(12\,\beta\,\lambda-5\,\alpha).$$

With the values (35) this condition becomes

$$(40)\qquad 10\,|A_0|+45\le 33\,|B_0|$$

which is satisfied in the numerical example.

7. We finally deal with (17) by the same method and find

$$(41)\qquad 2\,(n+1)\,|B_0|\cdot|C_{n+1}|\le 3\,(\gamma\,|A_0|+\alpha\,|C_0|)\,\frac{\lambda^n}{n\,(n+1)}+ \\ +6\,\alpha\,\gamma\,\lambda^n\,\frac{n-1+2\,s_{n-1}}{n\,(n+1)\,(n+2)}+2\,\beta\,\gamma\,\lambda^{n+1}\,\frac{n+2\,s_n}{(n+2)\,(n+3)}$$

which we require to be $\le 2\,(n+1)\,|B_0|\,\frac{\gamma\,\lambda^{n+1}}{(n+1)\,(n+2)}$. Hence, after multiplication by $(n+2)\,\lambda^{-n}$, the condition

$$(42)\qquad 3\,(\gamma\,|A_0|+\alpha\,|C_0|)\,\frac{n+2}{n\,(n+1)}+6\,\alpha\,\gamma\,T_n+2\,\beta\,\gamma\,W_n\,\lambda\le 2\,\gamma\,\lambda\,|B_0|.$$

Inserting the same limits as in the case of (38), we find as a sufficient condition

$$(43)\qquad 5\,\gamma\,|A_0|+5\,\alpha\,|C_0|+10\,\alpha\,\gamma+10\,\beta\,\gamma\,\lambda\le 8\,\gamma\,\lambda\,|B_0|.$$

With the values (35) this becomes

$$20\,|A_0| + \tfrac{5}{2}\,|C_0| + 46 + \tfrac{2}{3} \leq 64\,|B_0| \tag{44}$$

which condition is satisfied in the numerical example.

8. The result of the preceding investigation is that, if (19) is satisfied for $1 \leq \nu \leq 3$, and if the constants of integration A_0, B_0 and C_0 satisfy (34), (39) and (43), then (19) is satisfied for all $\nu \geq 1$, and the three power series (9) are convergent for $|x| \leq \frac{1}{\lambda}$.

In the numerical example there is, thus, at any rate convergence for $|x| \leq \frac{1}{2}$, that is, for $|\cos 2\,l| \leq \frac{1}{2}$.

If, in a given situation, it has been ascertained that (19) is satisfied over a greater range than $1 \leq \nu \leq 3$, better results may be obtained, because in that case the limits to the fractions (28) and (29) can be improved.

9. By the second and third of the equations (7) ε and ζ may evidently be eliminated from the first of the three equations. In this way we obtain a differential equation of the third order in η alone, but it is a non-linear equation of such complicated nature that nothing seems to be gained by this way of proceeding.