ABOUT THE VALUE DISTRIBUTION OF HOLOMORPHIC MAPS INTO THE PROJECTIVE SPACE

BY

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A First Main Theorem for holomorphic maps into the projective space was established in [10]. As an application, an equidistribution theorem for open holomorphic maps of maximal order was obtained. These results shall be extended to arbitrary order s. On a Stein manifold, they assume a special elegant form:

Let M be a non-compact, connected Stein manifold of dimension m. Let $h: M \to \mathbf{R}$ be a non-negative function of class C^∞ on M such that its Levi form (2) $\chi_1 = d^\perp dh$ is positive definite on M and such that for every $r > 0$ the open set $G_r = \{z \mid h(z) < r\}$ is not empty and relative compact. Such a function h exists on M if and only if M is a Stein manifold. Obviously, χ_1 is the exterior form of a Kaehler metric on M. Define $\chi_0 = 1$ and for s in $1 \leqslant s \leqslant m$ define

$$\chi_s = \frac{1}{s!} \chi_1 \wedge \ldots \wedge \chi_1$$

s-times.

Let V be a complex vector space of dimension $n+1 > 1$. Take a hermitian metric on V. It induces a Kaehler metric on the projective space $\mathbf{P}(V)$ associated to V, whose exterior form is denoted by $\ddot{\omega}_0$. Define $\ddot{\omega}_{00} = 1$ and

$$\ddot{\omega}_{0s} = \frac{1}{s!} \ddot{\omega}_0 \wedge \ldots \wedge \ddot{\omega}_0 \quad (s\text{-times})$$

$$W(s) = \frac{\pi^s}{s!}.$$

Let $f: M \to \mathbf{P}(V)$ be a holomorphic map. For $0 \leqslant s \leqslant \operatorname{Min}(n, m)$, define the characteristic of order s by

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(2) Define $d^\perp = i(\partial - \bar{\partial}) = -d^c$ where $d = \partial + \bar{\partial}$.

$$A_{s,f}(t) = \frac{1}{W(s)} \int_{G_t} f^*(\ddot{\omega}_{0,s}) \wedge \chi_{m-s}$$

$$T_{s,f}(r) = \int_0^r A_{s,f}(t)\,dt.$$

For $s=0$, $A_{0,f}(t) = M(t)$ is the volume of G_t.

Let $G_p(V)$ be the Grassmann manifold of p-dimensional linear projective subspace E of $\mathbf{P}(V)$. Suppose a neighborhood W of $E_0 \in G_{n-s}(V)$ and a neighborhood U of $z_0 \in M$ with $f(z_0) \in E_0$ exist such that $\dim_z f^{-1}(E) = m-s < m$ for all $z \in f^{-1}(E) \cap U$ and all $E \in W$. Then $T_{s,f}(r) \to \infty$ for $r \to \infty$. If

$$\frac{A_{s-1,f}(r)}{T_{s,f}(r)} \to 0 \quad \text{for } r \to \infty,$$

then the image $f(M)$ intersects almost every linear projective subspace $E \in G_{n-s}(V)$.

The mean value of the Levine form over the Grassmann manifold $G_p(V)$ has to be computed. This integration over the Grassmann manifold may be of independent interest. For instance, this method gives the degree of the Grassmann manifold easily.

Recently, Hirschfelder [3], Wu [13] and the author [11] obtained a First Main Theorem for holomorphic maps into compact Kaehler manifolds. However, the results there do not imply the results here because the Levine form is not a proper proximity form as obtained there.

Although this paper is a sequel to [10], it can be read independently. The beginning of § 3 provides a survey of the results obtained in [10]. The notation has been changed slightly, hopefully for the better.

1. Differential forms

Let V be a complex vector space of dimension $n+1$ with $n>0$. Let $\mathbf{P}(V)$ be the associated projective space. Let $\mathbf{P}\colon V-\{0\} \to \mathbf{P}(V)$ be the natural projection such that $\mathbf{P}(\mathfrak{z}) = \mathbf{P}(\mathfrak{w})$ if and only if $\mathfrak{z} \wedge \mathfrak{w} = 0$. The projection is denoted uniformly by $\mathbf{P}$ for all vector spaces. If the dependency of V shall be denoted, write $\mathbf{P} = \mathbf{P}_V$.

Associated are the exterior product $V[p] = V \wedge \ldots \wedge V$ (p-times) and the dual vector space V^*. For $0 \leqslant p \leqslant n$, define the *Grassmann cone* by

$$\tilde{G}_p(V) = \{\mathfrak{a}_0 \wedge \ldots \wedge \mathfrak{a}_p \mid \mathfrak{a}_\mu \in V\} \subseteq V[p+1].$$

The *Grassmann manifold* $G_p(V) = \mathbf{P}(\tilde{G}_p(V) - \{0\})$ is a smooth, compact, complex submanifold of $\mathbf{P}(V[p+1])$ and has dimension $(p+1)(n-p)$. For $0 \neq \mathfrak{a} \in \tilde{G}_p(V)$, the $(p+1)$-dimensional linear subspace

$$E(\mathfrak{a}) = \{\mathfrak{z} \in V \mid \mathfrak{z} \wedge \mathfrak{a} = 0\}$$

is defined. If $\mathfrak{a} = \mathfrak{a}_0 \wedge \ldots \wedge \mathfrak{a}_p \neq 0$, then $E(\mathfrak{a}) = \mathbf{C}\mathfrak{a}_0 + \ldots + \mathbf{C}\mathfrak{a}_p$. If $a \in G_p(V)$, then $E(\mathfrak{a}) = E(a)$ is the same for all $\mathfrak{a} \in \mathbf{P}^{-1}(a)$. Moreover, E maps $G_p(V)$ bijectively onto the set of all $(p+1)$-dimensional linear subspaces of V. If $a \in G_p(V)$, define

$$\ddot{E}(a) = \mathbf{P}_V(E(a) - \{0\}) = \mathbf{P}(E(a)) \subseteq \mathbf{P}(V).$$

Then $\ddot{E}$ maps $G_p(V)$ bijectively onto the set of all p-dimensional projective linear subspaces of V. Obviously, $G_0(V) = \mathbf{P}(V)$ and $G_{n-1}(V) \approx \mathbf{P}(V^*)$.

If $0 \leqslant p < n$ and $0 \leqslant q < n$, define

$$F_{p,q} = \begin{cases} \{(a,b) \in G_p(V) \times G_q(V) \mid E(a) \subseteq E(b)\} & \text{if } p \leqslant q \\ \{(a,b) \in G_p(V) \times G_q(V) \mid E(a) \supseteq E(b)\} & \text{if } p > q. \end{cases}$$

Let π: $F_{p,q} \to G_q(V)$ and τ: $F_{p,q} \to G_p(V)$ be the natural projections.

LEMMA 1.1. *$F_{p,q}$ is a connected, compact, smooth, complex submanifold of $G_p(V) \times G_q(V)$. The projections π and τ are proper, surjective, regular holomorphic maps.*

Proof. Obviously, $F_{p,q}$ is closed and locally given by holomorphic equations. Therefore, $F_{p,q}$ is a compact analytic subset of $G_p(V) \times G_q(V)$. Let $\mathrm{GL}(V) = \{\alpha \colon V \to V \mid \alpha \text{ linear isomorphism}\}$ be the general linear group on V. Then $\mathrm{GL}(V)$ acts on $G_p(V)$ by $\alpha(E(a)) = E(\alpha(a))$. Moreover, if $(a, b) \in F_{p,q}$ so is $(\alpha(a), \alpha(b)) \in F_{p,q}$. Hence $\mathrm{GL}(V)$ acts as a group of biholomorphic maps on $F_{p,q}$, and the action of $\mathrm{GL}(V)$ on $F_{p,q}$ is transitive. Because $F_{p,q}$ is smooth at at least one point, it is smooth. Obviously, the projections π and τ are surjective, proper, holomorphic and commute with the action of $\mathrm{GL}(V)$. By Sard's Theorem, π and τ are regular at least along one of its fibers; hence, considering the action of $\mathrm{GL}(V)$, they are regular. If $p \leqslant q$, then $F_{p,q}$ is a differentiable fiber bundle over the connected base space $G_p(V)$ with the connected fiber $G_p(E(b))$. Hence, $F_{p,q}$ is connected. If $p > q$, then $F_{p,q}$ is biholomorphically equivalent to $F_{q,p}$. Hence, $F_{p,q}$ is connected, q.e.d.

Let $(\cdot \mid \cdot)$ be a positive definite Hermitian product on V. With this product, V becomes a Hermitian vector space. Also, $V[p]$ and V^* become Hermitian vector spaces. If $0 \neq \mathfrak{x} \in V[p]$ and $0 \neq \mathfrak{y} \in V[q]$, then

$$\|\mathfrak{x} : \mathfrak{y}\| = \frac{|\mathfrak{x} \wedge \mathfrak{y}|}{|\mathfrak{x}|\ |\mathfrak{y}|}$$

is defined. If $x \in G_p(V)$ and $y \in G_q(V)$, then $\|x:y\|$ is well-defined by

$$\|x:y\| = \|\mathfrak{x} : \mathfrak{y}\| \quad \mathfrak{x} \in \mathbf{P}^{-1}(x) \quad \mathfrak{y} \in \mathbf{P}^{-1}(y).$$

Then $0 \leqslant \|x:y\| \leqslant 1$.

On any complex manifold, the exterior derivative d splits into $d=\partial+\bar{\partial}$. Define $d^{\perp}=i(\partial-\bar{\partial})=-d^c$. Define the forms υ_p and υ_{ps} on $V[p+1]$ and the forms ω_p and ω_{ps} on $V[p+1]-\{0\}$ by[1]

$$\upsilon_p(\mathfrak{x})=\tfrac{1}{4}d^{\perp}d|\mathfrak{x}|^2 \qquad \upsilon_{ps}=\frac{1}{s!}\,\upsilon\wedge\ldots\wedge\upsilon \quad (s\text{-times})$$

$$\omega_p(\mathfrak{x})=\tfrac{1}{4}d^{\perp}d\log|\mathfrak{x}|^2 \quad \omega_{ps}=\frac{1}{s!}\,\omega\wedge\ldots\wedge\omega \quad (s\text{-times}).$$

Observe that υ_0, υ_{0s}, ω_0, $\omega_{0,s}$ are forms on V. One and only one Kähler matric exists on $\mathbf{P}(V[p+1])$ with fundamental form $\ddot{\omega}_p$ such that $\mathbf{P}^*(\ddot{\omega}_p)=\omega_p$. Define $\ddot{\omega}_{ps}=(1/s!)\,\ddot{\omega}_p\wedge\ldots\wedge\ddot{\omega}_p$ (s-times). Then $\mathbf{P}^*(\ddot{\omega}_{ps})=\omega_{ps}$. Observe that[2]

$$W(n)=\int_{\mathbf{P}(V)}\ddot{\omega}_{0,n}=\frac{\pi^n}{n!}. \tag{1}$$

The Grassmann manifold $G_p(V)$ is a smooth, compact, complex submanifold of $\mathbf{P}(V[p+1])$ and has dimension $d_p=(p+1)(n-p)$. The pull back of the forms $\ddot{\omega}_{p,s}$ to the submanifold $G_p(V)$ will be denoted again by $\ddot{\omega}_{p,s}$. The volume of $G_p(V)$ is denoted by

$$W(n,p)=\int_{G_p(V)}\ddot{\omega}_{p,d_p} \tag{2}$$

and will be computed later.

For $a\in G_p(V)$ and $x\in\mathbf{P}(V)-\ddot{E}(a)$, the exterior product $x\wedge a$ is well defined by $x\wedge a=\mathbf{P}(\mathfrak{x}\wedge\mathfrak{a})$ where $\mathbf{P}(\mathfrak{x})=x$ and $\mathbf{P}(\mathfrak{a})=a$. A holomorphic map

$$\pi_a\colon\ \mathbf{P}(V)-\ddot{E}(a)\to G_{p+1}(V)$$

is defined by $\pi_a(x)=x\wedge a$. The map π_a is meromorphic on $\mathbf{P}(V)$. On $\mathbf{P}(V)-\ddot{E}(a)$, define

$$\Phi_p(a)=\pi_a^*(\ddot{\omega}_{p+1}).$$

If $a\in G_p(V)$, then $E^{\perp}(a)=\{\mathfrak{z}\in V\,|\,(\mathfrak{z}\,|\,\mathfrak{a})=0 \text{ for all } \mathfrak{a}\in E(a)\}$ is orthogonal to $E(a)$ and $V=E(a)\oplus E^{\perp}(a)$. Let $\tilde{\varrho}_a\colon V\to E^{\perp}(a)$ be the projection. Then $\varrho_a\colon \mathbf{P}(V)-\ddot{E}(a)\to\mathbf{P}(E^{\perp}(a))$ is well-defined by $\mathbf{P}\circ\tilde{\varrho}_a=\varrho_a\circ\mathbf{P}$. Let $j_a\colon\mathbf{P}(E^{\perp}(a))\to\mathbf{P}(V)$ be the inclusion. Then

$$\Phi_p(a)=\varrho_a^*j_a^*(\ddot{\omega}_0)$$

which implies
$$\Phi_p^r(a)=\underbrace{\Phi_p(a)\wedge\ldots\wedge\Phi_p(a)}_{(r\text{-times})}=0 \quad \text{if } r\geq n-p,$$

because $\mathbf{P}(E^{\perp}(a))$ has dimension $n-p-1$. Moreover,

$$\ddot{\omega}_0(x)-\Phi_p(a)(x)=\tfrac{1}{2}dd^{\perp}\log\|x:a\|$$

for $x\in\mathbf{P}(V)-\ddot{E}(a)$.

(1) For the proofs of the results mentioned here, see [10], § 3.

(2) See Lemma 2.1 for a proof.

If $s=n-p$ and $a\in G_p(V)$, define the *Levine form*

$$\hat{\Lambda}_s(a)=\frac{1}{(s-1)!}\sum_{\nu=0}^{s-1}\Phi_p(a)^\nu\wedge\ddot{\omega}_0^{s-1-\nu}$$

on $\mathbf{P}(V)-\ddot{E}(a)$. Obviously, $\hat{\Lambda}_s(a)$ is a non-negative, real-analytic form of bidegree $(s-1, s-1)$ with $\partial\hat{\Lambda}_s(a)=0$ and $\bar{\partial}\hat{\Lambda}_s(a)=0$. The *associate proximity form* $\Lambda_s(a)$ is defined by

$$\Lambda_s(a)(x)=\frac{1}{2s}\left(\log\frac{1}{\|x:a\|}\right)\hat{\Lambda}_s(a)(x)\geqslant 0$$

for $x\in\mathbf{P}(V)-\ddot{E}(a)$. Then

$$d^\perp d\Lambda_s(a)=\ddot{\omega}_{0s}.$$

It will be important to compute the integral average

$$\frac{1}{W(n,p)}\int_{a\in G_p(V)}\Lambda_s(a)\,\ddot{\omega}_{p,d_p}(a).$$

This has already been done in [10] for $p=0$, where the following identity was of importance: Take $\mathfrak{a}\in V$. On $V-E(\mathfrak{a})$, define $\xi_\mathfrak{a}$ by(1)

$$\xi_\mathfrak{a}(\mathfrak{w})=\frac{(d\mathfrak{w}\wedge\mathfrak{w}\,|\,\mathfrak{w}\wedge\mathfrak{a})}{|\mathfrak{w}|^2\,|\mathfrak{w}\wedge\mathfrak{a}|}=\frac{1}{|\mathfrak{w}\wedge\mathfrak{a}|}\left(\partial(\mathfrak{w}\,|\,\mathfrak{a})-\frac{(\mathfrak{w}\,|\,\mathfrak{a})}{|\mathfrak{w}|^2}\partial|\mathfrak{w}|^2\right).$$

Define $\tilde{\tau}_\mathfrak{a}=(i/2)\xi_\mathfrak{a}\wedge\bar{\xi}_\mathfrak{a}$ on $V-E(\mathfrak{a})$. If $a\in\mathbf{P}(V)$, then τ_a is welldefined on $V-E(a)$ by $\tau_a=\tilde{\tau}_\mathfrak{a}$ with $\mathfrak{a}\in\mathbf{P}^{-1}(a)$. Moreover, one and only one form $\ddot{\tau}_a$ of bidegree $(1,1)$ on $\mathbf{P}(V)-\ddot{E}(a)=\mathbf{P}(V)-\{a\}$ exists such that $\mathbf{P}^*(\ddot{\tau}_a)=\tau_a$. The form $\ddot{\tau}_a$ is non-negative and $\ddot{\tau}_a\wedge\ddot{\tau}_a=0$. Then

$$\|a:x\|^2\Phi_0(a)(x)=\ddot{\omega}_0(x)-\ddot{\tau}_a(x),$$

and

$$\|a:x\|^{2q}\Phi_0^q(a)(x)=\ddot{\omega}_0^q(x)-q\ddot{\tau}_a(x)\wedge\ddot{\omega}_0^{q-1}(x)\tag{3}$$

for $x\in\mathbf{P}(V)-\{a\}$ and $q\in\mathbf{N}$.

For a fixed integer p in $0\leqslant p\leqslant n$, consider the diagram

$$\begin{array}{ccc} F_{p-1,p} & \xrightarrow{\pi} & G_p(V) \\ \downarrow{\scriptstyle\tau} & & \\ G_{p-1}(V) & & \end{array}$$

In order to establish a fundamental identity for integration on Grassmann manifolds, the maps τ and π shall be expressed in local coordinates in a neighborhood of an arbitrary point $(a,b)\in F_{p-1,p}$ as follows:

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(1) For the proofs of the results mentioned here, see [10], § 5.

Observe $E(a) \subseteq E(b)$. Pick an orthonormal base $\mathfrak{a}_0, \dots, \mathfrak{a}_n$ of V such that

$$E(a) = \mathbf{C}\mathfrak{a}_0 + \dots + \mathbf{C}\mathfrak{a}_{p-1}$$

$$E(b) = \mathbf{C}\mathfrak{a}_0 + \dots + \mathbf{C}\mathfrak{a}_p.$$

Define $\mathfrak{a} = \mathfrak{a}_0 \wedge \dots \wedge \mathfrak{a}_{p-1}$ and $\mathfrak{b} = \mathfrak{a}_0 \wedge \dots \wedge \mathfrak{a}_p$. Then $a = \mathbf{P}(\mathfrak{a})$ and $b = \mathbf{P}(\mathfrak{b})$.

Consider $\mathbf{C}^{(n-p)p+n}$ as the vector space M of all matrices

$$z = \begin{Vmatrix} z_{0p} & \cdots & z_{0n} \\ \vdots & & \vdots \\ z_{pp} & \cdots & z_{pn} \end{Vmatrix} \quad \text{with } z_{pp} = 0. \tag{4}$$

Consider $\mathbf{C}^{(n-p+1)p}$ as the vector space M_0 of all matrices

$$x = \begin{Vmatrix} z_{0,p} & \cdots & z_{0,n} \\ \vdots & & \vdots \\ z_{p-1,p} & \cdots & z_{p-1,n} \end{Vmatrix}. \tag{5}$$

Consider $\mathbf{C}^{(n-p)(p+1)}$ as the vector space M_1 of all matrices

$$y = \begin{Vmatrix} z_{0,p+1} & \cdots & z_{0,n} \\ \vdots & & \vdots \\ z_{p,p+1} & \cdots & z_{p,n} \end{Vmatrix}. \tag{6}$$

Define the projection τ_0: $M \to M_0$ by $\tau_0(z) = x$ with z as in (4) and x as in (5). Define the surjective, regular holomorphic map by π_1: $M \to M_1$ by

$$\pi_1(z) = \begin{Vmatrix} u_{0,p+1} & \cdots & u_{0n} \\ \vdots & & \vdots \\ u_{p,p+1} & \cdots & u_{pn} \end{Vmatrix} \tag{7}$$

with $\quad u_{\mu\nu} = z_{\mu\nu} - z_{\mu p} z_{p\nu}$ for $0 \leqslant \mu \leqslant p-1$ and $p+1 \leqslant \nu \leqslant n \quad u_{p\nu} = z_{p\nu}$ for $p+1 \leqslant \nu \leqslant n$. (8)

For $\nu = 0, 1$, define ζ_ν: $M_\nu \to G_{p-1+\nu}(V)$ by

(a) $\nu = 0$: $x \in M_0$ as in (5):

$$\left.\begin{aligned} \mathfrak{e}_\mu &= \mathfrak{a}_\mu + \sum_{\nu=p}^{n} z_{\mu\nu}\mathfrak{a}_\nu \quad 0 \leqslant \mu \leqslant p-1 \\ \mathfrak{e} &= \mathfrak{e}_0 \wedge \dots \wedge \mathfrak{e}_{p-1} \\ \zeta_0(x) &= \mathbf{P}(\mathfrak{e}), \quad \text{then } \zeta_0(0) = a. \end{aligned}\right\} \tag{9}$$

(b) $\nu = 1$: $y \in M_1$ as in (6):

$$\left.\begin{aligned} \mathfrak{c}_\mu &= \mathfrak{a}_\mu + \sum_{\nu=p+1}^{n} z_{\mu\nu}\mathfrak{a}_\nu \\ \mathfrak{c} &= \mathfrak{c}_0 \wedge \dots \wedge \mathfrak{c}_p \\ \zeta_1(y) &= \mathbf{P}(\mathfrak{c}), \quad \text{then } \zeta_1(0) = b. \end{aligned}\right\} \tag{10}$$

According to [11] Lemma 2.1, ζ_ν is a biholomorphic map onto an open subset of $G_{p-1+\nu}(V)$. Define the holomorphic map $\zeta: M \to F_{p-1,p}$ by

$$\zeta(z) = (\mathbf{P}(\mathfrak{e}), \mathbf{P}(\mathfrak{e} \wedge \mathfrak{e}_p)) \in F_{p-1,p}$$

where $\mathfrak{e}$ and $\mathfrak{e}_p$ are defined as above. Then

$$\tau \circ \zeta = \zeta_0 \circ \tau_0.$$

Observe

$$\mathfrak{e} \wedge \mathfrak{c}_p = \left[\bigwedge_{0 \leqslant \mu \leqslant p-1} (\mathfrak{c}_\mu + z_{\mu p}\mathfrak{a}_p)\right] \wedge \left(\mathfrak{a}_p + \sum_{\nu=p+1}^{n} z_{p,\nu}\mathfrak{a}_\nu\right) = \left[\bigwedge_{0 \leqslant \mu \leqslant p-1} \left(\mathfrak{c}_\mu - z_{\mu p} \sum_{\nu=p+1}^{n} z_{p\nu}\mathfrak{a}_\nu\right)\right] \wedge \mathfrak{c}_p.$$

Hence, $\mathbf{P}(\mathfrak{e} \wedge \mathfrak{c}_p) = \zeta_1(\pi_1(z))$. Therefore, $\pi \circ \zeta = \zeta_1 \circ \pi_1$. Consequently, the map ζ is injective. Obviously, $\zeta(M)$ is contained in the open subset $F_{p-1,p} \cap (\zeta_0(M_0) \times \zeta_1(M_1))$ of F_{p-1}, p. Now, it will be shown that ζ maps M onto this subset. Take $(x, y) \in F_{p-1,p}$ with $x \in \zeta_0(M_0)$ and $y \in \zeta_1(M_1)$. Then $\tilde{x} = \zeta_0^{-1}(x)$ and $\tilde{y} = \zeta_1^{-1}(y)$ with

$$\tilde{x} = \begin{Vmatrix} x_{0p} & \dots & x_{0n} \\ \vdots & & \vdots \\ x_{p-1,p} & \dots & x_{p-1,n} \end{Vmatrix} \qquad \tilde{y} = \begin{Vmatrix} y_{0,p+1} & \dots & y_{0,n} \\ \vdots & & \vdots \\ y_{p,p+1} & \dots & y_{p,n} \end{Vmatrix}.$$

Define z as in (4) by $z_{\mu\nu} = x_{\mu\nu}$ for $0 \leqslant \mu \leqslant p-1$ and $p \leqslant \nu \leqslant n$ and $z_{p\nu} = y_{p\nu}$ for $p+1 \leqslant \nu \leqslant n$ and $z_{pp} = 0$. Then $\tau_0(z) = \tilde{x}$ and $\tau(\zeta(z)) = \zeta_0(\tau_0(z)) = \zeta_0(\tilde{x}) = x$. Denote $\pi_1(z)$ as in (7) and (8). Then $u_{p\nu} = y_{p\nu}$, if $p+1 \leqslant \nu \leqslant n$, and $u_{\mu\nu} = x_{\mu\nu} - x_{\mu p} y_{p\nu}$, if $0 \leqslant \mu \leqslant p-1$ and $p+1 \leqslant \nu \leqslant n$. Moreover,

$$\mathfrak{x}_\mu = \mathfrak{a}_\mu + \sum_{\nu=p}^{n} x_{\mu\nu}\mathfrak{a}_\nu \in E(x) \subseteq E(y) \quad \text{if } 0 \leqslant \mu \leqslant p-1,$$

$$\mathfrak{y}_\mu = \mathfrak{a}_\mu + \sum_{\nu=p+1}^{n} y_{\mu\nu}\mathfrak{a}_\nu \in E(y) \quad \text{if } 0 \leqslant \mu \leqslant p$$

where $\mathfrak{y}_0, \dots, \mathfrak{y}_p$ is a base of $E(y)$ over $\mathbf{C}$. Therefore, $c_{\mu\varrho} \in \mathbf{C}$ exist such that

$$\mathfrak{x}_\mu = \sum_{\varrho=0}^{p} c_{\mu\varrho}\mathfrak{y}_\varrho = \sum_{\varrho=0}^{p} c_{\mu\varrho}\mathfrak{a}_\varrho + \sum_{\nu=p+1}^{n} \left(\sum_{\varrho=0}^{p} c_{\mu\varrho} y_{\varrho\nu}\right)\mathfrak{a}_\nu$$

for $\mu = 0, \dots, p-1$. Hence, $c_{\mu\varrho} = 0$ for $\varrho \neq \mu$ and $\varrho \neq p$ with $c_{\mu\mu} = 1$ and $c_{\mu p} = x_{\mu p}$ for $\mu = 0, \dots, p-1$. Moreover,

$$x_{\mu\nu} = y_{\mu\nu} + x_{\mu p} y_{p\nu} \quad \text{if } p+1 \leqslant \nu \leqslant n$$

which implies $y_{\mu\nu} = u_{\mu\nu}$ if $0 \leqslant \mu \leqslant p-1$ and $p+1 \leqslant \nu \leqslant n$. Moreover, $y_{p\nu} = u_{p\nu}$ if $p+1 \leqslant \nu \leqslant n$. Therefore, $\pi_1(z) = \tilde{y}$ and

$$\pi(\zeta(z)) = \zeta_1(\pi_1(z)) = \zeta_1(\tilde{y}) = y$$

$$\zeta(z) = (\tau(\zeta(z)), \pi(\zeta(z))) = (x, y)$$

which implies

$$\zeta(M) = F_{p-1,p} \cap (\zeta_0(M_0) \times \zeta_1(M_1)).$$

Because $\zeta(M)$ is open and ζ injective, $\zeta\colon M \to F_{p-1,\,p}$ is a biholomorphic map onto an open neighborhood of (a, b). Because the complements of $\zeta_\nu(M_\nu)$ in $G_{p-1+\nu}(V)$ are thin analytic subsets, the complement of $\zeta(M)$ is a thin analytic subset and its intersection with each fiber of τ and π is a thin analytic subset of this fiber or the whole fiber. The following commutative diagram has been established:

$$\begin{tikzcd}
M \arrow[rr, "\pi_1"] \arrow[dr, "\zeta"] \arrow[dd, "\tau_0"'] & & M_1 \arrow[dr, "\zeta_1"] \arrow[d, phantom, "\circ"] & \\
& F_{p-1,p} \arrow[rr, "\pi"'] \arrow[dd, "\tau"] & {} & G_p(V) \\
M_0 \arrow[r, phantom, "\circ"] \arrow[dr, "\zeta_0"'] & {} & & \\
& G_{p-1}(V) & &
\end{tikzcd} \tag{11}$$

Especially, the dimension of $F_{p-1,\,p}$ is $n+p(n-p)$; the fiber dimension of τ is $n-p$ and the fiber dimension of π is p.

Lemma 1.2. *If* $b \in G_{p-1}(V)$, *then*

$$\int_{\pi^{-1}(b)} \tau^*(\ddot{\omega}_{p-1,\,p}) = W(p) = \frac{\pi^p}{p!}.$$

Proof. The diagram (11) implies

$$J = \int_{\pi^{-1}(b)} \tau^*(\ddot{\omega}_{p-1,\,p}) = \int_{\pi_1^{-1}(0)} \zeta^*\tau^*(\ddot{\omega}_{p-1,\,p}).$$

On $\pi_1^{-1}(0)$, the identities (9) read

$$\mathfrak{e}_\mu = \mathfrak{a}_\mu + z_{\mu p}\,\mathfrak{a}_p \quad \text{if } 0 \leqslant \mu \leqslant p-1$$

$$\mathfrak{e} = \mathfrak{e}_0 \wedge \ldots \wedge \mathfrak{e}_{p-1} = \mathfrak{a}_0 \wedge \ldots \wedge \mathfrak{a}_{p-1} + \sum_{\mu=0}^{p-1} z_{\mu p}\,\mathfrak{a}_0 \wedge \ldots \wedge \mathfrak{a}_{\mu-1} \wedge \mathfrak{a}_p \wedge \mathfrak{a}_{\mu+1} \wedge \ldots \wedge \mathfrak{a}_{p-1}$$

$$|\mathfrak{e}|^2 = 1 + \sum_{\mu=0}^{p-1} |z_{\mu p}|^2.$$

$$\zeta^*\tau^*(\ddot{\omega}_{p-1,\,p}) = \frac{1}{p!}\,\frac{1}{4}\, d^\perp d \log\left(1 + \sum_{\mu=0}^{p} |z_{\mu p}|\right)^p = \frac{1}{p!}\,\ddot{\omega}^p$$

where $\ddot{\omega}$ is the fundamental form of the Kähler metric of $\mathbf{P}(\mathbf{C}^{p+1})$ defined by the Hermitian product $(\mathfrak{x}\,|\,\mathfrak{y}) = \sum_{\mu=0}^{p} x_\mu \bar{y}_\mu$ on $\mathbf{C}^{p+1}$.

Hence,
$$J = \int_{\mathbf{P}(\mathbf{C}^{p+1})} \frac{1}{p!}\,\ddot{\omega}^p = W(p), \qquad \text{q.e.d.}$$

Consider the diagram (11). Differential forms from $G_{p-1}(V)$ and $G_p(V)$ can be pulled up to $F_{p-1,p}$ where the following important identity holds:

THEOREM 1.3. *Let $d_p=(p+1)(n-p)$ be the dimension of the Grassmann manifold $G_p(V)$. Then*

$$\pi^*(\ddot{\omega}_{p,d_p}) \wedge \tau^*(\ddot{\omega}_{p-1,p}) = \tau^*(\ddot{\omega}_{p-1,d_p-1}) \wedge \pi^*(\ddot{\omega}_{p,n-p}).$$

Proof. Pick $(a, b) \in F_{p-1,p}$ and construct the diagram (11). Then

$$\zeta_0^*(\ddot{\omega}_{p-1}) = \mathfrak{e}^*\mathbf{P}^*(\ddot{\omega}_{p-1}) = \mathfrak{e}^*(\omega_{p-1}) = \tfrac{1}{4}d^\perp d \log|\mathfrak{e}|^2$$

$$\zeta^*\tau^*(\ddot{\omega}_{p-1}) = \tau_0^*\zeta_0^*(\ddot{\omega}_{p-1}) = \tfrac{1}{4}d^\perp d \log|\mathfrak{e}\circ\tau_0|^2 = \tfrac{1}{4}d^\perp d \log|\mathfrak{e}|^2$$

if $\mathfrak{e}$ is regarded as a vector function on M. Moreover,

$$\zeta_1^*(\ddot{\omega}_p) = \mathfrak{c}^*\mathbf{P}^*(\ddot{\omega}_p) = \mathfrak{c}^*(\omega_p) = \tfrac{1}{4}d^\perp d \log|\mathfrak{c}|^2.$$

$$\zeta^*\pi^*(\ddot{\omega}_p) = \pi_1^*\zeta_1^*(\ddot{\omega}_p) = \tfrac{1}{4}d^\perp d \log|\mathfrak{c}\circ\pi_1|^2 = \tfrac{1}{4}d^\perp d \log|\mathfrak{e}\wedge\mathfrak{c}_p|^2$$

if $\mathfrak{e}$ and $\mathfrak{c}_p$ are regarded as vector functions on M. Hence,

$$\zeta^*\tau^*(\ddot{\omega}_{p-1}) = \frac{i}{2}\frac{1}{|\mathfrak{e}|^4}[|\mathfrak{e}|^2(d\mathfrak{e}\,|\,d\mathfrak{e}) - (d\mathfrak{e}\,|\,\mathfrak{e}) \wedge (\mathfrak{e}\,|\,d\mathfrak{e})]$$

$$\zeta^*\pi^*(\ddot{\omega}_p) = \frac{i}{2}\frac{1}{|\mathfrak{e}\wedge\mathfrak{c}_p|^4}[|\mathfrak{e}\wedge\mathfrak{c}_p|^2(d(\mathfrak{e}\wedge\mathfrak{c}_p)\,|\,d(\mathfrak{e}\wedge\mathfrak{c}_p)) - (d(\mathfrak{e}\wedge\mathfrak{c}_p)\,|\,\mathfrak{e}\wedge\mathfrak{c}_p)\wedge(\mathfrak{e}\wedge\mathfrak{c}_p\,|\,d(\mathfrak{e}\wedge\mathfrak{c}_p))].$$

Now, these differential forms shall be computed at $0 \in M$, which corresponds to (a, b):

$$d\mathfrak{e}_\mu = \sum_{\nu=p}^{n} dz_{\mu\nu}\mathfrak{a}_\nu \qquad \mathfrak{e}_\mu = \mathfrak{a}_\mu$$

$$d\mathfrak{e} = \sum_{\lambda=0}^{p-1}\mathfrak{e}_0 \wedge \ldots \wedge \mathfrak{e}_{\lambda-1}\wedge d\mathfrak{e}_\lambda \wedge \mathfrak{e}_{\lambda+1}\wedge \ldots \wedge \mathfrak{e}_{p-1}$$

$$= \sum_{\lambda=0}^{p-1}\sum_{\nu=p}^{n} dz_{\lambda\nu}\mathfrak{a}_0 \wedge \ldots \wedge \mathfrak{a}_{\lambda-1}\wedge\mathfrak{a}_\nu\wedge\mathfrak{a}_{\lambda+1}\wedge \ldots \wedge \mathfrak{a}_n$$

$$(d\mathfrak{e}\,|\,d\mathfrak{e}) = \sum_{\lambda=1}^{p-1}\sum_{\nu=p}^{n} dz_{\lambda\nu}\wedge d\bar{z}_{\lambda\nu} \qquad |\mathfrak{e}| = |\mathfrak{a}_0 \wedge \ldots \wedge \mathfrak{a}_{p-1}| = 1$$

$$(d\mathfrak{e}\,|\,\mathfrak{e}) = 0, \quad (\mathfrak{e}\,|\,d\mathfrak{e}) = 0.$$

Define

$$u = \frac{i}{2}\sum_{\lambda=0}^{p-1} dz_{\lambda p}\wedge d\bar{z}_{\lambda p}$$

$$v = \frac{i}{2}\sum_{\nu=p+1}^{n} dz_{p\nu}\wedge d\bar{z}_{p\nu}$$

$$w = \frac{i}{2}\sum_{\lambda=0}^{p-1}\sum_{\nu=p+1}^{n}\frac{i}{2}dz_{\lambda\nu}\wedge d\bar{z}_{\lambda\nu}.$$

Then
$$\zeta^*\tau^*(\ddot{\omega}_{p-1}) = u + w.$$

Observe $u^s = 0$ if $s > p$ and $w^q = 0$ if $q > p(n-p)$. Hence,

$$\zeta^*\tau^*(\ddot{\omega}_{p-1,d_{p-1}}) = \frac{1}{d_{p-1}!}(u+w)^{p(n-p)+p} = \sum_{\varrho=0}^{p} \frac{1}{\varrho!\,(p(n-p)+p-\varrho)!} u^\varrho w^{p(n-p)+p-\varrho}$$
$$= \frac{1}{p!\,(p(n-p))!} u^p w^{p(n-p)}.$$

Now,

$$d(\mathfrak{e} \wedge \mathfrak{c}_p) = (d\mathfrak{e}) \wedge \mathfrak{c}_p + \mathfrak{e} \wedge d\mathfrak{c}_p = d\mathfrak{e} \wedge \mathfrak{a}_p + \mathfrak{a}_0 \wedge \ldots \wedge \mathfrak{a}_{p-1} \wedge d\mathfrak{c}_p$$
$$= \sum_{\lambda=0}^{p-1} \sum_{\nu=p+1}^{n} dz_{\lambda\nu}\mathfrak{a}_0 \wedge \ldots \wedge \mathfrak{a}_{\lambda-1} \wedge \mathfrak{a}_\nu \wedge \mathfrak{a}_{\lambda+1} \wedge \ldots \wedge \mathfrak{a}_p + \sum_{\nu=p+1}^{n} dz_{p\nu}\mathfrak{a}_0 \wedge \ldots \wedge \mathfrak{a}_{p-1} \wedge \mathfrak{a}_\nu$$
$$= \sum_{\lambda=0}^{p} \sum_{\nu=p+1}^{n} dz_{\lambda\nu}\mathfrak{a}_0 \wedge \ldots \wedge \mathfrak{a}_{\lambda-1} \wedge \mathfrak{a}_\nu \wedge \mathfrak{a}_{\lambda+1} \wedge \ldots \wedge \mathfrak{a}_p.$$

Hence,
$$(d(\mathfrak{e} \wedge \mathfrak{c}_p) \,|\, d(\mathfrak{e} \wedge \mathfrak{c}_p)) = \sum_{\lambda=0}^{p} \sum_{\nu=p+1}^{n} dz_{\nu\lambda} \wedge d\bar{z}_{\nu\lambda}.$$

Moreover,
$$|\mathfrak{e} \wedge \mathfrak{c}_p|^2 = |\mathfrak{a}_0 \wedge \ldots \wedge \mathfrak{a}_p|^2 = 1$$
$$(d(\mathfrak{e} \wedge \mathfrak{c}_p) \,|\, \mathfrak{e} \wedge \mathfrak{c}_p) = (d(\mathfrak{e} \wedge \mathfrak{c}_p) \,|\, \mathfrak{a}_0 \wedge \ldots \wedge \mathfrak{a}_p) = 0 = (\mathfrak{e} \wedge \mathfrak{c}_p \,|\, d(\mathfrak{e} \wedge \mathfrak{c}_p)).$$

Therefore,
$$\zeta^*\pi^*(\ddot{\omega}_p) = \frac{i}{2} \sum_{\lambda=0}^{p} \sum_{\nu=p+1}^{n} dz_{\nu\lambda} \wedge d\bar{z}_{\nu\lambda} = v + w$$

where $v^s = 0$ if $s > n-p$ and $w^q = 0$ if $q > p(n-p)$. Hence,

$$\zeta^*\pi^*(\ddot{\omega}_{p,d_p}) = \frac{1}{d_p!}(v+w)^{p(n-p)+n-p} = \sum_{\varrho=0}^{n-p} \frac{1}{\varrho!\,(p(n-p)+n-p-\varrho)!} v^\varrho w^{p(n-p)+n-p-\varrho}$$
$$= \frac{1}{(n-p)!\,(p(n-p))!} v^{n-p} w^{p(n-p)} \tag{11}$$

Now,
$$\zeta^*(\pi^*(\ddot{\omega}_{p,d_p}) \wedge \tau^*(\ddot{\omega}_{p-1,p})) = \frac{1}{(n-p)!\,(p(n-p))!\,p!} v^{n-p} w^{p(n-p)} (u+w)^p$$
$$= \frac{1}{p!\,(n-p)!\,(p(n-p))!} u^p v^{n-p} w^{p(n-p)}$$

and
$$\zeta^*(\tau^*(\ddot{\omega}_{p-1,d_{p-1}}) \wedge \pi^*(\ddot{\omega}_{p,n-p})) = \frac{1}{p!\,(p(n-p))!\,(n-p)!} u^p w^{p(n-p)} (v+w)^{n-p}$$
$$= \frac{1}{p!\,(n-p)!\,(p(n-p))!} u^p v^{n-p} w^{p(n-p)}.$$

Therefore, the assertion of the theorem holds at the arbitrary point (a, b), q.e.d.

2. Integration over a Grassmann manifold

At first, some integrals over the projective space $\mathbf{P}(V)$ shall be computed. Let h be a measurable function on $\mathbf{P}(V)$ such that $h\ddot{\omega}_{0n}$ is integrable over $\mathbf{P}(V)$. Define

$$L(h) = L_V(h) = \frac{1}{W(n)} \int_{\mathbf{P}(V)} h\ddot{\omega}_{0n}.$$

If $\tilde{h} = h \circ \mathbf{P}$, then (1)

$$L(h) = \frac{1}{\pi^{n+1}} \int_V e^{-|\mathfrak{z}|^2} \tilde{h}(\mathfrak{z})\, \upsilon_{0,n+1}(\mathfrak{z}).$$

Define $I = \{t \in \mathbf{R} \mid 0 \leqslant t \leqslant 1\}$.

Lemma 2.1. *Let $g \geqslant 0$ be a measurable function on I. Take $w \in \mathbf{P}(V)$. Define $h: \mathbf{P}(V) \to \mathbf{R}$ almost everywhere on $\mathbf{P}(V)$ by $h(z) = g(\|w:z\|^2)$. Suppose that either $h\ddot{\omega}_{0n}$ is integrable over $\mathbf{P}(V)$ or $g(t)t^{n-1}$ is integrable over I. Then both are integrable and*

$$L(h) = n\int_0^1 g(t)t^{n-1}\,dt$$

is independent of $w \in \mathbf{P}(V)$.

Proof. Take $\mathfrak{w} \in V$ such that $|\mathfrak{w}| = 1$ and $w = \mathbf{P}(\mathfrak{w})$. Take an orthonormal base $\mathfrak{a}_0, \dots \mathfrak{a}_n$ of V such that $\mathfrak{a}_0 = \mathfrak{w}$. If $\mathfrak{z} = \sum_{\mu=0}^n z_\mu \mathfrak{a}_\mu \in V$, then

$$\mathfrak{w} \wedge \mathfrak{z} = \sum_{\mu=1}^n z_\mu \mathfrak{a}_0 \wedge \mathfrak{a}_\mu$$

$$\|\mathfrak{w} : \mathfrak{z}\| = \left(\sum_{\mu=1}^n |z_\mu|^2\right)\left(\sum_{\mu=0}^n |z_\mu|^2\right)^{-1}.$$

If $z_\nu = \sqrt{t_\nu}\, e^{i\varphi_\nu}$, $0 \leqslant t_\nu < \infty$, $0 \leqslant \varphi_\nu < 2\pi$, then

$$L(h) = \frac{1}{\pi^{n+1}} \int_V e^{-|\mathfrak{z}|^2} g(\|\mathfrak{w} : \mathfrak{z}\|^2)\, \upsilon_{0,n+1}(\mathfrak{z}) = \int_0^\infty \dots \int_0^\infty e^{-t_0 - \dots - t_n} g\left(\frac{t_1 + \dots + t_n}{t_0 + \dots + t_n}\right) dt_0 \dots dt_n.$$

Now, introduce the following change of variables:

$$t_0 = \tau(s_1 + \dots + s_n) \quad 0 < \tau < 1 \text{ and } 0 < s_\nu < \infty$$

$$t_\nu = (1 - \tau)\, s_\nu \quad \text{for } \nu = 1, \dots, n,$$

then

$$\tau = \frac{t_0}{t_0 + \dots + t_n}$$

$$s_\nu = t_\nu \frac{t_0 + \dots + t_n}{t_1 + \dots + t_n} \quad \text{for } \nu = 1, \dots, n$$

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(1) See [6] Hilfssatz 1.

with the identities
$$t_0+\ldots+t_n=s_1+\ldots+s_n$$
$$1-\tau=\frac{t_1+\ldots+t_n}{t_0+\ldots+t_n}$$
$$\frac{\partial(t_0,\ldots,t_n)}{\partial(\tau,s_1,\ldots,s_n)}=(s_1+\ldots+s_n)(1-\tau)^{n-1}.$$
Then
$$L(h)=\int_0^1 g(1-\tau)(1-\tau)^{n-1}d\tau\int_0^\infty\ldots\int_0^\infty e^{-s_1-\ldots-s_n}(s_1+\ldots+s_n)\,ds_1\ldots ds_n=n\int_0^1 g(t)\,t^{n-1}dt.$$

It was assumed that $L(h)$ exists. If $\int_0^1 g(t)t^{n-1}dt$ exists, the proof can be reversed because $g\geqslant 0$, q.e.d. Especially, $L(1)=1$, which proves $W(n)=\pi^n/n!$.

LEMMA 2.2. *Let $g\geqslant 0$ be a measurable function on I. For $\lambda\geqslant 1$, define*
$$I_\lambda(g)=\frac{1}{\pi^{n+1}}\int_{\mathbf{C}^{n+1}}e^{-|z_0|^2-\ldots-|z_n|^2}g\left(\frac{|z_1|^2+\ldots+|z_n|^2}{|z_0|^2+\ldots+|z_n|^2}\right)\frac{|z_\lambda|^2\upsilon_{0,n+1}}{|z_0|^2+\ldots+|z_n|^2}.$$
Then
$$I_\lambda(g)=\int_0^1 g(t)\,t^{n-1}dt$$
if either one of these integrals (hence both) exists.

Proof. The same changes of variables as in Lemma 2.1 imply
$$I_\lambda(g)=\int_0^\infty\ldots\int_0^\infty e^{-t_0-\ldots-t_n}g\left(\frac{t_1+\ldots+t_n}{t_0+\ldots+t_n}\right)\frac{t_\lambda\,dt_0\ldots dt_n}{t_1+\ldots+t_n}$$
$$=\int_0^1 g(1-\tau)(1-\tau)^{n-1}dt\int_0^\infty\ldots\int_0^\infty e^{-s_1-\ldots-s_n}s_\lambda\,ds_1\ldots ds_n=\int_0^1 g(t)\,t^{n-1}dt,\qquad \text{q.e.d.}$$

LEMMA 2.3. *Let $g\geqslant 0$ be a measurable function on I. Suppose that $g(t)\,t^{n-1}$ is integrable over I. Take integers $\varrho\geqslant 1$ and $\lambda\geqslant 1$ with $\varrho\neq\lambda$. Then the integral*
$$I_{\lambda\varrho}(g)=\frac{1}{\pi^{n+1}}\int_{\mathbf{C}^{n+1}}e^{-|z_0|^2-\ldots-|z_n|^2}g\left(\frac{|z_1|^2+\ldots+|z_n|^2}{|z_0|^2+\ldots+|z_n|^2}\right)\frac{z_\lambda\bar z_\varrho\upsilon_{0,n+1}}{|z_0|^2+\ldots+|z_n|^2}$$
exists and is zero.

Proof. Because $|z_\lambda z_\varrho|\leqslant|z_\lambda|^2+|z_\varrho|^2$, Lemma 2.2 implies the existence of $I_{\lambda\varrho}(g)$. The change of variables $u_\mu=z_\mu$ if $\mu\neq\lambda$ and $u_\lambda=-z_\lambda$ shows $I_{\lambda\varrho}(g)=-I_{\lambda\varrho}(g)$. Hence, $I_{\lambda\varrho}(g)=0$, q.e.d.

Let M be a complex manifold of dimension m. Suppose that for every $a\in\mathbf{P}(V)$ a differential form $\psi(a)$ of bidegree (p,q) on M is given. Let $z=(z_1,\ldots,z_m)$ be local holomorphic coordinates on an open subset U of M. Let $T(p, m)$ be the set of all injective, increasing maps of $\{1, ..., p\}$ into $\{1, ..., m\}$. For $\mu \in T(p, m)$ define

$$dz_\mu = dz_{\mu(1)} \wedge ... \wedge dz_{\mu(p)}$$

$$d\bar{z}_\mu = d\bar{z}_{\mu(1)} \wedge ... \wedge d\bar{z}_{\mu(p)}.$$

Then functions $\psi_{\mu\nu}(a)$ are uniquely determined on U such that

$$\psi(a) = \sum_{\mu \in T(p,m)} \sum_{\nu \in T(q,m)} \psi_{\mu\nu}(a)\, dz_\mu \wedge d\bar{z}_\mu.$$

If for every $x \in U$ all the integrals $L(\psi_{\mu\nu}(\cdot)(x))$ exist, then

$$L(\psi) = \sum_{\mu \in T(p,m)} \sum_{\nu \in T(q,m)} L(\psi_{\mu\nu}(a))\, dz_\mu \wedge d\bar{z}_\nu$$

is a well defined form of bidegree (p, q) on M. Thus, the average L extends from functions to forms.

LEMMA 2.4. *Let q be an integer with $0 \leqslant q \leqslant n$. Take $w \in \mathbf{P}(V)$. Let $g \geqslant 0$ be a measurable function on I such that $g(t)t^{n-1-q}$ is integrable over I. Define $\psi = \psi(a, w) = g(\|w:a\|^2)\Phi_0^q(a)(w)$. Then*

$$L(\psi) = (n-q)\int_0^1 g(t)\,(1-t)^{n-1-q}\, dt\, \ddot{\omega}_0^q.$$

Proof. Define $g_1(t) = g(t)t^{-q}$ for $0 < t \leqslant 1$ and $g_1(0) = 0$. Take $0 \neq \mathfrak{w} \in \mathbf{P}(V)$ with $w = \mathbf{P}(\mathfrak{w})$. Take $\mathfrak{z} \in V - E(w)$. Then

$$\xi_{\mathfrak{z}}(\mathfrak{w}) = |\mathfrak{w}|^{-2}|\mathfrak{w} \wedge \mathfrak{z}|^{-1}(d\mathfrak{w} \wedge \mathfrak{w} \,|\, \mathfrak{w} \wedge \mathfrak{z}).$$

Let $\mathfrak{a}_0, ..., \mathfrak{a}_n$ be an orthonormal base of V with $\mathfrak{w} = \mathfrak{a}_0|\mathfrak{w}|$. Then

$$\mathfrak{z} = \sum_{\nu=0}^n z_\nu \mathfrak{a}_\nu \qquad d\mathfrak{w} = \sum_{\nu=0}^n dw_\nu \mathfrak{a}_\nu$$

$$\mathfrak{w} \wedge \mathfrak{z} = |\mathfrak{w}| \sum_{\nu=1}^n z_\nu \mathfrak{a}_0 \wedge \mathfrak{a}_\nu \qquad |\mathfrak{w} \wedge \mathfrak{z}|^2 = |\mathfrak{w}|^2 \sum_{\nu=1}^n |z_\nu|^2$$

$$d\mathfrak{w} \wedge \mathfrak{w} = |\mathfrak{w}| \sum_{\nu=0}^n dw_\nu \mathfrak{a}_0 \wedge \mathfrak{a}_\nu$$

$$(d\mathfrak{w} \wedge \mathfrak{w} \,|\, \mathfrak{w} \wedge \mathfrak{z}) = |\mathfrak{w}|^3 \sum_{\nu=1}^n \bar{z}_\nu dw_\nu.$$

Therefore, $$\tau_{\mathfrak{z}}(\mathfrak{w}) = \frac{i}{2}\xi_{\mathfrak{z}}(\mathfrak{w}) \wedge \bar{\xi}_{\mathfrak{z}}(\mathfrak{w}) = \left(\sum_{\nu=1}^n |z_\nu|^2\right)^{-1} \sum_{\lambda,\varrho=1}^n z_\lambda \bar{z}_\varrho \frac{i}{2}\frac{dw_\varrho \wedge d\bar{w}_\lambda}{|\mathfrak{w}|^2}.$$

Moreover,

$$\omega_0(\mathfrak{w}) = \frac{i}{2}(|\mathfrak{w}|^{-4}(|\mathfrak{w}|^2(d\mathfrak{w}\,|\,d\mathfrak{w}) - (d\mathfrak{w}\,|\,\mathfrak{w}) \wedge (\mathfrak{w}\,|\,d\mathfrak{w}))$$

$$= \frac{i}{2}|\mathfrak{w}|^{-2}\left(\sum_{\varrho=0}^{n} dw_\varrho \wedge d\bar{w}_\varrho - dw_0 \wedge d\bar{w}_0\right) = \frac{i}{2}|\mathfrak{w}|^{-2}\sum_{\varrho=1}^{n} dw_\varrho \wedge d\bar{w}_\varrho.$$

Lemmas 2.2 and 2.3 imply:

$$\int_0^1 g(t)\, t^{n-1-q}\, dt\, \omega_0(\mathfrak{w})$$

$$= \frac{i}{2}\sum_{\varrho=1}^{n} |\mathfrak{w}|^{-2} dw_\varrho \wedge d\bar{w}_\varrho \int_0^1 g_1(t)\, t^{n-1}\, dt$$

$$= \frac{i}{2}\sum_{\lambda,\varrho=1}^{n} |\mathfrak{w}|^{-2} dw_\varrho \wedge d\bar{w}_\lambda \frac{1}{\pi^{n+1}}\int_{\mathbf{C}^{n+1}} e^{-|\mathfrak{z}|^2} g_1\left(\frac{|z_1|^2 + \ldots + |z_n|^2}{|z_0|^2 + \ldots + |z_n|^2}\right) \frac{z_\lambda \bar{z}_\varrho \upsilon_{0,n+1}(\mathfrak{z})}{|z_1|^2 + \ldots + |z_n|^2}$$

$$= \frac{1}{\pi^{n+1}}\int_{\mathfrak{z}\in V} e^{-|\mathfrak{z}|^2} g_1(\|\mathfrak{w} : \mathfrak{z}\|^2)\, \tau_{\mathfrak{z}}(\mathfrak{w})\, \upsilon_{0,n+1}(\mathfrak{z}).$$

Hence, $$L_1 = \frac{1}{W(n)}\int_{z\in\mathbf{P}(V)} g_1(\|w : z\|^2)\, \ddot{\tau}_z(w)\, \ddot{\omega}_{0n}(z) = \int_0^1 g(t)\, t^{n-1-q}\, dt\, \ddot{\omega}_0(w).$$

Moreover,

$$L_2 = \frac{1}{W(n)}\int_{z\in\mathbf{P}(V)} g_1(\|w : z\|^2)\, \ddot{\omega}_{0n}(z) = n\int_0^1 g_1(t)\, t^{n-1}\, dt = n\int_0^1 g(t)\, t^{n-1-q}\, dt.$$

Now (3) implies

$$(n-q)\int_0^1 g(t)\, t^{n-1-q}\, dt\, \ddot{\omega}_0^q(w)$$

$$= n\int_0^1 g(t)\, t^{n-1-q}\, dt\, \ddot{\omega}_0^q(w) - q\int_0^1 g(t)\, t^{n-1-q}\, \ddot{\omega}_0(w)) \wedge \ddot{\omega}_0^{q-1}(w)$$

$$= L_2\ddot{\omega}_0^q(w) - qL_1 \wedge \ddot{\omega}_0^{q-1}(w)$$

$$= \frac{1}{W(n)}\int_{\mathbf{P}(V)} g_1(\|w : z\|^2)\, [\ddot{\omega}_0^q(w) - q\ddot{\tau}_z(w) \wedge \ddot{\omega}_0^{q-1}(w)]\, \ddot{\omega}_{0,n}(z)$$

$$= \frac{1}{W(n)}\int_{\mathbf{P}(V)} [g(\|w : z\|^2)\, \Phi_0^q(z)\,(w)]\, \ddot{\omega}_{0n}(z) = L(\psi), \qquad \text{q.e.d.}$$

Let h: $G_p(V) \to \mathbf{C}$ be a function on the Grassmann manifold where $p \geq 1$. Then $h\ddot{\omega}_{p,n-p}$ is lifted to $\pi^*(h\ddot{\omega}_{p,n-p})$ on $F_{p-1,p}$. Pick $z \in G_{p-1}(V)$. Then $\tau^{-1}(z)$ is a compact, smooth, connected complex submanifold of dimension $n-p$ of $F_{p-1,p}$. Define

$$\varphi_p[h](z) = \frac{1}{W(n-p)}\int_{\tau^{-1}(z)} \pi^*(h\ddot{\omega}_{p,n-p})$$

if this integral exists. If it exists for all $z \in G_{p-1}(V)$, then $\varphi_p[h]$ is a function on $G_{p-1}(V)$. A partition of unity shows that $\varphi_p[h]$ is of class C^k if h is of class C^k. As L, also φ_p extends from functions to forms.

LEMMA 2.5. *Let p, q and s be integers with $0<p<n$ and $0\leqslant q<n-p=s$. Let g be a non-negative measurable function on I. Pick $a\in G_{p-1}(V)$ and $w\in \mathbf{P}(V)-\ddot{E}(a)$. Suppose that $g(\tau\|a:w\|^2)\tau^{s-q-1}$ is integrable over I. Define $h=h(z)=g(\|z:w\|^2)\Phi_p^q(z)(w)$ almost everywhere on $G_p(V)$. Then $\varphi_p[h]$ exists at a and*

$$\varphi_p[h]\,(a)=(s-q)\int_0^1 g(\tau\|a:w\|^2)\,\tau^{s-q-1}\,dt\,\Phi_{p-1}^q(a).$$

(Of course, if $q=0$, then w can be taken everywhere in $\mathbf{P}(V)$).

Proof. Let $\mathfrak{a}_0, \ldots, \mathfrak{a}_n$ be an orthonormal base of V with $E(a)=\mathbf{C}\mathfrak{a}_0+\ldots+\mathbf{C}\mathfrak{a}_{p-1}$. Define $\mathfrak{a}=\mathfrak{a}_0\wedge\ldots\wedge\mathfrak{a}_{p-1}$. Then

$$V=E(a)\oplus E^{\perp}(a).$$

Let $\tilde{j}_a\colon E^{\perp}(a)\to V$ and $j_a\colon \mathbf{P}(E^{\perp}(a))\to\mathbf{P}(V)$ be the inclusions. Then $E^{\perp}(a)$ is again a hermitian vector space (of dimension $s+1$) by restricting the scalar product to $E^{\perp}(a)$; the associated forms are $\tilde{j}^*(\omega_0)$ and $j^*(\ddot{\omega}_0)$. Obviously, $\mathbf{P}\circ\tilde{j}_a=j_a\circ\mathbf{P}$. Let $\tilde{\varrho}_a\colon V\to E^{\perp}(a)$ be the projection. Then $\varrho_a\colon \mathbf{P}(V)-\ddot{E}(a)\to\mathbf{P}(E^{\perp}(a))$ is well-defined by $\mathbf{P}\circ\tilde{\varrho}_a=\varrho_a\circ\mathbf{P}$ and

$$\Phi_{p-1}(a)=\varrho_a^* j_a^*(\ddot{\omega}_0).$$

Define
$$\tilde{\sigma}_0\colon E^{\perp}(a)\to\tilde{G}_p(V)$$

by $\tilde{\sigma}_0(\mathfrak{z})=\mathfrak{z}\wedge\mathfrak{a}$ for $\mathfrak{z}\in E^{\perp}(a)$. Then $|\tilde{\sigma}_0(\mathfrak{z})|=|\mathfrak{z}|$. Hence $\tilde{\sigma}_0$ is injective. Hence $\sigma_0\colon\mathbf{P}(E^{\perp}(a))\to G_p(V)$ is well-defined by $\mathbf{P}\circ\tilde{\sigma}_0=\sigma_0\circ\mathbf{P}$ and σ_0 is injective. Define $\sigma\colon\mathbf{P}(E^{\perp}(a))\to F_{p-1,p}$ by $\sigma(z)=(a,\sigma_0(z))$. Because $\dim\mathbf{P}(E^{\perp}(a))=n-p=\dim\tau^{-1}(a)$, the map

$$\sigma\colon\mathbf{P}(E^{\perp}(a))\to\tau^{-1}(a)$$

is biholomorphic. Moreover, $\pi\colon\tau^{-1}(a)\to G_p(V)$ is injective and $\pi\circ\sigma=\sigma_0$. Consequently, if $0\neq\mathfrak{z}\in E^{\perp}(a)$, then

$$\begin{aligned}\mathbf{P}^*\sigma^*\pi^*(\ddot{\omega}_p)\,(\mathfrak{z})&=\mathbf{P}^*\sigma_0^*(\ddot{\omega}_p)\,(\mathfrak{z})=\tilde{\sigma}_0^*\mathbf{P}^*(\ddot{\omega}_p)\,(\mathfrak{z})=\tilde{\sigma}_0^*(\omega_p)\,(\mathfrak{z})\\&=\tfrac{1}{4}d^{\perp}d\log|\mathfrak{z}\wedge\mathfrak{a}|^2=\tfrac{1}{4}d^{\perp}d\log|\mathfrak{z}|^2=\tilde{j}_a^*(\omega_0)\,(\mathfrak{z})=\mathbf{P}^*j_a^*(\ddot{\omega}_0)\,(\mathfrak{z})\end{aligned}$$

or
$$\sigma^*\pi^*(\ddot{\omega}_p)=j_a^*(\ddot{\omega}_0).$$

Take $\mathfrak{w}\in\mathbf{P}^{-1}(w)\subseteq V$. Define $\mathfrak{y}=\tilde{\varrho}_a(\mathfrak{w})\in E^{\perp}(a)$ and $\mathfrak{x}=\mathfrak{w}-\mathfrak{y}\in E(a)$. Since $\mathfrak{w}\notin E(a)$, the image $y=\mathbf{P}(\mathfrak{y})$ exists in $\mathbf{P}(E^{\perp}(a))$. If $z\in\mathbf{P}(E^{\perp}(a))$, take $\mathfrak{z}\in\mathbf{P}^{-1}(z)$. Then

$$\|\sigma_0(z):w\| = \frac{|\mathfrak{z} \wedge \mathfrak{a} \wedge \mathfrak{w}|}{|\mathfrak{z} \wedge \mathfrak{a}|\,|\mathfrak{w}|} = \frac{|\mathfrak{a} \wedge \mathfrak{z} \wedge \mathfrak{y}|}{|\mathfrak{z}|\,|\mathfrak{w}|} = \frac{|\mathfrak{z} \wedge \mathfrak{y}|}{|\mathfrak{z}|\,|\mathfrak{y}|}\,\frac{|\mathfrak{y}|}{|\mathfrak{w}|} = \|z:y\|\,\frac{|\mathfrak{a} \wedge \mathfrak{w}|}{|\mathfrak{a}|\,|\mathfrak{w}|} = \|z:y\|\;\|a|w\|.$$

Moreover,

$$\mathbf{P}^*\Phi_p(\sigma_0(z))\,(\mathfrak{w}) = \tfrac{1}{4}d^\perp d \log|\mathfrak{a} \wedge \mathfrak{z} \wedge \mathfrak{w}|^2 = \tfrac{1}{4}d^\perp d \log|\mathfrak{z} \wedge \mathfrak{y}|^2 = \mathbf{P}^*\Phi_0(z)\,(\mathfrak{y}) = \mathbf{P}^* j_a^*\Phi_0(z)\,(\mathfrak{w}).$$

Therefore $$\Phi_p(\sigma_0(z)) = j_a^*\Phi_0(z) \quad \text{if } z \in \mathbf{P}(E^\perp(a)).$$

Hence $$\sigma^*\pi^*(h\ddot{\omega}_{p,n-p})\,(z) = g(\|z:y\|^2\,\|a:w\|^2)\, j_a^*\Phi_0(z)\,(y)\, j_a^*(\ddot{\omega}_{0,n-p})\,(z).$$

Now, Lemma 2.4 implies

$$\begin{aligned}
\varphi_p[h]\,(a) &= \frac{1}{W(n-p)} \int_{u \in \pi^{-1}(a)} g(\|u:w\|^2)\,\Phi_p^q(u)\,(w)\,\ddot{\omega}_{p,n-p}(u) \\
&= \frac{1}{W(n-p)} \int_{z \in \mathbf{P}(E^\perp(a))} g(\|z:y\|^2\,\|a:w\|^2)\, j_a^*(\Phi_0^q(z))\,(y)\, j_a^*(\ddot{\omega}_{0,n-p})\,(z) \\
&= (s-q)\int_0^1 g(\tau\|a:w\|^2)\,\tau^{s-q-1}\,d\tau\; j_a^*(\ddot{\omega}_0^q)\,(y) \\
&= (s-q)\int_0^1 g(\tau\|a:w\|^2)\,\tau^{s-q-1}\,d\tau\; \varrho_a^* j_a^*(\ddot{\omega}_0^q) \\
&= (s-q)\int_0^1 g(\tau\|a:w\|^2)\,\tau^{s-q-1}\,d\tau\; \Phi_{p-1}^q(a)\,(w)
\end{aligned}$$

because $\varrho_a(w) = \varrho_a(\mathbf{P}(\mathfrak{w})) = \mathbf{P}(\tilde{\varrho}_a(\mathfrak{w})) = \mathbf{P}(\mathfrak{y}) = y$, q.e.d.

Taking $q=0$ and $g \equiv 1$ implies

$$\varphi_p[1] = 1.$$

Let $h: G_p(V) \to \mathbf{C}$ be a function on the Grassmann manifold with $p \geq 0$. Define the average by

$$L_p(h) = \frac{1}{W(n,p)} \int_{G_p(V)} h\ddot{\omega}_{p,d_p}$$

if this integral exists. Obviously, $L_p(1) = 1$ and $L_0(h) = L(h)$. As L, also L_p extends from functions to forms.

THEOREM 2.6. *Let h: $G_p(V) \to \mathbf{C}$ be a measurable function. Suppose $p \geq 1$. Suppose that $L_p(h)$ exists. Then*

$$L_p(h) = L_{p-1}(\varphi_p[h])$$

$$L_p(h) = \frac{1}{W(p)W(n,p)} \int_{F_{p-1,p}} (h \circ \pi)\,\pi^*(\ddot{\omega}_{p,d_p}) \wedge \tau^*(\ddot{\omega}_{p-1,p}).$$

Proof. Assume at first that $h \geqslant 0$. Lemma 1.2 implies

$$\infty \geqslant \int_{F_{p-1,p}} (h \circ \pi)\, \pi^*(\ddot{\omega}_{p,d_p}) \wedge \tau^*(\ddot{\omega}_{p-1,p})$$

$$= \int_{z \in G_p(V)} h(z) \left(\int_{\pi^{-1}(z)} \tau^*(\ddot{\omega}_{p-1,p}) \right) \ddot{\omega}_{p,d_p} = W(p)\, W(n,p)\, L_p(h) < \infty.$$

Hence the integral over $F_{p-1,p}$ exists. In the general case, the existence of $L_p(h)$ implies the existence of $L_p(|h|)$. Hence $|h \circ \pi| \pi^*(\ddot{\omega}_{p,d_p}) \wedge \tau^*(\ddot{\omega}_{p-1,p})$ is integrable over $F_{p-1,p}$. Therefore, the integral identity holds for general h. Theorem 1.3 implies

$$L_p(h) = \frac{1}{W(p)W(n,p)} \int_{F_{p-1,p}} (h \circ \pi)\, \pi^*(\ddot{\omega}_{p,d_p}) \wedge \tau^*(\ddot{\omega}_{p-1,p})$$

$$= \frac{1}{W(p)W(n-p)} \int_{F_{p-1,p}} (h \circ \pi)\, \tau^*(\ddot{\omega}_{p-1,d_{p-1}}) \wedge \pi^*(\ddot{\omega}_{p,n-p})$$

$$= \frac{W(n-p)}{W(p)W(n,p)} \int_{G_{p-1}(V)} \varphi_p[h]\, \ddot{\omega}_{p-1,d_{p-1}} = \frac{W(n-p)W(n,p-1)}{W(p)W(n,p)} L_{p-1}(\varphi_p[h]).$$

If $h=1$, then $L_p(1)=1$ and $L_{p-1}(\varphi_p[1]) = L_{p-1}(1) = 1$. Therefore,

$$W(n,p)\, W(p) = W(n,p-1)\, W(n-p)$$

and $L_p(h) = L_{p-1}(\varphi_p[h])$, q.e.d.

Of course, Theorem 2.6 extends to differential forms on a manifold depending on $a \in G_p(V)$ as a parameter.

PROPOSITION 2.7. (1) *The volume of the Grassmann manifold $G_p(V)$ is*

$$W(n,p) = \pi^{(p+1)(n-p)} \frac{p!\,(p-1)!\,\ldots\,1!}{(n-p)!\,(n-p+1)!\,\ldots\,n!}.$$

The degree of the Grassmann manifold $G_p(V)$ as an algebraic subvariety of $\mathbf{P}(V[p+1])$ is

$$\frac{p!\,(p-1)!\,\ldots\,1!}{(n-p)!\,(n-p+1)!\,\ldots\,n!}\,((p+1)(n-p))!.$$

Proof. The assertion is correct for $p=0$ as is well known (see also Lemma 2.1). Suppose the assertion is correct for $p-1<n$. Then

(1) The degree of a Grassmann manifold is well known. See Hodge–Pedoe [4], p. 366. If A is an algebraic variety of pure dimension q, then $(1/W(q)) \int_A \ddot{\omega}_{0,q}$ is the degree of A. See Thie [12].

$$W(n,p)=\frac{W(n-p)}{W(p)}\,W(n,p-1)=\pi^{p(n-p+1)+(n-p)-p}\,\frac{p!}{(n-p)!}\,\frac{(p-1)!\dots 1!}{(n-p+1)!\dots n!}$$

$$=\pi^{(p+1)(n-p)}\,\frac{p!\dots 1!}{(n-p)!\dots n!}\qquad\text{q.e.d.}$$

Lemma 2.5, Theorem 2.6 and Lemma 2.4 imply immediately:

PROPOSITION 2.8. *If* $q\geqslant 0$ *and* $n>p\geqslant 0$ *with* $s=n-p$, *then*

$$L_p(\Phi_p^q)=L_{p-1}(\Phi_{p-1}^q)=\dots=L_0(\Phi_0^q)=\ddot{\omega}_0^q$$

$$L_p(\hat{\Lambda}_s)=s\ddot{\omega}_{0,s}.$$

LEMMA 2.9. *Let* q *be an integer with* $0\leqslant q<n-p=s$. *Take* $a\in G_{p-1}(V)$ *and* $w\in\mathbf{P}(V)-\ddot{E}(a)$. *Define* h *by* $h(z)=\log\|z:w\|^{-2}\,\Phi_p^q(z)\,(w)$. *Then*

$$\varphi_p[h]\,(a)=\left(\frac{1}{s-q}+\log\frac{1}{\|a:w\|^2}\right)\Phi_{p-1}^q(a)\,(w).$$

Proof. Apply Lemma 2.5 with $g(t)=\log(1/t)$. Then

$$(s-q)\int_0^1\log\frac{1}{\tau\|a:w\|^2}\,\tau^{s-q-1}\,d\tau=\frac{1}{s-q}+\log\frac{1}{\|a:w\|^2}.\qquad\text{q.e.d.}$$

LEMMA 2.10. *Let* q *be an integer with* $0\leqslant q<n-p=s$. *Define* h *by* $h(z,w)=\log\|z:w\|^{-2}\,\Phi_p^q(z)\,(w)$ *if* $z\in G_p(V)$ *and* $w\in\mathbf{P}(V)-\ddot{E}(z)$. *Then*

$$L_p(h)=\sum_{\mu=0}^{p}\frac{1}{s-q+\mu}\,\ddot{\omega}_0^q\quad\textit{on}\ \mathbf{P}(V).$$

Proof. At first, the case $p=0$ shall be proved. Lemma 2.4 with $g(t)=\log(1/t)$ implies

$$L_0(h)=L(h)=(n-q)\int_0^1\log\frac{1}{\tau}\,(1-\tau)^{n-1-q}\,d\tau\,\ddot{\omega}_0^q=\frac{\ddot{\omega}_0^q}{n-q}.$$

Now, assume that the assertion is correct for $p-1$. Then

$$L_p(h)=L_{p-1}(\varphi_p[h])=L_{p-1}\left(\left(\frac{1}{s-q}+\log\frac{1}{\|a:w\|^2}\right)\Phi_{p-1}^q(a)\,(w)\right)$$

$$=\frac{1}{s-q}\,\ddot{\omega}_0^q+\sum_{\mu=0}^{p-1}\frac{1}{s+1-q+\mu}\,\ddot{\omega}_0^q=\sum_{\mu=0}^{p}\frac{1}{s-q+\mu}\,\ddot{\omega}_0^q,\qquad\text{q.e.d.}$$

THEOREM 2.11. *If* $0\leqslant p<n$ *and* $s=n-p$, *then*

$$L_p(\Lambda_s)=\frac{1}{4s}\sum_{\nu=1}^{s}\sum_{\mu=0}^{p}\frac{1}{\nu+\mu}\,\ddot{\omega}_{0,s-1}.$$

Proof. Lemma 2.10 implies

$$L_p(\Lambda_s) = L_p\left(\frac{1}{4s!}\log\frac{1}{\|w:a\|^2}\sum_{\nu=0}^{s-1}\Phi_p^\nu(a)\wedge\ddot{\omega}_0^{s-1-\nu}\right) = \frac{1}{4s!}\sum_{\nu=0}^{s-1}\sum_{\mu=0}^{p}\frac{1}{s-\nu+\mu}\ddot{\omega}_0^\nu\wedge\ddot{\omega}_0^{s-1-\nu}$$

$$= \frac{1}{4s}\sum_{\nu=1}^{s}\sum_{\mu=0}^{p}\frac{1}{\nu+\mu}\ddot{\omega}_{0,s-1}, \qquad \text{q.e.d.}$$

Since $G_p(V)$ is a symmetric space and the non-negative form $L_h(\Lambda_s)$ is invariant under all isometries, it can be concluded a priori that

$$L_h(\Lambda_s) = K\ddot{\omega}_{0,\,s-1}$$

where K is a non-negative constant which could be infinite. The importance of Theorem 2.11 consists of the fact that $K<\infty$.

3. The First Main Theorem

Let f: $M\to N$ be a holomorphic map of a pure m-dimensional complex manifold into a pure n-dimensional complex manifold with $m\geqslant n$. The rank of f at z is defined by

$$r_f(z) = m - \dim_z f^{-1}(f(z)).$$

The set $\{z\,|\,r_f(z)<p\}$ is analytic(1) for each integer p. The set $D_f=\{z\,|\,r_f(z)<n\}$ is called the *degeneracy* of f, and $f(D_f)$ is a set of measure zero (even almost thin set) in N. A point $z\in M$ belongs to $M-D_f$ if and only if an open neighborhood U of z exists such that $f\,|\,U$ is open. Hence, the multiplicity(2) $\nu_f(z)$ of f at $z\in M-D_f$ is defined. The map f is said to be *regular* at z if its jacobian matrix at z has rank n. The set R_f of regular points of f is open and contained in $M-D_f$. Moreover, $f(M-R_f)$ is a set of measure zero by Sard's theorem. Obviously, $\nu_f(z)=1$ if $z\in R_f$. The set $M-R_f$ is analytic.

Let V be a complex vector space of dimension $n+1>1$. Let M be a pure m-dimensional complex manifold. Let s be an integer with $0<s\leqslant m$ and $0<s\leqslant n$. Define $p=n-s$ and $q=m-s$. Let f: $M\to\mathbf{P}(V)$ be a holomorphic map. Define $F=F_f^s=f^*(F_{0,\,p})$ by

$$F = \{(z,a)\in M\times G_p(V)\,|\,f(z)\in\ddot{E}(a)\}.$$

Then F is a smooth, complex submanifold of $M\times G_p(V)$ with pure dimension $m+p(n-p)$. The projection σ: $F\to M$ is a surjective, proper, regular, holomorphic map. (3) A holomorphic

(1) See Remmert [5].

(2) See [8].

(3) For the proof of this and other results mentioned here, see [10].

map $\mathring{f}\colon F \to F_{0,p}$ is defined by $\mathring{f}(z, a) = (f(z), a)$ with $\tau \circ \mathring{f} = f \circ \sigma$. Define $\hat{f} = \pi \circ \mathring{f}\colon F \to G_p(V)$ as the projection. Then

$$\sigma\colon \hat{f}^{-1}(a) \to f^{-1}(\ddot{E}(a))$$

is biholomorphic for every $a \in G_p(V)$. The following commutative diagram is constructed

$$\begin{tikzcd} \hat{f}\colon\ F \arrow[r, "\mathring{f}"] \arrow[d, "\sigma"'] \arrow[dr, phantom, "\circ"] & F_{0,p} \arrow[r, "\pi"] \arrow[d, "\tau"] & G_p(V) \\ M \arrow[r, "f"] & \mathbf{P}(V) & \end{tikzcd}$$

The map f is said to be general of order s at $z \in M$ for $a \in G_p(V)$ if and only if open neighborhoods U of z in M and W of a in $G_p(V)$ exist such that $\dim_x f^{-1}(E(y)) = q$ for all $x \in f^{-1}(E(y))$ with $y \in W$, which is the case if and only if $(z, a) \in M \times G_p(V) - D_{\hat{f}}$. The map is said to be general for $a \in G_p(V)$ if and only if f is general of order s for a at every point of $f^{-1}(\ddot{E}(a))$, which is the case if and only if $a \in G_p(V) - \hat{f}(D_{\hat{f}})$, i.e., for almost all a. If f is general at $z \in M$ for $a \in G_p(V)$, the intersection number is defined by $\nu_f^a(z) = \nu_f(z, a) = \nu_{\hat{f}}(z; a)$ if $(z, a) \in F$ (i.e., $(z, a) \in F - D_{\hat{f}}$; i.e., $f(z) \in \ddot{E}(a)$) and by $\nu_f^a(z) = \nu_f(z; a) = 0$ if $(z, a) \notin F$ (i.e., $f(z) \notin \ddot{E}(a)$). Obviously, the support of ν_f^a is $f^{-1}(\ddot{E}(a))$.

If $p = 0$, then σ, τ and π are biholomorphic and $\nu_f(z; a) = \nu_{\hat{f}}(z, a)$ if $(z, a) \in F - D_{\hat{f}}$; i.e., $z \in (M - D_f) \cap f^{-1}(\ddot{E}(a))$.

Now, it shall be assumed that a non-negative exterior form χ of bidegree (q, q) and of class C^1 is given on M such that $d\chi = 0$ on M. Assume further that M is connected.

Let H be an open subset of M. A pure $(2m-1)$-dimensional oriented real manifold S of class C^k with $k \geqslant 1$ is said to be a *boundary manifold* of H if and only if

1. S is a relative open subset of $\bar{H} - H$ with the induced topology.
2. If $a \in S$, then an open neighborhood U of a in M and connected neighborhoods U'' of $0 \in \mathbf{R}^{m-1}$ and $0 \in \mathbf{R}$ and orientation preserving diffeomorphisms $\alpha\colon U \to I \times U''$ and $\beta\colon U \cap S \to U''$ exist such that $\alpha(a) = 0$ and

$$\alpha(x) = (g(x), \beta(x)) \quad \text{for } x \in U$$
$$U \cap H = \{x \in U \mid g(x) < 0\}$$
$$U \cap S = \{x \in U \mid g(x) = 0\}.$$

The collection $B = (G, \Gamma, g, \gamma, \psi)$ is said to be a *bump* on M if and only if G and g are relative compact, open subsets of M with $\bar{g} \subset G$ where $\Gamma = \bar{G} - G$ and $\gamma = \bar{g} - g$ are boundary manifolds of G, respectively g. Moreover, ψ: $M \to \mathbf{R}$ is a non-negative, continuous function on M with maximum $R>0$ such that $\psi|\bar{g} \equiv R$ and $\psi|(M-G) \equiv 0$ and such that $\psi|(\bar{G}-g)$ is of class C^2.

Then, $d\psi$ and $d^\perp\psi$ exist on $\bar{G}-g$ and are understood to be the limits from the interior on the boundary. For any such bump, define the *spherical image* of order s by

$$A_f(G) = A_{s,f}(G) = \frac{1}{W(s)} \int_G f^*(\ddot{\omega}_{0,s}) \wedge \chi \tag{12}$$

and the *characteristic* of order s by

$$T_f(G) = T_{s,f}(G) = \frac{1}{W(s)} \int_G \psi f^*(\ddot{\omega}_{0s}) \wedge \chi. \tag{13}$$

If $a \in G_p(V)$ and if f is general of order s at every point of $\bar{G} \cap f^{-1}(\ddot{E}(a))$, define the counting function of order s by

$$n_f(G, a) = \int_{G_a} \nu_f^a \chi \quad \text{with } G_a = G \cap f^{-1}(\ddot{E}(a)), \tag{14}$$

the *valence function* of order s by

$$N_f(G, a) = \int_{G_a} \psi \nu_f^a \chi, \tag{15}$$

the *proximity function* of order s by

$$m_f(\Gamma, a) = \frac{1}{W(s)} \int_\Gamma f^*(\Lambda_s(a)) \wedge d^\perp\psi \wedge \chi, \tag{16}$$

the *proximity remainder* of order s by

$$m_f(\gamma, a) = \frac{1}{W(s)} \int_\gamma f^*(\Lambda_s(a)) \wedge d^\perp\psi \wedge \chi, \tag{17}$$

and the *deficit* of order s by

$$D_f(G, a) = \frac{1}{W(s)} \int_{G-g} f^*(\Lambda_s(a)) \wedge dd^\perp\psi \wedge \chi. \tag{18}$$

All these integrals exist, and their integrands—with the exception of (18)—are non-negative. Observe that $\hat{f}(\sigma^{-1}(\bar{G}) \cap D_{\hat{f}})$ is the compact set of measure zero of all $a \in G_p(V)$ for which f is not general of order s at some point of $\bar{G} \cap f^{-1}(\ddot{E}(a))$. Hence, all the integrals (14) to (18) are defined on $G_p(V)$ with the exception of a compact set of measure zero. If f is general of order s at all points of $\bar{G} \cap f^{-1}(\ddot{E}(a))$, then

$$T_f(G) = N_f(G, a) + m_f(\Gamma, a) - m_f(\gamma, a) - D_f(G, a) \tag{19}$$

which is the *First Main Theorem.*

These results have been proved in [10] under a slightly stronger assumption. However, the results as stated here are obtained exactly the same way, only the neighborhood A in the proofs of Theorems 4.3 and 4.4, in the case $a \in \bar{H} \cap f^{-1}(\ddot{E}(\alpha))$, has to be taken so small that $\dim_z f^{-1}(\ddot{E}(x)) = q$ for all $z \in A \cap f^{-1}(\ddot{E}(x))$ with x in some neighborhood of α. This is possible by assumption.

Let $B = (G, \Gamma, g, \gamma, \psi)$ be a bump. For $0 \leqslant r \leqslant R$, define

$$G(r) = \{z \in M \mid R - \psi(z) < r\}$$

$$\Gamma(r) = \bar{G}(r) - G(r).$$

Then
$$g \subseteq G(0) \subset \bar{G}(0) \subset G(r) \subset \bar{G}(r) \subset G(R) \subseteq G$$

for $0 < r < R$, where $g \neq G(0)$ and $G(R) \neq G$ may be possible. However, $g = G(0)$ and $G(R) = G$ if $0 < \psi(z) < R$ for $z \in G - \bar{g}$. Define $\psi[r] = r - R + \psi$ on $G(r)$ and $\psi[r] = 0$ on $M - G(r)$. Then $B(r) = (G(r), \Gamma(r), g, \gamma, \psi[r])$ is a bump if $d\psi \neq 0$ on $\psi^{-1}(R - r) \cap (\bar{G} - g)$, which is true for almost all r in $0 < r \leqslant R$ by Sard's theorem. Then definitions (16) and (17) make sense for all those values of r, whereupon definitions (12), (13), (14), (15) and (18) make sense for all r in $0 \leqslant r \leqslant R$.

LEMMA 3.1. *Let $B = (G, \Gamma, g, \gamma, \psi)$ be a bump. Let S be a pure k dimensional analytic subset of G. Let φ be a differential form of bidegree (k, k) on G, which is integrable over S. Then*

$$\int_{G(r) \cap S} \psi[r]\, \varphi = \int_0^r \int_{\overline{G(t)} \cap S} \varphi\, dt \quad \text{if } 0 \leqslant r \leqslant R, \tag{20}$$

$$\int_{G \cap S} \psi\varphi = \int_0^R \int_{\overline{G(t)} \cap S} \varphi\, dt. \tag{21}$$

Proof. At first, assume that φ is non-negative at all simple points of S. Define $\lambda(z, t) = s$ if $R - \psi(z) \leqslant t$, and define $\lambda(z, t) = 0$ if $r - R + \psi(z) > t$. Then

$$\int_0^r \lambda(z, t)\, dt = r - R + \psi(z).$$

Hence,

$$\int_{G(r) \cap S} (r - R + \psi(z))\, \varphi = \int_{G(r) \cap S} \int_0^r \lambda(z, t)\, dt\, \varphi = \int_0^r \int_{G(r) \cap S} \lambda(z, t)\, \varphi\, dt = \int_0^r \int_{G(t) \cap S} \varphi\, dt.$$

In the general case, define $\mu^+(z) = 1$ (respectively $\mu^-(z) = 1$) if φ is non-negative (respectively negative) at the simple z of S. At all other points, define $\mu^+(z) = 0$ (respectively $\mu^-(z) = 0$). Then $\varphi = \mu^- \varphi + \mu^+ \varphi$ on S, and (20) holds for $\mu^+ \varphi$ (respectively $\mu^- \varphi$); hence by addition for φ. Now, (20) implies (21) because $\psi(z) = 0$ if $z \in G - G(R)$, q.e.d.

If Lemma 3.1 is applied with $S=G$ and $\varphi=f^*(\ddot{\omega}_{0,s})\cap\chi$, then

$$T_f(G(r))=\int_0^r A_f(t)\,dt \quad \text{for } 0\leqslant r\leqslant R$$

$$T_f(G)=T_f(G(R))=\int_0^R A_f(t)\,dt.$$

If Lemma 3.1 is applied with $S=G\cap f^{-1}(\ddot{E}(a))$ and $\varphi=\nu_f^a\chi$, then

$$N_f(G(r),a)=\int_0^r n_f(t,a)\,dt$$

$$N_f(G,a)=\int_0^R n_f(t,a)\,dt.$$

Obviously, A_f and $N_f(G(\cdot),a)$ are increasing functions continuous from the left if D^- is the left derivative. Then

$$D^-T_f(G(r))=A_f(G(r)) \quad \text{if } 0<r\leqslant R$$

$$D^-N_f(G(r),a)=n_f(G(r),a) \quad \text{if } 0<r\leqslant R.$$

Let $B=(G,g,\Gamma,\gamma,\psi)$ be a bump. The *average proximity function* of order s is defined by

$$\mu_f(\Gamma)=\mu_{s,f}(\Gamma)=\frac{1}{2\pi}\frac{1}{W(s-1)}\int_\Gamma f^*(\ddot{\omega}_{0,s-1})\wedge d^\perp\psi\wedge\chi.$$

The *average proximity remainder* of order s is defined by

$$\mu_f(\gamma)=\mu_{s,f}(\gamma)=\frac{1}{2\pi}\frac{1}{W(s-1)}\int_\gamma f^*(\ddot{\omega}_{0,s-1})\wedge d^\perp\psi\wedge\chi.$$

The *average deficit* of order s is defined by

$$\Delta_f(G)=\Delta_{s,f}(G)=\frac{1}{2\pi}\frac{1}{W(s-1)}\int_{G-\bar{g}} f^*(\ddot{\omega}_{0,s-1})\wedge dd^\perp\psi\wedge\chi.$$

Obviously, $\mu_f(\Gamma)$ and $\mu_f(\gamma)$ have non-negative integrands. Stoke's Theorem implies

$$\Delta_f(G)=\mu_f(\Gamma)-\mu_f(\gamma). \tag{22}$$

If u is a continuous form of bidegree $(1,1)$ on $\bar{G}-g$, define

$$D_f(G,a,u)=\frac{1}{W(s)}\int_{G-\bar{g}} f^*(\Lambda(a))\wedge u\wedge\chi.$$

This integral exists according to [10] Lemma 3.5. Define

$$\Delta_f(G, u) = \Delta_{sf}(G, u) = \frac{1}{2\pi} \frac{1}{W(s-1)} \int_{G-\bar{g}} f^*(\ddot{\omega}_{0,s-1}) \wedge u \wedge \chi.$$

The integrands of both of these integrals are non-negative.

For $s \geqslant 1$ and $p \geqslant 0$, define

$$c_{ps} = \tfrac{1}{2} \sum_{\nu=1}^{s} \sum_{\mu=0}^{p} \frac{1}{\nu+\mu}.$$

THEOREM 3.2. *Let $f: M \to \mathbf{P}(V)$ be a holomorphic map of the connected, m-dimensional, complex manifold M into the projective space $\mathbf{P}(V)$ of the $(n+1)$-dimensional hermitian vector space V. Let $0 < s \leqslant n$ and $0 < s \leqslant m$. Define $p = n - s$ and $q = m - s$. Let χ be a non-negative form of bidegree (q, q) and class C^1 on M with $d\chi = 0$. Let $B = (G, \Gamma, g, \gamma, \psi)$ be a bump on M. Then*

$$T_{sf}(G) = L_p(N_f(G, \cdot)) = \frac{1}{W(n,p)} \int_{G_p(V)} N_f(G, a)\, \ddot{\omega}_{p, d_p} \tag{23}$$

$$c_{ps}\mu_{sf}(\Gamma) = L_p(m_f(\Gamma, \cdot)) = \frac{1}{W(n,p)} \int_{G_p(V)} m_f(\Gamma, a)\, \ddot{\omega}_{p, d_p}, \tag{24}$$

$$c_{ps}\mu_{sf}(\gamma) = L_p(m_f(\gamma, \cdot)) = \frac{1}{W(n,p)} \int_{G_p(V)} m_f(\gamma, a)\, \ddot{\omega}_{p, d_p}, \tag{25}$$

$$c_{ps}\Delta_{sf}(G) = L_p(D_f(G, a)) = \frac{1}{W(n,p)} \int_{G_p(V)} D_f(G, a)\, \ddot{\omega}_{p, d_p}, \tag{26}$$

$$c_{ps}\Delta_{sf}(G, u) = L_p(D_f(G, a, u)) = \frac{1}{W(n,p)} \int_{G_p(V)} D_f(G, a, u)\, \ddot{\omega}_{p, d_p} \tag{27}$$

for every continuous form u of bidegree $(1, 1)$ on $\bar{G} - g$.

Proof. At first, assume that u is a continuous non-negative form of bidegree $(1, 1)$ on $\bar{G} - g$. Then $f^*(\Lambda(a)) \wedge u \wedge \chi \geqslant 0$ on $\bar{G} - g$. Hence

$$\begin{aligned}
L_p(D_f(G, a, u)) &= \frac{1}{W(n,p)} \frac{1}{W(s)} \int_{G_p(V)} \int_{G-\bar{g}} f^*(\Lambda_s(a)) \wedge u \wedge \chi\, \ddot{\omega}_{p, d_p}(a) \\
&= \frac{1}{W(s)} \int_{G-\bar{g}} \left(\frac{1}{W(n,p)} \int_{G_p(V)} f^*(\Lambda_s(a))\, \ddot{\omega}_{p, d_p}(a) \right) \wedge u \wedge \chi \\
&= \frac{1}{W(s)} \int_{G-\bar{g}} f^* \left(\frac{1}{W(n,p)} \int_{G_p(V)} \Lambda_s(a)\, \ddot{\omega}_{p, d_p}(a) \right) \wedge u \wedge \chi \\
&= \frac{1}{W(s)} \frac{1}{2s} \int_{G-\bar{g}} c_{ps} f^*(\ddot{\omega}_{0,s-1}) \wedge u \wedge \chi = c_{ps}\Delta_f(G, u).
\end{aligned}$$

Now, let u be any continuous form of bidegree (1, 1) on $\bar{G}-g$. Introduce a Hermitian metric on M. Let v be its exterior form of bidegree (1, 1) with $v>0$. A constant $K>0$ exists such that $\varphi=u+Kv>0$ on $\bar{G}-g$. Then

$$L_p(D_f(G, a, \varphi)) = c_{ps}\Delta_f(G, \varphi),$$

$$L_p(D_f(G, a, v)) = c_{ps}\Delta_f(G, v),$$

$$L_p(D_p(G, u)) = L_p(D_p(G, \varphi)) - KL_p(D_p(G, u)) = c_{ps}\Delta_f(G, \varphi) - c_{ps}K\Delta_f(G, v) = c_{ps}\Delta_f(G, u),$$

which proves (27) and implies (26) with $u=dd^{\perp}\psi$.

The mean value of the proximity form and proximity remainder is obtained the same way, observing that their integrands are non-negative. Now, (19) and (22) imply (23), q.e.d.

Differentiation implies

$$A_f(G(r)) = L_p(n_f(G(r), a))$$

for $0<r\leqslant R$. If $0<\psi$ on G, then $G(R)=G$ and

$$A_f(G) = L_p(n_f(G, a)).$$

(Using integration over the fibers, this could be proved directly, so providing an alternative proof for (23).)

Let N be a subset of M. Define

$$I_p(N, f) = I_p(N) = \{a\in G_p(V)\,|\,f(N)\cap \ddot{E}(a) \neq \varnothing\}.$$

Obviously, $I_p(N)=\hat{f}(\sigma^{-1}(N))$. Therefore, if N is compact, then $I_p(N)$ is compact. If N is measurable, then $I_p(N)$ is measurable. If N is measurable, define

$$b_f(N) = b_{s,f}(N) = \frac{1}{W(n, p)}\int_{I_p(N)} \ddot{\omega}_{p,d_p}.$$

Then $0\leqslant b_f(N)\leqslant 1$. Moreover, $1-b_f(N)$ is the measure of the set $\{a\in G_p(V)\,|\,f(N)\cap \ddot{E}(a)=\varnothing\}$. Hence, $b_f(M)=1$ if and only if $f^{-1}(\ddot{E}(a))\neq\varnothing$ for almost all $a\in G_p(V)$. Observe

$$b_f(N_1)\leqslant b_f(N_2)\leqslant b_f(M)\leqslant 1$$

if $N_1\subseteq N_2\subseteq M$.

PROPOSITION 3.3. *The assumptions of Theorem* 3.2 *are made. Moreover, let u be a continuous, non-negative form of bidegree* (1, 1) *on $\bar{G}-g$ with $dd^{\perp}\psi\wedge\chi\leqslant u\wedge\chi$. Then*

$$(1-b_{s,f}(G))\,T_{sf}(G) \leqslant c_{ps}(\Delta_{s,f}(G, u)+\mu_{s,f}(\gamma)).$$

Proof. The first main theorem implies

$$N_f(G, a) \leqslant T_f(G) + D_f(G, a, u) + m_f(\gamma, a)$$

for almost all $a \in G_p(V)$. Now

$$\frac{1}{W(n,p)} \int_{I_p(G)} N_f(G,a)\, \ddot{\omega}_{p,d_p} = \frac{1}{W(n,p)} \int_{G_p(V)} N_f(G,a)\, \ddot{\omega}_{p,d_p} = T_f(G),$$

$$\frac{1}{W(n,p)} \int_{I_p(G)} D_f(G,a,u)\, \ddot{\omega}_{p,d_p} \leqslant \frac{1}{W(n,p)} \int_{G_p(V)} D_f(G,a,u)\, \ddot{\omega}_{p,d_p} = c_{ps}\Delta_f(G,u),$$

$$\frac{1}{W(n,p)} \int_{I_p(G)} m_f(\gamma,a)\, \ddot{\omega}_{p,d_p} \leqslant \frac{1}{W(n,p)} \int_{G_p(V)} m_f(\gamma,a)\, \ddot{\omega}_{p,d_p} = c_{ps}\mu_f(\gamma).$$

Hence,
$$T_f(G) \leqslant b_f(G)\, T_f(G) + c_{ps}\Delta_f(G,u) + c_{ps}\mu_f(\gamma), \qquad \text{q.e.d.}$$

Now, divide by $T_f(G)$ and let G exhaust M. Then an estimate of $1 - b_f(M)$ is obtained. This will be done in the next section. At first, a condition will be given which implies $T_f(G) > 0$.

LEMMA 3.4. *Let $B = (G, \Gamma, g, \gamma, \psi)$ be a bump on M. Suppose that an open subset U of G exists such that $\psi\chi \mid U > 0$. Suppose that $a \in G_p(V)$ and $z_0 \in U$ with $f(z_0) \in \ddot{E}(a)$ exist such that f is general of order s at z_0 for a. Then $A_f(G) > 0$ and $T_f(G) > 0$.*

Proof. An open neighborhood U_0 of z_0 with $\bar{U}_0 \subseteq U$ and an open neighborhood U_0' of a exist such that $\hat{f} \mid W_0$ is open with $W_0 = F \cap (U_0 \times U_0')$. Here $(z_0, a) \in W_0$. An open neighborhood W_1 of (z_0, a) with $\overline{W}_1 \subseteq W_0$ and a biholomorphic map $\alpha\colon W_1 \to W_1'$ onto an open subset of $\mathbf{C}^{m+p(n-p)}$ exist with $\alpha(z_0, a) = 0$. Then a ball $W_2' = \{\mathfrak{z} \in \mathbf{C}^{m+p(n-p)} \mid |\mathfrak{z}| < r\}$ exists with $0 < r$ and $\overline{W}_2' \subseteq W_1'$. Define $W_2 = \alpha^{-1}(W_2')$. Then $(z_0, a) \in W_2 \subseteq \overline{W}_2 \subseteq W_1$. Moreover,

$$F_x = \sigma(\hat{f}^{-1}(x) \cap W_2) \subseteq f^{-1}(\ddot{E}(x)) \cap U_0 \quad \text{for } x \in U_0'.$$

[9], Theorem 3.9 implies that the fiber integral

$$L(x) = \int_{\hat{f}^{-1}(x) \cap W_1} \nu_{\hat{f}}\, \sigma^*(\chi) = \int_{F_x} \nu_f^x \chi \geqslant 0$$

is continuous on U_0'. Because $z_0 \in F_a \neq \varnothing$, the integral $L(a)$ is positive. Hence, a neighborhood U_1' of a and a constant $c > 0$ exist such that $L(x) \geqslant c > 0$ for $x \in U_1'$. Define

$$c_1 = \frac{1}{W(n,p)} \int_{U_1'} \ddot{\omega}_{p,d_p} > 0.$$

Because $\psi\chi \mid U > 0$, a constant r with $0 \leqslant r < R$ exists such that $R - \psi(z) < r$ for $z \in \bar{U}_0$, where $\bar{U}_0$ is the compact closure of U_0. Hence, $\bar{U}_0 \subseteq G(r)$, which implies

$$n_f(G(r), x) \geqslant L(x) \geqslant c \quad \text{for } x \in U_1'$$

and $$A_f(G) \geqslant A_f(G(r)) = \frac{1}{W(n,p)} \int_{G_p(V)} n_f(G(r), x)\, \ddot{\omega}_{p,d_p} \geqslant c_1 c > 0.$$

Moreover, $$T_f(G) = T_f(G(R)) \geqslant \int_0^R A_f(G(t)\, dt \geqslant (R-r)\, A_f(G(r)) > 0. \qquad \text{q.e.d.}$$

4. Equidistribution

Let M be a connected, noncompact, complex manifold of dimension m. Let N be a partially ordered set, such that for each $r_1 \in N$ and $r_2 \in N$ an element $r_3 \in N$ with $r_1 \leqslant r_3$ and $r_2 \leqslant r_3$ exists. Then N is a directed set. The net $\mathfrak{B} = \{B_r\}_{r \in N}$ is called an *exhaustion family* of bumps if and only if

1. The index set N is directed.
2. For each $r \in N$, the collection $B_r = (G_r, \Gamma_r, g, \gamma, \psi_r)$ is a bump where g and γ are the same for all $r \in N$.
3. For every compact subset K of M, an element $r_K \in N$ exists such that $\psi_r(z) > 0$ if $z \in K$ and if $r \geqslant r_K$. (Especially $G_r \supset K$ for $r \geqslant r_K$.)

A family $\mathfrak{A} = \{u_r\}_{r \in N}$ is said to be a *majorant to* $\mathfrak{B}$ if and only if

1. Each u_r is a non-negative continuous form of bidegree $(1, 1)$ on $\bar{G}_r - g$.
2. An element $r_0 \in N$ exists such that $u_r \wedge \chi \geqslant dd^{\perp}\psi_r \wedge \chi$ on $\bar{G}_r - g$ if $r \geqslant r_0$.

THEOREM 4.1. *Let M be a noncompact, connected, complex manifold of dimension m. Let $\mathfrak{B} = \{B_r\}_{r \in N}$ be an exhaustion family of bumps, and let $\mathfrak{A} = \{u_r\}_{r \in N}$ be a majorant to $\mathfrak{B}$. Let V be a hermitian vector space of dimension $n+1$. Let $m - q = n - p = s > 0$ where p and q are non-negative integers. Let $f: M \to \mathbf{P}(V)$ be a holomorphic map. Let χ be a non-negative form of bidegree (q, q) and of class C^1 on M with $d\chi = 0$. Suppose that an open, relative compact neighborhood U of $z_0 \in M$ and $a \in G_p(V)$ with $f(z_0) \in \ddot{E}(a)$ exist such that $\chi \mid U > 0$, and such that f is general of order s at z_0 for a.*

Then $r_0 \in N$ exists such that $T_{s,f}(G_r) > 0$ if $r > r_0$ and $r \in N$. Define the total defect by

$$\delta_{s,f} = \overline{\lim_{r \in N}} \frac{\Delta_{s,f}(G_r, u_r) + \mu_{s,f}(\gamma)}{T_{s,f}(G_r)}.$$

Then $$1 - b_{s,f}(M) \leqslant c_{ps}\, \delta_{s,f}.$$

Proof. Because $\bar{U}$ is compact, $r_0 \in N$ exists such that $\psi_r(z) > 0$ for $z \in \bar{U}$ and all $r \in N$ with $r \geqslant r_0$. Then $\psi_r \chi \mid U > 0$ for $r \in N$ with $r \geqslant r_0$. Lemma 3.4 implies $T_{s,f}(G_r) > 0$ if $r \geqslant r_0$. Hence, $\delta_{s,f}$ is defined. Proposition 3.3 implies

$$1 - b_{s,f}(M) \leqslant 1 - b_{s,f}(G_r) \leqslant c_{ps} \frac{\Delta_f(G_r, u_r) + \mu_{s,f}(\gamma)}{T_{s,f}(G_r)}.$$

Hence, $1 - b_{s,f}(M) \leqslant c_{ps}\, \delta_{sf}$ q.e.d.

REMARK 1. *The average proximity remainder $\mu_{s,f}(\gamma)$ may depend on r, although the notation does not show so.*

REMARK 2. *If $\delta_{sf}=0$, then $f(M)$ intersects $\ddot{E}(a)$ for almost every $a \in G_p(V)$.*

REMARK 3. *Theorem 4.1 and the equidistribution theorem stated in Remark 2 are not too significant unless the exhaustion family $\mathfrak{B}$ and the majorant $\mathfrak{A}$ can be chosen in a reasonable way such that $\Delta_{s,f}(G(r), u_r)$ can be better interpreted. This shall be done now in specific cases.*

1. CASE: *The order $s=1$.* (1) Here it is assumed that χ is positive definite form of class C^∞ and bidegree $(m-1, m-1)$ on all of M. Take an open, relative compact subset g of M with $\gamma=\bar{g}-g$ as boundary manifold of class C^∞ such that each component of $M-g$ is not compact. Let N be the set of all open, relative compact, connected subsets of M with $\Gamma=\bar{G}-G$ as boundary manifold of class C^∞ such that $G \supset \bar{g}$. Then N is a directed set. For every $G \in N$, a function φ_G of class C^∞ on $\bar{G}-g$ exists such that $\varphi_G|\gamma=1$ and $\varphi_G|\Gamma=0$ and $dd^\perp\varphi_G \wedge \chi=0$ on $G-\bar{g}$ because this is the Dirichlet problem of an elliptic differential equation. Each component of $G-\bar{g}$ has boundary points on Γ and on γ. The maximum principal implies $0<\varphi_G(z)<1$ for $z \in G-\bar{g}$. Define $\varphi_G=0$ on $M-G$ and $\varphi_G=1$ on g. Moreover,

$$C(G)=\frac{1}{2\pi}\int_\Gamma d^\perp\varphi_G \wedge \chi=\frac{1}{2\pi}\int_\gamma d^\perp\varphi_G \wedge \chi>0.$$

Define $R(G)=1/C(G)$ and $\psi_G=R(G)\varphi_G$. Then $B_G=(G, \Gamma, g, \gamma, \psi_G)$ is a bump with $\mu_f(\Gamma)=\mu_f(\gamma)=1$ and $D_f(G, a)=0$. Moreover, $\mathfrak{B}=\{B_G\}_{G\in N}$ is an exhaustion family of bumps with $\mathfrak{A}=\{u_G\}_{G\in N}$ as a majorant where $u_G=0$ for all $G \in N$.

Let $f: M \to \mathbf{P}(V)$ be a holomorphic map into the projective space of the hermitian vector space V of dimension $n+1>1$. It is no loss of generality to assume that $f(M)$ is not contained in any linear projective subspace of codimension 1 because the case of a constant map is uninteresting, and otherwise one can consider a lower dimensional projective space as image manifold. Since $f(M) \not\subset E(a)$ for each $a \in G_{n-1}(V)$, f is general of order 1 for every $a \in G_{n-1}(V)$ at every $z \in M$. Define

$$T_{1f}(M)=\sup\{T_{1f}(G)\,|\,G \in N\} \leqslant \infty,$$

$$R(M)=\sup\{R(G)\,|\,G \in N\} \leqslant \infty.$$

Because $\{T_{1f}(G)\}_{G\in M}$ and $\{R(G)\}_{G\in M}$ are increasing nets, they converge to $T_{1,f}(M)$ respectively $R(M)$. By Lemma 3.4, $T_{1f}(M) \geqslant T_{1f}(G)>0$ if $G \in M$. According to [7], $T_{1f}(M)=\infty$ if $R(M)=\infty$. Obviously,

(1) See [7] and [10], pp. 183–184.

$$c_{n-1,1} = \tfrac{1}{2}\sum_{\nu=1}^{n}\frac{1}{\nu}.$$

Hence, the following result has been established.

THEOREM 4.2. *Under the assumptions of this case,*

$$1 - b_{1,f}(M) \leq \tfrac{1}{2}\sum_{\nu=1}^{n}\frac{1}{\nu}\,\frac{1}{T_{1f}(M)}$$

if $T_{1f}(M) < \infty$. *If* $T_{1,f}(M) = \infty$, *which is always the case if* $R(M) = \infty$, *then* $b_{1f}(M) = 1$, *meaning that* $f(M)$ *intersects* $\ddot{E}(a)$ *for almost every* $a \in G_{n-1}(V)$. *(Observe that* $G_{n-1}(V)$ *is isomorphic to* $\mathbf{P}(V^*)$.)

The other cases use an exhaustion function. Again, let M be a connected, noncompact, complex manifold of dimension m. A proper map $h: M \to \mathbf{R}$ of class C^∞ with $\mathrm{Min}_{x\in M} h(x) = 0$ is called an exhaustion function. For $r > 0$, the sets

$$G_r = \{x \in M \mid h(x) < r\}, \quad \Gamma_r = \{x \in M \mid h(x) = r\}$$

are not empty. G_r is open and relative compact and Γ_r is compact. For every compact subset K, a number $r_K > 0$ exists such that $G_r \supset K$ for all $r \geq r_K$. Define $E_h = \{x \in M \mid (dh)(x) = 0\}$. Then $E'_h = h(E_h)$ is a set of measure zero in $\mathbf{R}$. If $0 < r \in \mathbf{R} - E'_h$, then $\Gamma_r = \bar{G}_r - G_r$ is a boundary manifold of class C^∞ of G_r. Take $0 < r_0 \in \mathbf{R} - E'_h$. Define $g = G_{r_0}$ and $\gamma = \Gamma_{r_0}$. For $r > r_0$, define ψ_r: $M \to \mathbf{R}$ by $\psi_r = 0$ on $M - G_r$ by $\psi_r = r - h$ on $G_r - g$ and by $\psi_r = r - r_0$ on g. Obviously, ψ_r is continuous and $\psi_r \mid \bar{G}_r - g$ is of class C^∞. On $\bar{G}_r - g$,

$$d^\perp \psi_r = -dh, \quad dd^\perp \psi = d^\perp dh. \tag{28}$$

Define $N = \{r \in \mathbf{R} \mid r_0 < r \notin E'_h\}$. For each $r \in N$,

$$B_r = (G_r, \Gamma_r, g, \gamma, \psi_r)$$

is a bump and $\mathfrak{B}_h = \{B_r\}_{r\in N}$ is an exhaustion family of bumps.

Let f: $M \to \mathbf{P}(V)$ be a holomorphic map into the projective space of the hermitian vector space V of dimension $n+1 > 1$. Take p, q, s as non-negative integers with $m - q = n - p = s > 0$. Let χ be a non-negative form of bidegree (q, q) and class C^1 on M with $d\chi = 0$. Suppose an open, relative compact subset U of M exists such that $\chi \mid U$ is positive. Suppose that $a \in G_p(V)$ and $z_0 \in U$ with $f(z_0) \in \ddot{E}(a)$ exist such that f is general for a at z_0. For $r > 0$, respectively $r \in N$, write $T_f(r) = T_f(G_r)$ and $A_f(r) = A_f(G_r)$ and $m_f(r, a) = m_f(\Gamma_r, a)$ and $m_f(r_0, a) = m_f(\gamma, a)$, etc. For $r_0 < r < R$, observe

$$G_r = \{z \in M \mid R - r_0 - \psi_R(z) < r - r_0\} = G_R(r - r_0), \quad \psi_R[r - r_0] = \psi_r.$$

If $r \in N$ and $R \in N$ with $r < R$, then $B_R(r - r_0) = B_r$. Therefore,

$$T_f(r) = \int_{r_0}^{r} A_f(t)\,dt,$$

$$N_f(r, a) = \int_{r_0}^{r} n_f(t, a)\,dt.$$

Here, A_f and $n_f(\cdot, a)$ are continuous from the left. Hence,

$$D^- T_f(r) = A_f(r) \quad \text{and} \quad D^- N_f(r, a) = n_f(r, a)$$

if $r > r_0$, and where T_f and N_f are differentiable at every $r \in N$. Observe that $m_f(r_0, a)$ and $\mu_f(r_0)$ are independent of r because of (28). Observe that $r_1 \in N$ exists such that $\bar{U} \subset G_r$ if $r \geqslant r_1$. Because $\psi_r > 0$ on G_r, also $\psi_r \chi \,|\, U$ is positive for $r \geqslant r_1$. Hence $A_f(r) > 0$ if $r \geqslant r_1$ and

$$T_f(r) \geqslant (r - r_1) A_f(r_1) > 0 \quad \text{if} \quad r > r_1.$$

Consequently, $T_f(r) \to \infty$ for $r \to \infty$, even

$$\varliminf_{r \to \infty} \frac{T_f(r)}{r} > 0.$$

2. Case: *Pseudoconcave manifolds.* Here it is assumed that an exhaustion function h on M exists such that its Levi form $d^\perp dh \leqslant 0$ is negative outside a compact set K. Suppose that such an exhaustion function is given. Obviously, r_0 can be taken so large that $K \subseteq G_{r_0}$. Construct $\mathfrak{B}_h$ with this number r_0. For each $r \in N$, define $u_r = 0$ on $\bar{G}_r - g$. Then $\mathfrak{A} = \{u_r\}_{r \in N}$ is a majorant and $\Delta_{sf}(G_r, u_r) = 0$. Because $T_{s,f}(r) \to \infty$ for $r \to \infty$ and because $\mu_{s,f}(r_0)$ is constant, $\delta_{s,f} = 0$. Hence, Theorem 4.1 implies:

Theorem 4.3. *If M is pseudoconcave and if the assumptions of this case are made, then $f(M)$ intersects $\ddot{E}_p(a)$ for almost every $a \in G_p(V)$.*

3. Case: *Pseudoconvex manifolds.* Here the existence of an exhaustion function h on M with $d^\perp dh \geqslant 0$ outside a compact set K is assumed. Suppose such an exhaustion function h is given. Construct $\mathfrak{B}_h$ with $r_0 > 0$ so large that $K \subseteq g = G_{r_0}$. Then $dd^\perp \psi_r = d^\perp dh \geqslant 0$ for $r \in N$. Hence $\mathfrak{A} = \{dd^\perp \psi_r\}_{r \in N}$ is a majorant with

$$\Delta_{s,f}(r) = \Delta_{s,f}(r, dd^\perp \psi_r) = \frac{1}{2\pi W(s-1)} \int_{G_r - g} f^*(\ddot{\omega}_{0,s-1}) \wedge d^\perp dh \wedge \chi.$$

Because $T_{s,f}(r) \to \infty$ for $r \to \infty$, and because $\mu_f(r_0)$ is constant, Theorem 4.1 implies

Theorem 4.4. *If M is pseudoconvex, if h is a pseudoconvex exhaustion and if the assumptions of this case are made, then*

$$1 - b_{s,f}(M) \leqslant c_{ps}\delta_{sf} = c_{ps} \varlimsup_{r\to\infty} \frac{\Delta_{s,f}(r)}{T_{s,f}(r)}.$$

If $\delta_{sf}=0$, *then* $f(M)$ *intersects* $\ddot{E}_p(a)$ *for almost every* $a \in G_p(V)$.

4. CASE: *Stein manifolds.* (1) If and only if M is a Stein manifold, then an exhaustion function h exists with $d^\perp dh > 0$ on all of M. Hence, Stein manifolds are pseudoconvex, and Case 3 applies. However, a better interpretation can be given on Stein manifolds by a convenient choice of χ. Therefore, let M be a Stein manifold and let h be an exhaustion function on M with $d^\perp dh > 0$. The couple (M, h) is called a *Levi manifold.* Construct $\mathfrak{B}_h$ and $\mathfrak{A} = \{dd^\perp \psi_r\}_{r\in N}$ as in Case 3. Observe that $d^\perp dh$ defines a Kähler metric on M. For each integer e in $0 \leqslant e \leqslant m$, define

$$\chi_e = \frac{1}{e!} d^\perp dh \wedge \ldots \wedge d^\perp dh \quad (e\text{-times})$$

where $\chi_0 = 1$. Then

$$\Delta_{s,f}(r) = \frac{1}{2\pi}\frac{q+1}{W(s-1)}\int_{G_r - g} f^*(\ddot{\omega}_{0,s-1}) \wedge \chi_{q+1}$$

where
$$A_{s,f}(r) = \frac{1}{W(s)}\int_{G_r} f^*(\ddot{\omega}_{0,s}) \wedge \chi.$$

For $s=0$, define
$$A_{0,p}(r) = M(r) = \int_{G_r} \chi_m$$

as the volume of G_r in the Kähler metric $d^\perp dh$. Then

$$\Delta_{s,f}(r) = \frac{q+1}{2\pi} A_{s-1,f}(r) - \frac{q+1}{2\pi} A_{s-1,f}(r_0).$$

Therefore, up to an additive and a multiplicative constant, the average deficit of order s equals the spherical image of order $s-1$, i.e., the derivative of the characteristic of order $s-1$. This gives a very instructive interpretation of the average deficit on Levi manifolds. Because $T_{s,f}(r) \to \infty$ for $r \to \infty$ and because $\mu_f(r_0)$ and $A_{s-1,f}(r_0)$ are constant, also the total defect receives a new interpretation:

$$\delta_{sf} = \varlimsup_{r\to\infty} \frac{\Delta_{s,f}(r)}{T_{s,f}(r)} = \frac{q+1}{2\pi} \varlimsup_{r\to\infty} \frac{A_{s-1,f}(r)}{T_{s,f}(r)} = \frac{q+1}{2\pi} \varlimsup_{\substack{r\to\infty\\ r\in N}} \frac{T'_{s-1,f}(r)}{T_{s,f}(r)}.$$

THEOREM 4.5. *Let* (M, h) *be a Levi manifold. Under the assumptions of this case,*

$$1 - b_{s,f}(M) \leqslant \frac{q+1}{2\pi} c_{ps} \varlimsup_{r\to\infty} \frac{A_{s-1,f}(r)}{T_{s,f}(r)}.$$

(1) For $s=n$, see [10] 3. Example, pp. 187–189.

Hence, if

$$\frac{A_{s-1,f}(r)}{T_{s,f}(r)} \to 0 \quad \textit{for } r \to \infty,$$

then $f(M)$ intersects $\ddot{E}(a)$ for almost every $a \in G_p(V)$.

Observe that if $s = 1$, then $A_{0,f}(r) = M(r)$ is the volume of G_r, and this result should be compared with Theorem 4.2.

Observe that Theorem 4.5 depends on M, h, V, f, p and the hermitian metric on V only, and that it is expressible with simplicity in explicit intrinsic terms. Only, the choice of h is not canonical, and the question remains as to how the theory depends on the choice of h on Stein manifolds.

Observe that Theorem 4.5 generalizes a result of Chern [2]. Also, the results of Bott and Chern [1] concerning equidistribution of the zero sets of sections can be obtained from this and generalized to the case where the fiber dimensions of the vector bundle is smaller than the dimension of the base space.

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