SUPPORTS AND SINGULAR SUPPORTS OF CONVOLUTIONS

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1. Introduction

If f is a distribution in R^n we write supp f (resp. sing supp f) for the smallest closed set outside which $f=0$ (resp. $f \in C^\infty$). Then the convolution theorem of Titchmarch [13], extended from one to n dimensions by Lions [10], states that

$$\text{ch supp } (f_1 * f_2) = \text{ ch supp } f_1 + \text{ ch supp } f_2; \quad f_1, f_2 \in \mathcal{E}'. \tag{1.1}$$

Here we have used the notation ch A for the convex hull of a set A in R^n and written

$$A+B=\{x+y;\ x \in A,\ y \in B\}$$

if A and B are subsets of R^n; below $A-B$ will be defined similarly.

The aim of this paper is to prove results similar to (1.1) where supports are replaced by singular supports. In Hörmander [5] it was proved in perfect analogy with (1.1) that

$$\text{ch sing supp } (f_1 * f_2) = \text{ ch sing supp } f_1 + \text{ ch sing supp } f_2 \tag{1.2}$$

provided that $f_1, f_2 \in \mathcal{E}'$ and either supp f_1 or supp f_2 consists of a finite number of points, a result due to F. John and B. Malgrange when the number of points is one. When f_2 is hypoelliptic in the sense of Ehrenpreis [4] it was also proved in Hörmander [6] that

$$\text{ch sing supp } f_1 \subset \text{ch sing supp } (f_1 * f_2) - \text{ ch sing supp } f_2, \tag{1.3}$$

which is a weakened form of the non-trivial part of (1.2) that the left-hand side of (1.2) contains the right-hand side. However, not even this weaker result can be valid for arbitrary f_2, for it may happen that $f_1 * f_2 \in C_0^\infty$ although neither f_1 nor f_2 is in C_0^∞. In fact, Ehrenpreis [4] has proved that every $f_1 \in \mathcal{E}'$ with $f_1 * f_2 \in C_0^\infty$ belongs to C_0^∞ if and only if the Fourier transform $\hat{f}_2$ of the distribution $f_2 \in \mathcal{E}'$ is slowly decreasing in the sense that for some constant A

$$\sup \{|\hat{f}_2(\eta)|;\ \eta \in R^n,\ |\eta - \xi| < A \log (2+|\xi|)\} \geqslant (A+|\xi|)^{-A} \quad (\xi \in R^n). \tag{1.4}$$

(A proof of this result is also given in Hörmander [5].)

We shall prove here that (1.3) is valid for arbitrary $f_1, f_2 \in \mathcal{E}'$ such that $\hat{f}_2$ satisfies (1.4). This result contains those of [6]. Moreover, we give necessary and sufficient conditions on the convex compact sets K_1 and K_3 in order that

$$\text{sing supp } (f_1 * f_2) \subset K_3 \Rightarrow \text{sing supp } f_1 \subset K_1.$$

For the statement of these results see section 5.

The proof of (1.3) is based on a study of the Laplace transforms of f_1 and f_2, combined with an analogue of the Paley–Wiener theorem for the singular supports given in Hörmander [6], which goes back to an idea of Ehrenpreis [3]. The estimates of analytic functions which we need are very closely related to those required to prove (1.1). However, we need an extension of these estimates to plurisubharmonic functions so we shall give complete proofs for them. The proof of (1.1) thus given is closely related to that of Koosis [8], the crucial point being an application of Harnack's inequality for positive harmonic functions. However, the formal presentation differs rather much. A similar use of Harnack's inequality was also made in Hörmander [6], following a suggestion by Malgrange, but the estimates given here are much more precise.

In section 2 we state the Paley–Wiener theorem and its analogue for singular supports. The facts concerning (pluri-)subharmonic functions which we shall use are given in section 3. There is no new result but we have found it difficult to find convenient references for all the facts we need. We then prove a slight extension of (1.1) in section 4, and using the estimates obtained there we prove results containing (1.3) in section 5. The consequences concerning convolution equations are discussed in section 6.

2. The Paley–Wiener theorem for supports and singular supports

If K is a convex compact subset of R^n, the supporting function H of K is defined by

$$H(\xi) = \sup_{x \in K} \langle x, \xi \rangle \quad (\xi \in R^n). \tag{2.1}$$

It is obvious that H is convex and positively homogeneous,

$$H(\xi+\eta) \leqslant H(\xi) + H(\eta) \quad (\xi, \eta \in R^n); \quad H(t\xi) = tH(\xi) \quad (\xi \in R^n,\ t \geqslant 0). \tag{2.2}$$

If K is empty we set $H = -\infty$; the last part of (2.2) then assumes that we define $0 \cdot (-\infty) = -\infty$. Conversely, every function H with values in $[-\infty, \infty)$ satisfying (2.2) is the supporting function of one and only one convex compact set K, and K is defined by

$$K = \{x;\ \langle x, \xi \rangle \leqslant H(\xi) \text{ for all } \xi \in R^n\}. \tag{2.3}$$

Therefore (2.1) and (2.3) give a one-to-one correspondence between the set $\mathcal{K}$ of convex compact subsets of R^n and the set $\mathcal{H}$ of functions satisfying (2.2). (Such functions $\neq -\infty$ are automatically continuous.) If K_1, K_2 are convex compact sets with supporting functions H_1, H_2, then the supporting function of the convex compact set $K_1 \pm K_2$ is $H_1(\xi) + H_2(\pm \xi)$. If H_α is the supporting function of K_α and $H = \sup_\alpha H_\alpha$ is finite everywhere then H is the supporting function of the closed convex hull of $\bigcup_\alpha K_\alpha$. For a proof of these elementary and classical facts we refer to Bonnesen and Fenchel [1].

The Paley–Wiener theorem can now be stated as follows:

THEOREM 2.1. *Let K be a convex compact subset of R^n with supporting function H. If f is a distribution with support contained in K, then the Fourier–Laplace transform $\hat{f}$ of f satisfies the estimate*

$$|\hat{f}(\zeta)| \leqslant C(1 + |\zeta|)^N e^{H(\operatorname{Im} \zeta)} \quad (\zeta \in C^n), \tag{2.4}$$

where N is the order of f. Conversely, every entire analytic function in C^n satisfying an estimate of the form (2.4) is the Fourier–Laplace transform of a distribution with support contained in K.

Proof. The theorem is proved in Schwartz [12] when K is a cube and in Hörmander [7] when K is a sphere. The modifications required in either of these proofs in the case of a general K are quite obvious and are left to the reader. Let us only note that (2.4) is trivial if f is a measure $d\mu$, for by definition we have

$$\hat{f}(\zeta) = \int e^{-i\langle x, \zeta \rangle}\, d\mu(x),$$

which implies that
$$|\hat{f}(\zeta)| \leqslant e^{H(\operatorname{Im} \zeta)} \int |d\mu(x)|.$$

This is the only case in which we use the necessity of (2.4). On the other hand, the quoted results imply that every entire function satisfying an estimate of the form (2.4) is the Fourier–Laplace transform of a distribution with support contained in an arbitrary sphere (parallelepiped) containing K. Since the intersection of all such spheres (parallelepipeds) is equal to K, the theorem follows.

THEOREM 2.2. *Let $f \in \mathcal{E}'(R^n)$ and let K be a convex non-empty compact subset of R^n. In order that sing supp $f \subset K$ it is necessary and sufficient that there be a constant N and a sequence of constants C_m, $m = 1, 2, \ldots$, such that*

$$|\hat{f}(\zeta)| \leqslant C_m (1 + |\zeta|)^N e^{H(\operatorname{Im}\zeta)}, \text{ if } |\operatorname{Im}\zeta| \leqslant m \log(|\zeta| + 1) \quad (m = 1, 2, \ldots). \tag{2.5}$$

In order that sing supp $f = \emptyset$ *it is necessary and sufficient that to any positive integers N and m one can find $C_{N,m}$ so that*

$$|\hat{f}(\zeta)| \leqslant C_{N,m} (1 + |\zeta|)^{-N}, \text{ if } |\operatorname{Im}\zeta| \leqslant m \log(|\zeta| + 1). \tag{2.5$'$}$$

Proof. The last statement follows at once from the form of the Paley–Wiener theorem which states that $f \in C_0^\infty$ if and only if one can find constants A and C_N, $N = 1, 2, \ldots$ such that

$$|\hat{f}(\zeta)| \leqslant C_N (1 + |\zeta|)^{-N} e^{A|\operatorname{Im}\zeta|} \quad (N = 1, 2, \ldots).$$

Apart from a translation of the coordinate system the first part of the theorem is identical with Theorem 1.7.8 in Hörmander [7] when K is a sphere. The necessity of (2.5) follows from Theorem 2.1 by following the proof of Theorem 1.7.8 in [7]. On the other hand, if $\hat{f}$ satisfies (2.5) we know from that result that sing supp f is contained in every sphere containing K, and the intersection of all such spheres is equal to K.

3. Preliminaries concerning subharmonic and plurisubharmonic functions

Let Ω be an open connected set in R^p, and set for $r > 0$

$$\Omega_r = \{x; |x - y| \leqslant r \Rightarrow y \in \Omega\},$$

where the norm denotes the Euclidian norm. If v is a measurable function in Ω which is bounded from above on compact subsets of Ω we set

$$v^r(x) = \int_{|y|<1} v(x + ry)\, dy \Big/ \int_{|y|<1} dy \quad (x \in \Omega_r).$$

A function v defined in Ω with values in $[-\infty, +\infty)$ is called *subharmonic* if

(a) v is semi-continuous from above,

(b) $v(x) \leqslant v^r(x)$ if $x \in \Omega_r$.

(It is convenient here not to require as usual that $v \not\equiv -\infty$. Except when $v \equiv -\infty$, however, v is finite almost everywhere and is in fact in $L_1^{loc}(\Omega)$.)

LEMMA 3.1. *Let v_k be a sequence of subharmonic functions in Ω which are uniformly bounded from above on every compact subset of Ω. Then the smallest upper semi-continuous majorant V of $v = \overline{\lim}\, v_k$ is subharmonic, and we have $V = v$ almost everywhere. If K is a compact subset of Ω and f is a continuous function on K, then*

$$\lim_{k \to \infty} \sup_K (v_k - f) \leqslant \sup_K (V - f). \tag{3.1}$$

Proof. Since we may replace Ω by arbitrary relatively compact subdomains containing K it is no restriction to assume that the sequence is uniformly bounded in Ω or even that $v_k \leqslant 0$ in Ω for every k. By Fatou's lemma we have

$$v(x) \leqslant \overline{\lim}\, v_k^r(x) \leqslant v^r(x) \quad (x \in \Omega_r). \tag{3.2}$$

Next note that if $x \in \Omega_r$ and $0 < \varepsilon < 1$ we can find δ so small that for every k

$$v_k(\xi) \leqslant (1-\varepsilon)\, v_k^r(x) \quad \text{if} \quad |\xi - x| < \delta. \tag{3.3}$$

In fact, since $v_k \leqslant 0$ we have if $|\xi - x| < \delta$ and $x \in \Omega_{r+2\delta}$

$$(r+\delta)^p\, v_k(\xi) \leqslant (r+\delta)^p\, v_k^{r+\delta}(\xi) \leqslant r^p v_k^r(x),$$

and if $r^p/(r+\delta)^p > 1 - \varepsilon$ we obtain (3.3). Combination of (3.2) and (3.3) now gives that if $a > v^r(x)$ and $0 < \varepsilon < 1$ then $v_k(\xi) < a(1-\varepsilon)$ if $|\xi - x| < \delta$ and $k > k_0$. Hence $V(x) \leqslant a(1-\varepsilon)$ which proves that

$$V(x) \leqslant v^r(x) \leqslant V^r(x) \tag{3.4}$$

so that V is subharmonic. If $V \equiv -\infty$ then $v \equiv -\infty$ but otherwise V is finite in a dense set and (3.4) shows that v is locally integrable. At every Lebesgue point for v we have

$$v(x) \leqslant V(x) \leqslant \lim_{r \to 0} v^r(x) = v(x)$$

which proves that $v = V$ almost everywhere.

To prove (3.1) finally, we take a and b so that $\sup_K (V - f) < b < a$. If $x \in K$ we have $V(x) < f(x) + b$ so that $v(\xi) < f(x) + b$ in a neighborhood of x, and so $v^r(x) < f(x) + b$

if r is sufficiently small. Hence we can by (3.2), (3.3) find k_0 and $\delta < 0$ so that $v_k(\xi) < f(x) + b$ if $|x - \xi| < \delta$ and $k > k_0$. Since f is continuous this implies that for some other $\delta > 0$ we have

$$v_k(\xi) < f(\xi) + a \quad \text{if} \quad |\xi - x| < \delta, \quad \xi \in K \quad \text{and} \quad k > k_0.$$

By the Borel–Lebesgue lemma this shows that $v_k(\xi) - f(\xi) < a$ in K for large k, which proves the lemma.

DEFINITION 3.1. *Let v and v_k $(k = 1, 2, \dots)$ be subharmonic functions in Ω. Then we say that $v_k \to v$ if*

$$\int v_k \varphi \, dx \to \int v \varphi dx \quad (k \to \infty), \tag{3.5}$$

for every $\varphi \in C_0^+(\Omega)$, the space of continuous non-negative functions with compact support in Ω.

Note that both sides of (3.5) are defined when $\varphi \in C_0^+(\Omega)$ even if v_k or v should be $\equiv -\infty$.

LEMMA 3.2. *Let v_k be a sequence of subharmonic functions in Ω which are uniformly bounded from above on every compact subset of Ω. Then there exists a subsequence v_{k_j} such that $v_{k_j} \to V$ where V is the smallest upper semi-continuous majorant of $\overline{\lim}_{j \to \infty} v_{k_j}$.*

Proof. It follows from Lemma 3.1 or Fatou's lemma that if $v_k(x)$ converges to $-\infty$ for every $x \in \Omega$ then $v_k \to -\infty$ in the sense of Definition 3.1. Passing if necessary to a subsequence we may therefore assume that $v_k(x)$ is bounded from below for some value of x. For arbitrary fixed $r > 0$ and $x \in \Omega_r$ it then follows that $v_k^r(x)$ is bounded from below. In fact, we could otherwise apply (3.3) to the functions v_k minus a common upper bound when $|\xi - x| < r + \varepsilon$ and conclude that there is a subsequence $v_{k'}$ such that $v_{k'}(\xi) \to -\infty$ in a neighborhood of x. But then Lemma 3.1 shows that $v_{k'}(x) \to -\infty$ for every x, which is a contradiction. The sequence v_k is therefore bounded in L_1^{loc} so a subsequence can be found which converges weakly to a measure $d\mu$. To simplify notations we may assume that the sequence v_k itself converges to $d\mu$, that is,

$$\int f v_k dx \to \int f \, d\mu \quad (f \in C_0(\Omega)).$$

By Lemma 3.1 the smallest upper semi-continuous majorant V of $v = \overline{\lim}\, v_k$ is subharmonic, and if $f \in C_0^+$ we have by Fatou's lemma

$$\int f\,d\mu = \lim_{k\to\infty}\int f v_k\,dx \leqslant \int fv\,dx = \int fV\,dx. \tag{3.6}$$

Now let $\varphi(x)$ be a continuous decreasing function of $|x|$ which is equal to 0 when $|x|>1$, assume that $\int \varphi(x)\,dx = 1$ and set $\varphi_\varepsilon(x) = \varepsilon^{-p}\varphi(x/\varepsilon)$. Then we have $w \leqslant w * \varphi_\varepsilon$ in Ω_ε for every subharmonic function w in Ω, which gives

$$v(x) = \overline{\lim}\; v_k(x) \leqslant \lim\,(v_k * \varphi_\varepsilon)(x) = (d\mu * \varphi_\varepsilon)(x) \quad (x \in \Omega_\varepsilon).$$

Since $\varphi_\varepsilon * d\mu$ is continuous in Ω_ε this proves that

$$V(x) \leqslant (d\mu * \varphi_\varepsilon)(x) \leqslant (V * \varphi_\varepsilon)(x) \quad (x \in \Omega_\varepsilon),$$

where the last inequality follows from (3.6). Now $V * \varphi_\varepsilon \to V$ in L_1 on every compact subset of Ω when $\varepsilon \to 0$, which proves that $d\mu * \varphi_\varepsilon \to V$ in L_1 on compact subsets of Ω, hence that V is the density of the measure $d\mu$. The proof is complete.

Now let Ω be a connected open set in C^n. If $\zeta \in C^n$ we write $D_\zeta = \{w\zeta;\ w \in C,\ |w| \leqslant 1\}$, and if v is a Borel measurable function in Ω which is bounded from above on compact subsets of Ω we define $v(z,\zeta)$ when $\{z\} + D_\zeta \subset \Omega$ as the average of v over $\{z\} + D_\zeta$, that is,

$$v(z,\zeta) = \frac{1}{\pi}\int_0^{2\pi}\int_0^1 v(z + re^{i\theta}\zeta)\, r\,dr\,d\theta.$$

A function v defined in Ω with values in $[-\infty, +\infty)$ is called *plurisubharmonic* if

(a) v is semi-continuous from above,

(b) $v(z) \leqslant v(z,\zeta)$ if $\{z\} + D_\zeta \subset \Omega$.

This class of functions is invariant for analytic coordinate transformations of the variables z_k (see Lelong [9]). It follows easily that a plurisubharmonic function is subharmonic if C^n is identified with R^{2n}, and when $n = 1$ the notions of subharmonic and plurisubharmonic functions coincide. When v is plurisubharmonic the function $v(z + w\zeta)$ of one complex variable w is of course subharmonic for arbitrary $z, \zeta \in C^n$ in the open set where it is defined.

LEMMA 3.3. *Let v_k be a sequence of plurisubharmonic functions in Ω which are uniformly bounded from above on every compact subset of Ω. Then the smallest upper semi-continuous majorant V of $v = \overline{\lim}\; v_k$ is plurisubharmonic and we have $V = v$ almost everywhere.*

Proof. By Fatou's lemma we have

$$\overline{\lim}\, v_k(z,\zeta) \leqslant v(z,\zeta) \quad \text{if} \quad \{z\} + D_\zeta \subset \Omega.$$

Hence

$$v(z) \leqslant V(z,\zeta) \quad \text{if} \quad \{z\} + D_\zeta \subset \Omega.$$

Now Fatou's lemma also shows that $V(z,\zeta)$ is an upper semi-continuous function of z in the open set of all z such that $\{z\} + D_\zeta \subset \Omega$. Hence $V(z) \leqslant V(z,\zeta)$, which proves that V is plurisubharmonic. The remaining part of the lemma follows from Lemma 3.1 since plurisubharmonic functions are also subharmonic.

Remark. Much more precise results than the previous lemmas are known; see Cartan [2].

If v is subharmonic in $\Omega \subset R^\nu$, if K is a compact subset of Ω and h a continuous function on K which is harmonic in the interior of K, then the maximum of $v - h$ in K is attained on the boundary of K. For a proof we refer to Radó [11], section 2.3, but we prove here the "three line theorem".

LEMMA 3.4. *Let v be subharmonic and bounded from above in a neighborhood of the strip $0 \leqslant \operatorname{Im} z \leqslant 1$ in C^1 and assume that for some constants C and A we have $v(z) \leqslant C + A \operatorname{Im} z$ on the boundary of the strip. Then this inequality holds also in the interior of the strip.*

Proof. The function $v(z) - C - A \operatorname{Im} z - \varepsilon \operatorname{Re}(1+z^2)$, where $\varepsilon > 0$, is $\leqslant 0$ on the boundary of the strip and tends to $-\infty$ at infinity. Hence it is $\leqslant 0$ in the whole strip, and when $\varepsilon \to 0$ this proves the assertion.

Before extending Lemma 3.4 to plurisubharmonic functions we note that Lemma 3.4 implies Liouville's theorem for plurisubharmonic functions.

LEMMA 3.5. *Let v be plurisubharmonic and bounded from above in C^n. Then v is a constant.*

Proof. First let $n = 1$. Take a fixed ζ and set

$$M(y) = \sup_{\operatorname{Im} z = y} v(\zeta + e^{-iz}).$$

Then Lemma 3.4 and the maximum principle show that $M(y)$ is a convex increasing function of y, and since $v(\zeta) \leqslant M(y)$ and v is semi-continuous from above we have $M(y) \to v(\zeta)$ when $y \to -\infty$. But an increasing bounded convex function must be a constant, so that $M(y) = v(\zeta)$ for every y, that is, $v(z) \leqslant v(\zeta)$ for every z. Since the roles of z and ζ may be interchanged, this proves that $v(z)=v(\zeta)$. If $n>1$ we can for arbitrary ζ, $z\in C^n$ apply the result just proved to the subharmonic function $v(\zeta+w(z-\zeta))$ of $w\in C^1$. Since this function must be constant we have $v(\zeta)=v(z)$, which proves the lemma.

Combination of Lemma 3.3 and Lemma 3.5 gives

LEMMA 3.6. *Let v_k be a sequence of plurisubharmonic functions in C^n which are uniformly bounded from above on every compact set. If $v=\overline{\lim}\, v_k$ is bounded from above in the whole of C^n, then $v(\zeta)=$ sup v almost everywhere.*

An extension of Lemma 3.4 to plurisubharmonic functions is given in the following theorem.

THEOREM 3.1. *Let ω be an open convex subset of R^n and let Ω be the tube defined by $\Omega=\{z;\, z\in C^n,\ \mathrm{Im}\ z\in\omega\}$. Let v be plurisubharmonic in Ω and assume that for every compact subset K of ω there is an upper bound for $v(z)$ when $\mathrm{Im}\ z\in K$. Then the function*

$$M(y)=\sup_x v(x+iy)\quad (y\in\omega),$$

where x varies in R^n, is a convex function of y.

Proof. Let $y_1, y_2\in\omega$, and let $x\in R^n$. Then the function of w

$$V(w)=v(x+iy_1+w(y_2-y_1))$$

is subharmonic and bounded from above in a neighborhood of the strip $0\leqslant \mathrm{Im}\ w\leqslant 1$. When $\mathrm{Im}\ w=0$ it is bounded by $M(y_1)$ and when $\mathrm{Im}\ w=1$ by $M(y_2)$. Hence Lemma 3.4 gives that

$$V(w)=v(x+iy_1+w(y_2-y_1))\leqslant(1-\mathrm{Im}\ w)\,M(y_1)+\mathrm{Im}\ w\,M(y_2).$$

If $0\leqslant\lambda_j$ $(j=1,2)$, and $\lambda_1+\lambda_2=1$, we obtain by setting $w=i\lambda_2$ that

$$v(x+i(\lambda_1 y_1+\lambda_2 y_2))\leqslant\lambda_1 M(y_1)+\lambda_2 M(y_2),$$

which proves that $M(\lambda_1 y_1+\lambda_2 y_2)\leqslant\lambda_1 M(y_1)+\lambda_2 M(y_2)$.

Now let v be plurisubharmonic in the whole of C^n and assume that

$$v(z)\leqslant C+A\,|\mathrm{Im}\ z| \tag{3.7}$$

for some constants C and A. Then Theorem 3.1 shows that

$$M(y)=\sup_{\mathrm{Im}\, z=y} v(z) \tag{3.8}$$

is a convex function in R^n. Hence the limit

$$H(y) = \lim_{t \to +\infty} \frac{M(ty)}{t} = \lim_{t \to +\infty} \frac{M(ty) - M(0)}{t} \tag{3.9}$$

exists for every y and $H(y) \leqslant A|y|$. That the difference quotient $(M(ty) - M(0))/t$ is increasing also shows that

$$M(y) \leqslant M(0) + H(y) \quad (y \in R^n). \tag{3.10}$$

Since H is the limit of convex functions it is clear that H is convex, and a substitution gives that $H(ay) = aH(y)$ if $a \geqslant 0$. Hence H belongs to the class $\mathcal{H}$ of supporting functions introduced in section 2, and *we shall call H the supporting function of v.* It is obvious from (3.10) that $H \not\equiv -\infty$ unless $v \equiv -\infty$.

THEOREM 3.2. *Let f be a measure with compact support in R^n and let H be the supporting function of* ch supp f. *Then the supporting function of the plurisubharmonic function* $\log |\hat{f}|$ *is also equal to H.*

Proof. By Theorem 2.1 we have for some constant C

$$\log |\hat{f}(\zeta)| \leqslant C + H(\operatorname{Im} \zeta).$$

Hence the supporting function H' of $\log |\hat{f}|$ is defined and $H' \leqslant H$ since $\log |\hat{f}(\zeta)| \leqslant C + tH(\eta)$ if $\operatorname{Im} \zeta = t\eta$. On the other hand, we have by (3.10) that

$$\log |\hat{f}(\zeta)| \leqslant C + H'(\operatorname{Im} \zeta)$$

so it follows from Theorem 2.1 that ch supp f is contained in the set whose supporting function is H', that is, $H \leqslant H'$.

When $n > 1$ the following theorem gives an important alternative characterization of $H(y)$ (compare Lions [10]).

THEOREM 3.3. *Let v be a plurisubharmonic function satisfying* (3.7) *and let M, H be defined by* (3.8) *and* (3.9). *Then we have for $y \in R^n$ and $\zeta \in C^n$*

$$\overline{\lim_{\operatorname{Im} w \to +\infty}} \frac{v(\zeta + wy)}{\operatorname{Im} w} \leqslant H(y), \tag{3.11}$$

with equality for almost every ζ when y is kept fixed.

Proof. By (3.10) we have when $\operatorname{Im} w > 0$

$$\frac{v(\zeta+wy)}{\operatorname{Im} w} \leqslant \frac{M(\operatorname{Im}\zeta+\operatorname{Im} wy)}{\operatorname{Im} w} \leqslant \frac{M(\operatorname{Im}\zeta)}{\operatorname{Im} w}+H(y), \tag{3.12}$$

which implies the inequality (3.11). Let

$$A=\sup_{\zeta} \varlimsup_{\operatorname{Im} w\to+\infty} \frac{v(\zeta+wy)}{\operatorname{Im} w}.$$

Then the three line theorem applied to the analytic function $v(\zeta+wy)$ of w which is $\leqslant M(\operatorname{Im}\zeta)$ for real w gives

$$v(\zeta+wy) \leqslant M(\operatorname{Im}\zeta)+A \operatorname{Im} w \quad (\operatorname{Im} w>0).$$

If we choose ζ real and take $w=it$ with $t>0$, this gives

$$M(ty) \leqslant M(0)+At,$$

hence $H(y) \leqslant A$. Since we have already seen that $A \leqslant H(y)$, this proves that $A=H(y)$. For an arbitrary $\varepsilon>0$ we can therefore find some ζ_0 and a sequence w_k with $\operatorname{Im} w_k \to +\infty$ such that

$$\varlimsup_{k\to\infty} \frac{v(\zeta_0+w_k y)}{\operatorname{Im} w_k}>H(y)-\varepsilon.$$

But in view of (3.12) we can apply Lemma 3.6 to the sequence of plurisubharmonic functions $v(\zeta+w_k y)/\operatorname{Im} w_k$ of ζ, which gives that

$$\varlimsup_{k\to\infty} \frac{v(\zeta+w_k y)}{\operatorname{Im} w_k}>H(y)-\varepsilon \text{ for almost every } \zeta.$$

If we use this result for a sequence of positive numbers ε converging to 0, the theorem follows.

4. The theorem on supports

If f_j $(j=1,2,3)$ are distributions with compact support such that $f_3=f_1 * f_2$, then $\hat{f}_3=\hat{f}_1\,\hat{f}_2$. From Theorem 2.1 it follows therefore that to prove the theorem on supports we have to examine how to estimate two analytic functions when an estimate for their product is known. We shall generalize this question slightly by studying estimates for two plurisubharmonic functions v_1 and v_2 when an estimate for $v_1+v_2=v_3$ is known. This extension of the results will be important in section 5.

Let Ω be a connected and simply connected open set in the complex plane, different from the whole plane. If $\zeta\in\Omega$ we let $z\to w(z,\zeta)$ be a conformal mapping of Ω onto the unit circle which maps ζ onto the origin. If u is a non-negative harmonic function in Ω, we then have

$$u(z) \leq u(\zeta)(1+|w(z,\zeta)|)(1-|w(z,\zeta)|)^{-1} \text{ if } z,\ \zeta \in \Omega. \tag{4.1}$$

In fact, if $z(w)$ is the inverse of $w(z,\zeta)$ for a fixed ζ, then (4.1) is just Harnack's inequality applied to $u(z(w))$ which is harmonic in the unit circle.

LEMMA 4.1. *Let v_j be subharmonic functions and h_j be harmonic functions in Ω $(j=1,2,3)$. If*

$$v_3 = v_1 + v_2;\ v_j \leq h_j \quad (j=1,2,3), \tag{4.2}$$

we have for arbitrary $z_j \in \Omega$ $(j=1,2,3)$,

$$\sum_1^2 (v_j(z_j) - h_j(z_j))(1+|w(z_3,z_j)|)(1-|w(z_3,z_j)|)^{-1} \leq h_3(z_3) - \sum_1^2 h_j(z_3). \tag{4.3}$$

Proof. We may assume that Ω is a circle and at first we also assume that the functions v_j and h_j are continuous in $\bar{\Omega}$. Let u_j $(j=1,2)$ be the harmonic function in Ω for which $u_j = h_j - v_j$ on the boundary of Ω. Then (4.2) and the maximum principle give that the inequalities

$$u_j \geq 0 \ (j=1,2);\ h_1 - u_1 + h_2 - u_2 \leq h_3 \tag{4.4}$$

are valid in Ω. For $j=1,2$ we now apply (4.1) to u_j with $z=z_3$ and $\zeta = z_j$. Adding the inequalities obtained and noting that $u_1 + u_2 \geq h_1 + h_2 - h_3$, we obtain (4.3) since $v_j - h_j \leq -u_j$.

In the general case where v_j and h_j are not continuous in Ω we introduce the averages $v_j^\varepsilon(z)$ and $h_j^\varepsilon(z)$ $(\varepsilon > 0)$ which are defined for every z with distance $>\varepsilon$ from $\complement\Omega$. If $\Omega_{2\varepsilon}$ is the circle consisting of points at distance $>2\varepsilon$ from $\complement\Omega$, then $v_j^\varepsilon(z)$ and $h_j^\varepsilon(z)$ are continuous in the closure of $\Omega_{2\varepsilon}$ unless $v_j \equiv -\infty$ and then the lemma is trivial. Since (4.2) implies that $v_3^\varepsilon(z) = v_1^\varepsilon(z) + v_2^\varepsilon(z)$ and that $v_j^\varepsilon(z) \leq h_j^\varepsilon(z) = h_j(z)$, we can apply (4.3) with v_j, h_j and Ω replaced by v_j^ε, h_j and $\Omega_{2\varepsilon}$. Since $v_j(z) \leq v_j^\varepsilon(z)$ we obtain (4.3) when $\varepsilon \to 0$.

LEMMA 4.2. *Let v_j $(j=1,2,3)$ be subharmonic when* $\operatorname{Im} z > 0$, *let $v_3 = v_1 + v_2$ and assume that*

$$v_j(z) \leq C_j + A_j \operatorname{Im} z, \quad \operatorname{Im} z > 0 \quad (j=1,2,3), \tag{4.5}$$

where C_j and A_j are constants. Then we have if $\operatorname{Im} z_j > 0$ $(j=1,2)$

$$\sum_1^2 \frac{v_j(z_j)}{\operatorname{Im} z_j} \leq \sum_1^2 \frac{C_j}{\operatorname{Im} z_j} + A_3. \tag{4.6}$$

Proof. We shall apply Lemma 4.1 with Ω equal to the half plane $\operatorname{Im} z > 0$. Then we have

$$w(z_3, z_j) = \frac{z_3 - z_j}{z_3 - \bar{z}_j} = 1 - \frac{2i \operatorname{Im} z_j}{z_3 - \bar{z}_j}.$$

If we let $\operatorname{Im} z_3 \to +\infty$ while $\operatorname{Re} z_3$ remains constant, we obtain

$$\operatorname{Im} z_3 (1 - |w(z_3, z_j)|) \to 2 \operatorname{Im} z_j. \tag{4.7}$$

Applying (4.3) with $h_j(z) = C_j + A_j \operatorname{Im} z$ and letting $\operatorname{Im} z_3 \to +\infty$ after division by $\operatorname{Im} z_3$ therefore gives

$$\sum_1^2 (v_j(z_j) - C_j - A_j \operatorname{Im} z_j)/\operatorname{Im} z_j \leqslant A_3 - A_1 - A_2.$$

This proves (4.6).

Note that (4.6) implies in particular that

$$\sum_1^2 \overline{\lim_{\operatorname{Im} z \to +\infty}} \frac{v_j(z)}{\operatorname{Im} z} \leqslant A_3. \tag{4.8}$$

THEOREM 4.1. *Let v_j $(j = 1, 2, 3)$ be plurisubharmonic functions in C^n such that $v_3 = v_1 + v_2$ and*

$$v_j(z) \leqslant C_j + A_j |\operatorname{Im} z| \quad (z \in C^n) \tag{4.9}$$

for some constants C_j and A_j. If H_j is the supporting function of v_j we then have

$$H_3 = H_1 + H_2. \tag{4.10}$$

Proof. Let $M_j(y)$ be the supremum of $v_j(z)$ when $\operatorname{Im} z = y$. By definition we have

$$H_j(y) = \lim_{t \to +\infty} \frac{M_j(ty)}{t},$$

and since $M_3 \leqslant M_1 + M_2$ it is clear that $H_3 \leqslant H_1 + H_2$. Now let $\zeta \in C^n$ and $y \in R^n$ be fixed and consider the subharmonic functions $v_j(\zeta + wy)$ of $w \in C^1$. We choose ζ so that

$$\overline{\lim_{\operatorname{Im} w \to +\infty}} \frac{v_j(\zeta + wy)}{\operatorname{Im} w} = H_j(y) \quad (j = 1, 2, 3) \tag{4.11}$$

which is possible by Theorem 3.3. Now we have if $\operatorname{Im} w > 0$,

$$v_j(\zeta + wy) \leqslant M_j(\operatorname{Im} \zeta + \operatorname{Im} wy) \leqslant M_j(0) + H_j(\operatorname{Im} \zeta) + \operatorname{Im} w H_j(y)$$

so we can apply Lemma 4.2. From (4.8) and (4.11) it then follows that

$$H_1(y)+H_2(y) \leqslant H_3(y),$$

which completes the proof.

We can now prove the convolution theorem of Titchmarsh and Lions.

THEOREM 4.2. *Let f_1, f_2 be distributions with compact support. Then*

$$\text{ch supp } (f_1 * f_2) = \text{ch supp } f_1 + \text{ch supp } f_2. \tag{4.12}$$

Proof. If f_1 and f_2 are measures, we obtain (4.12) by combining Theorem 4.1 and Theorem 3.2. To study the general case we note that it is trivial that the set on the left-hand side of (4.12) is contained in that on the right-hand side. Let K be a convex compact neighborhood of 0 and choose $\varphi \in C_0^\infty$ with support in K. Then

$$(f_1 * \varphi) * (f_2 * \varphi) = (f_1 * f_2) * \varphi * \varphi$$

and the support of the right-hand side is contained in ch supp $(f_1 * f_2) + 2K$. Hence

$$\text{ch supp } (f_1 * \varphi) + \text{ch supp } (f_2 * \varphi) \subset \text{ch supp } (f_1 * f_2) + 2K.$$

Now choose a sequence of sets K_j converging to $\{0\}$ and corresponding functions φ converging to the Dirac measure at the origin. When $j \to \infty$ we then obtain

$$\text{ch supp } f_1 + \text{ ch supp } f_2 \subset \text{ch supp } (f_1 * f_2),$$

which completes the proof.

5. The singular support of a convolution

Let $f \in \mathcal{E}'$ and consider for *real* ξ the plurisubharmonic function of z defined by

$$v_f(z; \xi) = \frac{\log |\hat{f}(\xi + z \log |\xi|)|}{\log |\xi|}. \tag{5.1}$$

If N is the order of f we have for some constants C and N (see Theorem 2.1)

$$|\hat{f}(\zeta)| \leqslant C(1+|\zeta|)^N e^{A\,|\mathrm{Im}\,\zeta|}.$$

This gives the estimate

$$v_f(z; \zeta) \leqslant \frac{\log C + N \log (1 + |\xi + z \log |\xi||)}{\log |\xi|} + A\,|\mathrm{Im}\, z|.$$

Hence $v_f(z; \xi)$ is bounded from above for z in any compact set when $\xi \to \infty$, and we have

$$\overline{\lim}_{\xi \to \infty} v_f(z; \xi) \leqslant N + A\,|\mathrm{Im}\, z|. \tag{5.2}$$

In virtue of Lemmas 3.2 and 3.3 we can therefore from every sequence $\xi_j \to \infty$ in R^n extract a subsequence ξ_{j_k} such that $v_f(z; \xi_{j_k})$ when $k \to \infty$ converges to a plurisubharmonic function bounded by $N + A|\mathrm{Im}\, z|$. With the limit there is associated (see section 3) a supporting function $\in \mathcal{H}$, which may be $-\infty$.

DEFINITION 5.1. *If $f_1, \ldots, f_k \in \mathcal{E}'$ we denote by $\mathcal{H}(f_1, \ldots, f_k)$ the set of k-tuples $(h_1, \ldots, h_k)$ of elements in $\mathcal{H}$ such that there is a sequence $\xi_\nu \to \infty$ in R^n for which $v_{f_j}(z; \xi_\nu)$ for every j converges to a plurisubharmonic function with supporting function h_j. The set of corresponding k-tuples of convex compact sets is denoted by $\mathcal{K}(f_1, \ldots, f_k)$.*

Let us first note a few obvious facts concerning $\mathcal{H}(f_1, \ldots, f_k)$.

LEMMA 5.1. *Let $f_1, \ldots, f_k \in \mathcal{E}'$. If $(h_1, \ldots, h_j) \in \mathcal{H}(f_1, \ldots, f_j)$, where $j < k$, one can choose $h_{j+1}, \ldots, h_k$ so that $(h_1, \ldots, h_k) \in \mathcal{H}(f_1, \ldots, f_k)$.*

Proof. Let ξ_ν be a sequence $\to \infty$ such that $v_{f_i}(z; \xi_\nu)$ converges when $\nu \to \infty$ to a plurisubharmonic function with supporting function h_i for every $i \leqslant j$. Passing if necessary to a subsequence we may assume that the sequences $v_{f_i}(z; \xi_\nu)$ also converge when $i = j+1, \ldots, k$. If we define h_i for these indices as the supporting functions of the corresponding limits, the lemma follows.

The lemma just proved means that knowing $\mathcal{H}(f_1, \ldots, f_k)$ we obtain $\mathcal{H}(f_1, \ldots, f_j)$ when $j < k$ by just eliminating the last $k - j$ components. We next prove that the singular support of $f \in \mathcal{E}'$ is determined by $\mathcal{H}(f)$.

LEMMA 5.2. *Let $f \in \mathcal{E}'$ and let H be the supporting function of* ch sing supp f. *Then we have for every $\xi \in R^n$*

$$H(\xi) = \sup\, \{h(\xi),\ h \in \mathcal{H}(f)\}.$$

Proof. If ch sing supp f is empty, that is, if $f \in C_0^\infty$ we can for every integer k find a constant C_k so that

$$|\hat{f}(\zeta)| \leqslant C_k (1 + |\zeta|)^{-k}\, e^{A\,|\mathrm{Im}\, \zeta|},$$

where A is independent of k. Hence $v_f(z, \xi) \to -\infty$ uniformly on every compact subset of C^n when $\xi \to \infty$ (compare (5.2)) so that $\mathcal{H}(f)$ only consists of the function $-\infty$. The lemma is therefore true in this case. Now assume that ch sing supp f is not empty. By using (2.5) we immediately obtain as in the proof of (5.2) that

$\overline{\lim}_{\xi\to\infty} v_f(z;\xi) \leqslant N + H(\operatorname{Im} z)$. Hence $h \leqslant H$ for every $h \in \mathcal{H}(f)$. On the other hand, let $-\infty \neq H' \geqslant h$, $h \in \mathcal{H}(f)$. If $H' \in \mathcal{H}$ we then claim that (2.5) is valid with H replaced by H', N replaced by $N+1$ and a suitable constant C_m. In fact, otherwise we can for some fixed m find a sequence $\zeta_\nu \to \infty$ such that $|\operatorname{Im} \zeta_\nu| \leqslant m \log (|\zeta_\nu|+1)$ and

$$|\hat{f}(\zeta_\nu)| \geqslant (1+|\zeta_\nu|)^{N+1} e^{H'(\operatorname{Im}\zeta_\nu)}. \tag{5.3}$$

By passing to a subsequence we may assume that $v_f(z; \operatorname{Re} \zeta_\nu)$ converges to a plurisubharmonic limit V. Since $V \leqslant N$ in R^n and the supporting function of V is $\leqslant H'$ we obtain (see (3.10)) that $V(z) \leqslant N + H'(\operatorname{Im} z)$ for all z. For large values of ν we have $|\operatorname{Im} \zeta_\nu| \leqslant (m+1) \log |\operatorname{Re} \zeta_\nu|$ and it follows from Lemma 3.1 that

$$v_f(z; \operatorname{Re} \zeta_\nu) < N+1+H'(\operatorname{Im} z) \quad \text{when} \quad |z| \leqslant m+1$$

if ν is large enough. But this implies if we take $z = i \operatorname{Im} \zeta_\nu / \log |\operatorname{Re} \zeta_\nu|$ that

$$|\hat{f}(\zeta_\nu)| < |\zeta_\nu|^{N+1} e^{H'(\operatorname{Im}\zeta_\nu)},$$

which contradicts (5.3). Hence (2.5) must in fact be valid with H replaced by H' which proves that $H \leqslant H'$. The proof is complete.

When studying the conditions in order that $-\infty \in \mathcal{H}(f)$ we need the following simple lemma.

Lemma 5.3 *Let $f \in \mathcal{E}'$ and let ξ_ν be a sequence $\to \infty$ in R^n such that $v_f(z; \xi_\nu) \to -\infty$ on an open subset of R^n. Then $v_f(z; \xi_\nu) \to -\infty$ uniformly on every compact subset of C^n.*

Proof. Let $x_0 \in R^n$ and $r > 0$ be such that $v_f(z; \xi_\nu) \to -\infty$ in the sphere with radius r and center at x_0 in R^n. If $y \in R^n$ and $|y| < r$ the subharmonic functions $v_f(x_0 + wy; \xi_\nu)$ of w in the half circle $|w| < 1$, $\operatorname{Im} w > 0$, are uniformly bounded from above for such w and $\to -\infty$ on a piece of the boundary. Hence the harmonic function in the half circle which has the same boundary values $\to -\infty$ when $\nu \to \infty$ which proves that $v_f(x_0 + wy; \xi_\nu) \to -\infty$ if w is inside the half circle. Varying y we find that $v_f(z; \xi_\nu) \to -\infty$ for every z in an open set in C^n, so Lemma 3.1 shows that $v_f(z; \xi_\nu) \to -\infty$ uniformly on every compact subset of C^n.

Lemma 5.4. *Let $f \in \mathcal{E}'$. If $\hat{f}$ is slowly decreasing in the sense that* (1.4) *is valid for some A, then $-\infty \notin \mathcal{H}(f)$. Conversely, if $-\infty \notin \mathcal{H}(f)$ one can for every $a > 0$ find A so that*

$$\sup \{|\hat{f}(\xi+\eta)|;\ |\eta| < a \log (2+|\xi|),\ \eta \in R^n\} > (A+|\xi|)^{-A} \quad (\xi \in R^n). \tag{5.4}$$

This follows immediately from Lemma 5.3.

The sets of supporting functions introduced in Definition 5.1 give a much more precise description than the singular support alone of the nature of the singularities of the distributions involved. This makes it easy to prove the following analogue of Theorem 4.2.

THEOREM 5.1. *Let* $f_1', f_1'', \ldots, f_k', f_k'' \in \mathcal{E}'$. *Then*

$$\mathcal{H}(f_1' * f_1'', \ldots, f_k' * f_k'') = \{(h_1' + h_1'', \ldots, h_k' + h_k'');\ (h_1', h_1'', \ldots, h_k', h_k'') \in \mathcal{H}(f_1', f_1'', \ldots, f_k', f_k'')\}.$$

Proof. Let ξ_ν be a sequence $\to \infty$ in R^n such that $v_{f_j}(z; \xi_\nu)$ converges for $j = 1, \ldots, k$. Here we have written $f_j = f_j' * f_j''$. Denote the limits by V_j. By passing to a subsequence we may assume that $v_{f_j'}(z; \xi_\nu)$ and $v_{f_j''}(z; \xi_\nu)$ also converge, and the limits are denoted by V_j' and V_j''. Then we have

$$V_j = V_j' + V_j''$$

so that Theorem 4.2 shows that the supporting function of V_j is for every j the sum of that of V_j' and that of V_j''. This proves that the left set in the theorem is included in the set to the right, and the opposite inclusion is equally trivial.

COROLLARY 5.1. *Let* $f_1, \ldots, f_k \in \mathcal{E}'$ *and let* $K_1, \ldots, K_k$ *be convex compact sets with supporting functions* $H_1, \ldots, H_k$, *such that*

$$h \in \mathcal{H},\ \ (h_1, \ldots, h_k) \in \mathcal{H}(f_1, \ldots, f_k),\ \ h + h_j \leqslant H_j\ \ (j = 2, \ldots, k) \Rightarrow h + h_1 \leqslant H_1. \tag{5.5}$$

Then we have

$$f \in \mathcal{E}',\ \text{sing supp}\ f * f_j \subset K_j\ \ (j = 2, \ldots, k) \Rightarrow \text{sing supp}\ f * f_1 \subset K_1. \tag{5.6}$$

Proof. This is a combination of Theorem 5.1 with Lemmas 5.1 and 5.2.

Example. When $f_1 = \delta$ only the function 0 belongs to $\mathcal{H}(f_1)$ and (5.5) means simply that if $h \in \mathcal{H}$ and $(h_2, \ldots, h_k) \in \mathcal{H}(f_2, \ldots, f_k)$, $h + h_j \leqslant H_j$ $(j = 2, \ldots, k)$, then $h \leqslant H_1$. In the conclusion we have sing supp $f \subset K_1$.

By further specialization of Corollary 5.1 we obtain with the same notations

COROLLARY 5.2. *If there is no* $(h_1, \ldots, h_k) \in \mathcal{H}(f_1, \ldots, f_k)$ *such that* $h_1 \neq -\infty$ *but* $h_2 = \ldots = h_k = -\infty$, *then*

$$f \in \mathcal{E}',\ \text{sing supp}\ f * f_j \subset K_j\ \ (j = 2, \ldots, k) \Rightarrow \text{sing supp}\ f * f_1 \subset K_1 \tag{5.7}$$

if the sets K_j are convex, compact and K_1 contains the sets

$$\text{ch sing supp } f_1 + K_j - \text{ch sing supp } f_j \ (j=2, \ldots, k).$$

Proof. If H_j denotes the supporting function of K_j and $h_j \in \mathcal{H}(f_j)$ $(j=1, \ldots, k)$, then by Lemma 5.2

$$H_1(\xi) \geq h_1(\xi) + H_j(\xi) + h_j(-\xi) \quad (j=2, \ldots, k).$$

If $h + h_j \leq H_j$ $(j=2, \ldots, k)$ we obtain

$$H_1(\xi) \geq h_1(\xi) + h(\xi) + h_j(\xi) + h_j(-\xi) \quad (j=2, \ldots, k).$$

Now if $h_j \neq -\infty$ for some j we have $h_j(\xi) + h_j(-\xi) \geq h_j(0) = 0$, hence $H_1 \geq h + h_1$. On the other hand, if $h_j = -\infty$ for $j=2, \ldots, k$, then $h_1 = -\infty$ if $(h_1, \ldots, h_k) \in \mathcal{H}(f_1 \ldots, f_k)$ so that $H_1 \geq h + h_1$ also in that case. The corollary now follows from Corollary 5.1.

Example. Take $f_1 = \delta$ and $k=2$. Then the hypothesis in the lemma means precisely that $\hat{f}_2$ is slowly decreasing in the sense of Ehrenpreis (see Lemma 5.4), so we have proved (1.3) in that case. This extends the results of [6].

Our next purpose is to construct examples which show that the results obtained are the best possible. In the constructions we first consider distributions f with sing supp $f = \{0\}$. This has the advantage that, as shown by Lemma 5.2, $\mathcal{H}(f)$ can only contain the two elements 0 and $-\infty$. In the next theorem we construct f so that $v_f(z; \xi)$ converges to $-\infty$ when $\xi \to \infty$ avoiding a very thin set. The construction depends on an idea of Ehrenpreis [4].

THEOREM 5.2. *Let ξ_j be a sequence $\to \infty$ in R^n and let E be a subset of R^n such that $d(\xi_j, E)/\log |\xi_j| \to \infty$ when $j \to \infty$. Here $d(\xi, E)$ denotes the distance from ξ to E. Then one can find $f \in \mathcal{E}'$ with* sing supp $f = \{0\}$ *so that*

$$v_f(z; \xi) \to -\infty \text{ when } E \ni \xi \to \infty,$$

the convergence being uniform on compact subsets of C^n, whereas $v_f(z; \xi_j)$ does not converge to $-\infty$.

Proof. We shall construct a continuous function f with compact support such that sing supp $f = \{0\}$ and

$$|\xi_j| \hat{f}(\xi_j) \text{ does not converge to 0 when } j \to \infty, \tag{5.8}$$

$$p_{N,m}(f) = \sup \{ |\hat{f}(\zeta)| \, |\xi|^N; \ \xi \in E, \ \zeta \in C^n, \ |\zeta - \xi| \leq m \log |\xi| \} < \infty \tag{5.9}$$

for all positive integers N and m. From (5.8) it follows that $v_f(0; \xi_j)$ does not converge to $-\infty$ when $j\to\infty$, and from (5.9) it follows that $v_f(z;\xi)+N=O(1/\log|\xi|)$ when $|z|\leqslant m$ and $\xi\to\infty$ in E. This will prove the theorem.

Suppose now that there is no continuous function f with sing supp $f=\{0\}$ and compact support such that (5.8) and (5.9) are valid. Put $\omega_\varrho=\{x;\,|x|<\varrho\}$, where $\varrho>0$, and consider the space $\mathfrak{F}$ of all continuous functions f with support contained in $\bar\omega_1$ such that $f\in C^\infty(\complement\{0\})$ and the semi-norms $p_{N,m}(f)$ defined by (5.9) are finite. In $\mathfrak{F}$ we introduce the topology defined by the semi-norms $p_{N,m}(f)$ together with $\sup|f|$ and $\sup_K|D^\alpha f|$ where K varies over all compact sets not containing 0 and α varies over all multi-indices. Then it is clear that $\mathfrak{F}$ is a Fréchet space. If (5.8) and (5.9) cannot be fulfilled, then the sequence $(|\xi_1|\hat f(\xi_1),\,|\xi_2|\hat f(\xi_2),\dots)$ belongs to l^∞ for every $f\in\mathfrak{F}$. We thus have a closed everywhere defined mapping of $\mathfrak{F}$ into l^∞, and the closed graph theorem shows that it must be continuous. For a suitable constant C, integers N', N and m and a compact set K not containing 0 we therefore have

$$\sup|\xi_\nu|\,|\hat f(\xi_\nu)|\leqslant C\{\sup|f|+\sum_{|\alpha|\leqslant N'}\sup_K|D^\alpha f|+p_{N,m}(f)\}\quad(f\in\mathfrak{F}).\tag{5.10}$$

In particular, this estimate holds when $f\in C_0^\infty(\omega_\delta)$, $0<\delta<1$, and we can choose δ so that $\omega_\delta\cap K=\emptyset$. Then we obtain

$$\sup|\xi_\nu|\,|\hat f(\xi_\nu)|\leqslant C\{\sup|f|+p_{N,m}(f)\}\quad(f\in C_0^\infty(\omega_\delta)).\tag{5.10$'$}$$

Now choose $\psi\in C_0^\infty(\omega_\delta)$ such that $\psi\geqslant 0$ and $\int\psi\,dx=1$, and set

$$\hat f_j(\zeta)=\left(\hat\psi\left(\frac{\zeta-\xi_j}{k_j}\right)\right)^{k_j},$$

where k_j is the largest integer $<\log|\xi_j|$, which is positive for large j. Then $f_j\in C_0^\infty(\omega_\delta)$ for it is the convolution of k_j functions with support in ω_{δ/k_j} and we have $|\hat f_j(\xi_j)|=1$ so that the left hand side of (5.10)$'$ with $f=f_j$ tends to infinity as fast as $|\xi_j|$ when $j\to\infty$. We have

$$\sup|f_j|\leqslant\int|\hat f_j|\,ds\leqslant\int\left|\hat\psi\left(\frac{\xi-\xi_j}{k_j}\right)\right|d\xi=Ck_j^n=o(|\xi_j|)\quad(j\to\infty).$$

If we can prove that $p_{N,m}(f_j)$ is bounded when $j\to\infty$, we will have a contradiction with (5.10)$'$ and the theorem will be proved.

Let $\xi\in E$ and $\zeta\in C^n$ satisfy the condition $|\zeta-\xi|\leqslant m\log|\xi|$ as in the definition of $p_{N,m}$, and put $z=(\zeta-\xi_j)/k_j$. Then we have

$$|\xi|\leqslant|\zeta-\xi_j|+|\zeta-\xi|+|\xi_j|<k_j|z|+m\log|\xi|+e^{k_j+1}.$$

For a sufficiently large constant C we have $m \log |\xi| < \frac{1}{2}|\xi| + C$, so we obtain with another constant $C > 1$

$$|\xi| \leqslant C e^{k_j}(1 + k_j|z|) < C(e(1 + |z|))^{k_j}. \tag{5.11}$$

Since $|\operatorname{Im} z| = |\operatorname{Im} \zeta|/k_j < m (\log |\xi|)/k_j$, we conclude that

$$e^{|\operatorname{Im} z|} < C^m (e(1 + |z|))^m. \tag{5.12}$$

Next we claim that $|z|$ has a lower bound which $\to \infty$ with j. To prove this we consider two different cases:

1. If $|\xi - \xi_j| \leqslant 2|\xi_j|$ we have $k_j|z| = |\zeta - \xi_j| \geqslant |\xi - \xi_j| - |\zeta - \xi| \geqslant d(\xi_j, E) - m \log |\xi| \geqslant d(\xi_j, E) - m \log (3|\xi_j|)$. Since $d(\xi_j, E)/k_j \to \infty$ when $j \to \infty$, we find that $|z|$ has a lower bound which $\to \infty$ with j.
2. If $|\xi - \xi_j| > 2|\xi_j|$ we have $|\xi| \leqslant |\xi - \xi_j| + |\xi_j| < \frac{3}{2}|\xi - \xi_j|$, so we obtain

$$k_j|z| > \tfrac{2}{3}|\xi| - m \log |\xi| > \tfrac{1}{2}|\xi| > \tfrac{1}{2}|\xi_j|$$

if j is sufficiently large. Hence $|z|$ again has a lower bound which $\to \infty$ with j.

With the same assumptions on ξ_j, ζ, ξ, and z as above we have by (5.11),

$$|\hat{f}_j(\zeta)|\,|\xi|^N \leqslant C^N |\hat{\psi}(z) e^N (1 + |z|)^N|^{k_j}. \tag{5.13}$$

Since z satisfies (5.12) we have $|\hat{\psi}(z) e^N (1 + |z|)^N| < 1$ if $|z|$ is sufficiently large, which proves that $p_{N,m}(f_j) \to 0$ when $j \to \infty$. The proof is complete.

To use this theorem we need a lemma.

LEMMA 5.5. *Let $f \in \mathcal{E}'$, let $\xi_j \to \infty$ in R^n and assume that $v_f(z; \xi_j)$ converges to a plurisubharmonic function V with supporting function h. Then there exists a set E such that $d(\xi_j, E)/\log |\xi_j| \to \infty$ and the supporting function of $\overline{\lim}_{\complement E \ni \eta \to \infty} v_f(z, \eta)$ is equal to h.*

Proof. We assume in the proof that $h \not\equiv -\infty$; the case $h = -\infty$ is handled in the same way. Let N be the order of f. Since $V(z) \leqslant N + h(\operatorname{Im} z)$ we can for every integer k find j_k so that

$$v_f(z; \xi_j) < N + 1 + h(\operatorname{Im} z) \quad \text{if} \quad |z| < 2k \text{ and } j \geqslant j_k.$$

If we introduce the inverse of the function $k \to j_k$ this means that we can find $R_j \to \infty$ when $j \to \infty$ so that

$$v_f(z; \xi_j) < N + 1 + h(\operatorname{Im} z) \quad \text{if} \quad |z| < 2R_j.$$

We may of course choose R_j so that $R_j \log |\xi_j| = o(|\xi_j|)$ when $j \to \infty$.

Now let E be the set of all ξ such that

$$|\xi - \xi_j| \geqslant R_j \log |\xi_j| \text{ for every } j. \tag{5.14}$$

Then $d(\xi_j, E)/\log |\xi_j| \geqslant R_j \to \infty$ when $j \to \infty$. If $|\xi - \xi_j| < R_j \log |\xi_j|$ we note that since

$$|\hat{f}(\zeta)| < |\xi_j|^{N+1} e^{h(\operatorname{Im}\zeta)} \quad \text{if} \quad |\zeta - \xi_j| < 2R_j \log |\xi_j|,$$

it follows that $$|\hat{f}(\zeta)| < |\xi_j|^{N+1} e^{h(\operatorname{Im}\zeta)} \quad \text{if} \quad |\zeta - \xi| < R_j \log |\xi_j|.$$

Since $|\xi - \xi_j| < R_j \log |\xi_j| = o(|\xi_j|)$, the quotient $|\xi_j|/|\xi|$ approaches 1 if j tends to infinity. Hence

$$\overline{\lim_{\complement E \ni \xi \to \infty}} v_f(z; \xi) \leqslant N + 1 + h(\operatorname{Im} z),$$

which proves the lemma.

THEOREM 5.3. *Let* $f_1, \ldots, f_k \in \mathcal{E}'$ *and let* $(h_1, \ldots, h_k) \in \mathcal{H}(f_1, \ldots, f_k)$. *Then one can find* $f \in \mathcal{E}'$ *with* sing supp $f = \{0\}$ *so that the supporting function of* ch sing supp $f * f_j$ *is equal to* h_j *for* $j = 1, \ldots, k$, *and moreover* $(0, h_1, \ldots, h_k) \in \mathcal{H}(f, f_1, \ldots, f_k)$.

Proof. Let $\xi_\nu \to \infty$ be such that $v_{f_j}(z; \xi_\nu)$ for $j = 1, \ldots, k$ converges to a plurisubharmonic function with supporting function h_j. For every j we choose a set E_j according to Lemma 5.5 and set $E = \bigcup_1^k E_j$. Then we choose f according to Theorem 5.2. For a suitable subsequence of the sequence ξ_ν we then have that $v_f(z; \xi_\nu)$ converges to a plurisubharmonic function with the supporting function 0, which proves that $(0, h_1, \ldots, h_k) \in \mathcal{H}(f, f_1, \ldots, f_k)$. On the other hand, let η_ν be an arbitrary sequence $\to \infty$ in R^n. Passing if necessary to a subsequence we may assume either that $\eta_\nu \in E$ for every ν or that $\eta_\nu \in \complement E$ for every ν. In the former case we have $v_f(z; \eta_\nu) \to -\infty$, in the latter case the supporting function of $\lim v_{f_j}(z; \eta_\nu)$ if the limit exists is $\leqslant h_j$. Hence it follows from Theorem 5.1 that $(H_1, \ldots, H_k) \in \mathcal{H}(f * f_1, \ldots, f * f_k)$ implies that $H_j \leqslant h_j$ for every j. Since we have just seen that equality takes place when the sequence η_ν is a suitable subsequence of the sequence ξ_ν, the theorem follows from Lemma 5.2 and Lemma 5.3.

COROLLARY 5.3. *Suppose* K_j *are convex compact sets such that* (5.6) *holds. Then* (5.5) *must be fulfilled.*

Proof. From Theorem 5.3 we know that for every x and $(h_1, \ldots, h_k) \in \mathcal{H}(f_1, \ldots, f_k)$ one can choose f with sing supp $f = \{x\}$ so that ch sing supp $f * f_j$ has the supporting function $\langle x, \xi \rangle + h_j(\xi)$ for $j = 1, \ldots, k$. If (5.6) holds it follows that

$$\langle x, \xi \rangle + h_j(\xi) \leqslant H_j(\xi) \quad (j=2, \ldots, k) \quad \Rightarrow \langle x, \xi \rangle + h_1(\xi) \leqslant H_1(\xi).$$

If $h \in \mathcal{H}$ and $h + h_j \leqslant H_j$ $(j=2, \ldots, k)$ we therefore obtain $\langle x, \xi \rangle + h_1(\xi) \leqslant H_1(\xi)$ if $\langle x, \xi \rangle \leqslant h(\xi)$, and since $h(\xi)$ is the least upper bound of smaller linear functions we obtain finally that $h + h_1 \leqslant H_1$, which proves (5.5).

THEOREM 5.4. *Let $f_1, \ldots, f_k$ be elements in $\mathcal{E}'$ with disjoint singular supports and let $f = f_1 + \ldots + f_k$. For every $h \in \mathcal{H}(f)$ one can choose $(h_1, \ldots, h_k) \in \mathcal{H}(f_1, \ldots, f_k)$ such that*

$$\sup_j h_j \leqslant h \tag{5.15}$$

and for arbitrary $(h_1, \ldots, h_k) \in \mathcal{H}(f_1, \ldots, f_k)$ one can choose $h \in \mathcal{H}(f)$ so that

$$h \leqslant \sup_j h_j. \tag{5.16}$$

Proof. Let $h \in \mathcal{H}(f)$ and choose $f_0 \in \mathcal{E}'$ with sing supp $f_0 = \{0\}$ so that ch sing supp $f_0 * f$ has the supporting function h (Theorem 5.3). We have

$$f_0 * f = \sum_1^k f_0 * f_j \tag{5.17}$$

and since $f_0 * f_j$ have disjoint singular supports we obtain

$$\text{ch sing supp } (f_0 * f) = \text{ch} \bigcup_1^k \text{sing supp } (f_0 * f_j). \tag{5.18}$$

Choose h_j so that $(0, h_1, \ldots, h_k) \in \mathcal{H}(f_0, f_1, \ldots, f_k)$, which implies that $(0, h_j) \in \mathcal{H}(f_0, f_j)$ for $j = 1, \ldots, k$. Then Theorem 5.1 shows that the supporting function of ch sing supp $f_0 * f_j$ is larger than or equal to h_j, so that (5.18) gives $h_j \leqslant h$ for every j.

On the other hand, given $(h_1, \ldots, h_k) \in \mathcal{H}(f_1, \ldots, f_k)$ we can choose $f_0 \in \mathcal{E}'$ with sing supp $f_0 = \{0\}$ so that h_j is the supporting function of ch sing supp $f_0 * f_j$ for $j = 1, \ldots, k$ (Theorem 5.3). Then (5.17) gives that the supporting function of ch sing supp $f_0 * f$ is $\leqslant \sup_j h_j$, and if we choose h so that $(0, h) \in \mathcal{H}(f_0, f)$, we obtain (5.16).

COROLLARY 5.4. *If under the assumptions of Theorem 5.4 at least one of the functions $\hat{f}_j$ is slowly decreasing, then $\hat{f}$ is slowly decreasing. If $\mathcal{H}(f_j)$ only contains the supporting function of* ch sing supp f_j *for $j = 1, 2, \ldots, k$, then $\mathcal{H}(f)$ only contains the supporting function of* ch sing supp f.

Example. If f is a distribution with supp $f = \{0\}$ then $\hat{f}$ is a polynomial and it is trivial that $\hat{f}$ is slowly decreasing so that $\mathcal{H}(f)$ only contains the function 0. By

Corollary 5.4 we therefore conclude that if supp f consists of a finite number of points then $\hat{f}$ is slowly decreasing and $\mathcal{H}(f)$ consists of the supporting function of supp f only, hence Theorem 5.1 gives that (1.2) is valid. This was also proved in [5].

6. Existence theorems for convolution equations

Combination of the results obtained in section 5 with those of Hörmander [5] immediately gives existence theorems for the convolution equation

$$S * u = f \tag{6.1}$$

when $S \in \mathcal{E}'$. Let Ω_1 and Ω_2 be two open sets in R^n such that

$$\Omega_1 - \operatorname{supp} S \subset \Omega_2, \tag{6.2}$$

which implies that $S * u \in \mathcal{D}'(\Omega_1)$ for every $u \in \mathcal{D}'(\Omega_2)$.

THEOREM 6.1. *Let Ω_1, Ω_2 be convex. Then* (6.1) *has a solution $u \in \mathcal{D}'(\Omega_2)$ for every $f \in \mathcal{D}'(\Omega_1)$ if and only if $\hat{S}$ is slowly decreasing and every $x \in R^n$ such that $\{x\} - k \subset \Omega_2$ for some $k \in \mathcal{K}(S)$ is in fact in Ω_1.*

Proof. Choose a fixed $k \in \mathcal{K}(S)$ and set $K = \{x; \{x\} - k \subset K_2\}$, where K_2 is a convex compact subset of Ω_2 so large that $\Omega_1 \cap K \neq \emptyset$ (cf. (6.2)). Then K is convex. In view of Theorem 5.3 we can for every $x \in K \cap \Omega_1$ choose $\varphi \in \mathcal{E}'$ with sing supp $\varphi = \{x\}$ so that sing supp $\varphi * \check{S} = \{x\} - k \subset K_2$, and after multiplying φ by a function in $C_0^\infty(\Omega_1)$ one may assume that $\varphi \in \mathcal{E}'(\Omega_1)$. If the equation (6.1) has a solution $u \in \mathcal{D}'(\Omega_2)$ for every $f \in \mathcal{D}'(\Omega_1)$ it now follows from Theorems 4.1 and 4.2 in Hörmander [5] that $K \cap \Omega_1$ is relatively compact in Ω_1, and since K is convex this shows that K is a compact subset of Ω_1. In particular this implies that $\varphi \notin \mathcal{K}(S)$, so that $\hat{S}$ is slowly decreasing according to Lemma 5.4. Thus the necessity of the conditions in the theorem is proved. To prove their sufficiency, let K_2 be a compact subset of Ω_2 and let $\varphi \in \mathcal{E}'(\Omega_1)$, sing supp $\varphi * \check{S} \subset K_2$. If the distance from K_2 to $\complement\Omega_2$ is δ, then the compact set

$$M = \{x; \{x\} - k \subset K_2\} \tag{6.3}$$

also has distance at least δ to $\complement\Omega_1$ since $M + \{x; |x| \leqslant \varepsilon\} \subset \Omega$, if $\varepsilon < \delta$. Furthermore, M is contained in ch sing supp $S + K_2$ since $\varnothing \neq k \subset$ ch sing supp S. Hence the closed convex hull of all sets (6.3) with $k \in \mathcal{K}(f)$ is a compact subset K_1 of Ω_1 with distance $\geqslant \delta$ to $\complement\Omega_1$, and we have

$$\langle x, \xi \rangle \leqslant H_1(\xi) \quad \text{if} \quad \langle x, \xi \rangle + \check{h}(\xi) \leqslant H_2(\xi) \quad \text{for some} \quad h \in \mathcal{H}(S),$$

where H_1 and H_2 are the supporting functions of K_1 and of K_2. Since functions in $\mathcal{H}$ are upper bounds of families of linear functions, this means that

$$h_1 \in \mathcal{H},\ h \in \mathcal{H}(\check{S}),\ h_1 + h \leqslant H_2 \text{ implies } h_1 \leqslant H_1.$$

From Corollary 5.1 with $f_1 = \delta$ and $f_2 = \check{S}$ it follows therefore that ch sing supp $\check{S} * \varphi \subset K_2$ implies ch sing supp $\varphi \subset K_1$, when $\varphi \in \mathcal{E}'$, which is one of the requirements in the definition of a strongly S-convex pair given in Hörmander [5]. Since $k \subset$ ch supp S we also have

$$\{x\} - \text{ch supp } S \subset K_2 \Rightarrow x \in K_1$$

so the theorem of supports (Theorem 4.2) gives that supp $\varphi \subset K_1$ if $\varphi \in \mathcal{E}'$ and supp $\varphi * \check{S} \subset K_2$. Hence the pair (Ω_1, Ω_2) is strongly S-convex and the theorem follows from Theorem 4.5 in Hörmander [5].

COROLLARY 6.1. *Let $0 \neq S \in \mathcal{E}'$, and assume that $\mathcal{H}(S)$ consists of the supporting function of* ch supp S *alone. If Ω_2 is convex and Ω_1 is the largest open set such that* (6.2) *holds, then the equation* (6.1) *has a solution $u \in \mathcal{D}'(\Omega_2)$ for every $f \in \mathcal{D}'(\Omega_1)$.*

An example where Corollary 6.1 can be applied is that where the support of S consists of a finite number of points (see the example at the end of section 5). This case was also discussed in [5].

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