CAUCHY'S THEOREM AND ITS CONVERSE

BY

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1. Let C be a simple closed contour which has a central point z_0. By a 'central point' of a simple closed contour, we mean a point within the contour, such that every radius vector drawn from it to the contour lies wholly in the closed domain bounded by the contour and intersects it in only one point.

The existence of a central point z_0 imposes the restriction that the inside of C be a star with respect to z_0. Among such star domains many, including all convex domains, have the required property for all points z_0.

We shall first prove a form of Cauchy's theorem which imposes restrictions, both on the form of the contour and on the derivative of the function. We then remove these restrictions later on.

The point of affix

$$\zeta = z_0 + \lambda (z - z_0),$$

when z lies on C; and $0 < \lambda < 1$, lies on a similar closed contour lying within C and having z_0 as its central point. Call this contour C_λ.

Let us further suppose that

(i) $f(z)$ is a function of z, which has got a definite finite value at every point of the closed domain which consists of all the straight lines drawn from z_0 to the contour C; and of all contours C_λ, $0 \leq \lambda \leq 1$, save possibly at the point z_0;

(ii) $f(z)$ is one-valued and continuous along every contour C_λ, $0 \leq \lambda \leq 1$; and differentiable along every contour C_λ, $0 < \lambda < 1$, at every point of C_λ;

(iii) the maximum-modulus of $f(z)$ on the contour C_λ is bounded, when λ tends to zero and also when λ tends to unity;

(iv) $f(z)$ is continuous along every straight-line joining z_0 to the contour C, at every point of the straight line;

(v) $f(z)$ is differentiable along every straight line, joining z_0 to the contour C, at every point of the straight line, save possibly at one or both of its end points;

(vi) the derivative of $f(\zeta)$, at any point ζ of the contour C_λ, is the same whether taken along C_λ or along the straight line joining ζ to z_0;

(vii) the derivative of $f(\zeta)$, at the point ζ, taken along the straight line passing through ζ and z_0 is uniformly bounded with respect to z and λ, when z lies on C and λ lies in any closed interval (λ_1, λ_2), $0<\lambda_1<\lambda_2<1$.

Then

$$\int_C f(z)\,dz=0.$$

Proof. We define the Lebesgue integral of a function $f(z)$, round a simple closed contour C, by the relation

$$\int_C f(z)\,dz=\int_{t_0}^{T} \operatorname{Re} \dot{z} f(z)\,dt+i\int_{t_0}^{T} \operatorname{Im} \dot{z} f(z)\,dt,$$

where z is a function of a real parameter t, when z lies on C; and the two integrals on the right-hand side are Lebesgue integrals.

Now, if λ_1 and λ_1+h_1 be any two points in the open interval (0, 1), by the condition (vii) there exists a positive number M, depending only on λ_1 and h_1, such that $|f'(\zeta)|<M$, when λ lies in the closed interval whose end-points are λ_1 and λ_1+h_1, $f'(\zeta)$ being the derivative of $f(\zeta)$ along the contour C_λ.

Therefore, by the Fundamental theorem of the Lebesgue integration, we have

$$\left|\frac{f(z_0+(\lambda_1+h)(z-z_0))-f(z_0+\lambda_1(z-z_0))}{h}\right|=\left|\frac{1}{h}\int_{z_0+\lambda_1(z-z_0)}^{z_0+(\lambda_1+h)(z-z_0)} f'(\zeta)\,d\zeta\right|<M\cdot l, \qquad \text{(A)}$$

when λ_1+h lies in the closed interval whose end-points are λ_1 and λ_1+h_1; and z lies on C. The path of integration is a straight line; and l is the greatest distance of the point z_0 from C. By the condition (v), we have

$$\lim_{h\to 0}\frac{f(z_0+(\lambda_1+h)(z-z_0))-f(z_0+\lambda_1(z-z_0))}{h}=(z-z_0)\,f'(z_0+\lambda_1(z-z_0))$$

when z lies on C.

Consequently, applying Lebesgue's convergence theorem to the real and imaginary parts of the following integral on the left-hand side, we can easily show that

$$\lim_{h\to 0}\int_C \frac{f(z_0+(\lambda_1+h)(z-z_0))-f(z_0+\lambda_1(z-z_0))}{h}\,dz=\int_C (z-z_0)\,f'(z_0+\lambda_1(z-z_0))\,dz, \qquad \text{(B)}$$

where we make h tend to zero through an enumerable sequence.

Also, by the relation (A), we have

$$\lim_{h\to 0}\int_C f(z_0+(\lambda_1+h)(z-z_0))\,dz=\int_C f(z_0+\lambda_1(z-z_0))\,dz. \tag{C}$$

Now, let

$$\psi(\lambda)=\int_C \lambda f(z_0+\lambda(z-z_0))\,dz,$$

where $0<\lambda<1$; and the left-hand side is a Lebesgue integral.

Combining (B) and (D), we have

$$\psi'(\lambda_1)=\lim_{h\to 0}\frac{\psi(\lambda_1+h)-\psi(\lambda_1)}{h}=\int_C \{f(z_0+\lambda_1(z-z_0))+\lambda_1(z-z_0)f'(z_0+\lambda_1(z-z_0))\}\,dz$$

$$=\frac{1}{\lambda_1}\int_{C_{\lambda_1}}\{f(\zeta)+(\zeta-z_0)f'(\zeta)\}\,d\zeta,$$

where ζ lies on C_{λ_1}.

But the inequality $|f'(\zeta)|<M$ holds at all points of C_{λ_1}; and $f(\zeta)$ is continuous along C_{λ_1}, therefore by the Fundamental theorem of the Lebesgue integration, we have

$$\psi'(\lambda_1)=\frac{1}{\lambda_1}\cdot\int_{C_{\lambda_1}}\{f(\zeta)+(\zeta-z_0)f'(\zeta)\}\,d\zeta$$

$$=\frac{1}{\lambda_1}\cdot\int_{C_{\lambda_1}}\frac{d}{d\zeta}\{(\zeta-z_0)f(\zeta)\}\,d\zeta$$

$$=\frac{1}{\lambda_1}\cdot[(\zeta-z_0)f(\zeta)]_{C_{\lambda_1}}$$

$$=0.$$

Proving thereby that the derivative $\psi'(\lambda)$ of the function $\psi(\lambda)$ vanishes, when $0<\lambda<1$. Therefore $\psi(\lambda)$ is independent of λ.

By the conditions (iii) and (iv) and by Lebesgue's convergence theorem, we can very easily prove that $\psi(\lambda)$ is continuous at the points $\lambda=0$ and $\lambda=1$. But $\psi(0)$ is zero; and hence

$$\int_C f(z)\,dz=0.$$

2. If $\mathfrak{r}$ and $\mathfrak{r}_\lambda$ be a pair of corresponding arcs of the contours C and C_λ respectively; and if $f(z)$ satisfy all the conditions of § 1, with respect to the similar

contours $\mathfrak{r}_\lambda$, $\lambda_1 \leq \lambda \leq 1$; and the segments of straight lines drawn from z_0, which lie between r and r_{λ_1}, then

$$\int_\Delta f(z)\,dz = 0,$$

where Δ is the closed contour formed by $\mathfrak{r}$, $\mathfrak{r}_{\lambda_1}$ and the segments of straight lines, joining the corresponding end-points of $\mathfrak{r}$ and $\mathfrak{r}_{\lambda_1}$.

Proof. Let a and b be the end-points of $\mathfrak{r}$; and let

$$\psi(\lambda) = \int_{\mathfrak{r}} \lambda f(z_0 + \lambda(z - z_0))\,dz,$$

where $\lambda_1 \leq \lambda \leq 1$.

Replacing C by $\mathfrak{r}$ in (B) and (C) of § 1, we have

$$\psi'(\lambda) = \frac{1}{\lambda} \cdot [(\zeta - z_0) f(\zeta)]_{\mathfrak{r}_\lambda}$$
$$= (b - z_0) f(z_0 + \lambda(b - z_0)) - (a - z_0) f(z_0 + \lambda(a - z_0)),$$

where $\lambda_1 < \lambda < 1$.

Integrating each side of this relation, between the limits λ_1 and 1, we get

$$\psi(1) - \psi(\lambda_1) = \int_{\lambda_1}^{1} (b - z_0) f(z_0 + \lambda(b - z_0))\,d\lambda - \int_{\lambda_1}^{1} (a - z_0) f(z_0 + \lambda(a - z_0))\,d\lambda$$
$$= \int_{b_1}^{b} f(z)\,dz - \int_{a_1}^{a} f(z)\,dz,$$

where a_1 and b_1 are the end-points of $\mathfrak{r}_{\lambda_1}$; and the last two integrals are taken along segments of straight lines drawn from z_0.

Hence we have

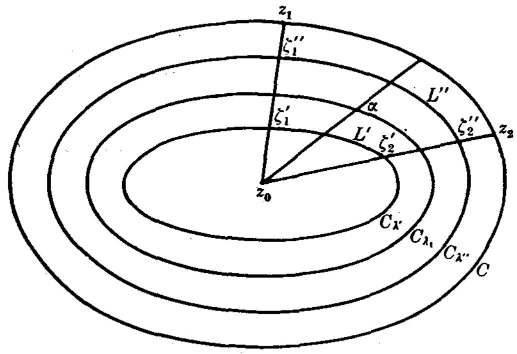
$$\int_\Delta f(z)\,dz = \int_{\mathfrak{r}} f(z)\,dz - \int_{b_1}^{b} f(z)\,dz - \int_{\mathfrak{r}_{\lambda_1}} f(z)\,dz + \int_{a_1}^{a} f(z)\,dz$$
$$= 0.$$

3. If a function $f(z)$ satisfy all the conditions of § 1, with respect to a simple closed contour C; and if α be any point within C other than the central point z_0, then

$$f(\alpha) = \frac{1}{2\pi i} \int_C \frac{f(z)\,dz}{z - \alpha},$$

where $f(\alpha)$ is the value of $f(z)$, at the point α, as defined in § 1.

Proof. Let us suppose that the point α lies on a similar contour C_{λ_1}. Consider the following figure:



Let L' and L'' denote the arcs of $C_{\lambda'}$ and $C_{\lambda''}$ respectively, which join ζ_1' to ζ_2'; and ζ_1'' to ζ_2''; and let r' and r'' denote the remaining portions of $C_{\lambda'}$ and $C_{\lambda''}$ respectively. Let R_1 and R_2 be the segments of straight lines joining ζ_1' to ζ_1'' and ζ_2' to ζ_2'', respectively.

Now, if $F(z) = \dfrac{f(z) - f(\alpha)}{z - \alpha}$, by § 2, we have

$$\int_{r''} F(z)\,dz - \int_{r'} F(z)\,dz = \int_{R_2} F(z)\,dz - \int_{R_1} F(z)\,dz.$$

Therefore

$$\int_{C_{\lambda''}} F(z)\,dz - \int_{C_{\lambda'}} F(z)\,dz = \int_{\Delta} F(z)\,dz, \tag{D}$$

where Δ denotes the closed contour formed by L', L'', R_1 and R_2.

Let $\Delta, \Delta_1, \Delta_2, \ldots \Delta_n, \ldots$ be a sequence of closed contours, each contained in its predecessor, such that every one of them contains α; and is formed by the arcs of similar contours C_λ; and the segments of straight lines drawn from z_0. Let us assume that the second derivative of $f(\zeta)$ exists along every contour of the sequence $\Delta, \Delta_1, \Delta_2, \ldots$, at every point of it; and is uniformly bounded with respect to z and λ, when z lies on an arc of C joining z_1 and z_2; and λ lies in the closed interval (λ', λ'').

Integrating by parts, we have

$$\int_\Delta F(z)\,dz = [\{f(z) - f(\alpha) - (z-\alpha) f'(z)\} \log (z-\alpha)]_\Delta + \int_\Delta f''(z) \{(z-\alpha) \log (z-\alpha) - (z-\alpha)\}\,dz.$$

In the last equation, the integrated part becomes

$$2\pi i\,[f(z) - f(\alpha) - (z-\alpha) f'(z)]_\Delta.$$

Here the second term in the bracket tends to zero, since $f'(z)$ is bounded.

The first term tends to zero if, in evaluating the variation of the integrated part, the initial point of Δ is taken to be a point where the contour C_{λ_1} or the straight line through z_0 and α cuts Δ. This is possible because the second integral in the equation tends to zero independently of the initial point, in virtue of the hypothesis on $f''(z)$.

We have thus shown that

$$\int_{\Delta} F(z)\,dz \to 0, \tag{E}$$

when we make z_1 tend to z_2; and each of λ' and λ'' tend to λ_1, by taking the sequence of contours $\Delta, \Delta_1, \Delta_2, \ldots$.

Now, by the method of § 1, we can prove that the function

$$\psi(\lambda) = \int_C \lambda F(z_0 + \lambda(z - z_0))\,dz$$

is independent of λ, when $0 \leq \lambda < \lambda_1$; and also when $\lambda_1 < \lambda \leq 1$. But, by (D) and (E), $\psi(\lambda)$ is continuous for $\lambda = \lambda_1$; therefore $\psi(\lambda)$ is constant in the closed interval (0, 1). Proving thereby that

$$\psi(1) = \int_C \frac{f(z) - f(\alpha)}{z - \alpha}\,dz = 0.$$

Hence

$$f(\alpha) = \frac{1}{2\pi i}\int_C \frac{f(z)\,dz}{z - \alpha}.$$

We can easily deduce from this formula that

$$f^n(\alpha) = \frac{\underline{|n}}{2\pi i}\int_C \frac{f(z)\,dz}{(z-\alpha)^{n+1}},$$

when $f^n(\alpha)$ is the nth derivative of $f(z)$ at α; and this derivative is independent of the path along which it is taken.

We, now, prove that it is unnecessary to assume the existence and the uniform boundedness of the second derivative of $f(z)$.

Let $\varphi(z) = \int_{z_0}^{z} f(z)\,dz$, where $f(z)$ satisfies all the conditions of § 1 with respect to C; and the path of integration is the straight line joining z_0 and z. If ζ_1 and ζ_2 be any pair of points lying on a contour C_λ, by § 2, we have

$$\lim_{\zeta_2 \to \zeta_1} \frac{\varphi(\zeta_2) - \varphi(\zeta_1)}{\zeta_2 - \zeta_1} = \lim_{\zeta_2 \to \zeta_1} \frac{\int_{\zeta_1}^{\zeta_2} f(\zeta)\,d\zeta}{\zeta_2 - \zeta_1}$$
$$= f(\zeta_1),$$

where the integral on the right-hand side is taken along an arc of C_λ, whose end-points are ζ_1 and ζ_2. Therefore, the function $\varphi(z)$ satisfies all the conditions of § 1. The second derivative $\varphi''(\zeta)$ of $\varphi(\zeta)$ at any point ζ of the contour C_λ, exists along C_λ; and is equal to its second derivative along the straight line passing through ζ and z_0. Since $\varphi''(\zeta) = f'(\zeta)$, $\varphi''(\zeta)$ is uniformly bounded with respect to z and λ, when z lies on C; and λ lies in any closed interval (λ_1, λ_2), $0 < \lambda_1 < \lambda_2 < 1$.

Consequently, if α lies within C_λ, $0 < \lambda < 1$, we have

$$\varphi'(\alpha) = \frac{1}{2\pi i} \int\limits_{C_\lambda} \frac{\varphi(z)\, dz}{(z-\alpha)^2},$$

where $\varphi'(\alpha)$ denotes the derivative of $\varphi(z)$, at the point α. Proving thereby that

$$f(\alpha) = \frac{1}{2\pi i} \int\limits_{C_\lambda} \frac{f(z)\, dz}{z-\alpha}.$$

Hence, making λ tend to unity, we have

$$f(\alpha) = \frac{1}{2\pi i} \int\limits_{C} \frac{f(z)\, dz}{z-\alpha}.$$

Corollary 1. If $f(z)$ satisfies all the conditions of § 1, with respect to a simple closed contour C which has a central point z_0; and if $f(z)$ has got the value $f(z_0)$ at the point z_0, then

$$f(z_0) = \frac{1}{2\pi i} \int\limits_{C} \frac{f(z)\, dz}{z-z_0}$$

and

$$f^n(z_0) = \frac{\underline{|n}}{2\pi i} \int\limits_{C} \frac{f(z)\, dz}{(z-z_0)^{n+1}},$$

where $f^n(z_0)$ denotes the nth derivative of $f(z)$ at the point z_0 taken along any path.

Proof. If α be any point within C other than z_0, then by § 3, we have

$$f(\alpha) = \frac{1}{2\pi i} \int\limits_{C} \frac{f(z)\, dz}{z-\alpha}.$$

Now, making α tend to z_0, along the straight line passing through α and z_0, we obtain the first formula; and the second formula is easily deducible from this.

Corollary 2. If $f(z)$ satisfies all the conditions of § 1, with respect to a simple closed contour C which has a central point z_0, then $f(z)$ is an analytic function of z, regular within C.

4. Let us suppose that a function $f(z)$ satisfies the following conditions:

(i) $f(z)$ is continuous along every straight line joining any point z of a simple closed contour C to its central point z_0, at every point of the straight line, save possibly at its end-point z;

(ii) $f(z)$ is one-valued and continuous along every similar contour C_λ;

(iii) the integral of $f(z)$, round every contour formed by an arc of any contour C_λ and the straight lines joining the end-points of the arc to z_0, vanishes;

(iv) the maximum-modulus of $f(\zeta)$ on C_λ is bounded in every closed interval (λ_1, λ_2), where $0<\lambda_1<\lambda_2<1$. Then $f(z)$ is regular within C.

Proof. Let $\varphi(z)$ be a function of z, defined by the relation

$$\varphi(z)=\int_{z_0}^{z} f(z)\,dz,$$

where the path of integration is the straight line joining z_0 and z.

By hypothesis, we can easily show that the function $\varphi(z)$ is one-valued and continuous along every similar contour C_λ; and also along every straight line joining z_0 to any point z of the contour C_λ.

Also, if z_1 and z_2 be any two points of a contour C_λ, then, by hypothesis, we have

$$\frac{\varphi(z_2)-\varphi(z_1)}{z_2-z_1}=\frac{\int_{z_1}^{z_2} f(z)\,dz}{z_2-z_1},$$

where the integral on the right-hand side is taken along an arc of C_λ whose end-points are z_1 and z_2.

Consequently, we have

$$\varphi'(z_1)=\lim_{z_2\to z_1}\frac{\varphi(z_2)-\varphi(z_1)}{z_2-z_1}=f(z_1).$$

Proving thereby that the derivative of $\varphi(z)$ at any point z of the contour C_λ, exists along C_λ; and is equal to its derivative at the same point, taken along the straight line passing through z_0 and z.

Moreover, by the condition (iv), the maximum-modulus of this derivative on C_λ is bounded in every closed interval (λ_1, λ_2), $0<\lambda_1<\lambda_2<1$.

We have thus shown that $f(z)$ satisfies all the conditions of § 1, with respect to C. Hence, by Corollary 2, $f(z)$ is regular within C.

5. If $f(z)$ satisfies all the conditions of § 1, except (vii), and if $f(z)$ is bounded in the open domain enclosed by C, then

$$\int_C f(z)\,dz = 0.$$

Proof. Let λ and $\lambda + h_n$ be any two points of the open interval $(0, 1)$; and let z be a function of a real parameter t, $t_0 \leq t \leq T$, when z lies on C.

Consider the function

$$\varphi_n(\lambda, t) = \left| \frac{f(z_0 + (\lambda + h_n)(z - z_0)) - f(z_0 + \lambda(z - z_0))}{h_n} \right|.$$

The function $\varphi_n(\lambda, t)$ may be expressed by $\nu(\lambda, t, \eta)$, a function of the three variables λ, t, η, where $\eta = \frac{1}{n}$. The function $\nu(\lambda, t, \eta)$ is in the first instance defined only for values of η, of the form $\frac{1}{n}$, but it may be extended to the case in which η has all values in the interval $0 < \eta \leq 1$, by such a rule as that, when η is in the interval $\left(\frac{1}{n+1}, \frac{1}{n}\right)$,

$$\nu(\lambda, t, \eta) = \nu\left(\lambda, t, \frac{1}{n}\right) + \frac{\frac{1}{n} - \frac{1}{\eta}}{\frac{1}{n} - \frac{1}{n+1}} \left\{ \nu\left(\lambda, t, \frac{1}{n+1}\right) - \nu\left(\lambda, t, \frac{1}{n}\right) \right\}.$$

The function $\nu(\lambda, t, \eta)$, so defined for the three-dimensional set of points $0 < \lambda < 1$, $t_0 \leq t \leq T$ and $0 < \eta \leq 1$ is, everywhere continuous with respect to each variable. Therefore, by a theorem of Baire ([1], p. 422, ex. 2) there must be points in every domain lying in the plane $\eta = 0$, at which $\nu(\lambda, t, \eta)$ is continuous with respect to (λ, t, η); and therefore with respect to (λ, t). Consequently $|f'(z_0 + \lambda(z - z_0))|$, which is $\lim_{\eta \to 0} \nu(\lambda, t, \eta)$, is point-wise discontinuous with respect to (λ, t). It follows that the points of infinite discontinuity of the derivative $f'(\zeta)$ of $f(\zeta)$ at any point ζ of the contour C_λ, taken along C_λ, form a set which is non-dense in the open domain bounded by the contour C.

Now, if a be any point within C, which is not an infinite discontinuity of $f'(\zeta)$, there exists a closed contour Δ of the same form as that of § 2, such that no point of infinite discontinuity of $f'(\zeta)$ lies within or on it; and a is an interior point of it.

Let $\varphi(z) = \int_b^z f(z)\,dz$, where b is a fixed point within Δ, z is any point within or on it; and the integral is taken along a path which consists of two parts: (i) the segment of the straight line drawn from z_0 through b, joining b to the point z_1 where this straight line intersects the contour C_λ on which z lies; and (ii) the arc of the contour C_λ, lying inside Δ, whose end-points are z and z_1.

If ζ_1 and ζ_2 be any two points within or on Δ, lying on the same straight line through z_0, then by § 2, we have

$$\lim_{\zeta_2 \to \zeta_1} \frac{\varphi(\zeta_2) - \varphi(\zeta_1)}{\zeta_2 - \zeta_1} = \lim_{\zeta_2 \to \zeta_1} \frac{\int_{\zeta_1}^{\zeta_2} f(\zeta)\, d\zeta}{\zeta_2 - \zeta_1}$$
$$= f(\zeta_1),$$

where the integral on the right-hand side is taken along the straight line.

Also, if ζ_1 and ζ_2 lie on the same contour C_λ, by the definition of $\varphi(z)$, we have

$$\lim_{\zeta_2 \to \zeta_1} \frac{\varphi(\zeta_2) - \varphi(\zeta_1)}{\zeta_2 - \zeta_1} = \lim_{\zeta_2 \to \zeta_1} \frac{\int_{\zeta_1}^{\zeta_2} f(\zeta)\, d\zeta}{\zeta_2 - \zeta_1}$$
$$= f(\zeta_1),$$

where the path of integration is an arc of C_λ.

Since $f(z)$ is bounded in the closed domain enclosed by Δ, $\varphi(z)$ satisfies all the conditions of § 2 with respect to Δ. Moreover, the second derivative of $\varphi(\zeta)$ at any point ζ within or on Δ, taken in the sense of § 1 along a path inside Δ, is $f'(\zeta)$ which is bounded. Therefore, by the method of § 3, we can easily prove that $\varphi(z)$ is regular within Δ. Consequently, $f(z)$ is regular within Δ.

We have thus proved that $f(z)$ is regular in a neighbourhood of every point within C, with the possible exception of a non-dense set.

Let Γ be a closed contour formed by an arc of any contour C_λ, $0 < \lambda < 1$, and the straight lines joining the end-points of the arc to z_0. The set of points within or on Γ, which are not infinite discontinuities of $f'(\zeta)$, is open. It can be covered by an enumerable set of closed contours Δ.

Since no boundary point of Δ is an infinite discontinuity of $f'(\zeta)$, the function $f(z)$ can be continued analytically outside Δ. We take a point on Δ and draw a closed contour of the same form as Δ, corresponding to this point. We then repeat the same process at the common boundary points of this contour and Δ; and so on. It should be observed that, in this process of analytical continuation, isolated points or unclosed curves of infinite discontinuities of $f'(\zeta)$ can not occur. For an end-point of such a curve will be a singularity of the analytic function $f'(z)$ and consequently of $f(z)$; which is untenable, under our hypothesis. Since $f(z)$ is one-valued and bounded within C, we can easily show that $f(z)$ can be represented by a Cauchy's integral formula, at all points in a small neighbourhood of such an end-point, which do not lie on the curve. By the conditions (ii) and (iv) of § 1, this integral formula can be proved to be valid also for the points of the curve in the small neighbourhood. Proving thereby that $f(z)$ is regular at all points in this neighbourhood.

So there are two possibilities: either $f(z)$ is regular within $\mathfrak{r}$ or in continuing $f(z)$ outside Δ, we reach a natural boundary of $f(z)$ which is composed of arcs of similar contours C_λ and segments of straight lines drawn from z_0. We call such a closed contour Δ'. Any point of infinite discontinuity of $f'(\zeta)$ within or on $\mathfrak{r}$, lies on a contour Δ'; and the closed domain bounded by $\mathfrak{r}$ is thus covered by an enumerable set of non-overlapping contours Δ'.

Moreover, the closed domain bounded by a contour Δ' can be divided up into a finite number of contours Δ; and therefore, by § 2, the integral of $f(z)$ round Δ' vanishes. Since $f(z)$ is one-valued and bounded, the integrals of $f(z)$ along an arc of C_λ or along a segment of a straight line drawn from z_0, taken in opposite directions cancel. Consequently, if we exclude the portions of the contours Δ' lying outside $\mathfrak{r}$, the integral of $f(z)$ round $\mathfrak{r}$ vanishes.

Hence, by § 4, $f(z)$ is regular within C.

Finally, we can remove the restriction on the type of the contour C in two ways: (i) these theorems can be applied to a closed contour C, the inside of which can be divided up into a finite number of sub-domains, such that each sub-domain has a central point, provided that $f(z)$ satisfies all the conditions of these theorems with respect to each sub-domain; and is one-valued and bounded within C; and (ii) the inside of C can be represented conformally ([2], § 8.2) on a domain which is a star with respect to one or more of its interior points.

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References

[1]. E. W. Hobson, *Theory of functions of a real variable*, Vol. I, Cambridge, 1921.
[2]. E. T. Copson, *Functions of a complex variable*, Oxford, 1935.