THE COEFFICIENTS OF QUASICONFORMALITY OF DOMAINS IN SPACE

BY

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Dedicated to Professor C. Loewner on his seventieth birthday

1. Introduction

1.1. *Main problem.* Quasiconformal mappings in Euclidean n-space, $n > 2$, have been studied rather intensively in recent years by several authors. See, for example, Gehring [4], [5], [6]; Krivov [8]; Loewner [9]; Šabat [14]; Väisälä [17], [18]; and Zorič [20], [21]. It turns out that these mappings have many properties similar to those of plane quasiconformal mappings. On the other hand, there are also striking differences. Probably the most important of these is that there exists no analogue of the Riemann mapping theorem when $n > 2$. This fact gives rise to the following two problems. Given a domain D in Euclidean n-space, does there exist a quasiconformal homeomorphism f of D onto the n-dimensional unit ball B^n? Next, if such a homeomorphism f exists, how small can the dilatation of f be?

Complete answers to these questions are known when $n = 2$. For a plane domain D can be mapped quasiconformally onto the unit disk B^2 if and only if D is simply connected and has at least two boundary points. The Riemann mapping theorem then shows that if D satisfies these conditions, there exists a conformal homeomorphism f of D onto B^2.

The situation is very much more complicated in higher dimensions, and this paper is devoted to the study of these two questions in the case where $n = 3$.

1.2. *Notation.* We let R^3 denote Euclidean 3-space with a fixed orthonormal basis (e_1, e_2, e_3), and we let $\bar{R}^3$ denote the Möbius space obtained by adding the point ∞ to R^3.

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Unless otherwise stated, all point sets considered in this paper are assumed to lie in $\bar{R}^3$. Finite points will usually be designated by capital letters P and Q, or by small letters x and y. In the latter case, x_1, x_2, x_3 will denote the coordinates for x, relative to the basis (e_1, e_2, e_3), and similarly for y. Points are treated as vectors and $|P|$ and $|x|$ will denote the norms of P and x, respectively.

Given a finite point P and $t > 0$ we let $B^3(P,t)$ denote the 3-dimensional ball $|x-P| < t$ and $S^2(P,t)$ its 2-dimensional boundary sphere $|x-P| = t$. We will also employ the abbreviations

$$B^3(t) = B^3(0,t), \quad B^3 = B^3(1), \quad S^2(t) = S^2(0,t), \quad S^2 = S^2(1),$$

where 0 denotes the origin. Next for each set $E \subset \bar{R}^3$ we let $\bar{E}$, ∂E, and $C(E)$ denote the closure, boundary, and complement of E, all taken with respect to $\bar{R}^3$. When $E \subset R^3$, we let $\Lambda(E)$, $\Lambda^2(E)$, and $m(E)$ denote respectively the linear or 1-dimensional Hausdorff measure, the 2-dimensional Hausdorff measure, and the 3-dimensional Lebesgue measure of E. (See [10] and [15].)

By a homeomorphism f of a domain $D \subset R^3$ we mean a homeomorphism of D onto a domain $D' \subset R^3$. For each quantity Δ associated with D, such as a subset of D or a family of arcs contained in D, we let Δ' denote its image under f.

1.3. *Modulus of a ring.* We say that a domain $R \subset R^3$ is a *ring* if $C(R)$ has exactly two components, say C_0 and C_1. Then following Loewner we define the *conformal capacity* of R as

$$\operatorname{cap} R = \inf_u \iiint_R |\nabla u|^3 d\omega, \tag{1.1}$$

where the infimum is taken over all functions u which are continuously differentiable in R with boundary values 0 on C_0 and 1 on C_1, and where $|\nabla u|$ is the norm of the gradient vector $(\partial u/\partial x_1, \partial u/\partial x_2, \partial u/\partial x_3)$. It is easy to see that $0 \leq \operatorname{cap} R < \infty$. We then define the *modulus* of R by means of the relation

$$\operatorname{mod} R = \left(\frac{4\pi}{\operatorname{cap} R}\right)^{\frac{1}{2}}. \tag{1.2}$$

This modulus behaves in many ways like the familiar modulus of a plane ring, usually defined by means of conformal mapping. For example, if R is the domain bounded by concentric spheres of radii a and b, $a < b$, then

$$\operatorname{mod} R = \log \frac{b}{a}.$$

1.4. *Modulus of a family of arcs.* The conformal capacity of a ring can also be expressed in terms of extremal lengths, or more precisely, as the modulus of a certain family of arcs.

A set $\gamma \subset R^3$ is said to be an *arc* if it is homeomorphic to a linear interval which may be open, half open, or closed. If E_0 and E_1 are two sets which meet the closure of a set D, then an arc (or more generally a connected set) γ is said to *join* E_0 *and* E_1 *in* D if $\gamma \subset D$ and if $\bar{\gamma} \cap E_0 \neq \emptyset$, $\bar{\gamma} \cap E_1 \neq \emptyset$.

Suppose next that γ is an arc and that ϱ is a non-negative Borel measurable function defined on some set containing γ. We define the line integral of ϱ along γ by means of linear measure,

$$\int_\gamma \varrho ds = \int_\gamma \varrho d\Lambda. \tag{1.3}$$

(See p. 19 and p. 53 in [15].) When γ is rectifiable, it is not difficult to see that the integral in (1.3) is equal to the usual line integral taken over γ with respect to arclength. (When ϱ is constant, this follows from [1]. The general case is then obtained by a simple limiting argument.)

Suppose that Γ is a family of arcs in R^3. We denote by $F(\Gamma)$ the family of functions ϱ which are non-negative and Borel measurable in R^3 and for which

$$\int_\gamma \varrho ds \geqslant 1$$

for each arc $\gamma \in \Gamma$. We then define the *modulus* of the arc family Γ as

$$M(\Gamma) = \inf_\varrho \iiint_{R^3} \varrho^3 d\omega, \tag{1.4}$$

where the infimum is taken over all functions $\varrho \in F(\Gamma)$. It is clear that $0 \leqslant M(\Gamma) \leqslant \infty$ and that $M(\Gamma) = 0$ whenever the family Γ is empty.

It is important to observe that the nonrectifiable arcs have no influence on the modulus of a given arc family. That is, if Γ_1 denotes the subfamily of rectifiable arcs in a family Γ, then

$$M(\Gamma) = M(\Gamma_1). \tag{1.5}$$

See [17] for this and other properties of the modulus $M(\Gamma)$.

Now suppose that R is a ring and that Γ is the family of arcs which join the components of ∂R in R. Then it follows from (1.5) and Theorem 1 of [3] that

$$\operatorname{cap} R = M(\Gamma). \tag{1.6}$$

1.5. *Dilatations of a homeomorphism.* Suppose that f is a homeomorphism of a domain $D \subset R^3$. If R is a bounded ring with $\bar{R} \subset D$, then R' is a bounded ring with $\bar{R}' \subset D'$. We define the *inner dilatation* $K_I(f)$ and the *outer dilatation* $K_0(f)$ of the homeomorphism f as

$$K_I(f) = \sup_R \frac{\operatorname{mod} R}{\operatorname{mod} R'}, \qquad K_0(f) = \sup_R \frac{\operatorname{mod} R'}{\operatorname{mod} R}, \tag{1.7}$$

where the suprema are taken over all bounded rings R with $\bar{R} \subset D$ for which $\operatorname{mod} R$ and $\operatorname{mod} R'$ are not both infinite. We call

$$K(f) = \max(K_I(f), K_0(f)) \tag{1.8}$$

the *maximal dilatation* of f. Obviously

$$K_I(f) = K_0(f^{-1}), \quad K_0(f) = K_I(f^{-1}), \quad K(f) = K(f^{-1}). \tag{1.9}$$

Moreover, it follows from Theorem 5 of [4] or Theorem 6.11 of [17] that

$$K_I(f) \leq K_0(f)^2, \qquad K_0(f) \leq K_I(f)^2. \tag{1.10}$$

Thus the three dilatations of a homeomorphism f are simultaneously finite or infinite. In the former case, f is said to be *quasiconformal*; f is said to be *K-quasiconformal* if $K(f) \leq K$ where $1 \leq K < \infty$.

One can also define the dilatations of a homeomorphism in purely analytic terms. Suppose that f is a homeomorphism of a domain $D \subset R^3$. For each $P \in D$ we let

$$L(P) = \limsup_{x \to P} \frac{|f(x) - f(P)|}{|x - P|}, \qquad l(P) = \liminf_{x \to P} \frac{|f(x) - f(P)|}{|x - P|},$$
$$J(P) = \limsup_{t \to 0} \frac{m(B')}{m(B)}, \tag{1.11}$$

where $B = B^3(P, t)$. At a point of differentiability, $L(P)$ and $l(P)$ are just the maximum and minimum stretching under f, and $J(P)$ is the absolute value of the Jacobian. Next we say that f is *absolutely continuous on lines*, or simply ACL, in D if for each ball B with $\bar{B} \subset D$, f is absolutely continuous on almost all line segments in B which are parallel to the coordinate axes.

LEMMA 1.1. *Suppose that f is a homeomorphism of a domain $D \subset R^3$. If f is not differentiable with $J > 0$ a.e. in D or if f is not* ACL *in D, then*

$$K_I(f) = K_0(f) = K(f) = \infty.$$

If f is differentiable with $J > 0$ a.e. in D and if f is ACL *in D, then*

$$K_I(f)^2 = \operatorname*{ess\,sup}_{P \in D} \frac{J(P)}{l(P)^3}, \qquad K_0(f)^2 = \operatorname*{ess\,sup}_{P \in D} \frac{L(P)^3}{J(P)}.$$

This result follows from Theorems 4 and 6 of [4] and also from Theorems 6.10, 6.13 and 6.16 of [17].

If we apply Lemma 1.1 to the affine mapping

$$f(x_1, x_2, x_3) = (K^2 x_1, K^2 x_2, x_3), \quad K > 1,$$

we obtain $K_I(f) = K^2$ and $K_0(f) = K$. Hence the inequalities in (1.10) are best possible.

We see from (1.2) and (1.6) that it is possible to define the dilatations of a homeomorphism f in terms of what happens to the moduli of certain arc families under f, namely the families of arcs which join the boundary components of bounded rings with closure in D. Surprisingly enough, if we know what happens to the moduli of this particular class of arc families under f, we know what happens to the moduli of *all* arc families under f. In particular, combining Lemma 1.1 with Theorem 6.5 of [17], we obtain the following result.

LEMMA 1.2. *Suppose that f is a homeomorphism of a domain $D \subset R^3$. Then*

$$K_I(f)^2 = \sup_{\Gamma} \frac{M(\Gamma')}{M(\Gamma)}, \qquad K_0(f)^2 = \sup_{\Gamma} \frac{M(\Gamma)}{M(\Gamma')},$$

where the suprema are taken over all arc families Γ which lie in D and for which $M(\Gamma)$ and $M(\Gamma')$ are not simultaneously equal to 0 or ∞.

1.6. *Coefficients of quasiconformality.* Suppose that D is a domain in R^3 which is homeomorphic to the unit ball B^3. We set

$$K_I(D) = \inf_f K_I(f), \quad K_0(D) = \inf_f K_0(f), \quad K(D) = \inf_f K(f), \tag{1.12}$$

where the infima are taken over all homeomorphisms f of D onto B^3. We call these three numbers $K_I(D)$, $K_0(D)$, and $K(D)$ the *inner, outer,* and *total coefficients of quasiconformality* of D. From (1.10) we obtain

$$K_I(D) \leqslant K_0(D)^2, \qquad K_0(D) \leqslant K_I(D)^2, \tag{1.13}$$

while from (1.8) and (1.10) it follows that

$$\max(K_I(D), K_0(D)) \leqslant K(D) \leqslant \min(K_I(D), K_0(D))^2. \tag{1.14}$$

Thus these three coefficients are simultaneously finite or infinite. In the former case we say that D is *quasiconformally equivalent* to a ball.

1.7. *Summary of results.* We can now formulate in a more precise manner the two problems with which our paper is concerned. First determine what kinds of domains D

are quasiconformally equivalent to a ball. Next given such a domain D, determine the coefficients $K_I(D)$, $K_0(D)$, and $K(D)$. Needless to say, both of these problems are fairly difficult and we give only rather fragmentary contributions to the solution of each.

The aim of this paper is, therefore, to obtain bounds for the coefficients of certain domains. To obtain upper bounds for a given domain D, it is only necessary to construct a suitable homeomorphism f of D onto B^3 and calculate the dilatations of f by means of Lemma 1.1. The problem of obtaining significant lower bounds is much more difficult, since one must find lower bounds for the various dilatations of all homeomorphisms f of D onto B^3. We do this by considering what happens to certain arc families under each homeomorphism f and then appealing to Lemma 1.2.

We begin in section 2 by giving a few general properties of the coefficients $K_I(D)$, $K_0(D)$, and $K(D)$. Next in section 3 we derive bounds for the moduli of some arc families. In section 4 we show that if D and D' have sufficiently smooth boundaries, each quasiconformal mapping f of D onto D' induces a homeomorphism f^* of ∂D onto $\partial D'$ which is quasiconformal with maximal dilatation

$$K(f^*) \leq \min(K_I(f), K_0(f))^2. \tag{1.15}$$

We use this sharp bound in sections 8 and 9 where we actually calculate the outer coefficients of an infinite circular cylinder and of a convex circular cone. In section 5 we obtain an upper bound for the coefficients of a certain class of starshaped domains; here the homeomorphism f may be chosen as a simple radial mapping. In section 6 we give asymptotically best possible lower bounds for the inner coefficient of another class of domains which are characterized by a certain separation property. Then in section 7 we calculate the inner coefficient of a convex dihedral wedge.

It is natural to assume that the values of the coefficients of a domain depend strongly upon how smooth the boundary of the domain is. In section 10 we study what can be said about the coefficients of D when ∂D contains a spire or a ridge. It turns out that the presence of a spire and the presence of a ridge have quite different effects on the coefficients, and that these effects depend also on whether the spire or the ridge is directed into or out of D. Finally in section 11 we prove that the space of all domains which are quasiconformally equivalent to a ball has a natural metric, and that this metric space is complete and nonseparable.

1.8. *Definitions for quasiconformality.* The terminology concerning quasiconformal mappings in space is rather confused. The class of K-quasiconformal mappings considered here is the same as the class studied by Gehring in [4], [5], and [6]. It also coincides with the class of K^2-quasiconformal mappings studied by Väisälä in [17] and [18]. In particular, the numbers $K_I(f)^2$ and $K_0(f)^2$ were called the *inner* and *outer* dilatations of the homeomorphism f in [17].

According to Šabat [14], a homeomorphism f of D is K-quasiconformal if f is C^1 and if $J>0$ and $L \leqslant Kl$ everywhere in D. Such mappings are K-quasiconformal by our definition. In the other direction, if f is K-quasiconformal by our definition and if f is C^1 with $J>0$ in D, then f is $K^{4/3}$-quasiconformal by Šabat's definition. The affine mapping

$$f(x_1, x_2, x_3) = (K^{4/3} x_1, K^{2/3} x_2, x_3)$$

shows that the constant $K^{4/3}$ cannot be improved.

2. General properties of the coefficients of domains

2.1. *Lower semicontinuity of the dilatations.* We need the following result to establish the existence of extremal mappings and to obtain a similar continuity result for the coefficients of a domain.

LEMMA 2.1. *Suppose that $\{f_n\}$ is a sequence of homeomorphisms of domains $D_n \subset R^3$, that each compact subset of a domain $D \subset R^3$ is contained in all but a finite number of D_n, and that the f_n converge uniformly on each compact subset of D to a homeomorphism f of D. Then*

$$K_I(f) \leqslant \liminf_{n\to\infty} K_I(f_n), \qquad K_0(f) \leqslant \liminf_{n\to\infty} K_0(f_n), \tag{2.1}$$

and similarly for the maximal dilatation $K(f)$.

Proof. Let R be a bounded ring with $\bar{R} \subset D$. Then $\bar{R} \subset D_n$ for $n \geqslant n_0(R)$. If R'_n and R' denote the images of R under f_n and f, then the hypotheses imply that each component of $\partial R'_n$ converges uniformly to the corresponding component of $\partial R'$, in the sense of Lemma 6 of [4]. Hence $\operatorname{mod} R'_n$ converges to $\operatorname{mod} R'$ and we have

$$\operatorname{mod} R \leqslant \liminf_{n\to\infty} (K_I(f_n) \operatorname{mod} R'_n) = \operatorname{mod} R' \liminf_{n\to\infty} K_I(f_n).$$

Since this inequality holds for all such R, we obtain the first half of (2.1). The proof for the second half is similar, and (2.1) then implies the analogous inequality for $K(f)$.

2.2. *Extremal mappings.* We see next that there exist extremal quasiconformal mappings for each domain with finite coefficients.

LEMMA 2.2. *If D is a domain in R^3 which is quasiconformally equivalent to a ball, then there exist extremal homeomorphisms f_I, f_0, f of D onto B^3 for which*

$$K_I(f_I) = K_I(D), \quad K_0(f_0) = K_0(D), \quad K(f) = K(D). \tag{2.2}$$

Proof. Fix K so that $K_I(D) < K < \infty$. Then we may choose a sequence of homeomorphisms $\{f_n\}$ of D onto B^3 such that

$$K_I(D) = \lim_{n\to\infty} K_I(f_n) \tag{2.3}$$

and such that $K_I(f_n) \leqslant K$ for all n. By composing f_n with a suitable Möbius transformation of B^3 onto itself, we may further assume that $f_n(P) = P'$, where P and P' are fixed points in D and B^3, respectively. Now (1.10) implies that the f_n are all K^2-quasiconformal, and hence by Corollary 7 of [4] there exists a subsequence $\{f_{n_k}\}$ which converges to a homeomorphism f_I of D onto B^3, uniformly on compact subsets of D. Combining Lemma 2.1 with (2.3) yields

$$K_I(D) \leqslant K_I(f_I) \leqslant \liminf_{k\to\infty} K_I(f_{n_k}) = K_I(D),$$

and hence f_I satisfies the first part of (2.2). The proofs for the existence of f_0 and f follow exactly the same lines.

2.3. *Lower semicontinuity of the coefficients.* Suppose that $\{D_n\}$ is a sequence of domains in R^3 which contain a fixed point P. We define the *kernel D at P* of the sequence $\{D_n\}$ as follows [6].

(i) If there exists no fixed neighborhood U of P which is contained in all of the D_n, then D consists only of the point P.

(ii) If there exists a fixed neighborhood U of P which is contained in all of the D_n, then D is the unique domain with the following properties.

(a) $P \in D$.

(b) Each compact set $E \subset D$ lies in all but a finite number of D_n.

(c) If Δ is a domain satisfying (a) and (b), then $\Delta \subset D$.

Next the sequence $\{D_n\}$ is said to *converge to its kernel D at P* if for each subsequence $\{n_k\}$, the sequence of domains $\{D_{n_k}\}$ also has D as its kernel at P.

Using this notion of convergence, we obtain the following continuity property for the coefficients of a domain.

THEOREM 2.1. *Suppose that $\{D_n\}$ is a sequence of domains in R^3 which contain the point P, that the D_n converge to their kernel D at P, and that $D \neq \{P\}, R^3$. Then*

$$K_I(D) \leqslant \liminf_{n\to\infty} K_I(D_n), \qquad K_0(D) \leqslant \liminf_{n\to\infty} K_0(D_n), \tag{2.4}$$

and similarly for the coefficient $K(D)$.

Proof. We establish the first half of (2.4). For this let

$$K = \liminf_{n\to\infty} K_I(D_n).$$

We may assume that $K<\infty$, for otherwise there is nothing to prove. Next for each n, let f_n be one of the extremal homeomorphisms of D_n onto B^3 for which $K_I(f_n)=K_I(D_n)$. By choosing a subsequence and relabeling, we may assume that

$$K=\lim_{n\to\infty} K_I(f_n)$$

and that $K_I(f_n)\leqslant K+1$ for all n. Furthermore, by composing the f_n with suitable Möbius transformations of B^3 onto itself, we may assume that $f_n(P)=0$. Now the f_n are $(K+1)^2$-quasiconformal, $D\subset R^3$, and D has a finite boundary point. Hence we may apply Theorem 3 of [6] to obtain a subsequence $\{f_{n_k}\}$ which converges to a homeomorphism f of D onto B^3, uniformly on each compact subset of D. Then with Lemma 2.1 we have

$$K_I(D)\leqslant K_I(f)\leqslant \liminf_{k\to\infty} K_I(f_{n_k})=K$$

as desired. The proofs for $K_0(D)$ and $K(D)$ follow similarly.

The hypothesis that $D\neq R^3$ is essential. For if we let $D_n=B^3(P,n)$, then all coefficients of each D_n are equal to 1. On the other hand, $D=R^3$ and hence all coefficients of D are infinite [9].

2.4. *Range of the coefficients.* The inequalities (1.13) and (1.14) imply that

$$K_I(D)\geqslant 1,\quad K_0(D)\geqslant 1,\quad K(D)\geqslant 1 \tag{2.5}$$

for each domain $D\subset R^3$, and that there is simultaneously equality or inequality in (2.5). It then follows from Lemma 2.2 and Theorem 15 of [4] that the coefficients of a domain D are equal to 1 if and only if D is either a ball or a half space. Hence we see that

$$K_I(D)>1,\quad K_0(D)>1,\quad K(D)>1$$

for essentially all domains $D\subset R^3$.

2.5. *Influence of the boundary.* The main task in this paper is to obtain some significant lower bounds for the coefficients of certain domains; by significant lower bounds, we mean bounds which exceed 1. Up to now, the only general result of this kind is the following one [18] which considers the topological nature of the boundary.

THEOREM 2.2. *If D is a domain in R^3, if D is locally connected at each point of its boundary ∂D, and if ∂D is not homeomorphic to S^2, then all the coefficients of D are infinite.* (1)

(1) A domain D is said to be *locally connected* at a boundary point if for each neighborhood U of the point there exists a second neighborhood V of the point such that each pair of points in $V\cap D$ can be joined by an arc $\gamma\subset U\cap D$.

It is obvious from Theorem 2.2 that the coefficients of a given domain depend strongly on the global nature of ∂D. We show now how one can obtain lower bounds for the coefficients by examining D in a neighborhood of a fixed finite boundary point.

We say that a domain $\Delta \subset R^3$ is *raylike* at a point Q if for each point P, $P \in \Delta$ if and only if $Q + t(P-Q) \in \Delta$ for $0 < t < \infty$. That is, Δ is raylike at Q if each open ray from Q lies either in Δ or in $C(\Delta)$.

THEOREM 2.3. *Suppose that D is a domain in R^3, that U is a neighborhood of $Q \in \partial D$, and that $D \cap U = \Delta \cap U$, where Δ is a domain that is raylike at Q. Then*

$$K_I(D) \geqslant K_I(\Delta), \quad K_0(D) \geqslant K_0(\Delta), \quad K(D) \geqslant K(\Delta). \tag{2.6}$$

Proof. We may assume without loss of generality that $Q=0$. Choose $a>0$ so that $B^3(a) \subset U$, and for each positive integer n let $D_n = \{x: x/n \in D\}$. If $x \in \Delta$ and $n > |x|/a$, then because Δ is raylike at the origin

$$\frac{x}{n} \in \Delta \cap U \subset D, \quad x \in D_n. \tag{2.7}$$

Hence if we fix $P \in \Delta$ with $|P| < a$, we see that $P \in D_n$ for all n.

Now let D' denote the kernel of the D_n at P. Arguing as in (2.7) it follows that each compact subset of Δ must lie in all but a finite number of D_n, and hence we see that $\Delta \subset D'$. If $x \in D'$, then $x \in D_n$ for $n > n_0$. Thus $n > n_0, |x|/a$ implies that

$$\frac{x}{n} \in D \cap U \subset \Delta, \quad x \in \Delta,$$

since Δ is raylike at the origin. Thus $\Delta = D'$, and repeating the above argument with a subsequence $\{n_k\}$, we conclude that the D_n converge to their kernel Δ at P. Finally since $0 \in \partial\Delta$, we can apply Theorem 2.1 to obtain

$$K_I(\Delta) \leqslant \liminf_{n\to\infty} K_I(D_n) = K_I(D),$$

and similarly for the two other coefficients.

2.6. *An example.* We conclude this section with an example which will motivate the separation property discussed in section 6. Let D be the domain

$$D = \{x: (x_1^2 + x_2^2)^{\frac{1}{2}} < \infty, \quad |x_3| < 1\},$$

let $\{b_n\}$ be any sequence of positive numbers which approach ∞, and for each n let D_n be the right circular cylinder

$$D_n = \{x: (x_1^2 + x_2^2)^{\frac{1}{2}} < b_n, \quad |x_3| < 1\}.$$

Then D satisfies the hypotheses of Theorem 2.2 and hence has infinite coefficients [18]. On the other hand, the D_n also converge to their kernel D at 0, and since $D \neq R^3$, we see from Theorem 2.1 that

$$\lim_{n\to\infty} K_I(D_n) = \lim_{n\to\infty} K_0(D_n) = \lim_{n\to\infty} K(D_n) = \infty.$$

Thus the coefficients of a right circular cylinder approach ∞ as the ratio of its radius to height approaches ∞. We will give bounds in section 6 to show how fast the inner coefficient grows.

3. Estimates for the moduli of certain arc families

3.1. *Spherical cap inequality.* We begin with an inequality which is required later in the proof of a symmetry principle for the moduli of arc families. (Cf. Lemma 1 in [4] and Theorem 3.6 in [17].)

LEMMA 3.1. *Suppose that S is a sphere of radius t, that D is an open half space, that $\Sigma = S \cap D$, and that ϱ is a non-negative Borel measurable function in S. Then each pair of points P and Q in $\overline{\Sigma}$ can be joined by a circular arc $\alpha \subset \Sigma$ for which*

$$\left(\int_\alpha \varrho\, ds\right)^3 \leqslant At \iint_\Sigma \varrho^3 d\sigma, \tag{3.1}$$

where A is the absolute constant $16q^2/\pi$ and

$$q = q\left(\frac{\pi}{2}\right) = \int_0^{\frac{\pi}{2}} (\sin u)^{-\frac{1}{2}} du. \tag{3.2}$$

Proof. Since the inequality (3.1) is invariant under similarity transformations of R^3 onto itself, we may assume that S is the unit sphere S^2 and that P is the point $(0,0,1)$. Let f map S stereographically onto the extended complex plane Z. Then P corresponds to $z=\infty$ and Q to some point $z=a \neq \infty$. Moreover, if $S-\Sigma$ is nonempty, this set corresponds to a closed disk or half plane E which does not contain a or ∞ as interior points. Since E is convex, we can find an angle β such that for $\beta < \theta < \beta + \pi$, the ray

$$z = a + ue^{i\theta}, \quad 0 < u < \infty,$$

does not meet E. Hence this ray corresponds to a circular arc $\alpha(\theta)$ which joins P and Q in Σ. For each such arc we see that

$$\int_{\alpha(\theta)} \varrho(x)\, ds = 2\int_0^\infty \varrho_1(z) \frac{du}{1+|z|^2}, \quad z = a + ue^{i\theta}, \quad \varrho_1 = \varrho \circ f^{-1},$$

and thus we may choose a particular circular arc α joining P and Q in Σ for which

$$\int_\alpha \varrho(x)\,ds \leqslant \frac{2}{\pi}\iint_\Omega \frac{\varrho_1(z)}{|z-a|}\,\frac{d\sigma}{1+|z|^2}, \tag{3.3}$$

where Ω is the half plane $\beta < \arg(z-a) < \beta+\pi$. Now Hölder's inequality implies that the cube of the right-hand side of (3.3) is majorized by

$$\frac{8}{\pi^3}\left(\iint_\Omega |z-a|^{-\frac{3}{2}}(1+|z|^2)^{-\frac{1}{2}}\,d\sigma\right)^2\left(\iint_\Omega \varrho_1(z)^3(1+|z|^2)^{-2}\,d\sigma\right).$$

If we appeal to the argument in the proof of Lemma 1 in [4] or to Theorem 7.2 in [12], we obtain

$$\iint_\Omega |z-a|^{-\frac{3}{2}}(1+|z|^2)^{-\frac{1}{2}}\,d\sigma \leqslant \iint_Z |z-a|^{-\frac{3}{2}}(1+|z|^2)^{-\frac{1}{2}}\,d\sigma \leqslant \iint_Z |z|^{-\frac{3}{2}}(1+|z|^2)^{-\frac{1}{2}}\,d\sigma = 2^{\frac{3}{2}}\pi q.$$

Finally we see that

$$4\iint_\Omega \varrho_1(z)^3(1+|z|^2)^{-2}\,d\sigma \leqslant \iint_\Sigma \varrho(x)^3\,d\sigma,$$

and if we combine the above inequalities, we obtain (3.1) as desired.

3.2. Suppose that D is an open half space and that E_0 and E_1 are disjoint continua in $\overline{D}$.(1) Next for small $t>0$ let Γ and $\Gamma(t)$ be the families of arcs which join E_0 to E_1 and $E_0(t)$ to $E_1(t)$ in D, respectively, where $E_i(t)$ denotes the closed set of points which lie within distance t of E_i for $i=0,1$.

The following result yields an important relation between the families of functions $F(\Gamma)$ and $F(\Gamma(t))$. (Cf. Lemma 2 in [3] and Lemma 2 in [19].)

LEMMA 3.2. *If $\varrho \in F(\Gamma)$ and if ϱ is L^3-integrable, then for each $a>1$ there exists a $t>0$ such that $a\varrho \in F(\Gamma(t))$.*

Proof. Choose $b>0$ so that $a(1-2b)=1$, let $c>0$ denote the minimum of the diameters of E_0 and E_1, and let $d>0$ denote the distance between E_0 and E_1. Next for $P\in\overline{D}$ and $t>0$ let $\Sigma(P,t)=S^2(P,t)\cap D$. Since ϱ is L^3-integrable, we can choose t, $0<t<c/4, d/6$, such that

$$\iiint_{B^3(P,2t)} \varrho^3\,d\omega \leqslant \frac{\log 2}{A}\,b^3$$

for all $P\in\overline{D}$, where A is the constant of Lemma 3.1. This means that for each $P\in\overline{D}$ we can find a spherical cap $\Sigma(P)=\Sigma(P,u)$ such that $t<u=u(P)<2t$ and

$$Au\iint_{\Sigma(P)} \varrho^3\,d\sigma \leqslant b^3. \tag{3.4}$$

(1) By a *continuum* we mean a compact connected set in $\overline{R}^3$ which contains more than one point.

To complete the proof of Lemma 3.2 we must show that

$$\int_\gamma \varrho ds \geqslant 1-2b \tag{3.5}$$

for all $\gamma \in \Gamma(t)$. Choose $\gamma \in \Gamma(t)$. There are two cases to consider.

Suppose first that there exist finite points $P_i \in \bar{\gamma} \cap E_i(t)$ for $i=0,1$. Then since $P_i \in \bar{\gamma}$, since the diameter of γ exceeds that of $\Sigma(P_i)$, and since γ is a connected set in D, γ must meet $\Sigma(P_i)$ for $i=0,1$. Next because E_i is a connected set in $\bar{D}$, a similar argument shows that E_i must meet $\bar{\Sigma}(P_i)$ for $i=0,1$. We conclude from Lemma 3.1 and (3.4) that for $i=0,1$ there exists a circular arc α_i which joins γ and E_i in $\Sigma(P_i) \subset D$ and for which

$$\int_{\alpha_i} \varrho ds \leqslant b. \tag{3.6}$$

It is then easy to show that $\bar{\alpha}_0 \cup \bar{\alpha}_1 \cup \gamma$ contains an arc β which joins E_0 to E_1 in D and hence

$$\int_\gamma \varrho ds \geqslant \int_\beta \varrho ds - \int_{\alpha_0} \varrho ds - \int_{\alpha_1} \varrho ds \geqslant 1-2b.$$

Suppose next that one of the sets, say $\bar{\gamma} \cap E_1(t)$, contains only the point ∞. Then $\bar{\gamma} \cap E_0(t)$ contains a finite point P_0, and arguing as above, we can find a circular arc α_0 which joins γ to E_0 in $\Sigma(P_0) \subset D$ and for which (3.6) holds with $i=0$. Since $\infty \in E_1$, $\bar{\alpha}_0 \cup \gamma$ contains an arc β which joins E_0 to E_1 in D and we obtain

$$\int_\gamma \varrho ds \geqslant \int_\beta \varrho ds - \int_{\alpha_0} \varrho ds \geqslant 1-b > 1-2b.$$

Thus the proof for Lemma 3.2 is complete.

3.3. *Remark.* Now suppose that D is an arbitrary open set and that E_0 and E_1 are bounded continua in D which lie at a positive distance from each other. Then the argument given above, or Lemma 2 of [19], shows that Lemma 3.2 is again valid if for small $t>0$, we let Γ and $\Gamma(t)$ denote the families of arcs which join E_0 to E_1 and $E_0(t)$ to $E_1(t)$ in D, respectively.

3.4. *Symmetry principle.* We next use Lemma 3.2 to establish the following symmetry principle for the moduli of arc families.

LEMMA 3.3. *Suppose that D is an open half space, that E_0 and E_1 are disjoint continua in $\bar{D}$, and that $\tilde{E}_0$ and $\tilde{E}_1$ are the symmetric images of E_0 and E_1 in the plane ∂D. If Γ is the family of arcs which join E_0 and E_1 in D and Γ_1 the family of arcs which join $E_0 \cup \tilde{E}_0$ and $E_1 \cup \tilde{E}_1$ in R^3, then*

$$M(\Gamma) = \tfrac{1}{2} M(\Gamma_1). \tag{3.7}$$

Proof. We may assume, for convenience of notation, that D is the half space $x_3>0$. If we let $\tilde{\Gamma}$ denote the family of arcs which join $\tilde{E}_0$ and $\tilde{E}_1$ in $\tilde{D}$, the half space $x_3<0$, then Γ and $\tilde{\Gamma}$ are separate families and $\Gamma\cup\tilde{\Gamma}\subset\Gamma_1$. Obviously $M(\Gamma)=M(\tilde{\Gamma})$ and hence

$$2M(\Gamma)=M(\Gamma)+M(\tilde{\Gamma})=M(\Gamma\cup\tilde{\Gamma})\leqslant M(\Gamma_1)$$

by Lemma 2.1 of [17].

Next let $\bar{\Gamma}$ denote the family of arcs which join E_0 and E_1 in $\bar{D}$, let f be the continuous mapping of R^3 into $\bar{D}$ given by

$$f(x_1,x_2,x_3)=(x_1,x_2,|x_3|),$$

and let $\varrho\in F(\bar{\Gamma})$. Set $\varrho_1=\varrho\circ f$ and choose $\gamma_1\in\Gamma_1$. Then $f[\gamma_1]$, the image of γ_1 under f, contains an arc $\gamma\in\bar{\Gamma}$ and hence

$$\int_{\gamma_1}\varrho_1(x)\,ds=\int_{\gamma_1}\varrho(f(x))\,d\Lambda\geqslant\int_{f[\gamma_1]}\varrho(x)\,d\Lambda\geqslant\int_{\gamma}\varrho(x)\,ds\geqslant 1.$$

Thus $\varrho_1\in F(\Gamma_1)$,

$$M(\Gamma_1)\leqslant\iiint_{R^3}\varrho_1^3\,d\omega=2\iiint_{\bar{D}}\varrho^3\,d\omega\leqslant 2\iiint_{R^3}\varrho^3\,d\omega,$$

and we conclude that $M(\Gamma_1)\leqslant 2M(\bar{\Gamma})$.

To complete the proof of (3.7) we must show that

$$M(\bar{\Gamma})\leqslant M(\Gamma). \tag{3.8}$$

Now the fact that E_0 and E_1 are disjoint implies that $M(\Gamma)<\infty$. Fix $a>1$ and choose $\varrho\in F(\Gamma)$ so that ϱ is L^3-integrable. By Lemma 3.2 we can choose $t>0$ so that $a\varrho\in F(\Gamma(t))$. Set $\varrho_1(x)=a\varrho(x+te_3)$, let $\gamma_1\in\bar{\Gamma}$, and let γ be the arc γ_1 translated through the vector te_3. Then $\gamma\in\Gamma(t)$ and we have

$$\int_{\gamma_1}\varrho_1(x)\,ds=\int_{\gamma}a\varrho(x)\,ds\geqslant 1.$$

Hence $\varrho_1\in F(\bar{\Gamma})$,

$$M(\bar{\Gamma})\leqslant\iiint_{R^3}\varrho_1^3\,d\omega=a^3\iiint_{R^3}\varrho^3\,d\omega,$$

and taking the infimum over all such ϱ yields

$$M(\bar{\Gamma})\leqslant a^3M(\Gamma).$$

Finally if we let $a\to 1$, we obtain (3.8) as desired.

3.5. *Continuity of moduli.* The following continuity property for the moduli of arc families is an easy consequence of the preceding arguments. (Cf. Lemma 6 in [4].)

LEMMA 3.4. *Suppose that D is an open set, that E_0 and E_1 are disjoint bounded continua in D, and that for small $t>0$, Γ and $\Gamma(t)$ are the families of arcs which join E_0 to E_1 and $E_0(t)$ to $E_1(t)$ in D, respectively. Then*

$$M(\Gamma)=\lim_{t\to 0} M(\Gamma(t)). \tag{3.9}$$

Proof. If $0<t_1<t_2$, then $\Gamma\subset\Gamma(t_1)\subset\Gamma(t_2)$ and hence

$$M(\Gamma)\leqslant M(\Gamma(t_1))\leqslant M(\Gamma(t_2)).$$

Thus the limit in (3.9) exists and

$$M(\Gamma)\leqslant \lim_{t\to 0} M(\Gamma(t)). \tag{3.10}$$

Since E_0 and E_1 are disjoint continua, $0<M(\Gamma)<\infty$. Fix $a>1$ and choose $\varrho\in F(\Gamma)$ such that

$$\iiint_{R^3} \varrho^3\, d\omega \leqslant aM(\Gamma).$$

By Lemma 3.2 and the remark in section 3.3, we can choose $t>0$ so that $a\varrho\in F(\Gamma(t))$. Thus

$$M(\Gamma(t))\leqslant a^3\iiint_{R^3} \varrho^3 d\omega\leqslant a^4 M(\Gamma),$$

and we obtain

$$\lim_{t\to 0} M(\Gamma(t))\leqslant a^4 M(\Gamma).$$

If we let $a\to 1$, then the resulting inequality and (3.10) imply (3.9), and the proof is complete.

3.6. *Extremal problem.* Now suppose that D is the half space $x_3>0$, that E_0 and E_1 are continua in $\bar{D}$, and that $P_0, 0\in E_0$ and $P_1,\infty\in E_1$, where $P_0\neq 0$ and $P_1\neq\infty$. We want to find a sharp lower bound for $M(\Gamma)$, where Γ is the family of arcs which join E_0 and E_1 in D.

For this let E_0^* denote the segment $-|P_0|\leqslant x_1\leqslant 0$, $x_2=x_3=0$ and E_1^* the ray $|P_1|\leqslant x_1\leqslant\infty$, $x_2=x_3=0$, and let Γ^* denote the family of arcs which join E_0^* and E_1^* in D. We shall show that the family Γ^* has the following extremal property.

THEOREM 3.1. $M(\Gamma)\geqslant M(\Gamma^*)$.

The proof of Theorem 3.1 depends upon an analogous extremal property of the Teichmüller ring in space. In order to make use of this property we must first establish the following result.

LEMMA 3.5. *If E_0 and E_1 are disjoint continua, then there exist a domain D and disjoint continua C_0 and C_1 such that C_0 and C_1 are the components of $C(D)$ and $\partial C_i\subset E_i\subset C_i$ for $i=0,1$.*

Proof. Choose $Q_0 \in E_0$ and $Q_1 \in E_1$ so that $|Q_0 - Q_1|$ is equal to the distance between E_0 and E_1, let $Q = \frac{1}{2}(Q_0 + Q_1)$, and let D be the component of $C(E_0 \cup E_1)$ which contains Q. Then each component Δ of $C(\bar{D})$ is a domain with a connected boundary. (See p. 123 and p. 137 in [11].) Next since $E_0 \cap E_1 = \emptyset$ and since

$$\partial\Delta \subset \partial C(\bar{D}) \subset \partial D \subset E_0 \cup E_1,$$

either $\partial\Delta \subset E_0$ or $\partial\Delta \subset E_1$. Now let

$$C_i = (\bigcup \bar{\Delta}) \cup E_i, \quad i = 0, 1,$$

where for each i the union is taken over all components Δ of $C(\bar{D})$ for which $\partial\Delta \subset E_i$. Then it is not difficult to see that C_0 and C_1 are continua and that $C(D) = C_0 \cup C_1$. Since $\partial D \subset E_0 \cup E_1$ and $Q_i \in \partial D \cap E_i$ for $i = 0, 1$, ∂D is not connected and hence C_0 and C_1 are the components of $C(D)$. Finally we see that

$$\partial C_i \subset \partial D \cap C_i \subset E_i \subset C_i$$

for $i = 0, 1$, and the proof of Lemma 3.5 is complete.

Proof of Theorem 3.1. We first observe that $M(\Gamma) = \infty$ whenever $E_0 \cap E_1 \neq \emptyset$. (Cf. p. 31 in [17].) For fix $P \in E_0 \cap E_1$. Then since E_0 and E_1 are nondegenerate, we can find $a > 0$ such that the closure of $\Sigma(t) = S^2(P, t) \cap D$ meets E_0 and E_1 for $0 < t < a$. Choose $\varrho \in F(\Gamma)$. Then

$$\int_\alpha \varrho \, ds \geq 1$$

for each circular arc α which joins E_0 and E_1 in $\Sigma(t)$, and Lemma 3.1 implies that

$$\iiint_{R^3} \varrho^3 \, d\omega \geq \int_0^a \left(\iint_{\Sigma(t)} \varrho^3 \, d\sigma \right) dt \geq \frac{1}{A} \int_0^a \frac{dt}{t} = \infty.$$

Hence $M(\Gamma) = \infty$ and the desired inequality follows trivially.

Suppose now that E_0 and E_1 are disjoint, let $\tilde{E}_0$ and $\tilde{E}_1$ be the symmetric images of E_0 and E_1 in ∂D, and let Γ_1 be the family of arcs which join $E_0 \cup \tilde{E}_0$ to $E_1 \cup \tilde{E}_1$ in R^3. Lemma 3.5 implies there exists a ring R which has C_0 and C_1 as the components of its complement, where $\partial C_i \subset E_i \subset C_i$ for $i = 0, 1$. Hence the family of arcs which join the components of ∂R in R is a subfamily of Γ_1, and we obtain

$$M(\Gamma) = \tfrac{1}{2} M(\Gamma_1) \geq \tfrac{1}{2} \operatorname{cap} R$$

from Lemma 3.3 and (1.6). Since $P_0, 0 \in C_0$ and $P_1, \infty \in C_1$, we can now apply Theorem 1 of [2] to conclude that

$$\operatorname{cap} R \geq \operatorname{cap} R^*,$$

where R^* is the ring bounded by the continua E_0^* and E_1^*. Because these sets are symmetric in ∂D, it follows that

$$M(\Gamma^*) = \tfrac{1}{2} M(\Gamma_1^*) = \tfrac{1}{2} \operatorname{cap} R^*, \tag{3.11}$$

where Γ_1^* is the family of arcs joining E_0^* to E_1^* in R^3, and we obtain $M(\Gamma) \geqslant M(\Gamma^*)$.

3.7. *Some applications.* For each $u > 0$ we let $\psi(u)$ denote the modulus of the family of arcs which join the segment $-1 \leqslant x_1 \leqslant 0$, $x_2 = x_3 = 0$ to the ray $u \leqslant x_1 \leqslant \infty$, $x_2 = x_3 = 0$ in the half space $x_3 > 0$. From (3.11) it follows that

$$\psi(u) = 2\pi (\log \Psi(u))^{-2}, \tag{3.12}$$

where $\Psi(u)$ is the function described in [2]. If we combine Lemmas 6 and 8 in [2] with the Corollary in [3] and with the estimates due to Hersch [7] and Teichmüller [16] for the modulus of the plane Teichmüller ring, we obtain

$$2\pi (\log \lambda^2 (u+1))^{-2} \leqslant \psi(u) \leqslant 2\pi (\log (16u+1))^{-2}, \tag{3.13}$$

where λ is an absolute constant, $4 \leqslant \lambda \leqslant 12.4 \ldots$.

Theorem 3.1 now yields the following lower bounds for the moduli of three different families of arcs in D, the half space $x_3 > 0$.

COROLLARY 3.1. *Suppose that E_0 and E_1 are continua in $\bar{D}$, that both E_0 and E_1 meet $S^2(a)$ where $a > 0$, and that $0 \in E_0$ and $\infty \in E_1$. If Γ is the family of arcs which join E_0 and E_1 in D, then*

$$M(\Gamma) \geqslant \psi(1).$$

COROLLARY 3.2. *Suppose that E_0 and E_1 are continua in $\bar{D}$, that E_0 separates 0 and ∞ in ∂D, and that $0, \infty \in E_1$. If Γ is the family of arcs which join E_0 and E_1 in D, then*

$$M(\Gamma) \geqslant \psi(\tfrac{1}{2}).$$

COROLLARY 3.3. *Suppose that P_1, P_2, P_3, P_4 are distinct points in ∂D and that E_1, E_2, E_3, E_4 are continua in $\bar{D}$ which join P_1 to P_2, P_2 to P_3, P_3 to P_4, P_4 to P_1, respectively. If Γ_1 and Γ_2 are the families of arcs which join E_1 to E_3 and E_2 to E_4 in D, respectively, then*

$$M(\Gamma_1) \geqslant \psi(1) \quad \textit{or} \quad M(\Gamma_2) \geqslant \psi(1).$$

Proof of Corollary 3.1. By hypothesis there exist points $P_0 \in S^2(a) \cap E_0$ and $P_1 \in S^2(a) \cap E_1$, and hence by Theorem 3.1

$$M(\Gamma) \geqslant \psi\left(\frac{|P_1|}{|P_0|}\right) = \psi(1).$$

Proof of Corollary 3.2. Since E_0 separates 0 and ∞ in ∂D, we can find a pair of points P_0, $P_1 \in E_0 \cap \partial D$ such that $0 < |P_0| \leqslant \frac{1}{2}|P_1 - P_0|$. For example, we may take P_0 as one of the points of $E_0 \cap \partial D$ nearest to 0 and then let P_1 be any point in the intersection of E_0 with the ray from 0 through $-P_0$. Then after the change of variables $y = x - P_0$, Theorem 3.1 yields

$$M(\Gamma) \geqslant \psi\left(\frac{|P_0|}{|P_1 - P_0|}\right) \geqslant \psi(\tfrac{1}{2}),$$

since $\psi(u)$ is nonincreasing in u.

Proof of Corollary 3.3. By performing a preliminary Möbius transformation of D onto itself, we may assume, without loss of generality, that $P_2 = 0$ and $P_4 = \infty$. Then since $P_1, 0 \in E_1$ and $P_3, \infty \in E_3$, we have

$$M(\Gamma_1) \geqslant \psi\left(\frac{|P_3|}{|P_1|}\right).$$

Next since P_3, $0 \in E_2$ and P_1, $\infty \in E_4$,

$$M(\Gamma_2) \geqslant \psi\left(\frac{|P_1|}{|P_3|}\right).$$

Thus $M(\Gamma_1) \geqslant \psi(1)$ if $|P_1| \geqslant |P_3|$ and $M(\Gamma_2) \geqslant \psi(1)$ if $|P_1| \leqslant |P_3|$.

3.8. *Remarks.* The bounds in Corollaries 3.1 and 3.3 are sharp. For example, we have $M(\Gamma_1) = M(\Gamma_2) = \psi(1)$ in Corollary 3.3 if E_i denotes the arc of the circle $x_1^2 + x_2^2 = 1$, $x_3 = 0$ which is contained in the closed i-th quadrant of $x_3 = 0$, $i = 1, 2, 3, 4$.

On the other hand, we are sure that the bound in Corollary 3.2 is not best possible, and we conjecture that under the hypotheses of Corollary 3.2,

$$M(\Gamma) \geqslant \left(\frac{\pi}{q}\right)^2 = 1.43\ldots, \tag{3.14}$$

where q is as in (3.2). There is equality in (3.14) when E_0 is the circle $x_1^2 + x_2^2 = 1$, $x_3 = 0$ and E_1 the ray $x_1 = x_2 = 0$, $0 \leqslant x_3 \leqslant \infty$. From the second half of (3.13) we obtain

$$\psi(\tfrac{1}{2}) \leqslant 2\pi(\log 9)^{-2} = 1.30\ldots,$$

and hence the conjectured lower bound in (3.14) is greater than $\psi(\frac{1}{2})$.

3.9. We consider next the asymptotic behaviour of a particular family of arcs. We use this result to establish the bound given in (1.15) for the boundary mapping f^* induced by a quasiconformal mapping f.

LEMMA 3.6. *For each* $a>0$, *let* $p(a)$ *denote the modulus of the family of arcs which join the segments* $x_1=\pm a$, $|x_2|\leqslant 1$, $x_3=0$ *in* R^3. *Then* $ap(a)$ *is nondecreasing in* a *and*

$$\lim_{a\to 0} ap(a)=A, \tag{3.15}$$

where $0<A<\infty$.

Proof. Fix $a>0$ and let Γ be the family of arcs which join the above segments in R^3. Then for $a'>a$,

$$f(x_1, x_2, x_3)=\left(\frac{a'}{a}x_1, x_2, \frac{a'}{a}x_3\right)$$

is a homeomorphism of R^3 onto itself and we obtain

$$p(a)=M(\Gamma)\leqslant K_0(f)^2 M(\Gamma')=\frac{a'}{a}p(a').$$

Thus $ap(a)$ is nondecreasing in a and the limit in (3.15) exists with $A\leqslant p(1)<\infty$. To show that $A>0$, choose $\varrho\in F(\Gamma)$. Then Theorem 3.5 in [17] implies that

$$\iiint_{R^3}\varrho^3 d\omega\geqslant\int_{-1}^{1}\left(\iint_{x_2=u}\varrho^3 d\sigma\right)du>\int_{-1}^{1}\frac{1}{21a}du=\frac{2}{21a}.$$

Hence $ap(a)>2/21$ and we conclude that $A\geqslant 2/21$.

3.10. *Cylinder and cone inequalities.* We conclude this section by obtaining sharp lower bounds for the moduli of two more families of arcs. These estimates will be used in sections 8 and 9 when we calculate the outer coefficients of an infinite cylinder and of a convex cone.

LEMMA 3.7. *Suppose that* $a<b$, *that* C *is the finite part of the cylinder* $x_1^2+x_2^2<1$ *which is bounded by the planes* $x_3=a$ *and* $x_3=b$, *and that* E *is a connected set in* C *which joins the bases of* C. *If* Γ *is the family of arcs in* C *which join* E *to the lateral surface of* C, *then*

$$M(\Gamma)\geqslant\tfrac{1}{2}\pi(b-a). \tag{3.16}$$

There is equality in (3.16) *if* E *is the segment* $x_1=x_2=0$, $a<x_3<b$.

Proof. Choose $\varrho\in F(\Gamma)$. For $a<u<b$, the plane $x_3=u$ meets both E and the lateral surface of C, and we can apply Theorem 3.4 of [17] to obtain

$$\iiint_{R^3}\varrho^3 d\omega\geqslant\int_a^b\left(\iint_{x_3=u}\varrho^3 d\sigma\right)du\geqslant\int_a^b\tfrac{1}{2}\pi\,du=\tfrac{1}{2}\pi(b-a).$$

This yields (3.16) as desired.

Next suppose that E is the segment $x_1=x_2=0$, $a<x_3<b$, and set $\varrho(x)=\frac{1}{2}r^{-\frac{1}{3}}$ in C and $\varrho(x)=0$ in $C(C)$, where r denotes the distance from x to the x_3-axis. Then $\varrho\in F(\Gamma)$ and

$$\iiint_{R^3}\varrho^3\,d\omega=\tfrac{1}{2}\pi(b-a).$$

Hence in this case there is equality in (3.16).

Finally we have the following cone analogue of Lemma 3.7.

LEMMA 3.8. *Suppose that $0<\alpha\leqslant\frac{1}{2}\pi$ and $0<a<b$, that C is the part of the cone $x_3>(\cot\alpha)(x_1^2+x_2^2)^{\frac{1}{2}}$ which is bounded by the spheres $S^2(a)$ and $S^2(b)$, and that E is a connected set in C which joins the spherical bases of C. If Γ is the family of arcs in C which join E to the lateral surface of C, then*

$$M(\Gamma)\geqslant 2\pi q(\alpha)^{-2}\log\frac{b}{a}, \tag{3.17}$$

where

$$q(\alpha)=\int_0^\alpha(\sin u)^{-\frac{1}{2}}\,du. \tag{3.18}$$

There is equality in (3.17) *if E is the segment $x_1=x_2=0$, $a<x_3<b$.*

Proof. Choose $\varrho\in F(\Gamma)$ and for each $t>0$ let $\Sigma(t)=S^2(t)\cap C$. We first show that

$$\iint_{\Sigma(t)}\varrho^3\,d\sigma\geqslant\frac{2\pi}{t}q(\alpha)^{-2} \tag{3.19}$$

for $a<t<b$.

For this fix t, $a<t<b$. Since E joins $S^2(a)$ and $S^2(b)$ in C, we can find a point $Q\in E\cap\Sigma(t)$. Next let T be any fixed plane containing 0 and Q, let $T(\theta)$ denote the plane through 0 and Q which meets T at an angle θ, and for $P\in\Sigma(t)$ let $\varphi=\varphi(P)$ denote the angle formed by the segments $0P$ and $0Q$, $0\leqslant\varphi<2\alpha$. For each θ, $\beta(\theta)=T(\theta)\cap\Sigma(t)$ contains a pair of circular arcs which join E to the lateral surface of C. Thus

$$\int_{\beta(\theta)}\varrho\,ds\geqslant 2,$$

and with Hölder's inequality we obtain

$$\left(\int_{\beta(\theta)}\varrho^3\sin\varphi\,ds\right)\left(\int_{\beta(\theta)}(\sin\varphi)^{-\frac{1}{2}}\,ds\right)^2\geqslant 8.$$

Since the length of $\beta(\theta)$ does not exceed $2\alpha t\leqslant\pi t$, it is easy to see that

$$\int_{\beta(\theta)}(\sin\varphi)^{-\frac{1}{2}}\,ds=t\int_{\beta(\theta)}(\sin\varphi)^{-\frac{1}{2}}\,|d\varphi|\leqslant 2tq(\alpha).$$

Thus we have
$$\int_{\beta(\theta)} \varrho^3 t^2 \sin\varphi\,|d\varphi| \geqslant \frac{2}{t} q(\alpha)^{-2},$$

and we obtain (3.19) by integrating both sides of this inequality over $0 \leqslant \theta \leqslant \pi$.

Next if we integrate both sides of (3.19) over the interval $a < t < b$, we get

$$\iiint_{R^3} \varrho^3 d\omega \geqslant \int_a^b \left(\iint_{\Sigma(t)} \varrho^3 d\sigma \right) dt \geqslant 2\pi q(\alpha)^{-2} \log \frac{b}{a},$$

and hence (3.17) follows.

Finally suppose that E is the segment $x_1 = x_2 = 0$, $a < x_3 < b$, and let

$$\varrho(x) = t^{-1} (\sin\varphi)^{-\frac{1}{3}} q(\alpha)^{-1}$$

in C and $\varrho(x) = 0$ in $C(C)$, where $t = |x|$ and φ is the acute angle between the segment $0x$ and the positive x_3-axis. Then $\varrho \in F(\Gamma)$ and

$$\iiint_{R^3} \varrho^3 d\omega = 2\pi\, q(\alpha)^{-2} \log \frac{b}{a}.$$

Hence in this case there is equality in (3.17).

4. Boundary correspondence induced by quasiconformal mappings

4.1. *Introduction.* If f is a quasiconformal mapping of D onto a half space D' and if D is locally connected at each point of its boundary, then f induces a homeomorphism f^* of ∂D onto $\partial D'$ by Theorem 1 in [18]. Moreover, by Theorem 10 in [4], this boundary mapping is a two-dimensional quasiconformal mapping whenever ∂D is itself a plane.

We show in this section that this result remains valid when ∂D is, for example, a smooth free surface, and we obtain a sharp bound for the maximal dilatation of the boundary mapping f^* in terms of the inner and outer dilatations of f.

4.2. *Quasi-isometries.* We introduce the notion of a quasi-isometry in order to describe a certain class of surfaces. Suppose that f is a homeomorphism of a domain $D \subset R^3$. We say that f is a *C-isometry*, $1 \leqslant C < \infty$, if

$$C^{-1} |P_1 - P_2| \leqslant |f(P_1) - f(P_2)| \leqslant C |P_1 - P_2| \tag{4.1}$$

for all $P_1, P_2 \in D$. A homeomorphism is a *quasi-isometry* if it is a C-isometry for some C. We define $C(f)$, the *maximal distortion* of f in D, as the smallest constant C for which (4.1) holds for all $P_1, P_2 \in D$.

If f is a C-isometry, then it follows from Lemma 1.1 that

$$K_I(f) \leqslant C^2, \qquad K_0(f) \leqslant C^2. \tag{4.2}$$

Thus f is quasiconformal. On the other hand, it is clear that a quasiconformal mapping need not be a quasi-isometry.

4.3. *Admissible surfaces.* A connected set $S \subset R^3$ is said to be an *admissible surface* if to each point $P \in S$ there corresponds a quasi-isometry i_P with the following properties. For each $\varepsilon > 0$ there exists a neighborhood U_P of P, in which i_P is defined, such that i_P maps $S \cap U_P$ onto a plane domain T_P and such that the maximal distortion $C(i_P)$ of i_P in U_P satisfies the inequalities

$$\sup_{P \in S} C(i_P) < \infty, \qquad \operatorname{ess\,sup}_{P \in S} C(i_P) \leqslant 1 + \varepsilon. \tag{4.3}$$

Here, and throughout the rest of section 4, the essential suprema and infima over S and S' are taken with respect to the Λ^2-measure.

We want a simple geometric condition which implies that S is an admissible surface. Suppose that a point $P \in S$ has a neighborhood V such that $S \cap V$ is homeomorphic to an open disk, suppose that n is a fixed unit vector, and suppose that for each pair of points $Q_1, Q_2 \in S \cap V$, the acute angle which the segment $Q_1 Q_2$ makes with n is never less than $\alpha > 0$. Then there exists a neighborhood U of P such that $U \subset V$ and each point $x \in U$ has a unique representation of the form

$$x = Q + un,$$

where $Q \in S \cap U$ and u is real. For each such x we let

$$i(x) = i(Q) + un,$$

where $i(Q)$ is the projection of Q onto the plane through P which has n as its normal. Then i maps $S \cap U$ onto a plane domain T, and it follows from Corollary 5.1 that i is a quasi-isometry of U with maximal distortion

$$C(i) \leqslant \cot \alpha + 1.$$

Thus a connected set $S \subset R^3$ is an admissible surface if to each point P there corresponds a unit vector n_P with the following property. For each $\varepsilon > 0$ there exists a neighborhood U_P of P such that $S \cap U_P$ is homeomorphic to an open disk and such that for each pair of points $Q_1, Q_2 \in S \cap U_P$, the acute angle between the segment $Q_1 Q_2$ and the vector n_P is never less than α_P, where

$$\inf_{P \in S} \alpha_P > 0, \qquad \operatorname{ess\,inf}_{P \in S} \alpha_P \geqslant \tfrac{1}{2}\pi - \varepsilon. \tag{4.4}$$

For example, a two-dimensional manifold $S \subset R^3$ is an admissible surface if it has a well defined continuously turning tangent plane at each point $P \in S$.

4.4. *Quasiconformal mappings between admissible surfaces.* Suppose that S and S' are admissible surfaces and that f is a homeomorphism of S onto S'. Next for each $P \in S$, let $P' = f(P)$ and let i_P and $i_{P'}$ be the quasi-isometries associated with P and P'. We say that f is *K-quasiconformal*, $1 \leqslant K < \infty$, if for each $\varepsilon > 0$ there exist neighborhoods U_P of P and $U_{P'}$ of P' with the following properties. The quasi-isometries i_P and $i_{P'}$ map $S \cap U_P$ and $S' \cap U_{P'}$ onto plane domains T_P and $T_{P'}$ respectively, f maps $S \cap U_P$ into $S' \cap U_{P'}$, and

$$\sup_{P \in S} K(g_P) < \infty, \qquad \operatorname*{ess\,sup}_{P \in S} K(g_P) \leqslant K + \varepsilon, \tag{4.5}$$

where $K(g_P)$ denotes the maximal dilatation of the plane homeomorphism

$$g_P = i_{P'} \circ f \circ i_P^{-1}. \tag{4.6}$$

We say that f is *quasiconformal* if it is K-quasiconformal for some K, and we define $K(f)$, the *maximal dilatation* of f, as the smallest number K for which f is K-quasiconformal.

LEMMA 4.1. *Suppose that S and S' are admissible surfaces and that f is a quasiconformal mapping of S onto S'. If $E \subset S$ and if $\Lambda^2(E) = 0$, then $\Lambda^2(E') = 0$.*

Proof. Suppose that f is K-quasiconformal, and for $\varepsilon = 1$ and each $P \in S$, let U_P and $U_{P'}$ be the neighborhoods of the above definition. By Lindelöf's covering theorem, we can choose a sequence of points $P_n \in S$ so that the neighborhoods U_{P_n} cover E. Set $E_n = E \cap U_{P_n}$. Then

$$E' = \bigcup_n E'_n$$

and $\Lambda^2(E_n) = 0$. Since i_{P_n} and $i_{P'_n}$ are quasi-isometries and since g_{P_n} is a plane quasiconformal mapping, it follows that $\Lambda^2(E'_n) = 0$. Hence $\Lambda^2(E') = 0$ as desired.

From Lemma 4.1 it follows that if f is a K-quasiconformal mapping of S onto S', then f^{-1} is a K-quasiconformal mapping of S' onto S.

4.5. *Surface modulus of a family of arcs.* The above definition for quasiconformal mappings is awkward since it involves the quasi-isometries i_P and $i_{P'}$. We shall give two other equivalent definitions, but first we must introduce the notion of the surface modulus of an arc family.

Suppose that S is an admissible surface and that Γ is a family of arcs in S. As in section 1.4 we let $F(\Gamma)$ denote the family of functions ϱ which are non-negative and Borel measurable in S and for which

$$\int_\gamma \varrho\, ds \geqslant 1$$

for each arc $\gamma \in \Gamma$. We then define the *surface modulus* of the arc family Γ as

$$M^S(\Gamma) = \inf_{\varrho} \iint_S \varrho^2 d\sigma,$$

where the integral is defined by means of the Λ^2-measure and where the infimum is taken over all $\varrho \in F(\Gamma)$.

The surface modulus of a family of arcs in an admissible surface behaves like the familiar plane modulus of a family of arcs in a plane domain. In particular, $M^S(\Gamma)$ reduces to this modulus when S is a plane domain. Next it is easy to see that all of the assertions of Lemma 2.1 in [17] hold with M_2 replaced by M^S. Finally we can argue as in the proof of Theorem 2.3 in [17] to show that the surface modulus of the family of all compact nonrectifiable arcs in an admissible surface S is equal to zero. This means that the arcs of a family Γ, which are not locally rectifiable, have no influence on $M^S(\Gamma)$. That is, if Γ_1 is the subfamily of locally rectifiable arcs in Γ, then

$$M^S(\Gamma) = M^S(\Gamma_1). \tag{4.7}$$

We could also have used the following inequality to reduce the proof of (4.7) to the special case where S is a plane domain.

LEMMA 4.2. *Suppose that S is an admissible surface, that i is a C-isometry of U which maps $S \cap U$ onto a plane domain T, that Γ is a family of arcs in $S \cap U$, and that Γ' is the image of Γ under i. Then*

$$C^{-4} M^S(\Gamma) \leqslant M^T(\Gamma') \leqslant C^4 M^S(\Gamma). \tag{4.8}$$

Proof. If $\varrho \in F(\Gamma)$, then $C\varrho' \in F(\Gamma')$, where $\varrho' = \varrho \circ i^{-1}$, and

$$M^T(\Gamma') \leqslant \iint_T C^2 \varrho'^2 \, d\sigma \leqslant C^4 \iint_S \varrho^2 \, d\sigma.$$

This yields the second half of (4.8). The first half follows similarly.

4.6. *Analytic characterization.* Suppose that f is a homeomorphism of an admissible surface S. For each $P \in S$ we let

$$L(P) = \limsup_{x \to P} \frac{|f(x) - f(P)|}{|x - P|}, \qquad J^S(P) = \limsup_{t \to 0} \frac{\Lambda^2((S \cap B)')}{\Lambda^2(S \cap B)}, \tag{4.9}$$

where $B = B^3(P,t)$. Next we say that f is *absolutely continuous on arcs*, or simply ACA, in S if $M^S(\Gamma) = 0$, where Γ is the family of all locally rectifiable arcs in S which contain a compact subarc on which f is not absolutely continuous. (1)

(1) Suppose that S is a plane domain. If f is ACA in S, then f is clearly ACL in S. Conversely, if f is ACL in S and if the partial derivatives of f are locally L^2-integrable in S, then f is ACA by Lemma 4.1 of [17].

We now have the following analytic characterization of quasiconformal mappings between admissible surfaces.

THEOREM 4.1. *Suppose that S and S' are admissible surfaces and that f is a homeomorphism of S onto S'. Then f is K-quasiconformal, $1 \leqslant K < \infty$, if and only if f is* ACA *in S and*

$$L(P)^2 \leqslant K J^S(P) \tag{4.10}$$

Λ^2-a.e. in S.

Proof. Suppose that f is K-quasiconformal. We wish to show that f is ACA in S and that (4.10) holds for $P \in S - E$ where $\Lambda^2(E) = 0$. Fix $\varepsilon > 0$. Because S and S' are admissible surfaces we can choose for each $P \in S$ and $P' = f(P)$ neighborhoods U_P and $U_{P'}$ such that

$$\begin{aligned} &\sup_{P \in S} C(i_P) < \infty, \qquad \operatorname*{ess\,sup}_{P \in S} C(i_P) \leqslant 1 + \varepsilon, \\ &\sup_{P' \in S'} C(i_{P'}) < \infty, \qquad \operatorname*{ess\,sup}_{P' \in S'} C(i_{P'}) \leqslant 1 + \varepsilon, \end{aligned} \tag{4.11}$$

where $C(i_P)$ and $C(i_{P'})$ denote the maximal distortions of i_P and $i_{P'}$ in U_P and $U_{P'}$, respectively. Next because f is K-quasiconformal, we can choose these neighborhoods so that, in addition, f maps $S \cap U_P$ into $S' \cap U_{P'}$ and

$$\sup_{P \in S} K(g_P) < \infty, \qquad \operatorname*{ess\,sup}_{P \in S} K(g_P) \leqslant K(1 + \varepsilon), \tag{4.12}$$

where $K(g_P)$ is the maximal dilatation of the plane homeomorphism g_P given in (4.6).

We show first that f is ACA. Let Γ denote the family of all locally rectifiable arcs in S which contain compact subarcs on which f is not absolutely continuous. Next choose a sequence of points $P_n \in S$ so that the corresponding neighborhoods U_{Pn} cover S. Each $\gamma \in \Gamma$ has a compact subarc β on which f is not absolutely continuous. A bisection argument then shows that for some n, β has a compact subarc $\alpha \subset S \cap U_{Pn}$ on which f is not absolutely continuous. These arcs α form a family Γ_0 which minorizes Γ,(1) and hence

$$M^S(\Gamma) \leqslant M^S(\Gamma_0).$$

Let Γ_n be the subfamily of arcs of Γ_0 which lie in $S \cap U_{Pn}$, and let Γ_n' denote the image of Γ_n under the quasi-isometry i_{Pn}. The analytic definition for plane quasiconformal mappings implies that g_{Pn} is ACA in T_{Pn}, and since g_{Pn} is not absolutely continuous on any arc of Γ_n', it follows that the plane modulus of Γ_n' is equal to zero. Hence $M^S(\Gamma_n) = 0$ by Lemma 4.2 and we conclude that

$$M^S(\Gamma_0) \leqslant \sum_n M^S(\Gamma_n) = 0.$$

Thus $M^S(\Gamma) = 0$ and f is ACA in S.

(1) An arc family Γ_1 is said to minorize an arc family Γ_2 if for each $\gamma_2 \in \Gamma_2$ there exists a $\gamma_1 \in \Gamma_1$ such that $\gamma_1 \subset \gamma_2$.

We turn next to the inequality (4.10). Lemma 4.1, (4.11), and (4.12) imply there exists a set $E_1 \subset S$ such that $\Lambda^2(E_1)=0$ and

$$C(i_P) \leqslant 1+\varepsilon, \quad C(i_{P'}) \leqslant 1+\varepsilon, \quad K(g_P) \leqslant K(1+\varepsilon)$$

for $P \in S - E_1$. Fix such a point P and let $y = i_P(x)$ for $x \in S \cap U_P$. Then $g = g_P$ is a plane $K(1+\varepsilon)$-quasiconformal mapping of $T = T_P$ and we have, in an obvious notation,

$$L(x)^2 \leqslant (1+\varepsilon)^4 L_g(y)^2 \leqslant K(1+\varepsilon)^5 J_g^T(y) \leqslant K(1+\varepsilon)^9 J^S(x) \tag{4.13}$$

Λ^2-a.e. in T_P, and hence Λ^2-a.e. in $S \cap U_P$. Since there exists a sequence of points $P_n \in S - E_1$ whose neighborhoods U_{P_n} cover $S - E_1$, we can find a set E_2 such that $\Lambda^2(E_2)=0$ and

$$L(P)^2 \leqslant K(1+\varepsilon)^9 J^S(P)$$

for $P \in S - E_2$. Finally let E be the union of the exceptional sets E_2 for $\varepsilon = 1/n$, $n = 1, 2, \dots$. Then $\Lambda^2(E)=0$ and (4.10) holds for $P \in S - E$. This completes the proof of the necessity part of Theorem 4.1.

For the sufficiency part fix $\varepsilon > 0$. Then for $P \in S$ and $P' = f(P)$ choose neighborhoods U_P and $U_{P'}$ so that (4.11) holds and so that f maps $S \cap U_P$ into $S' \cap U_{P'}$. We show first that the homeomorphism g_P of (4.6) is a plane quasiconformal mapping with maximal dilatation

$$K(g_P) \leqslant K C(i_P)^4 C(i_{P'})^4. \tag{4.14}$$

Fix $P \in S$, let Γ' be the family of all locally rectifiable arcs in T_P which contain a compact subarc on which g_P is not absolutely continuous, and let Γ be the image of Γ' under i_P^{-1}. Since f is by hypothesis ACA in S, $M^S(\Gamma)=0$ and hence $M^{T_P}(\Gamma')=0$ by Lemma 4.2. Thus g_P is ACA and, a fortiori, ACL in T_P. Next arguing as in (4.13) we see from (4.10) that

$$L_g(y)^2 \leqslant K C(i_P)^4 C(i_{P'})^4 J_g^T(y), \quad g = g_P \text{ and } T = T_P,$$

Λ^2-a.e. in T_P, and we obtain (4.14) from the analytic definition for plane quasiconformal mappings.

Now (4.11) and (4.14) imply that

$$\sup_{P \in S} K(g_P) < \infty,$$

and hence that f is quasiconformal. Then Lemma 4.1, (4.11), and (4.14) imply that

$$\operatorname*{ess\,sup}_{P \in S} K(g_P) \leqslant K(1+\varepsilon)^8.$$

Thus f is K-quasiconformal and the proof of Theorem 4.1 is complete.

4.7. *Modulus characterization.* We can also characterize quasiconformal mappings between admissible surfaces by means of the surface moduli of arc families.

THEOREM 4.2. *Suppose that S and S' are admissible surfaces and that f is a homeomorphism of S onto S'. Then f is K-quasiconformal, $1 \leqslant K < \infty$, if and only if*

$$M^{S'}(\Gamma') \leqslant K M^{S}(\Gamma) \tag{4.15}$$

for each family of arcs Γ in S.

Proof. Suppose that (4.15) holds for all arc families Γ in S, fix $\varepsilon > 0$, and choose U_P, $U_{P'}$, and g_P as in the last part of the proof of Theorem 4.1. Then (4.15) and Lemma 4.2 imply that

$$M^{T_{P'}}(\Gamma') \leqslant K C(i_P)^4 C(i_{P'})^4 M^{T_P}(\Gamma)$$

for each arc family Γ in T_P, where Γ' is the image of Γ under g_P. Thus we obtain (4.14) by virtue of the geometric definition for plane quasiconformal mappings, and the proof that f is K-quasiconformal is concluded as in the proof of Theorem 4.1.

Suppose now that f is K-quasiconformal. Since f^{-1} is K-quasiconformal, (4.15) will follow if we can show that

$$M^{S}(\Gamma) \leqslant K M^{S'}(\Gamma') \tag{4.16}$$

for each arc family Γ in S. For this let Γ be any family of arcs in S, let Γ_1 be the family of arcs in Γ which are locally rectifiable, and let Γ_2 be the family of arcs in Γ_1 on each compact subarc of which f is absolutely continuous. Then (4.7) and the fact that f is ACA in S imply that

$$M^{S}(\Gamma) = M^{S}(\Gamma_1) = M^{S}(\Gamma_2). \tag{4.17}$$

Choose $\varrho' \in F(\Gamma')$, set

$$\varrho(x) = \varrho'(f(x)) L(x)$$

for all $x \in S$, and pick $\gamma \in \Gamma_2$. If β is any compact subarc of γ, then β is rectifiable, f is absolutely continuous on β, and we obtain

$$\int_{\gamma} \varrho\, ds \geqslant \int_{\beta} \varrho\, ds = \int \varrho' L\, ds \geqslant \int_{\beta'} \varrho'\, ds.$$

(Cf. p. 24 in [17].) Since this inequality holds for all such β,

$$\int_{\gamma} \varrho\, ds \geqslant \sup_{\beta'} \int_{\beta'} \varrho'\, ds = \int_{\gamma'} \varrho'\, ds \geqslant 1.$$

Because ϱ is Borel measurable, we conclude that $\varrho \in F(\Gamma_2)$. Thus

$$M^{S}(\Gamma_2) \leqslant \iint_S \varrho^2\, d\sigma = \iint_S (\varrho' L)^2\, d\sigma \leqslant K \iint_S \varrho'^2 J^S\, d\sigma \leqslant K \iint_{S'} \varrho'^2\, d\sigma,$$

and hence
$$M^{S}(\Gamma_2) \leqslant K M^{S'}(\Gamma').$$
This, together with (4.17), yields (4.16) as desired.

4.8. *Boundary correspondence theorem.* Suppose that D is a domain in R^3. We say that a two-dimensional manifold S is a *free boundary surface* of D if

$$S \subset \partial \overline{D} \quad \text{and} \quad S \cap (\overline{\partial D - S}) = \emptyset. \tag{4.18}$$

Suppose next that f is a function defined on D. Then for each $P \in \partial D$ we denote by $C(f,P)$ the cluster set of f at P, that is the set of limit points of all sequences $\{f(P_n)\}$, where $P_n \to P$ in D.

Our objective, in this section, is to establish the following result on the boundary correspondence induced by quasiconformal mappings. (Cf. Theorem 10 in [4].)

THEOREM 4.3. *Suppose that f is a quasiconformal mapping of a domain $D \subset R^3$, that S and S'' are free admissible boundary surfaces of D and D', respectively, and that*

$$C(f,P) \cap S'' \neq \emptyset \tag{4.19}$$

for each $P \in S$. Then f can be extended to be a homeomorphism of $D \cup S$ onto $D' \cup S'$, where S' is an admissible surface contained in S''. The induced boundary mapping f^* is a quasiconformal mapping of S onto S' with maximal dilatation

$$K(f^*) \leqslant \min(K_I(f), K_0(f))^2. \tag{4.20}$$

This bound for $K(f^*)$ is sharp.

The proof of Theorem 4.3 depends upon the following four lemmas.

LEMMA 4.3. *If f is continuous in D and if D is locally connected at $P \in \partial D$, then $C(f,P)$ is a closed connected set.*

Proof. If for each n we let $E_n = D \cap B^3(P, 1/n)$, then it follows that

$$C = C(f,P) = \bigcap_n \overline{f[E_n]}.$$

Clearly C is closed. Because D is locally connected at P, for each n we can find an m such that each pair of points in E_m can be joined by an arc in E_n. Thus each pair of points in C can be joined by a connected set in $\overline{f[E_n]}$. If C were not connected, we could find a bounded open set G such that both G and $C(G)$ would contain points of C while $\partial G \cap C = \emptyset$. But $\partial G \cap \overline{f[E_n]} \neq \emptyset$ for all n, whence

$$\partial G \cap C = \bigcap_n (\partial G \cap \overline{f[E_n]}) \neq \emptyset.$$

Hence C is connected.

LEMMA 4.4. *Suppose that S is a free admissible boundary surface of D and that U is a neighborhood of $P \in S$. Then P has a neighborhood $V \subset U$ such that the quasi-isometry i_P maps $D \cap V$ onto a hemiball H and $S \cap V$ onto the plane part of ∂H.*

Proof. By definition P has a neighborhood U_P, in which i_P is defined, such that i_P maps $S \cap U_P$ onto a plane domain T_P. Let $Q = i_P(P)$ and $B = B^3(Q,t)$. By (4.18), P lies at a positive distance from $\partial D - S$. Hence we may choose $t > 0$ so that T_P divides B into two open hemiballs, H_0 and H_1, and so that

$$B \subset i_P[U \cap U_P], \quad B \cap i_P[(\partial D - S) \cap U_P] = \emptyset. \tag{4.21}$$

Let V and W_i be the images of B and H_i under i_P^{-1}. Then $V \subset U$ and (4.18) implies that W_0 or W_1, say W_1, contains a point of $C(\bar{D})$. Now (4.21) implies that $\partial D \cap W_1 = \emptyset$ and hence $D \cap W_1 = \emptyset$. Thus $D \cap V = W_0$ and i_P maps $D \cap V$ onto $H = H_0$ as desired.

Lemma 4.4 shows that a domain is locally connected at each point of a free admissible boundary surface.

LEMMA 4.5. *Suppose that S is a free admissible boundary surface of D, that E_0 and E_1 are nondegenerate connected sets in D, and that $\bar{E}_0 \cap \bar{E}_1$ contains a point $P \in S$. If Γ is the family of arcs which join E_0 and E_1 in D, then $M(\Gamma) = \infty$.*

Proof. Let V be the neighborhood of Lemma 4.4 with $U = R^3$, and let Γ_1 be the family of arcs which join $F_0 = E_0 \cap V$ and $F_1 = E_1 \cap V$ in $D \cap V$. Then

$$M(\Gamma) \geq M(\Gamma_1) \geq C(i_P)^{-4} M(\Gamma_1'), \quad \Gamma_1' = i_P[\Gamma_1]$$

by (4.2), where $C(i_P)$ is the maximal distortion of the quasi-isometry i_P in V. Since the sets E_0 and E_1 are connected, we can find $a > 0$ such that the hemisphere $\Sigma(t) = S^2(i_P(P), t) \cap H$ meets both $i_P[F_0]$ and $i_P[F_1]$ for $0 < t < a$. Hence we can argue as in the first part of the proof of Theorem 3.1, or as on p. 31 in [17], to conclude that $M(\Gamma_1') = \infty$. Thus $M(\Gamma) = \infty$.

LEMMA 4.6. *Suppose that D and D' are domains in the half space $x_3 > 0$, that T and T' are plane domains in $x_3 = 0$ which are free boundary surfaces of D and D', respectively, and that g is a homeomorphism of $D \cup T$ onto $D' \cup T'$ which is quasiconformal in D. Then the boundary mapping g^* is a plane quasiconformal mapping of T onto T' with maximal dilatation*

$$K(g^*) \leq \min(K_I(g), K_0(g))^2. \tag{4.22}$$

Proof. Since T and T' are free boundary surfaces, $D_1 = D \cup T \cup \tilde{D}$ and $D_1' = D' \cup T' \cup \tilde{D}'$ are domains, where $\tilde{D}$ and $\tilde{D}'$ denote the symmetric images of D and D' in $x_3 = 0$. We can next extend g by reflection to obtain a quasiconformal mapping g_1 of D_1 onto D_1'

with $K_I(g_1) = K_I(g)$ and $K_0(g_1) = K_0(g)$. (See Corollary 5 in [4].) Then arguing as in the proof of Theorem 10 in [4], we conclude that g^* is actually a plane quasiconformal mapping of T onto T'.

To complete the proof of (4.22), it is sufficient to show that g^* has maximal dilatation

$$K(g^*) \leqslant K_0(g)^2,$$

for then, by symmetry, it will follow that

$$K(g^*) = K(g^{*-1}) \leqslant K_0(g^{-1})^2 = K_I(g)^2.$$

Next by virtue of the analytic definition for plane quasiconformal mappings, it is sufficient to show that

$$L(P)^2 \leqslant K_0(g)^2 J^T(P)$$

at each point $P \in T$, where g^* is differentiable with $J^T > 0$; here L and J^T are the distortion functions of (4.9) with g^* and T in place of f and S. Fix such a point P. By performing preliminary similarity mappings in the plane $x_3 = 0$, we may assume without loss of generality that $P = 0$, that $g^*(P) = 0$, and that

$$g^*(x_1, x_2) = (ax_1, x_2) + o(|x_1| + |x_2|), \tag{4.23}$$

where $a \geqslant 1$. We then must show that

$$a \leqslant K_0(g)^2. \tag{4.24}$$

For this, fix $\varepsilon > 0$, choose $0 < b < 1$ so that

$$abp(ab) \leqslant A + \varepsilon, \tag{4.25}$$

where p and A are as in Lemma 3.6, and choose $c > 0$ so that $B^3(c) \subset D_1$. Then for $0 < u < 2^{-\frac{1}{2}}c$ let E_0 and E_1 be the segments $x_1 = \pm bu$, $|x_2| \leqslant u$, $x_3 = 0$, and let Γ_1 and Γ_2 be the families of arcs which join E_0 and E_1 in D_1 and R^3, respectively. Then each arc $\gamma \in \Gamma_2 - \Gamma_1$ must contain a subarc which joins $S^2(2^{\frac{1}{2}}u)$ to $S^2(c)$. Hence

$$M(\Gamma_2 - \Gamma_1) \leqslant 4\pi \left(\log 2^{-\frac{1}{2}} \frac{c}{u}\right)^{-2},$$

and we may choose u_1 so that

$$p(b) = M(\Gamma_2) \leqslant M(\Gamma_1) + \frac{\varepsilon}{ab} \tag{4.26}$$

for $0 < u < u_1$.

Next for $u > 0$ and $0 < t < \frac{1}{2}ab$, let F_0 and F_1 be the sets of points which lie within a distance of tu of the segments $x_1 = \pm abu$, $|x_2| \leqslant u$, $x_3 = 0$, and let Γ_3 be the family of arcs which join F_0 and F_1 in R^3. Then by Lemma 3.4,

$$\lim_{t \to 0} M(\Gamma_3) = p(ab),$$

and we may fix $t>0$ so that

$$M(\Gamma_3) \leqslant p(ab) + \frac{\varepsilon}{ab} \tag{4.27}$$

for all $u>0$.

Finally by (4.23), we may choose $u_2>0$ so that $E_0' \subset F_0$ and $E_1' \subset F_1$ for $0<u<u_2$, and hence so that

$$M(\Gamma_1') \leqslant M(\Gamma_3) \tag{4.28}$$

for $0<u<u_2$, where E_i' and Γ_1' are the images of E_i and Γ_1 under the homeomorphism g_1. Fix u so that $0<u<u_1, u_2$. Then we can combine inequalities (4.25) through (4.28) with the inequalities

$$A \leqslant bp(b), \quad M(\Gamma_1) \leqslant K_0(g)^2 M(\Gamma_1')$$

to obtain

$$aA \leqslant K_0(g)^2 A + (2K_0(g)^2+1)\varepsilon.$$

Letting $\varepsilon \to 0$ then yields (4.24), and the proof of Lemma 4.6 is complete.

Proof of Theorem 4.3. We begin by showing that $C(f,P)$ reduces to a single point for each $P \in S$. Fix $P \in S$ and suppose that $C(f,P)$ contains two distinct points. Then Lemmas 4.3 and 4.4 imply that $C(f,P)$ is a continuum, and by (4.19) we can find a pair of distinct points $P_0', P_1' \in C(f,P) \cap S''$. Hence there exist sequences $\{P_{i,n}\}$ in D such that $P_{i,n} \to P$ and $P_{i,n}' \to P_i'$ for $i=0,1$. Since $P_0', P_1' \in S''$, we can use Lemma 4.4 to construct two nondegenerate connected sets E_0', E_1' in D' such that $\bar{E}_0' \cap \bar{E}_1' = \emptyset$ and such that E_i' contains all but a finite number of $P_{i,n}'$, $i=0,1$. Let Γ' be the family of arcs joining E_0' and E_1' in D'. Then clearly $M(\Gamma')<\infty$. On the other hand, we see that $P \in \bar{E}_0 \cap \bar{E}_1$, and hence $M(\Gamma)=\infty$ by Lemma 4.5. This contradicts the fact that f is a quasiconformal mapping, and we conclude that $C(f,P)$ must reduce to a point $P' \in S''$ for each $P \in S$.

We now extend f by setting $f(P)=P'$ for $P \in S$. Then f is continuous in $D \cup S$. Let $S'=f[S]$. Then $S' \subset S''$ and for each $P' \in S'$ we have

$$P \in C(f^{-1},P') \cap S \neq \emptyset.$$

The above argument shows that $C(f^{-1},P')$ reduces to the point P, and we conclude that f is a homeomorphism of $D \cup S$ onto $D' \cup S'$. It is then clear that S' is a free admissible boundary surface of D'.

We must now show that the induced boundary mapping f^* is a quasiconformal mapping of S onto S' with maximal dilatation satisfying (4.20). Fix $\varepsilon>0$ and for $P \in S$ and $P'=f^*(P)$ choose neighborhoods U_P and $U_{P'}$ so that (4.11) holds. Next let V_P and $V_{P'}$ be the neighborhoods of Lemma 4.4, chosen so that $V_P \subset U_P$, $V_{P'} \subset U_{P'}$ and so that f maps

$(D\cup S)\cap V_P$ into $(D'\cup S')\cap V_{P'}$. Finally let H and H'' be the hemiballs corresponding to $D\cap V_P$ and $D'\cap V_{P'}$, and let T and T'' be the plane parts of ∂H and $\partial H''$. Then

$$g_P = i_{P'}\circ f\circ i_P^{-1}$$

is a homeomorphism of $H\cup T$ onto $H'\cup T'\subset H''\cup T''$ which is quasiconformal in H. Since T and $T'\subset T''$ are plane domains which are free boundary surfaces, we have essentially the situation in Lemma 4.6. Thus the boundary mapping g_P^* is a plane quasiconformal mapping of T onto T' with maximal dilatation

$$\begin{aligned} K(g_P^*) &\leqslant \min(K_I(g_P), K_0(g_P))^2 \\ &\leqslant C(i_P)^4 C(i_{P'})^4 \min(K_I(f), K_0(f))^2. \end{aligned}$$

Then arguing as in the last part of the proof of Theorem 4.1, we obtain

$$\sup_{P\in S} K(g_P^*) < \infty, \quad \operatorname*{ess\,sup}_{P\in S} K(g_P^*) \leqslant (1+\varepsilon)^8 \min(K_I(f), K_0(f))^2,$$

and hence f^* is a quasiconformal mapping of S onto S' whose maximal dilatation satisfies (4.20). Moreover, if we let

$$f(x_1, x_2, x_3) = (K^2 x_1, x_2, x_3), \quad K>1,$$

then f maps $x_3>0$ onto itself with $K_I(f)=K$ and $K_0(f)=K^2$, while the boundary mapping f^* sends $x_3=0$ onto itself with $K(f^*)=K^2$. Thus the bound in (4.20) cannot be improved.

5. Upper bounds for the coefficients of certain domains

5.1. We shall derive in this section upper bounds for the coefficients of bounded starlike domains. To do this, we need only find some appropriate quasiconformal mappings, for given any homeomorphism f of D onto B^3, we obviously have

$$K_I(D)\leqslant K_I(f), \quad K_0(D)\leqslant K_0(f), \quad K(D)\leqslant K(f).$$

All of our estimates are based upon the following homeomorphism.

5.2. *Projection mapping.* Suppose that $S\subset R^3$ is homeomorphic to a plane domain and that, for all $Q_1, Q_2\in S$, the acute angle which the segment Q_1Q_2 makes with the basis vector e_3 is never less than $\alpha>0$. Next let T denote the projection of S onto the plane $x_3=0$ and let D denote the set of all points P of the form

$$P = Q + ue_3, \tag{5.1}$$

where $Q\in S$ and u is real. Then for each $P\in D$ the representation (5.1) is unique and we define

$$f(P) = f(Q) + aue_3, \tag{5.2}$$

where $f(Q)$ denotes the projection of Q onto $x_3=0$ and a is some fixed positive number.

LEMMA 5.1. *The mapping f is a homeomorphism of D onto itself which maps S onto T, and*

$$\frac{a}{A}\,|P_1-P_2| \leqslant |f(P_1)-f(P_2)| \leqslant A\,|P_1-P_2| \tag{5.3}$$

for all $P_1, P_2 \in D$, where

$$A=\tfrac{1}{2}((a\csc\alpha)^2+2a+1)^{\frac{1}{2}}+\tfrac{1}{2}((a\csc\alpha)^2-2a+1)^{\frac{1}{2}}. \tag{5.4}$$

Proof. Fix points $P_1, P_2 \in D$,

$$P_1=Q_1+u_1e_3, \quad P_2=Q_2+u_2e_3,$$

and let β and γ denote the acute angles which the segments P_1P_2 and Q_1Q_2 make with e_3. From (5.2) it follows that

$$\begin{aligned}|f(P_1)-f(P_2)|^2 &= |f(Q_1)-f(Q_2)|^2+a^2(u_1-u_2)^2\\ &=((\sin\beta)^2+a^2(\cos\beta\pm\cot\gamma\sin\beta)^2)\,|P_1-P_2|^2\\ &=B^2|P_1-P_2|^2, \quad B>0,\end{aligned} \tag{5.5}$$

and it is then not difficult to verify that $a/C \leqslant B \leqslant C$, where C is equal to the right hand side of (5.4) with γ in place of α. Now the hypotheses on S imply that $\alpha \leqslant \gamma \leqslant \pi/2$. Hence $C \leqslant A$ and (5.3) follows from (5.5).

COROLLARY 5.1. *If $a=1$ in (5.2), then f is a quasi-isometry with maximal distortion*

$$C(f) \leqslant \cot\alpha+1,$$

and a quasiconformal mapping with

$$K(f) \leqslant (\tfrac{1}{2}((\cot\alpha)^2+4)^{\frac{1}{2}}+\tfrac{1}{2}\cot\alpha)^{\frac{3}{2}} \leqslant (\cot\alpha+1)^{\frac{3}{2}}.$$

COROLLARY 5.2. *If $a=\sin\alpha$ in (5.2), then f is a quasiconformal mapping with*

$$K_I(f)^2 \leqslant K(f)^2 \leqslant 2^{-\frac{1}{2}} \cot\frac{\alpha}{2} \csc\frac{\alpha}{2},$$

$$K_0(f)^2 \leqslant 2^{\frac{1}{2}} \cot\frac{\alpha}{2} \cos\frac{\alpha}{2}.$$

Proofs. If we set $a=1$ in (5.4), we have

$$A=\tfrac{1}{2}((\cot\alpha)^2+4)^{\frac{1}{2}}+\tfrac{1}{2}\cot\alpha \leqslant \cot\alpha+1, \tag{5.6}$$

while if we set $a=\sin\alpha$ in (5.4), we get

$$A=\tfrac{1}{2}(2+2\sin\alpha)^{\frac{1}{2}}+\tfrac{1}{2}(2-2\sin\alpha)^{\frac{1}{2}}=2^{\frac{1}{2}}\cos\frac{\alpha}{2}. \tag{5.7}$$

We see from (5.3) that

$$L(P) \leqslant A, \quad l(P) \geqslant \frac{a}{A}, \quad J(P)=a$$

for all $P \in D$, and hence Corollaries 5.1 and 5.2 follow from Lemma 1.1 and (5.6) or (5.7), respectively.

5.3. *Starlike domains.* A domain $D \subset R^3$ is said to be *starlike* at a point $Q \in D$ if the closed segment PQ lies in D whenever $P \in D$. Suppose next that D is a domain which is bounded and starlike at the origin 0, and that $Q \in \partial D$. For each $P \in \partial D$, $P \neq Q$, we let $\alpha(P,Q)$ denote the acute angle which the segment PQ makes with the ray from 0 through Q, and we define

$$\alpha(Q) = \liminf_{P \to Q} \alpha(P, Q), \quad 0 \leqslant \alpha(Q) \leqslant \tfrac{1}{2}\pi. \tag{5.8}$$

If ∂D has a tangent plane at Q whose normal forms an acute angle β with the ray from 0 through Q, then $\alpha(Q) = \frac{1}{2}\pi - \beta$.

THEOREM 5.1. *Suppose that D is a domain which is bounded and starlike at the origin, and that $\alpha(Q) \geqslant \alpha > 0$ for all $Q \in \partial D$. Then*

$$\begin{aligned} K_I(D)^2 &\leqslant K(D)^2 \leqslant 2^{-\frac{1}{2}} \cot \frac{\alpha}{2} \csc \frac{\alpha}{2}, \\ K_0(D)^2 &\leqslant 2^{\frac{1}{2}} \cot \frac{\alpha}{2} \cos \frac{\alpha}{2}. \end{aligned} \tag{5.9}$$

Proof. Fix $a > 0$. Since D is bounded and starlike at the origin and since $\alpha(Q) > 0$ for all $Q \in \partial D$, each point $P \in D$, $P \neq 0$, has a unique representation of the form $P = uQ$, where $Q \in \partial D$ and $0 < u < 1$. For each such P we define

$$f(P) = u^a f(Q), \qquad f(Q) = \frac{Q}{|Q|}, \tag{5.10}$$

and we let $f(0) = 0$. Then f is a homeomorphism of R^3 onto itself which carries D onto B^3, and a tedious but elementary argument, similar to the one given in the proof of Lemma 5.1, shows that

$$L(P) \leqslant A \frac{|f(P)|}{|P|}, \quad l(P) \geqslant \frac{a}{A} \frac{|f(P)|}{|P|}, \quad J(P) = a \frac{|f(P)|^3}{|P|^3}, \tag{5.11}$$

for all $P \in D$, $P \neq 0$, where A is as in (5.4). The bound on $L(P)$ implies that f is ACL and that f is differentiable a.e. in D. Since $J > 0$ in D, we conclude from (5.11) and Lemma 1.1 that

$$K_I(f)^2 \leqslant \frac{A^3}{a^2}, \qquad K_0(f)^2 \leqslant \frac{A^3}{a}.$$

Finally if we set $a = \sin\alpha$, we obtain, as in Corollary 5.2,

$$K_I(f)^2 \leq K(f)^2 \leq 2^{-\frac{1}{2}} \cot\frac{\alpha}{2} \csc\frac{\alpha}{2},$$

$$K_0(f)^2 \leq 2^{\frac{1}{2}} \cot\frac{\alpha}{2} \cos\frac{\alpha}{2},$$

and this implies (5.9).

5.4. *Convex domains.* We can use Theorem 5.1 to obtain the following upper bounds for the coefficients of convex domains.

THEOREM 5.2. *Suppose that* $0 < a \leq b$, *that* D *is a convex domain, and that* $B^3(a) \subset D \subset B^3(b)$. *Then* (5.9) *holds with* $\alpha = \arcsin(a/b)$. *In particular,*

$$K_I(D) \leq K(D) < 8^{\frac{1}{2}} \frac{b}{a}, \qquad K_0(D) < 8^{\frac{1}{4}} \left(\frac{b}{a}\right)^{\frac{1}{2}}. \tag{5.12}$$

Proof. The hypotheses imply that D is bounded and starlike at the origin. Hence (5.9) will follow if we can show that

$$\alpha(Q) \geq \arc\sin\frac{a}{b}$$

for all $Q \in \partial D$. For this fix $Q \in \partial D$, let C_1 be the finite cone which consists of the union of all open segments PQ with $P \in B^3(a)$, and let C_2 be the symmetric image of C_1 in Q. Since D is convex and $B^3(a) \subset D$, it follows that $C_1 \subset D$ and $C_2 \subset C(\bar{D})$. Thus

$$\partial D \cap C_1 = \emptyset, \qquad \partial D \cap C_2 = \emptyset,$$

and since $D \subset B^3(b)$, we conclude that

$$\alpha(Q) \geq \arc\sin\frac{a}{|Q|} \geq \arc\sin\frac{a}{b}.$$

Finally if $\alpha = \arc\sin(a/b)$, then

$$2^{-\frac{1}{2}} \cot\frac{\alpha}{2} \csc\frac{\alpha}{2} < 8^{\frac{1}{2}} (\sin\alpha)^{-2} = 8^{\frac{1}{2}} \left(\frac{b}{a}\right)^2,$$

$$2^{\frac{1}{2}} \cot\frac{\alpha}{2} \cos\frac{\alpha}{2} < 8^{\frac{1}{2}} (\sin\alpha)^{-1} = 8^{\frac{1}{2}} \frac{b}{a},$$

and we obtain the less precise but simpler bounds in (5.12).

6. Lower bounds for the coefficients of certain domains

6.1. In this section we shall obtain some lower bounds for the inner coefficient of domains which have a certain separation property. In general, it is much more difficult to obtain a significant lower bound for a coefficient of a given domain D than it is to obtain an upper bound, since one must find a lower bound for the corresponding dilatation of each homeomorphism f of D onto B^3. We shall accomplish this by studying what happens to the modulus of a certain arc family Γ under f and then appealing to Lemma 1.2.

6.2. *Main theorem.* We observed in section 2.6 that the coefficients of a right circular cylinder D approach ∞ as the ratio of its radius to height approaches ∞. We establish now a rather general result which gives a lower bound for the order of growth of the inner coefficient of the cylinder D.

THEOREM 6.1. *Suppose that $0<a<b$, that D is a domain in R^3, and that $C(D) \cap B^3(b)$ has at least two components which meet $S^2(a)$. Then*

$$K_I(D) \geq A \log \frac{b}{a}, \tag{6.1}$$

where A is the absolute constant

$$A = \left(\frac{\psi(\frac{1}{2})}{4\pi}\right)^{\frac{1}{2}} \geq .129\ldots, \tag{6.2}$$

and ψ is as in (3.12).

Proof. Let f be any homeomorphism of D onto B^3. We must show that

$$K_I(f) \geq A \log \frac{b}{a}. \tag{6.3}$$

Since the right hand side of (6.3) is continuous in b, it is sufficient to establish (6.3) under the slightly stronger hypothesis that the closed set $H = C(D) \cap \overline{B^3(b)}$ has at least two components which meet $S^2(a)$.

We consider first the special case where f can be extended to be a homeomorphism of $\overline{D}$ onto $\overline{B^3}$. By hypothesis, there exist points $Q_1, Q_2 \in S^2(a)$ which belong to different components of H, and hence we can find disjoint compact sets H_1 and H_2 such that $H = H_1 \cup H_2$ and $Q_1 \in H_1$, $Q_2 \in H_2$. Let F_0 be the closed segment Q_1Q_2, let P_1 be the last point in $F_0 \cap H_1$ as we move from Q_1 toward Q_2 along F_0, let P_2 be the first point in $F_0 \cap H_2$ as we move from P_1 toward Q_2 along F_0, and let E_0 be the closed segment P_1P_2. Then $E_0 \subset \overline{D}$ and P_1, P_2 are points of ∂D which lie in different components of H.

Next let $F_1 = \partial D \cap C(B^3(b))$ and let C be any connected set in ∂D which contains both P_1 and P_2. Since P_1 and P_2 belong to different components of H, $F_1 \cap C \neq \emptyset$. Hence F_1

separates P_1 and P_2 in ∂D, and because ∂D is homeomorphic to S^2, we can find a continuum $E_1 \subset F_1$ which separates P_1 and P_2 in ∂D. (See p. 123 in [11].)

Now let Γ be the family of arcs which join E_0 and E_1 in D. Since $E_0 \subset \overline{B^3(a)}$ and $E_1 \subset C(B^3(b))$, Γ is minorized by the family of arcs which join $S^2(a)$ and $S^2(b)$ in R^3, and hence

$$M(\Gamma) \leqslant 4\pi \left(\log \frac{b}{a}\right)^{-2}. \tag{6.4}$$

On the other hand, we see that E_0' joins P_1' and P_2' in B^3, that E_1' separates P_1' and P_2' in S^2, and that Γ' is the family of arcs which join E_0' and E_1' in B^3. Hence if we map B^3 conformally onto $x_3 > 0$ so that P_1' and P_2' map onto 0 and ∞, we can apply Corollary 3.2 to conclude that

$$M(\Gamma') \geqslant \psi(\tfrac{1}{2}). \tag{6.5}$$

We then obtain (6.3) from Lemma 1.2, (6.4), and (6.5).

We consider now the general case. For each positive integer n let D_n denote the image of $B^3(n/(n+1))$ under f^{-1}, let $H_n = C(D_n) \cap \overline{B^3(b)}$, and let f_n denote the restriction of f to D_n. The hypotheses imply there exist points $Q_1, Q_2 \in S^2(a)$ which belong to different components of H. Let C and C_n denote the components of H and H_n which contain Q_1. Then the C_n are nonincreasing in n,

$$C = \bigcap_n C_n,$$

and since $Q_2 \notin C$, there exists an n such that $Q_2 \notin C_n$. Thus Q_1 and Q_2 lie in different components of H_n, and we can appeal to what was proved above to conclude that

$$K_I(f) \geqslant K_I(f_n) \geqslant A \log \frac{b}{a}.$$

This completes the proof for Theorem 6.1.

6.3. *An alternative formulation.* There is a useful inverted form of Theorem 6.1 which we will need for studying what effect the presence of a spire in ∂D has on the coefficients of D.

THEOREM 6.2. *Suppose that $0 < a < b$, that D is a domain in R^3, and that $C(D) \cap C(B^3(a))$ has at least two components which meet $S^2(b)$. Then*

$$K_I(D) \geqslant A \log \frac{b}{a}, \tag{6.6}$$

where A is the absolute constant in (6.2).

Proof. Let f be a homeomorphism of D onto B^3. We want to show that (6.3) holds; the last argument in the proof of Theorem 6.1 shows we may assume that f can be extended to be a homeomorphism of $\bar{D}$ onto $\overline{B^3}$. By hypothesis we can find $Q_1, Q_2 \in S^2(b)$ which belong to different components of $H = C(D) \cap C(B^3(a))$. Let F_0 be any closed arc in $S^2(b)$ joining these points. Arguing as before, we can find a closed subarc $E_0 \subset \bar{D}$ with endpoints P_1, P_2 which lie in different components of H. Let $F_1 = \partial D \cap \overline{B^3(a)}$. Then F_1 separates P_1 and P_2 in ∂D, and hence a continuum $E_1 \subset F_1$ separates these points in ∂D. If we let Γ denote the family of arcs which join E_0 and E_1 in D, then (6.4) and (6.5) hold, and we obtain (6.3) as before.

6.4. *Bound for convex domains.* If we apply Theorem 6.1 to a right circular cylinder D with radius b and height h, we obtain

$$K_I(D) \geqslant A \log \frac{(4b^2+h^2)^{\frac{1}{2}}}{h} \geqslant A \log \frac{2b}{h}.$$

This is a rather poor estimate for the order of growth of $K_I(D)$, since the class of domains considered in Theorem 6.1 is so large. For the domains in this class which are also convex, we have the following sharper bound.

THEOREM 6.3. *Suppose that* $0 < a < b$, *that* D *is a convex domain in* R^3, *and that* $C(D) \cap B^3(b)$ *has at least two components which meet* $S^2(a)$. *Then*

$$K_I(D) \geqslant 2^{\frac{1}{2}} A \left(\left(\frac{b}{a} \right)^2 - 1 \right)^{\frac{1}{2}}, \tag{6.7}$$

where A *is the absolute constant in* (6.2).

Proof. Let f be any homeomorphism of D onto B^3. We must show that

$$K_I(f) \geqslant 2^{\frac{1}{2}} A \left(\left(\frac{b}{a} \right)^2 - 1 \right)^{\frac{1}{2}}. \tag{6.8}$$

As in the proof of Theorem 6.1, it is sufficient to establish (6.8) under the hypothesis that $H = C(D) \cap \overline{B^3(b)}$ has at least two components which meet $S^2(a)$.

Consider first the special case where f can be extended to be a homeomorphism of $\bar{D}$ onto $\overline{B^3}$. By hypothesis there exist points $Q_1, Q_2 \in S^2(a)$ which belong to different components of H, and since D is convex, we can find planes T_1, T_2 such that $Q_i \in T_i$ and $T_i \subset C(D)$ for $i = 1, 2$. Let F_0 be the union of the two closed segments from 0 drawn perpendicular to T_1 and T_2. Now the parts of T_1 and T_2 in $\overline{B^3(b)}$ must belong to different components of H. Hence F_0 has a closed subarc $E_0 \subset \bar{D}$ with endpoints P_1, P_2 which lie in different components of H. Then, as in the proof of Theorem 6.1, there exists a continuum

$E_1 \subset \partial D \cap C(B^3(b))$ which separates P_1 and P_2 in ∂D, and if we let Γ denote the family of arcs which join E_0 and E_1 in D, we have

$$M(\Gamma') \geqslant \psi(\tfrac{1}{2}). \tag{6.9}$$

We need an upper bound for $M(\Gamma)$. Let G denote the set of points in D which lie within a distance of $(b^2-a^2)^{\frac{1}{2}}$ from E_0, and set $\varrho=(b^2-a^2)^{-\frac{1}{2}}$ in G and $\varrho=0$ in $C(G)$. Since D lies between the planes T_1 and T_2, it is not difficult to show that the distance between E_0 and E_1 is not less than $(b^2-a^2)^{\frac{1}{2}}$ and that

$$m(G) \leqslant 2\pi a(b^2-a^2).$$

Thus $\varrho \in F(\Gamma)$,

$$M(\Gamma) \leqslant \iiint_{R^3} \varrho^3 d\omega \leqslant 2\pi a(b^2-a^2)^{-\frac{1}{2}}, \tag{6.10}$$

and we obtain (6.8) from Lemma 1.2, (6.9), and (6.10).

For the general case let D_n be the image of $B^3(n/(n+1))$ under f^{-1}, let

$$H_n = C(D_n) \cap \overline{B^3(b)},$$

and let f_n denote the restriction of f to D_n. Next pick points $Q_1, Q_2 \in S^2(a)$ which belong to different components of H. Then there exists an n such that Q_1, Q_2 belong to different components of H_n. Since D_n is a subdomain of the convex domain D, we can find planes T_1, T_2 such that $Q_i \in T_i$ and $T_i \subset C(D_n)$ for $i=1,2$. The above argument then shows that

$$K_I(f) \geqslant K_I(f_n) \geqslant 2^{\frac{1}{2}} A\left(\left(\frac{b}{a}\right)^2-1\right)^{\frac{1}{2}},$$

and this completes the proof of Theorem 6.3.

6.5. *Remarks.* It is not difficult to verify that

$$\log x < 2^{\frac{1}{2}}(x^2-1)^{\frac{1}{2}}$$

for $1 < x < \infty$, and hence Theorem 6.3 yields a better lower bound for the inner coefficient of a convex domain than that given by Theorem 6.1. Moreover, if the conjectured inequality (3.14) were true, we could take

$$A = \frac{\pi^{\frac{1}{2}}}{2q} = .337\ldots$$

in Theorems 6.1, 6.2, and 6.3. On the other hand, we should point out that these three theorems give sharp bounds for the *order* of growth of $K_I(D)$ as $b/a \to \infty$.

To see this in the case of Theorem 6.1, for $1<b<\infty$ let $\mathcal{D}(b)$ denote the class of domains $D\subset R^3$ such that $C(D)\cap B^3(b)$ has at least two components which meet S^2, and let

$$g(b)=\inf_D K_I(D), \tag{6.11}$$

where the infimum is taken over all $D\in\mathcal{D}(b)$. If $1<b'<b$, then the mapping

$$f(x)=x\,|x|^{c-1}, \quad c=\frac{\log b'}{\log b},$$

is a homeomorphism of R^3 onto itself such that for each domain $D\subset R^3$, $D\in\mathcal{D}(b)$ if and only if $D'\in\mathcal{D}(b')$. From Lemma 1.1 it follows that $K_I(f)=c^{-1}$, and we obtain

$$g(b)\leqslant\frac{\log b}{\log b'}g(b').$$

Thus $g(b)/\log b$ is nonincreasing in $1<b<\infty$ and

$$\lim_{b\to\infty}\frac{g(b)}{\log b}=B\geqslant A>0. \tag{6.12}$$

Hence $g(b)\sim B\log b$ as $b\to\infty$, and we see that the lower bound for the order of growth of $K_I(D)$ given in Theorem 6.1 cannot be improved. The above argument shows that the same is true of Theorem 6.2.

We exhibit a particular domain D to show that the order is right in Theorem 6.3. For $0<\alpha<\pi$, let $P_1=(\cos\frac{1}{2}\alpha,0,0)$ and $P_2=(-\cos\frac{1}{2}\alpha,0,0)$, and let

$$D=B^3(P_1,1)\cap B^3(P_2,1).$$

Then D is a lens shaped domain which can be mapped by means of an inversion onto a convex wedge D', bounded by two half planes which meet at an angle α. Hence from Theorem 7.1 we obtain

$$K_I(D)=K_I(D')=\left(\frac{\pi}{\alpha}\right)^{\frac{1}{2}}. \tag{6.13}$$

Next it is easy to see that D is itself convex and that $C(D)\cap B^3(b)$ has two components which meet $S^2(a)$, where

$$a=1-\cos\frac{\alpha}{2}, \quad b=\sin\frac{\alpha}{2}, \quad \frac{b}{a}=\cot\frac{\alpha}{4}. \tag{6.14}$$

From (6.13) and (6.14) it follows that

$$K_I(D)=\frac{\pi^{\frac{1}{2}}}{2}\left(\operatorname{arc\,cot}\frac{b}{a}\right)^{-\frac{1}{2}}\sim\frac{\pi^{\frac{1}{2}}}{2}\left(\frac{b}{a}\right)^{\frac{1}{2}}$$

as $b/a \to \infty$, and thus the order of the lower bound in Theorem 6.3 cannot be improved. This example also yields the upper bound

$$A \leqslant \left(\frac{\pi}{8}\right)^{\frac{1}{2}} = .626...$$

for the constant in Theorem 6.3.

6.6. *Bounds for a right circular cylinder.* We conclude this section by determining how fast the inner coefficient of a right circular cylinder grows as the ratio of its radius to height approaches ∞.

Suppose that $0 < h < 2b$ and that D is the right circular cylinder

$$D = \left\{x\colon (x_1^2 + x_2^2)^{\frac{1}{2}} < b, \quad |x_3| < \frac{h}{2}\right\}.$$

Then from Theorem 6.3 we obtain

$$K_I(D) \geqslant 2A\left(\frac{b}{h}\right)^{\frac{1}{2}} > .259\left(\frac{b}{h}\right)^{\frac{1}{2}}, \tag{6.15}$$

where A is as in (6.2). Next the homeomorphism

$$f(x_1, x_2, x_3) = \left(\frac{x_1}{b}, \frac{x_2}{b}, \frac{2x_3}{h}\right)$$

maps D onto a right circular cylinder D', where $B^3(1) \subset D' \subset B^3(2^{\frac{1}{2}})$. Theorem 5.2 implies that

$$K_I(D')^2 \leqslant 2^{-\frac{1}{2}} \cot\frac{\pi}{8} \csc\frac{\pi}{8} = 4.46...,$$

and since $K_I(f)^2 = 2b/h$, we obtain

$$K_I(D) < 2.99\left(\frac{b}{h}\right)^{\frac{1}{2}}. \tag{6.16}$$

Neither of the constants given in (6.15) and (6.16) is best possible. For example an independent argument, based on Corollary 3.3, shows that we can replace the constant .259... in (6.15) by .408... . Moreover, if the conjectured inequality (3.14) were true, we could improve this constant to .667... . Similarly, by making a more judicious choice for a in (5.10), we can improve the bound for $K_I(D')$ and thus reduce the constant in (6.16). Nevertheless, these inequalities do show that the order of growth for the inner coefficient of a right circular cylinder is equal to the square root of the ratio of its radius to height.

7. The inner coefficient of a dihedral wedge

7.1. *Introduction.* In the last two sections we obtained lower and upper bounds for coefficients of various domains. In the next three sections we will calculate coefficients of three different domains.

To calculate a given coefficient of a domain $D \subset R^3$, we first must obtain a lower bound for the corresponding dilatation of each homeomorphism f of D onto B^3. Then we must show that this bound is actually assumed by some extremal homeomorphism of D onto B^3. Clearly it is the sharp lower bounds which are most difficult to obtain. We use two different methods. The first involves selecting a certain extremal family of arcs Γ in D and then comparing $M(\Gamma)$ and $M(\Gamma')$. In the second, we choose arc families $\Gamma_1 \subset D$ and $\Gamma_2 \subset S \subset \partial D$ and then compare the relations between $M(\Gamma_1)$ and $M^S(\Gamma_2)$ and between $M(\Gamma_1')$ and $M^{S'}(\Gamma_2')$.

7.2. *Dihedral wedge.* Let (r,θ,x_3) be cylindrical coordinates in R^3. We say that a domain D is a *dihedral wedge of angle* α, $0<\alpha\leqslant 2\pi$, if it can be mapped by means of a similarity transformation f onto the domain

$$D_\alpha = \{x=(r,\theta,x_3):\quad 0<\theta<\alpha, \quad |x|<\infty\}. \tag{7.1}$$

The image of the x_3-axis under f^{-1} is said to be the *edge* of the dihedral wedge D. We shall calculate here the inner coefficient of a convex dihedral wedge. But first we require the following preliminary result.

LEMMA 7.1. *Suppose that* $0<\alpha\leqslant 2\pi$, *that* E_0 *is the segment* $r=0$, $-1\leqslant x_3\leqslant 0$, *and that* E_1 *is the ray* $r=0$, $1\leqslant x_3\leqslant\infty$. *If* Γ_α *is the family of arcs which join* E_0 *to* E_1 *in* D_α, *then*

$$M(\Gamma_\alpha)=\frac{\alpha}{\pi}\,\psi(1),$$

where ψ *is as in* (3.12).

Proof. Suppose that $0<\alpha<\beta\leqslant 2\pi$. Then the homeomorphism

$$f(r,\theta,x_3)=\left(r,\frac{\beta}{\alpha}\theta,x_3\right)$$

maps D_α onto D_β, Γ_α onto Γ_β, and since $K_I(f)^2=\beta/\alpha$, we obtain

$$M(\Gamma_\beta)\leqslant\frac{\beta}{\alpha}M(\Gamma_\alpha). \tag{7.2}$$

Suppose next that $0<\alpha<\beta\leqslant 2\pi$, that β/α is a positive integer n, and for $m=1,2,\ldots,n$ let Γ_β^m be the family of arcs which join E_0 and E_1 in the dihedral wedge

$$D_\beta^m=\left\{x=(r,\theta,x_3):\ \frac{m-1}{n}\beta<\theta<\frac{m}{n}\beta,\ |x|<\infty\right\}.$$

Then the Γ_β^m are separate families, $\Gamma_\beta \supset \Gamma_\beta^1 \cup \ldots \cup \Gamma_\beta^n$, and hence by Lemma 2.1 of [17]

$$M(\Gamma_\beta) \geqslant M(\Gamma_\beta^1) + \ldots + M(\Gamma_\beta^n) = \frac{\beta}{\alpha} M(\Gamma_\alpha). \tag{7.3}$$

With the aid of (7.2) and (7.3) it is now easy to show that

$$M(\Gamma_\beta) = \frac{\beta}{\alpha} M(\Gamma_\alpha), \tag{7.4}$$

whenever $0 < \alpha, \beta < 2\pi$ and β/α is rational. Then since $M(\Gamma_\alpha)$ is nondecreasing in α, an elementary limiting argument, together with (7.2), gives (7.4) even when β/α is irrational. Finally if we set $\beta = \pi$ in (7.4), we obtain

$$M(\Gamma_\alpha) = \frac{\alpha}{\pi} M(\Gamma_\pi) = \frac{\alpha}{\pi} \psi(1)$$

as desired.

7.3. *The inner coefficient.* We now calculate the inner coefficient of a convex dihedral wedge.

THEOREM 7.1. *Suppose that D is a convex dihedral wedge of angle α. Then*

$$K_I(D) = \left(\frac{\pi}{\alpha}\right)^{\frac{1}{2}}. \tag{7.5}$$

Proof. We may assume, for convenience of notation, that D is the dihedral wedge D_α in (7.1). Then since D_α is convex, $0 < \alpha \leqslant \pi$, and we see that the folding mapping

$$f(r, \theta, x_3) = \left(r, \frac{\pi}{\alpha}\theta, x_3\right)$$

is a homeomorphism of D_α onto the half space D_π with $K_I(f)^2 = \pi/\alpha$. Since we can map D_π onto B^3 by means of a Möbius transformation g, we have

$$K_I(D_\alpha)^2 \leqslant K_I(g \circ f)^2 = K_I(f)^2 = \frac{\pi}{\alpha}.$$

To complete the proof of (7.5), it is sufficient to show that

$$K_I(f)^2 \geqslant \frac{\pi}{\alpha} \tag{7.6}$$

for each quasiconformal mapping f of D_α onto D_π. For this let E_1 and E_2 be the segments $r=0$, $-1 \leqslant x_3 \leqslant 0$ and $r=0$, $0 \leqslant x_3 \leqslant 1$, let E_3 and E_4 be the rays $r=0$, $1 \leqslant x_3 \leqslant \infty$ and $r=0$,

$-\infty \leqslant x_3 \leqslant -1$, and let Γ_1 and Γ_2 be the families of arcs which join E_1 to E_3 and E_2 to E_4 in D_α. Then by Lemma 7.1

$$M(\Gamma_1) = M(\Gamma_2) = \frac{\alpha}{\pi}\psi(1). \tag{7.7}$$

Since D_α is locally connected at each point of its boundary, f can be extended to be a homeomorphism of $\bar{D}_\alpha$ onto $\bar{D}_\pi$ by Theorem 1 of [18]. Then E_1', E_2', E_3', E_4' are continua in ∂D_π which satisfy the hypotheses of Corollary 3.3. Hence

$$M(\Gamma_1') \geqslant \psi(1) \quad \text{or} \quad M(\Gamma_2') \geqslant \psi(1), \tag{7.8}$$

and (7.6) follows from Lemma 1.2, (7.7), and (7.8).

7.4. *The other coefficients.* We have not been able to calculate the other coefficients of a convex dihedral wedge. However, the following estimates are easily obtained.

THEOREM 7.2. *Suppose that D is a convex dihedral wedge of angle α. Then*

$$\left(\frac{\pi}{\alpha}\right)^{\frac{1}{4}} \leqslant K_0(D) \leqslant \left(\frac{\pi}{\alpha}\right)^{\frac{1}{2}}, \qquad \left(\frac{\pi}{\alpha}\right)^{\frac{1}{2}} \leqslant K(D) \leqslant \left(\frac{\pi}{\alpha}\right)^{\frac{3}{4}}. \tag{7.9}$$

Proof. The lower bounds follow directly from (1.13), (1.14), and (7.5). The upper bounds result from the fact that the mappings

$$f(r, \theta, x_3) = \left(r, \frac{\pi}{\alpha}\theta, \frac{\pi}{\alpha}x_3\right), \qquad g(r, \theta, x_3) = \left(r, \frac{\pi}{\alpha}\theta, \left(\frac{\pi}{\alpha}\right)^{\frac{1}{2}} x_3\right)$$

are homeomorphisms of D_α onto D_π with $K_0(f) = (\pi/\alpha)^{\frac{1}{2}}$ and $K(g) = (\pi/\alpha)^{\frac{3}{4}}$. We conjecture that

$$K_0(D) = \left(\frac{\pi}{\alpha}\right)^{\frac{1}{2}}.$$

7.5 *Some lower bounds.* We can combine these results with Theorem 2.3 to obtain the following lower bounds for the coefficients of a large class of domains.

THEOREM 7.3. *Suppose that D is a domain in R^3, that U is a neighborhood of a point $Q \in \partial D$, and that $D \cap U = \Delta \cap U$, where Δ is a dihedral wedge of angle α which has Q as a point of its edge. Then the coefficients of D are not less than the corresponding coefficients of Δ. In particular if Δ is convex,*

$$K_I(D) \geqslant \left(\frac{\pi}{\alpha}\right)^{\frac{1}{2}}, \qquad K_0(D) \geqslant \left(\frac{\pi}{\alpha}\right)^{\frac{1}{4}}.$$

Proof. Since Q is on the edge of Δ, Δ is raylike at Q, and the results follow from (2.6), (7.5), and (7.9).

Theorem 7.3 yields lower bounds for the coefficients of all polyhedra. For example if D is a convex polyhedron with n faces, then the planes of a pair of adjacent faces must bound a dihedral wedge Δ which contains D and is of angle α, $0<\alpha\leqslant(n-3)(n-1)^{-1}\pi$ and we obtain

$$K_I(D)\geqslant\left(\frac{n-1}{n-3}\right)^{\frac{1}{2}},\qquad K_0(D)\geqslant\left(\frac{n-1}{n-3}\right)^{\frac{1}{4}}.$$

Or if D is a rectangular parallelepiped, then (7.9) gives

$$K_I(D)\geqslant 2^{\frac{1}{2}},\qquad K_0(D)\geqslant 2^{\frac{1}{4}}.$$

We can also use Theorem 7.3 to obtain lower bounds for the coefficients of a domain with a piecewise smooth boundary. For example, suppose that D is the right circular cylinder

$$D=\{x=(r,\theta,x_3):\quad 0\leqslant r<b,\quad 0<x_3<h\},$$

fix $0<a<b,h$, and let $g(u)=(b^2-u^2)^{\frac{1}{2}}$ in $|u|\leqslant a$ and $g(u)=(b^2-a^2)^{\frac{1}{2}}$ in $|u|>a$. From Corollary 5.1 it follows that

$$f(x_1,x_2,x_3)=(x_1+g(x_2),x_2,x_3)$$

is a quasiconformal mapping of R^3 onto itself and that $K(f)\to 1$ as $a\to 0$. Now $D'\cap U=\Delta\cap U$, where $U=B^3(a)$ and Δ is the quarter space $x_1>0$, $x_3>0$. Hence

$$K_I(D)\,K(f)\geqslant K_I(D')\geqslant 2^{\frac{1}{2}},\qquad K_0(D)\,K(f)\geqslant K_0(D')\geqslant 2^{\frac{1}{4}},$$

and letting $a\to 0$ yields

$$K_I(D)\geqslant 2^{\frac{1}{2}},\qquad K_0(D)\geqslant 2^{\frac{1}{4}}.$$

8. The outer coefficient of an infinite cylinder

8.1. *Infinite cylinder.* Let (r,θ,x_3) be cylindrical coordinates in R^3. We say that a domain is an *infinite circular cylinder* if it can be mapped by means of a similarity transformation onto the domain

$$D=\{x=(r,\theta,x_3):\ 0\leqslant r<1,\ |x|<\infty\}.\tag{8.1}$$

We shall calculate in this section the outer coefficient of an infinite circular cylinder. For this we require the following preliminary result.

LEMMA 8.1. *Suppose that D is the cylinder in* (8.1), *that D' is the half space $x_3>0$, that f is a homeomorphism of $\bar{D}-\{\infty\}$ onto $\bar{D}'-\{0\}-\{\infty\}$, and that*

$$\lim_{x_3\to-\infty} f(x)=0,\qquad \lim_{x_3\to+\infty} f(x)=\infty.\tag{8.2}$$

Then for each $a' > 0$, the image of the hemisphere $S^2(a') \cap \bar{D}'$ under f^{-1} lies between two planes $x_3 = a_0$ and $x_3 = a_1$, where

$$0 \leqslant a_1 - a_0 \leqslant A K_I(f) \tag{8.3}$$

and A is an absolute constant.

Proof. Fix $a' > 0$, let $\Sigma' = S^2(a') \cap \bar{D}'$, and set

$$a_0 = \inf_{x \in \Sigma} x_3, \qquad a_1 = \sup_{x \in \Sigma} x_3.$$

Clearly we may assume that $a_0 < a_1$, for otherwise there is nothing to prove. Next let

$$E_0 = \{x: x \in \bar{D}, -\infty < x_3 \leqslant a_0\}, \quad E_1 = \{x: x \in \bar{D}, a_1 \leqslant x_3 < \infty\},$$

and let Γ be the family of arcs which join E_0 and E_1 in D. Then it is easy to see that

$$M(\Gamma) = \pi(a_1 - a_0)^{-2}. \tag{8.4}$$

Next it follows from (8.2) that $\bar{E}_0' = E_0' \cup \{0\}$ and that $\bar{E}_1' = E_1' \cup \{\infty\}$. Hence Γ' is the family of arcs which join the continua $\bar{E}_0'$ and $\bar{E}_1'$ in D'. Finally since $\bar{E}_0'$ and $\bar{E}_1'$ both meet $S^2(a')$, we have

$$M(\Gamma') \geqslant \psi(1) \tag{8.5}$$

by virtue of Corollary 3.1, and (8.3) follows from Lemma 1.2, (8.4), and (8.5) with

$$A = \left(\frac{\pi}{\psi(1)}\right)^{\frac{1}{2}}.$$

8.2. *The outer coefficient.* We calculate now the outer coefficient of an infinite circular cylinder.

THEOREM 8.1. *Suppose that D is an infinite circular cylinder. Then*

$$K_0(D) = \left(\frac{q}{2}\right)^{\frac{1}{2}} = 1.14\ldots, \tag{8.6}$$

where as in (3.2)

$$q = \int_0^{\pi/2} (\sin u)^{-\frac{1}{2}} du.$$

Proof. We may assume, for convenience of notation, that D is the cylinder in (8.1). Next let (t, θ, φ) be spherical coordinates in R^3, where the polar angle φ is measured from the positive half of the x_3-axis, let D' be the half space $x_3 > 0$, and set

$$f_1(r, \theta, x_3) = (t, \theta, \varphi), \tag{8.7}$$

where

$$r = \left(\frac{1}{q}\int_0^{\varphi} (\sin u)^{-\frac{1}{2}} du\right)^2, \quad x_3 = \frac{2}{q} \log t. \tag{8.8}$$

Then f_1 is a continuously differentiable homeomorphism of D onto D' which maps each infinitesimal sphere onto an infinitesimal ellipsoid whose axes are proportional to

$$\frac{dt}{dx_3}=\frac{qt}{2},\quad \frac{t\,d\varphi}{dr}=\frac{qt}{2}\left(\frac{\sin\varphi}{r}\right)^{\frac{1}{2}},\quad \frac{t\sin\varphi\,d\theta}{r\,d\theta}=t\frac{\sin\varphi}{r}. \tag{8.9}$$

It is easy to show by means of elementary calculus that

$$\left(\frac{r}{\sin\varphi}\right)^{\frac{1}{2}}=\frac{1}{q}(\sin\varphi)^{-\frac{1}{2}}\int_0^{\varphi}(\sin u)^{-\frac{1}{2}}du$$

increases from $2/q$ to 1 as φ increases from 0 to $\pi/2$. Hence from Lemma 1.1 and (8.9) it follows that

$$K_0(f_1)^2=\sup_{P\in D}\frac{L(P)^3}{J(P)}=\frac{q}{2},$$

and since we can map D' onto B^3 by means of a Möbius transformation g, we have

$$K_0(D)\leqslant K_0(g\circ f_1)=K_0(f_1)=\left(\frac{q}{2}\right)^{\frac{1}{2}}.$$

To complete the proof for (8.6), it is sufficient to show that

$$K_0(f)\geqslant\left(\frac{q}{2}\right)^{\frac{1}{2}} \tag{8.10}$$

for every quasiconformal mapping f of D onto D', the half space $x_3>0$. Choose such a mapping f and let f_1 be the mapping given in (8.7) and (8.8). Then $f\circ f_1^{-1}$ is a quasiconformal mapping of D' onto D' which can be extended to be a homeomorphism of $\bar{D}'$ onto $\bar{D}'$. We can next choose a Möbius transformation g such that $h=g\circ f\circ f_1^{-1}$ is a homeomorphism of $\bar{D}'$ onto $\bar{D}'$ with $h(0)=0$ and $h(\infty)=\infty$. Since $g\circ f=h\circ f_1$, it follows from the properties of f_1 that we can extend $g\circ f$ to be a homeomorphism of $\bar{D}-\{\infty\}$ onto $\bar{D}'-\{0\}-\{\infty\}$ such that

$$\lim_{x_3\to-\infty} g\circ f(x)=0,\qquad \lim_{x_3\to+\infty} g\circ f(x)=\infty.$$

Finally since $K_0(g\circ f)=K_0(f)$, we may assume, without loss of generality, that the given mapping f satisfies the hypotheses of Lemma 8.1.

Now choose $0<a'<b'$, and let C', S', and E' be the parts of D', $\partial D'$, and the positive x_3-axis bounded by $S^2(a')$ and $S^2(b')$. Next let Γ_1' be the family of arcs which join E' to S' in C' and let Γ_2' be the family of arcs which join $S^2(a')$ to $S^2(b')$ in S'. Then by virtue of Lemma 3.8,

$$M(\Gamma_1')=\frac{2\pi}{q^2}\log\frac{b'}{a'}, \tag{8.11}$$

while a familiar calculation yields

$$M^{S'}(\Gamma_2') = 2\pi\left(\log\frac{b'}{a'}\right)^{-1}. \tag{8.12}$$

Lemma 8.1 implies that f^{-1} maps $S^2(a')\cap\bar{D}'$ and $S^2(b')\cap\bar{D}'$ into $a_0\leqslant x_3\leqslant a_1$ and $b_0\leqslant x_3\leqslant b_1$, respectively, where

$$0\leqslant a_1-a_0,\, b_1-b_0\leqslant AK_I(f),\quad a_0<b_1. \tag{8.13}$$

Hence we obtain
$$M(\Gamma_1)\geqslant\frac{\pi}{2}(b_0-a_1) \tag{8.14}$$

from Lemma 3.7, while a direct calculation shows that

$$M^S(\Gamma_2)\geqslant 2\pi(b_1-a_0)^{-1}. \tag{8.15}$$

Now S and S' are free admissible boundary surfaces of D and D', respectively. Hence by Theorem 4.3, f^*, the restriction of f to S, is a quasiconformal mapping of S onto S' with maximal dilatation

$$K(f^*)\leqslant\min(K_I(f),K_0(f))^2,$$

and we have
$$M^S(\Gamma_2)\leqslant K(f^*)\,M^{S'}(\Gamma_2')\leqslant K_0(f)^2M^{S'}(\Gamma_2') \tag{8.16}$$

from Theorem 4.2. If we combine the above inequalities with Lemma 1.2, we obtain

$$\pi^2\frac{b_0-a_1}{b_1-a_0}\leqslant M(\Gamma_1)\,M^S(\Gamma_2)\leqslant K_0(f)^4\,M(\Gamma_1')\,M^{S'}(\Gamma_2')=\frac{4\pi^2}{q^2}K_0(f)^4. \tag{8.17}$$

Now (8.17) holds for all $0<a'<b'$, while (8.2) and (8.13) imply that

$$\lim_{\substack{a'\to 0\\ b'\to\infty}}\frac{b_0-a_1}{b_1-a_0}=1. \tag{8.18}$$

Combining (8.17) and (8.18) yields (8.10), and the proof for Theorem 8.1 is complete.

8.3. *The inner coefficient.* We have not been able to calculate the other coefficients for an infinite circular cylinder. However, we have obtained the following bounds for the inner coefficient.

THEOREM 8.2. *Suppose that D is an infinite circular cylinder. Then*

$$2^{1/6}\leqslant K_I(D)\leqslant 2^{\frac{1}{2}}. \tag{8.19}$$

Proof. Assume that D is the cylinder in (8.1), that D' is the half space $x_3>0$, and let (r,θ,x_3) and (t,θ,φ) be the cylindrical and spherical coordinate systems given in section 8.2. Next set

$$f(r,\theta,x_3)=(t,\theta,\varphi),$$

where
$$r=2^{-\frac{1}{2}}\frac{\sin\varphi}{\sin(\varphi+\pi/4)},\quad x_3=2^{-\frac{1}{2}}\log t.$$

Then f is a continuously differentiable homeomorphism of D onto D' which maps each infinitesimal sphere onto an infinitesimal ellipsoid whose axes are proportional to

$$\frac{dt}{dx_3}=2^{\frac{1}{2}}t,\quad \frac{td\varphi}{dr}=2t(\sin(\varphi+\pi/4))^2,\quad \frac{t\sin\varphi\, d\theta}{r\,d\theta}=2^{\frac{1}{2}}t\sin(\varphi+\pi/4). \tag{8.20}$$

Since
$$2^{\frac{1}{2}}\sin(\varphi+\pi/4)\leqslant 2(\sin(\varphi+\pi/4))^2,\ 2^{\frac{1}{2}},$$

it follows that
$$K_I(f)^2=\sup_{P\in D}\frac{J(P)}{l(P)^3}=2^{\frac{1}{2}}.$$

Then because we can map D' onto B^3 by means of a Möbius transformation g, we have

$$K_I(D)\leqslant K_I(g\circ f)=K_I(f)=2^{\frac{1}{4}}.$$

To establish the left hand part of (8.19), we must show that

$$K_I(f)\geqslant 2^{1/6} \tag{8.21}$$

for each quasiconformal mapping f of D onto D', the half space $x_3>0$. As in the proof of Theorem 8.1, we may assume that f satisfies the hypotheses of Lemma 8.1. Fix $0<a'<b'$ so that $a_1<b_0$, let C' and S' be the parts of D' and $\partial D'$ bounded by $S^2(a')$ and $S^2(b')$, and let Γ_1' and Γ_2' be the families of arcs which join $S^2(a')$ to $S^2(b')$ in C' and S', respectively. Then it is easy to verify that

$$M(\Gamma_1')=2\pi\left(\log\frac{b'}{a'}\right)^{-2}, \tag{8.22}$$

while as in (8.12)
$$M^{S'}(\Gamma_2')=2\pi\left(\log\frac{b'}{a'}\right)^{-1}. \tag{8.23}$$

Lemma 8.1 implies that the images of $S^2(a')\cap\bar{D}'$ and $S^2(b')\cap\bar{D}'$ under f^{-1} lie in $a_0\leqslant x_3\leqslant a_1$ and in $b_0\leqslant x_3\leqslant b_1$, respectively, where (8.13) holds. Hence it follows that

$$M(\Gamma_1)\leqslant\pi(b_0-a_1)^{-2},$$

while as in (8.15)
$$M^S(\Gamma_2)\geqslant 2\pi(b_1-a_0)^{-1}.$$

Now S and S' are free admissible boundary surfaces of D and D'. Thus

$$M(\Gamma_1')\leqslant K_I(f)^2M(\Gamma_1),\qquad M^S(\Gamma_2)\leqslant K_I(f)^2M^{S'}(\Gamma_2')$$

by virtue of Lemma 1.2 and Theorems 4.2 and 4.3. If we combine all of these inequalities, we obtain

$$\frac{1}{2\pi} = M(\Gamma_1')\, M^{S'}(\Gamma_2')^{-2} \leqslant K_I(f)^6\, M(\Gamma_1)\, M^S(\Gamma_2)^{-2} \leqslant \frac{1}{4\pi}\left(\frac{b_1 - a_0}{b_0 - a_1}\right)^2 K_I(f)^6.$$

Again (8.18) holds, and letting $a' \to 0$, $b' \to \infty$ yields (8.21).

8.4. *Some lower bounds.* We shall require the following analogue of Theorem 7.3 in order to derive some lower bounds for the coefficients of a domain which has a spire in its boundary.

THEOREM 8.3. *Suppose that D is a domain in R^3, that U is a half space, and that $D \cap U = \Delta \cap U$, where Δ is an infinite circular cylinder whose axis is perpendicular to ∂U. Then the coefficients of D are not less than the corresponding coefficients of Δ. In particular,*

$$K_I(D) \geqslant 2^{1/6}, \qquad K_0(D) \geqslant \left(\frac{q}{2}\right)^{\frac{1}{2}}.$$

Proof. Assume, for convenience of notation, that U is the half space $x_3 > 0$, choose $P \in \Delta \cap U$, and for each positive n let

$$D_n = \{x : x + ne_3 \in D\}.$$

Then as in the proof of Theorem 2.3, it is easy to show that the D_n converge to their kernel Δ at P. Hence

$$K_I(D) = \liminf_{n \to \infty} K_I(D_n) \geqslant K_I(\Delta),$$

and similarly for the other coefficients, by virtue of Theorem 2.1. The rest follows from (8.6) and (8.19).

8.5. *Folding of an infinite cylinder.* We shall also need the following homeomorphism, which folds an infinite cylinder onto a semi-infinite cylinder, in our study of spires.

LEMMA 8.2. *Suppose that D is the cylinder in* (8.1) *and that D' and E are the parts of D and ∂D which lie in the half space $x_3 < 0$. Then there exists a homeomorphism f of $D \cup E$ onto $D' \cup E$ such that $f(x) = x$ for $x \in E$,*

$$K(f) \leqslant 2\left(\frac{q}{2}\right)^{\frac{3}{2}} = 3.00\ldots \tag{8.24}$$

in D, and

$$L(x) \leqslant \tfrac{3}{2} q^2 = 10.3\ldots \tag{8.25}$$

in D'.

Proof. Let f_1 be the homeomorphism given in (8.7) and (8.8), and let U denote the half space $x_3>0$. Then there exists a Möbius transformation f_2 which carries U onto the dihedral wedge D_π and $U \cap B^3$ onto the dihedral wedge $D_{\pi/2}$, where for $0<\alpha<2\pi$, D_α is as defined in (7.1). We see that $f_2 \circ f_1$ is a homeomorphism of D onto D_π which maps D' onto $D_{\pi/2}$ and E into the closed half plane

$$T=\{x=(r,\theta,x_3):\quad \theta=0\}.$$

Now let f_3 be the folding mapping

$$f_3(r,\theta,x_3)=\left(r,\frac{\theta}{2},x_3\right).$$

Then f_3 maps D_π onto $D_{\pi/2}$ and $f_3(x)=x$ for all $x\in T$. Hence the homeomorphism

$$f=f_1^{-1}\circ f_2^{-1}\circ f_3\circ f_2\circ f_1$$

maps $D\cup E$ onto $D'\cup E$, $f(x)=x$ for $x\in E$, and

$$K(f)\leqslant K_I(f_1)\,K_0(f_1)\,K(f_3)=2\left(\frac{q}{2}\right)^{\frac{3}{2}}$$

in D. Finally from (8.9) we see that

$$|f_1(x)|\leqslant l_{f_1}(x)\leqslant L_{f_1}(x)\leqslant\left(\frac{q}{2}\right)^2|f_1(x)| \tag{8.26}$$

for $x\in D$, while a direct calculation yields

$$L_g(x)\leqslant 1 \quad\text{and}\quad |g(x)|\geqslant\frac{|x|}{6} \tag{8.27}$$

for $x\in B^3\cap U$, where $g=f_2^{-1}\circ f_3\circ f_2$. Inequality (8.25) follows from (8.26) and (8.27).

9. The outer coefficient of a cone

9.1. *Cone.* Let (t,θ,φ) be spherical coordinates in R^3, where the polar angle φ is measured from the positive half of the x_3-axis. We say that a domain is a *circular cone of angle* α, $0<\alpha<\pi$, if it can be mapped by means of a similarity transformation onto the domain

$$D=\{x=(t,\theta,\varphi):\quad 0\leqslant\varphi<\alpha,\quad 0<t<\infty\}. \tag{9.1}$$

We have the following cone analogue of Lemma 8.1.

LEMMA 9.1. *Suppose that D is the cone in* (9.1), *that D' is the half space $x_3>0$, that f is a homeomorphism of $\bar{D}$ onto $\bar{D}'$, and that $f(0)=0$ and $f(\infty)=\infty$. Then for each $a'>0$,* *the image of the hemisphere $S^2(a') \cap \bar{D}'$ under f^{-1} lies between two spheres $S^2(a_0)$ and $S^2(a_1)$, where*

$$1 \leqslant \frac{a_1}{a_0} \leqslant e^{AK_I(f)} \tag{9.2}$$

and A is an absolute constant.

Proof. Fix $a' > 0$, let $\Sigma' = S^2(a') \cap \bar{D}'$, and set

$$a_0 = \inf_{x \in \Sigma} |x|, \qquad a_1 = \sup_{x \in \Sigma} |x|.$$

We may assume that $a_0 < a_1$, for otherwise there is nothing to prove. Next let

$$E_0 = \{x: \quad x \in \bar{D}, \quad |x| \leqslant a_0\}, \qquad E_1 = \{x: \quad x \in \bar{D}, \quad |x| \geqslant a_1\},$$

and let Γ be the family of arcs which join E_0 to E_1 in D. Then Γ is minorized by the family of arcs which join $S^2(a_0)$ to $S^2(a_1)$ in R^3, and hence

$$M(\Gamma) \leqslant 4\pi \left(\log \frac{a_1}{a_0}\right)^{-2}. \tag{9.3}$$

Next since $0 \in E_0'$ and $\infty \in E_1'$, Corollary 3.1 implies that

$$M(\Gamma') \geqslant \psi(1), \tag{9.4}$$

and we obtain (9.2) from Lemma 1.2, (9.3), and (9.4) with

$$A = \left(\frac{4\pi}{\psi(1)}\right)^{\frac{1}{2}}.$$

9.2. *The outer coefficient.* We calculate now the outer coefficient of a convex circular cone.

THEOREM 9.1. *Suppose that D is a convex circular cone of angle α. Then*

$$K_0(D) = \left(\frac{q}{q(\alpha)}\right)^{\frac{3}{2}} (\sin \alpha)^{\frac{1}{2}}, \tag{9.5}$$

where $q = q(\pi/2)$ and

$$q(\alpha) = \int_0^\alpha (\sin u)^{-\frac{1}{2}} du. \tag{9.6}$$

Proof. Assume, for convenience of notation, that D is the cone in (9.1) and let D' be the half space $x_3 > 0$. Next set

$$f(t, \theta, \varphi) = (t', \theta, \varphi'),$$

where

$$t' = t^{a(\sin \alpha)^{-\frac{1}{2}}}, \quad q(\varphi') = a q(\varphi), \quad a = \frac{q}{q(\alpha)}. \tag{9.7}$$

Then f is a continuously differentiable homeomorphism of D onto D' which maps each infinitesimal sphere onto an infinitesimal ellipsoid whose axes are proportional to

$$\frac{dt'}{dt}=\frac{t'}{t}a(\sin\alpha)^{-\frac{1}{2}},\quad \frac{t'd\varphi'}{t\,d\varphi}=\frac{t'}{t}a\left(\frac{\sin\varphi'}{\sin\varphi}\right)^{\frac{1}{2}},\quad \frac{t'\sin\varphi'}{t\sin\varphi}. \tag{9.8}$$

Moreover, since $0<\alpha\leqslant\pi/2$, it is not difficult to show that

$$(\sin\alpha)^{-\frac{1}{2}}\leqslant\left(\frac{\sin\varphi'}{\sin\varphi}\right)^{\frac{1}{2}}\leqslant\frac{q(\varphi')}{q(\varphi)}=a$$

for $0<\varphi\leqslant\alpha$. Hence

$$K_0(f)^2=\sup_{P\in D}\frac{L(P)^3}{J(P)}=\frac{q}{q(\alpha)}(\sin\alpha)^{\frac{1}{2}}$$

by virtue of Lemma 1.1, (9.7), and (9.8). Then since D' is conformally equivalent to B^3, we obtain

$$K_0(D)\leqslant K_0(f)=\left(\frac{q}{q(\alpha)}\right)^{\frac{1}{2}}(\sin\alpha)^{\frac{1}{4}}.$$

To complete the proof for (9.5), we must show that

$$K_0(f)\geqslant\left(\frac{q}{q(\alpha)}\right)^{\frac{1}{2}}(\sin\alpha)^{\frac{1}{4}} \tag{9.9}$$

for each quasiconformal mapping f of D onto the half space D'. Choose any such mapping f. Then arguing as in the proof of Theorem 8.1, we see we may assume that f satisfies the hypotheses of Lemma 9.1. Fix $0<a'<b'$, let C', S', and E' be the parts of D', $\partial D'$, and the positive x_3-axis bounded by $S^2(a')$ and $S^2(b')$, and let Γ_1' and Γ_2' be the families of arcs which join E' to S' in C' and $S^2(a')$ to $S^2(b')$ in S', respectively. Then as in (8.11) and (8.12) we have

$$M(\Gamma_1')=\frac{2\pi}{q^2}\log\frac{b'}{a'},\qquad M^{S'}(\Gamma_2')=2\pi\left(\log\frac{b'}{a'}\right)^{-1}.$$

Lemma 9.1 implies that the images of $S^2(a')\cap\bar{D}'$ and $S^2(b')\cap\bar{D}'$ under f^{-1} lie between $S^2(a_0)$ and $S^2(a_1)$ and between $S^2(b_0)$ and $S^2(b_1)$, respectively, where

$$1\leqslant\frac{a_1}{a_0},\ \frac{b_1}{b_0}\leqslant e^{AK_I(f)},\quad a_0<b_1. \tag{9.10}$$

Hence we obtain

$$M(\Gamma_1)\geqslant 2\pi q(\alpha)^{-2}\log\frac{b_0}{a_1}$$

from Lemma 3.8, while a direct calculation yields

$$M^S(\Gamma_2)\geqslant 2\pi\sin\alpha\left(\log\frac{b_1}{a_0}\right)^{-1}. \tag{9.11}$$

Again S and S' are free admissible boundary surfaces of D and D', and hence

$$M(\Gamma_1) \leqslant K_0(f)^2 M(\Gamma_1'), \qquad M^S(\Gamma_2) \leqslant K_0(f)^2 M^{S'}(\Gamma_2').$$

If we combine the above inequalities, we have

$$\frac{q^2 \sin \alpha}{q(\alpha)^2} \frac{\log \dfrac{b_0}{a_1}}{\log \dfrac{b_1}{a_0}} \leqslant K_0(f)^4. \tag{9.12}$$

Finally if we let $a' \to 0, b' \to \infty$, then $b_1/a_0 \to \infty$ and (9.9) follows from (9.10).

9.3. *The inner coefficient.* We have obtained the following bounds for the inner coefficient of a convex circular cone.

THEOREM 9.2. *Suppose that D is a convex circular cone of angle α. Then*

$$(1+\cos\alpha)^{1/6} \leqslant K_I(D) \leqslant (1+\cos\alpha)^{\frac{1}{4}}. \tag{9.13}$$

Proof. Assume that D is the cone in (9.1) and that D' is the half space $x_3 > 0$. Next set

$$f(t, \theta, \varphi) = (t', \theta, \varphi'),$$

where

$$t' = t^a, \quad \cot \varphi' = \frac{\sin(\alpha - \varphi)}{\sin \varphi}, \quad a = (1-\cos\alpha)^{-\frac{1}{2}}.$$

Then f is a continuously differentiable homeomorphism of D onto D'. A direct calculation shows that

$$K_I(f)^2 = \sup_{P \in D} \frac{J(P)}{l(P)^3} = (1+\cos\alpha)^{\frac{1}{2}},$$

and since D' is conformally equivalent to B^3, we conclude that

$$K_I(D) \leqslant K_I(f) = (1+\cos\alpha)^{\frac{1}{4}}.$$

For the left hand side of (9.13) we must show that

$$K_I(f) \geqslant (1+\cos\alpha)^{1/6} \tag{9.14}$$

for each quasiconformal mapping f of D onto the half space D'. As in the proof of Theorem 9.1, we may assume f satisfies the hypotheses of Lemma 9.1. Fix $0 < a' < b'$ so that $a_1 < b_0$, let C' and S' be the parts of D' and $\partial D'$ bounded by $S^2(a')$ and $S^2(b')$, and let Γ_1' and Γ_2' be the families of arcs which join $S^2(a')$ to $S^2(b')$ in C' and S', respectively. Then as in (8.22) and (8.23),

$$M(\Gamma_1') = 2\pi \left(\log \frac{b'}{a'}\right)^{-2}, \qquad M^{S'}(\Gamma_2') = 2\pi \left(\log \frac{b'}{a'}\right)^{-1}.$$

Lemma 9.1 implies that the images of $S^2(a') \cap \bar{D}'$ and $S^2(b') \cap \bar{D}'$ under f^{-1} lie between $S^2(a_0)$ and $S^2(a_1)$ and between $S^2(b_0)$ and $S^2(b_1)$, respectively, where (9.10) holds. Direct calculation yields

$$M(\Gamma_1) \leqslant 2\pi(1-\cos\alpha)\left(\log\frac{b_0}{a_1}\right)^{-2},$$

while as in (9.11)

$$M^S(\Gamma_2) \geqslant 2\pi\sin\alpha\left(\log\frac{b_1}{a_0}\right)^{-1}.$$

Again S and S' are free admissible boundary surfaces of D and we obtain

$$M(\Gamma_1') \leqslant K_I(f)^2 M(\Gamma_1), \qquad M^S(\Gamma_2) \leqslant K_I(f)^2 M^{S'}(\Gamma_2').$$

Combining all of the above inequalities, we have

$$(1+\cos\alpha)\left(\frac{\log\dfrac{b_0}{a_1}}{\log\dfrac{b_1}{a_0}}\right)^2 \leqslant K_I(f)^6. \tag{9.15}$$

Finally if we let $a' \to 0$, $b' \to \infty$, then $b_0/a_1 \to \infty$ and (9.14) follows from (9.10).

9.4. *Remark.* Suppose that D is the cylinder in (8.1) and that for $0<\alpha<\pi/2$, D_α is the cone in (9.1) translated through the vector $-(\cot\alpha)e_3$. Then the D_α converge to their kernel D at 0 as $\alpha \to 0$, and we may think of D as a cone of angle 0. In particular, since

$$\left(\frac{q}{2}\right)^{\frac{1}{3}} = \lim_{\alpha\to 0}\left(\frac{q}{q(\alpha)}\right)^{\frac{1}{3}}(\sin\alpha)^{\frac{1}{3}},$$

(8.6) is what we get by formally letting $\alpha \to 0$ in (9.5). Similarly the bounds in (8.19) are the limits of those given in (9.13) as $\alpha \to 0$.

9.5. *Some lower bounds.* We conclude this section with the following cone analogue of Theorem 8.3.

THEOREM 9.3. *Suppose that D is a domain in R^3, that U is a neighborhood of a point $Q \in \partial D$, and that $D \cap U = \Delta \cap U$, where Δ is a circular cone of angle α which has Q as its vertex. Then the coefficients of D are not less than the corresponding coefficients of Δ. In particular if Δ is convex,*

$$K_I(D) \geqslant (1+\cos\alpha)^{1/6}, \qquad K_0(D) \geqslant \left(\frac{q}{q(\alpha)}\right)^{\frac{1}{3}}(\sin\alpha)^{\frac{1}{3}}.$$

Proof. Since Q is the vertex of Δ, Δ is raylike at Q, and the results follow from (2.6), (9.5), and (9.13).

10. Spires and ridges

10.1. *Introduction.* In view of the results of section 2.5, it is natural to assume that for a given domain D, the presence of a spire or a ridge in ∂D has a strong influence on the coefficients of D. We shall study this question in detail. It turns out that if ∂D has a spire which is directed into D or a ridge which is directed out of D, then $K(D)=\infty$. In the reverse situations, $K(D)$ may be finite.

10.2. *Spires.* A point set in R^3 is said to be a *spire* if it can be mapped by means of a similarity transformation f onto

$$S=\{x=(r,\theta,x_3):\quad r=g(x_3),\quad 0<x_3\leqslant a\}, \tag{10.1}$$

where $a<\infty$ and g is subject to the following restrictions:

$$\left.\begin{array}{l}\text{(i) } g(u) \text{ is continuous in } 0\leqslant u\leqslant a \text{ and } g(a)=0,\\ \text{(ii) } g'(u) \text{ is continuous and increasing in } 0<u<a,\\ \text{(iii) } \lim\limits_{u\to a} g'(u)=0.\end{array}\right\} \tag{10.2}$$

These conditions imply that $g(u)>0$ in $0\leqslant u<a$ and that

$$\int_0^a \frac{du}{g(u)}=\infty. \tag{10.3}$$

The image of the point $Q=(0,0,a)$ under f^{-1} is called the *vertex* of the spire, the image of the basis vector e_3 is its *direction*, and the image of the disk

$$B=\{x=(r,\theta,x_3):\quad 0\leqslant r<g(0),\quad x_3=0\}$$

is its *base*.

A domain $D\subset R^3$ is said to have a spire in its boundary if some point $Q\in\partial\bar{D}$ has a neighborhood U such that $S=\partial D\cap U$ is a spire with vertex at Q. Let n be the direction of S. Then the points $Q+un$ do not belong to S, and hence not to ∂D, for small $u>0$. Thus there exists a constant $b>0$ such that either $Q+un\in D$ for $0<u<b$ or $Q+un\in C(\bar{D})$ for $0<u<b$. We say that the spire S is *inward directed* in the first case and *outward directed* in the second case.

10.3. *Inward directed spires.* The following result answers a question raised by B. V. Šabat.

THEOREM 10.1. *If D is a domain in R^3 whose boundary contains an inward directed spire, then $K(D)=\infty$.*

Proof. By performing a preliminary similarity transformation, we may assume that the vertex of the spire S is the origin, that its direction is $-e_3$, and that for some $a>0$

$$S \cap B^3(a) = \partial D \cap B^3(a).$$

Then S splits $B^3(a)$ into two domains, and since S is inward directed, $C(\bar{D}) \cap B^3(a)$ is the component of $B^3(a)-S$ which contains the interval $r=0$, $0<x_3<a$. Fix $0<c<1$. Because S is a spire, we can choose b, $0<b<\frac{1}{2}a$, such that $S^2(be_3, bc)$ separates 0 from ∞ in $C(D)$. Hence $C(D) \cap C(B^3(be_3, bc))$ has two components which meet $S^2(be_3, b)$, and we conclude from Theorem 6.2 that

$$K_I(D) \geqslant A \log \frac{1}{c}, \tag{10.4}$$

where A is an absolute constant. Letting $c \to 0$ in (10.4) yields $K_I(D)=\infty$, whence $K(D)=\infty$.

10.4. *Outward directed spires.* In contrast to the above situation, there exist domains with outward directed spires in their boundaries and finite coefficients. We require first the following result.

LEMMA 10.1. *Suppose that $g(u)>0$ for $0 \leqslant u < a \leqslant \infty$ and that*

$$|g(u)-g(v)| \leqslant b|u-v|, \quad b<\infty, \tag{10.5}$$

for $0 \leqslant u, v < a$. Suppose next that D is the domain

$$D=\{x=(r,\theta,x_3): \quad 0 \leqslant r < g(x_3), \quad 0<x_3<a\},$$

that D' is the circular cylinder

$$D'=\left\{x=(r,\theta,x_3): \quad 0 \leqslant r<g(0), \quad 0<x_3<g(0)\int_0^a \frac{du}{g(u)}\right\},$$

and that B is the common base of D and D',

$$B=\{x=(r,\theta,x_3): \quad 0 \leqslant r<g(0), \quad x_3=0\}.$$

Then there exists a homeomorphism f of $D \cup B$ onto $D' \cup B$ such that $f(x)=x$ for $x \in B$ and

$$K(f) \leqslant (\tfrac{1}{2}(b^2+4)^{\frac{1}{2}}+\tfrac{1}{2}b)^{\frac{4}{3}} \leqslant (b+1)^{\frac{4}{3}}. \tag{10.6}$$

Proof. Let

$$f(r,\theta,x_3)=(rh(x_3),\theta,j(x_3)),$$

where

$$h(x_3)=\frac{g(0)}{g(x_3)}, \qquad j(x_3)=g(0)\int_0^{x_3}\frac{du}{g(u)}.$$

Then f is a homeomorphism of $D \cup B$ onto $D' \cup B$ and $f(x)=x$ for $x \in B$. Since g satisfies (10.5), f is ACL and a.e. differentiable in D. Next an easy computation shows that at each point $x=(r,\theta,x_3) \in D$ where $g'(x_3)$ exists,

$$\frac{J(x)}{l(x)^3}=\frac{L(x)^3}{J(x)}=\left(\tfrac{1}{2}(c^2+4)^{\frac{1}{2}}+\tfrac{1}{2}c\right)^3 \leqslant (c+1)^3, \tag{10.7}$$

where
$$c=\frac{r}{g(x_3)}\,|g'(x_3)| \leqslant |g'(x_3)| \leqslant b$$

by virtue of (10.5). Hence (10.6) follows from (10.7) and Lemma 1.1.

Now for $0<a<\infty$ set $g(u)=(a-u)^2$ in $0 \leqslant u \leqslant a$. Then g satisfies the hypotheses of Lemma 10.1 with $b=2a$, and

$$D=\{x=(r,\theta,x_3): \quad 0 \leqslant r < g(|x_3|), \quad 0 \leqslant |x_3| < a\}$$

is a domain with a pair of outward directed spires in its boundary. Since g satisfies (10.3), we can use the mapping of Lemma 10.1 to construct a homeomorphism f of D onto an infinite circular cylinder D' with $K(f) \leqslant (2a+1)^{\frac{3}{2}}$. Finally since a may be chosen arbitrarily small, we obtain the following result.

THEOREM 10.2. *For each $\varepsilon>0$ there exists a domain $D \subset R^3$ whose boundary contains an outward directed spire and whose coefficients are within ε of the corresponding coefficients of an infinite circular cylinder.*

10.5. *An example.* We consider next the class of domains D which are obtained by adding an arbitrary number of outward directed spires to a half space. More precisely, let T be the plane $x_3=0$, let $\{B_n\}$ be a collection of disjoint open disks in T, and for each n let S_n be a spire with base B_n and direction e_3. Then

$$(T-\bigcup_n B_n) \cup (\bigcup_n S_n)$$

is a surface which divides R^3 into two domains. By Theorem 10.1, the upper domain has infinite coefficients. Let D be the lower domain. One might think that $K(D)$ could be made arbitrarily large by making the spires S_n very sharp or by adjusting their positions on T. We show, however, that this is not the case.

THEOREM 10.3. *For each such domain D, $K(D) \leqslant 4.5$.*

Proof. Let D_n denote the points of D which lie below S_n,

$$D_n=\{x=P-ue_3: \quad P \in S_n, \quad 0<u<\infty\},$$

and let D'_n and E_n denote the parts of D_n and ∂D_n which lie in the half space $x_3 < 0$. The proof of Theorem 10.3 depends upon the following result.

LEMMA 10.2. *For each n there exists a homeomorphism f_n of $D_n \cup E_n$ onto $D'_n \cup E_n$ such that $f_n(x) = x$ for $x \in E_n$,*

$$K(f_n) \leq 4.5 \tag{10.8}$$

in D_n, and

$$L_{f_n}(x) \leq 10.4 \tag{10.9}$$

in D'_n.

We now define a mapping f of D by setting

$$f(x) = \begin{cases} f_n(x) & \text{if} \quad x \in D_n, \\ x & \text{if} \quad x \in F = D - (\bigcup_n D_n). \end{cases}$$

By (10.8) and (10.9), f is a homeomorphism which is ACL and a.e. differentiable in D. Each point of $D - F$ has a neighborhood in which $K(f) \leq 4.5$. Since $f(x) = x$ in F and since almost every point of F is a point of linear density in the directions of the coordinate axes [15],

$$L(x) = l(x) = J(x) = 1$$

a.e. in F. We conclude from Lemma 1.1 that f is a 4.5-quasiconformal mapping of D, and hence that $K(D) \leq 4.5$ as desired.

10.6. *Proof of Lemma* 10.2. Fix n and for convenience of notation write $S = S_n$, $B = B_n$, $D = D_n$, $D' = D'_n$, and $E = E_n$. By performing a preliminary translation, we may assume that S is the spire in (10.1). We now define a homeomorphism f of $D \cup E$ onto $D' \cup E$, such that $f(x) = x$ for $x \in E$ and

$$K(f) \leq 4.5 \text{ in } D, \qquad L(x) \leq 10.4 \text{ in } D', \tag{10.10}$$

as follows.

Suppose first that $|g'(u)| < \frac{1}{2}$ in $0 < u < a$, let D_1 be the part of D in $x_3 > 0$, and let D'_1 be the symmetric image of D' in $x_3 = 0$. Since g satisfies (10.3) and the hypotheses of Lemma 10.1 with $b = \frac{1}{2}$, there exists a homeomorphism f_1 of $D_1 \cup B$ onto $D'_1 \cup B$ such that $f_1(x) = x$ for $x \in B$ and

$$K(f_1) \leq \left(\frac{17^{\frac{1}{2}} + 1}{4}\right)^{\frac{3}{2}} < 1.45 \tag{10.11}$$

in D_1. Now let $D_2 = D' \cup B \cup D'_1$. By Lemma 8.2 we can find a homeomorphism f_2 of $D_2 \cup E$ onto $D' \cup E$ such that $f_2(x) = x$ for $x \in E$ and

$$K(f_2) < 3.01 \text{ in } D_2, \qquad L_{f_2}(x) \leq 10.4 \text{ in } D'. \tag{10.12}$$

Now set

$$f(x) = \begin{cases} f_2 \circ f_1(x) & \text{if} \quad x \in D_1 \cup B, \\ f_2(x) & \text{if} \quad x \in D' \cup E. \end{cases}$$

Then f is a homeomorphism of $D \cup E$ onto $D' \cup E$, $f(x)=x$ for $x \in E$, and f satisfies (10.10) by virtue of (10.11), (10.12), and Corollary 5 of [4].

Suppose next that there exists a number b, $0<b<a$, such that $g'(b)=-\frac{1}{2}$, let C be the infinite circular cylinder $0 \leqslant r < g(b)$, and set

$$f_1(x)=\begin{cases} x-h(r)\,e_3 & \text{if} \quad x \in (D \cup E)-\bar{C}, \\ x-be_3 & \text{if} \quad x \in D \cap \bar{C}, \end{cases}$$

where h is the inverse function of g. Then f_1 is a homeomorphism of $D \cup E$ and $f_1(x)=x$ for $x \in E$. Moreover, since $|g'(u)|>\frac{1}{2}$ in $0<u<b$, $|h'(u)|<2$ in $g(b)<u<g(0)$, and we conclude from Corollary 5.1, with $\cot\alpha=2$, that

$$K(f_1) \leqslant (2^{\frac{1}{2}}+1)^{\frac{3}{2}} < 3.76, \qquad L_{f_1}(x) \leqslant 3, \tag{10.13}$$

in D. Now f_1 translates $D \cap C$ onto a domain D_1 which lies below the spire

$$S_1=\{x=(r,\theta,x_3): \quad r=g(x_3+b), \quad 0<x_3 \leqslant a-b\}.$$

Let D_1' and E_1 denote the parts of D_1 and ∂D_1 which lie in $x_3<0$. Since $|g'(u+b)|<\frac{1}{2}$ in $0<u<a-b$, by what was proved above we can find a homeomorphism f_2 of $D_1 \cup E_1$ onto $D_1' \cup E_1$ such that $f_2(x)=x$ for $x \in E_1$ and

$$K(f_2) \leqslant 4.5 \text{ in } D_1, \qquad L_{f_2}(x) \leqslant 10.4 \text{ in } D_1'. \tag{10.14}$$

Finally set

$$f(x)=\begin{cases} f_1(x) & \text{if} \quad x \in (D \cup E)-\bar{C}, \\ f_2 \circ f_1(x) & \text{if} \quad x \in D \cap \bar{C}. \end{cases}$$

Then f is a homeomorphism of $D \cup E$ onto $D' \cup E$, $f(x)=x$ for $x \in E$, and (10.10) holds by virtue of (10.13), (10.14), and Corollary 5 of [4]. Hence the proof of Lemma 10.2 is complete.

10.7. *Inaccessible boundary points.* We can use Theorem 10.3 to show that there exists a domain which has finite coefficients and some inaccessible boundary points. For choose a sequence of disjoint open disks $\{B_n\}$ which converge to the origin, erect a spire S_n of height 1 on each B_n, and let D be the corresponding domain, as defined in section 10.5. Then each point of the segment $r=0$, $0<x_3 \leqslant 1$ is an inaccessible boundary point of D, while $K(D) \leqslant 4.5$ by Theorem 10.3. Another such example has been given by Zorič [21].

It is clear how the above construction can be slightly modified to yield a domain with finite coefficients, for which the set of inaccessible boundary points has positive 3-dimensional measure.

10.8. *A lower bound.* Finally we have the following sharp lower bound for the coefficients of a domain whose boundary contains a spire.

THEOREM 10.4. *If D is a domain in R^3 whose boundary contains a spire, then the coefficients of D are not less than the corresponding coefficients of an infinite circular cylinder. In particular,*

$$K_I(D) \geqslant 2^{1/6}, \qquad K_0(D) \geqslant \left(\frac{q}{2}\right)^{\frac{1}{2}}.$$

Proof. By performing a preliminary similarity transformation, we may assume that the vertex of the spire S is the origin and that its direction is $-e_3$. Next by Theorem 10.1 we may assume that S is outward directed. Finally by definition we can choose $a>0$ so that

$$S \cap B^3(a) = \partial D \cap B^3(a).$$

Then S splits $B^3(a)$ into two domains and $D \cap B^3(a)$ is the component of $B^3(a) - S$ which contains the segment $r=0$, $0<x_3<a$. Let f_1 denote inversion in $S^2(a)$, let D_1 denote the image of D under f_1, and let U_1 denote the half space $x_3>a$. Since S is a spire, it follows that

$$D_1 \cap U_1 = \{x=(r,\theta,x_3): \quad 0 \leqslant r < g(x_3), \quad a < x_3 < \infty\},$$

where $g'(u)$ is continuous in $a<u<\infty$ and

$$\lim_{u\to\infty} g'(u) = 0. \tag{10.15}$$

Fix $\varepsilon>0$, choose $b>a$ so that $|g'(u)|<\varepsilon$ in $b<u<\infty$, let U be the half space $x_3>b$, and let Δ be the infinite circular cylinder

$$\Delta = \{x=(r,\theta,x_3): \quad 0 \leqslant r < g(b), \quad |x| < \infty\}.$$

Since $|g'(u)|<\varepsilon$ in $b<u<\infty$,

$$\int_b^\infty \frac{du}{g(u)} = \infty,$$

and hence by Lemma 10.1 there exists a homeomorphism f_2 of $D_1 \cap \bar{U}$ onto $\Delta \cap \bar{U}$ such that $f_2(x)=x$ in $D_1 \cap \partial U$ and $K(f_2) \leqslant (1+\varepsilon)^{\frac{3}{2}}$ in $D_1 \cap U$. Set

$$f_3(x) = \begin{cases} f_2(x) & \text{if} \quad x \in D_1 \cap \bar{U}, \\ x & \text{if} \quad x \in D_1 - \bar{U}. \end{cases}$$

Then $f = f_3 \circ f_1$ is a homeomorphism of D onto a domain D', $D' \cap U = \Delta \cap U$, and $K(f) \leqslant (1+\varepsilon)^{\frac{3}{2}}$. The desired lower bounds are now obtained by first applying Theorem 8.3 to D' and then letting $\varepsilon \to 0$. Theorem 10.2 shows that these bounds cannot be improved.

10.9. *Ridges.* A point set in R^3 is said to be a *ridge* if it can be mapped by means of a similarity transformation f onto

$$S = \{x = (x_1, x_2, x_3): \quad |x_2| = g(x_1), \quad 0 < x_1 \leqslant a, \quad |x_3| < b\}, \tag{10.16}$$

where $a < \infty$, $b \leqslant \infty$, and g satisfies the conditions in (10.2). The image of the line segment

$$E = \{x = (x_1, x_2, x_3): \quad x_1 = a, \quad x_2 = 0, \quad |x_3| < b\}$$

under f^{-1} is called the *edge* of the ridge and the image of the vector e_1 is its *direction.*

A domain $D \subset R^3$ is said to have a ridge in its boundary if some point $Q \in \partial \bar{D}$ has a neighborhood U such that $S = \partial D \cap U$ is a ridge with Q a point of its edge E. Let n be the direction of S. As in the case of spires, there exists a constant $c > 0$ such that either $Q + un \in D$ for $0 < u < c$ or $Q + un \in C(\bar{D})$ for $0 < u < c$. The ridge S is said to be *inward directed* in the first case and *outward directed* in the second case.

10.10. *Outward directed ridges.* We have the following analogue of Theorem 10.1 for ridges.

THEOREM 10.5. *If D is a domain in R^3 whose boundary contains an outward directed ridge, then* $K(D) = \infty$.

Proof. By performing a preliminary similarity transformation, we may assume that the edge of the ridge S is the line segment $x_1 = x_2 = 0$, $|x_3| < 1$, that its direction is $-e_1$, that $Q = 0$, and that for some $a > 0$

$$S \cap B^3(a) = \partial D \cap B^3(a).$$

Then S divides $B^3(a)$ into two domains, and since S is outward directed, $D \cap B^3(a)$ is the component of $B^3(a) - S$ which contains the interval $0 < x_1 < a$, $x_2 = x_3 = 0$. Because S is a ridge, given $0 < c < 1$, we can choose $0 < b < \frac{1}{2}a$ so that D separates $(b, bc, 0)$ from $(b, -bc, 0)$ in $B^3(be_1, b)$. Thus $C(D) \cap B^3(be_1, b)$ has two components which meet $S^2(be_1, bc)$, and we conclude from Theorem 6.1 that

$$K_I(D) \geqslant A \log \frac{1}{c}. \tag{10.17}$$

Letting $c \to 0$ in (10.17) yields $K_I(D) = \infty$, whence $K(D) = \infty$.

10.11. *Inward directed ridges.* In contrast to the above situation, there exist domains with inward directed ridges in their boundaries and finite coefficients. For example, given $0 < a < \infty$, set $g(u) = \min(u^2, a^2)$ and let

$$T = \{x = (x_1, x_2, x_3): \quad |x_2| = g(x_1), \quad x_1 \geqslant 0\}.$$

Then T bounds a domain $D \subset R^3$ which has an inward directed ridge in its boundary. For $x=(x_1,x_2,x_3) \in D$ let

$$f_1(x)=\begin{cases} x-g(x_1)(\operatorname{sgn} x_2)\,e_2 & \text{if} \quad x_1 \geq 0, \\ x & \text{if} \quad x_1 < 0, \end{cases}$$

where the function $\operatorname{sgn} u$ is defined to be $u/|u|$ when $u \neq 0$ and 0 when $u=0$. Then f is a homeomorphism of D onto a dihedral wedge D' of angle 2π,

$$D'=\{x=(r,\theta,x_3): \quad 0<\theta<2\pi, \quad |x|<\infty\},$$

and $K(f_1) \leq (2a+1)^{\frac{2}{3}}$ by virtue of Corollary 5.1. Now D' has finite coefficients, since

$$f_2(r,\theta,x_3)=(r,\tfrac{1}{2}\theta,(\tfrac{1}{2})^{\frac{1}{2}}x_3)$$

maps D' onto a half space with $K(f_2)=2^{\frac{2}{3}}$. Finally because a may be chosen arbitrarily small, we obtain the following analogue of Theorem 10.2.

THEOREM 10.6. *For each $\varepsilon>0$ there exists a domain $D \subset R^3$ whose boundary contains an inward directed ridge and whose coefficients are within ε of the corresponding coefficients of a dihedral wedge of angle 2π.*

10.12. *An example.* We consider next a class of domains analogous to those studied in section 10.5. Let g be any function which satisfies (10.2). Next let T be the plane $x_1=0$, S the ridge

$$S=\{x=(x_1,x_2,x_3): \quad |x_2|=g(x_1), \quad 0<x_1 \leq a\},$$

and B the base of S,

$$B=\{x=(x_1,x_2,x_3): \quad |x_2|<g(0), \quad x_1=0\}.$$

Then $(T-B) \cup S$ is a surface which divides R^3 into two domains. The domain which contains the negative half of the x_1-axis has infinite coefficients by Theorem 10.5. Let D be the other domain. We show that the coefficients of D remain bounded no matter how sharp we make the ridge S.

THEOREM 10.7. *For each such domain D, $K(D) \leq 2.6$.*

Proof. Set

$$f_1(x_1,\, x_2,\, x_3)=(a-x_1,\, x_2,\, x_3), \quad f_2(r,\,\theta,\,x_3)=(r,\,\tfrac{3}{4}\theta,\,(\tfrac{3}{4})^{\frac{1}{2}}x_3).$$

Then $f_2 \circ f_1$ is a homeomorphism of D onto a domain D_1, which lies in the dihedral wedge $0<\theta<\frac{3}{2}\pi$, and

$$K(f_2 \circ f_1)=K(f_2)=(\tfrac{4}{3})^{\frac{2}{3}}<1.25.$$

Now for each pair of points $Q_1, Q_2 \in \partial D_1$, the angle between the segment Q_1Q_2 and the vector $e_2 - e_1$ is never less than $\pi/4$. Hence Corollary 5.1 yields a homeomorphism f_3 of D_1 onto the half space $x_2 - x_1 > 0$ with

$$K(f_3) \leqslant \left(\frac{5^{\frac{1}{2}}+1}{2}\right)^{\frac{3}{2}} < 2.06,$$

and we conclude that

$$K(D) \leqslant K(f_3 \circ f_2 \circ f_1) < 2.6.$$

10.13. *A lower bound.* We conclude this section with the following implicit sharp lower bound for the coefficients of a domain whose boundary contains a ridge.

THEOREM 10.8. *If D is a domain in R^3 whose boundary contains a ridge, then the coefficients of D are not less than the corresponding coefficients of a dihedral wedge of angle 2π.*

Proof. Suppose that D contains a ridge in its boundary, and for $0 < a < \infty$, let U be the open cube bounded by the planes $x_1 = a \pm a$, $x_2 = \pm a$, $x_3 = \pm a$. By performing a preliminary similarity transformation, we may choose a so that

$$\partial D \cap U = \{x = (x_1, x_2, x_3): \quad |x_2| = g(x_1), \quad 0 < x_1 \leqslant a, \quad |x_3| < a\},$$

where g satisfies (10.2). Next by Theorem 10.5, we may assume that the ridge is inward directed and hence that

$$C(D) \cap U = \{x = (x_1, x_2, x_3): \quad |x_2| \leqslant g(x_1), \quad 0 < x_1 \leqslant a, \quad |x_3| < a\}.$$

Now fix b so that $\frac{1}{2}a < b < a$, set $h(u)$ equal to $g(u)$ for $b \leqslant u \leqslant a$ and 0 for $u > a$, and extend h so that $h(u) = h(2b - u)$ for all u. Next for $x = (x_1, x_2, x_3) \in D$ let

$$f(x) = \begin{cases} x - h(x_1)\,(\operatorname{sgn} x_2)\, e_2 & \text{if } \; |x_3| < a - b, \\ x - h(x_1)\,(\operatorname{sgn} x_2) \left(\dfrac{a - |x_3|}{b}\right) e_2 & \text{if } \; a - b \leqslant |x_3| \leqslant a, \\ x & \text{if } \; |x_3| > a. \end{cases}$$

Then f is a homeomorphism of D onto a domain D' and

$$D' \cap B^3(Q, t) = \Delta' \cap B^3(Q, t),$$

where $Q = (a, 0, 0)$, $0 < t < a - b$, and Δ' is a dihedral wedge of angle 2π. Using Corollary 5.1, we can show that $K(f) \to 1$ as $b \to a$, and hence the desired conclusion follows from Theorem 7.3.

11. The space of domains quasiconformally equivalent to a ball

11.1. *Space of domains.* Let $\mathfrak{D}$ denote the class of all domains $D \subset R^3$ with $K(D) < \infty$. Next given $D, D' \in \mathfrak{D}$, we define the *distance* between D and D' as

$$d(D, D') = \inf_f (\log K(f)), \tag{11.1}$$

where the infimum is taken over all homeomorphisms f of D onto D'. We identify two domains D and D' whenever $d(D, D') = 0$. Then it is trivial to show that d is a metric on $\mathfrak{D}$. In this final section we show that $\mathfrak{D}$ is complete and nonseparable under d.

11.2. *Completeness.* The completeness is equivalent to the following result.

THEOREM 11.1. *Suppose that $\{D_n\}$ is a sequence of domains in $\mathfrak{D}$ and that*

$$\lim_{m,\, n \to \infty} d(D_m, D_n) = 0. \tag{11.2}$$

Then there exists a domain $D_0 \in \mathfrak{D}$ such that

$$\lim_{n \to \infty} d(D_n, D_0) = 0. \tag{11.3}$$

Proof. By virtue of (11.2) we may choose a subsequence $\{n_m\}$ such that

$$d(D_{n_m}, D_{n_{m+1}}) < 2^{-m}$$

for $m = 1, 2, \ldots$. Next fix a pair of distinct points $P_0, P_1 \in D_{n_1}$, let f_m be a homeomorphism of D_{n_m} onto $D_{n_{m+1}}$ with

$$\log K(f_m) < 2^{-m}, \tag{11.4}$$

and let φ_m be a Möbius transformation of D_{n_m} onto a domain $D'_m \subset R^3$, chosen so that $g_m(P_0) = P_0$ and $g_m(P_1) = P_1$ where

$$g_m = \varphi_m \circ f_{m-1} \circ \ldots \circ f_1.$$

Then g_m is a homeomorphism of D_{n_1} onto D'_m and (11.4) implies that

$$\log K(g_m) < 1$$

for all m. Hence by Lemma 5 of [6], the g_m are uniformly bounded and equicontinuous on each compact subset of D_{n_1}, and there exists a subsequence $\{m_k\}$ such that

$$\lim_{k \to \infty} g_{m_k}(x) = g(x) \tag{11.5}$$

uniformly on each compact subset of D_{n_1}. Since $g(P_0) = P_0$ and $g(P_1) = P_1$, Lemma 7 of [6] implies that g is a homeomorphism of D_{n_1} onto a domain $D_0 \subset R^3$. Fix m and for $m' > m$ set

$$h_{m'} = g_{m'} \circ f_1^{-1} \circ \ldots \circ f_{m-1}^{-1} = \varphi_{m'} \circ f_{m'-1} \circ \ldots \circ f_m.$$

Then $h_{m'}$ is a homeomorphism of D_{nm} for which

$$\log K(h_{m'}) < 2^{-m+1}, \tag{11.6}$$

and from (11.5) it follows that

$$\lim_{k\to\infty} h_{mk}(x) = h(x) \tag{11.7}$$

uniformly on each compact subset of D_{nm}, where

$$h = g \circ f_1^{-1} \circ \ldots \circ f_{m-1}^{-1}.$$

Thus h is a homeomorphism of D_{nm} onto D_0, and from (11.6), (11.7), and Lemma 2.1 it follows that

$$\log K(h) \leqslant 2^{-m+1}.$$

Hence

$$d(D_{nm}, D_0) \leqslant 2^{-m+1}, \tag{11.8}$$

and (11.3) follows from (11.2) and (11.8).

11.3. *Lower bounds for the dilatations of a homeomorphism.* We require the following result in the proof that $\mathfrak{D}$ is not separable.

THEOREM 11.2. *Suppose that D and D' are domains in R^3, that U and U' are neighborhoods of $Q \in \partial D$ and $Q' \in \partial D'$, and that $D \cap U = \Delta \cap U$ and $D' \cap U' = \Delta' \cap U'$, where Δ is a dihedral wedge with Q a point of its edge and Δ' is a half space. If f is a homeomorphism of D onto D' and if $f(P) \to Q'$ as $P \to Q$ in D, then*

$$K_I(f) \geqslant K_I(\Delta), \quad K_0(f) \geqslant K_0(\Delta), \quad K(f) \geqslant K(\Delta). \tag{11.9}$$

Proof. We may assume that $K(f) < \infty$, for otherwise there is nothing to prove. Next by performing preliminary translations, we may assume that $Q = Q' = 0$. Choose $a > 0$ so that $B^3(a) \subset U$ and fix $P \in D$ with $|P| < a$. For each n let

$$f_n(x) = a_n f\left(\frac{x}{n}\right) + Q_n',$$

where

$$a_n = \left| f\left(\frac{P}{n}\right)\right|^{-1}, \quad Q_n' = P' - a_n f\left(\frac{P}{n}\right), \quad P' = f(P).$$

Then $f_n(P) = P'$ and $|Q_n'| \leqslant |P'| + 1$. Moreover, since $f(x) \to 0$ as $x \to 0$ in D,

$$\lim_{n\to\infty} a_n = \infty. \tag{11.10}$$

Now let

$$D_n = \left\{x:\ \frac{x}{n} \in D\right\}, \qquad D_n' = \{a_n x + Q_n':\ x \in D'\}.$$

As in the proof of Theorem 2.3, $P \in D_n$ for all n and the D_n converge to their kernel Δ at P. Since f_n is a homeomorphism of D_n onto D'_n, $P' \in D'_n$ for all n. Next because the Q'_n are bounded, by choosing a subsequence and relabeling, we may assume that

$$\lim_{n\to\infty} Q'_n = Q''. \tag{11.11}$$

Using Theorem 11 of [4], one can prove that P' has a neighborhood which is contained in all the D'_n. Then with (11.10) and (11.11) it is easy to show that the D'_n converge to their kernel Δ'' at P', where Δ'' is the half space Δ' translated through Q''. Since $K(f_n) = K(f) < \infty$, Theorem 3 of [6] implies there exists a subsequence $\{n_m\}$ such that

$$\lim_{m\to\infty} f_{n_m}(x) = g(x)$$

uniformly on each compact subset of Δ, where g is a homeomorphism of Δ onto Δ''. Hence we obtain

$$K_I(\Delta) \leqslant K_I(g) \leqslant \liminf_{m\to\infty} K_I(f_{n_m}) = K_I(f)$$

from Lemma 2.1, and the rest of (11.9) follows similarly.

11.4. *Nonseparability.* Finally we show that $\mathfrak{D}$ is nonseparable by establishing the following result.

THEOREM 11.3. *Given $0 < a < \infty$, we can associate with each b, $0 < b < 1$, a domain $D_b \in \mathfrak{D}$ such that*

$$d(D_b, B^3) \leqslant a \tag{11.12}$$

for $0 < b < 1$ and such that

$$d(D_b, D_{b'}) \geqslant c \tag{11.13}$$

for $0 < b, b' < 1$, $b \neq b'$, where c is a positive constant which depends only on a.

Proof. Pick $m > 0$ so that

$$\log (m+1)^{\frac{2}{3}} = a. \tag{11.14}$$

With each b, $0 < b < 1$, we can associate a sequence $\{b_n\}$ such that $b_n = 0$ or 1 for each n and

$$b = \sum_1^\infty b_n 2^{-n}. \tag{11.15}$$

Next let $c_0 = 0$ and

$$c_n = (c_{n-1} + 1)\, e^{b_n + 1} + 1 > c_{n-1} + 2 \tag{11.16}$$

for $n = 1, 2, \ldots$. Then for $u \geqslant 0$ set

$$g_b(u) = \begin{cases} \dfrac{m}{2}(1 - (u - c_n)^2) & \text{if } \; |u - c_n| \leqslant 1 \text{ for some } n \geqslant 0, \\ 0 & \text{if } \; |u - c_n| > 1 \text{ for all } n \geqslant 0, \end{cases}$$

and let D_b be the domain

$$D_b = \{x = (r,\theta,x_3):\ \ g_b(r) < x_3 < \infty,\quad 0 \leqslant r < \infty\}.$$

It is not difficult to show that, for each pair of points $Q_1, Q_2 \in \partial D_b$, the acute angle between the segment Q_1Q_2 and the vector e_3 is not less than arc cot m. Hence Corollary 5.1 yields a homeomorphism f of D_b onto the half space $x_3 > 0$ for which

$$K(f) \leqslant (m+1)^{\frac{3}{2}}, \tag{11.17}$$

and (11.12) follows from (11.14) and (11.17).

Let $\{b_n'\}$, $\{c_n'\}$, and $D_{b'}$ be the sequences and domain corresponding to a second number $b' \neq b$, $0 < b' < 1$. To complete the proof of Theorem 11.3, we shall show that (11.13) holds with $c = \log M$, where

$$M = \left(1 - \frac{\arctan m}{\pi}\right)^{-\frac{1}{2}} > 1. \tag{11.18}$$

Suppose this is not the case. Then there exists a homeomorphism f of D_b onto $D_{b'}$ with

$$K(f) < M < 2^{\frac{1}{2}}. \tag{11.19}$$

Since D_b and $D_{b'}$ are Jordan domains in $\overline{R}^3$, f induces a homeomorphism f^* of ∂D_b onto $\partial D_{b'}$ [18]. Let E_b be the union of the circles $r=1$, $x_3=0$ and $r=c_n \pm 1$, $x_3=0$, $n=1,2,\ldots$ in ∂D_b, and let $E_{b'}$ be the corresponding set in $\partial D_{b'}$. We prove first that, because of (11.19), f^* maps E_b onto $E_{b'}$.

Choose $Q \in E_b$ and suppose that $f^*(Q)$ is a finite point $Q' \in \partial D_{b'} - E_{b'}$. Then $\partial D_{b'}$ has a tangent plane at Q'. Fix $\varepsilon > 0$. Arguing essentially as in section 7.5, we can find $(1+\varepsilon)$-quasiconformal mappings h and h' of R^3 onto itself with the following properties: h carries D_b onto D, h' carries $D_{b'}$ onto D', and the points $h(Q)$ and $h'(Q')$ have neighborhoods U and U' such that $D \cap U = \Delta \cap U$ and $D' \cap U' = \Delta' \cap U'$, where Δ is a dihedral wedge of angle $\pi -$ arc tan m with $h(Q)$ as a point of its edge and where Δ' is a half space. From (7.5), (11.9), and (11.18) it follows that

$$K_I(h' \circ f \circ h^{-1}) \geqslant K_I(\Delta) = M,$$

and hence

$$K(f) \geqslant K_I(f) \geqslant (1+\varepsilon)^{-2} M. \tag{11.20}$$

Since (11.20) holds for all $\varepsilon > 0$, we can let $\varepsilon \to 0$ to obtain an inequality which contradicts (11.19).

Suppose next that $f^*(Q) = \infty$, let C be the circle of E_b which contains Q, and let C' be the image of C under f^*. Then $C' - \{\infty\}$ is connected, and by what was proved above,

$$C' - \{\infty\} \subset E_{b'}.$$

This means that $C' - \{\infty\}$ must lie in one of the circles of $E_{b'}$. Hence $C' - \{\infty\}$ is bounded and this contradicts the assumption that $f^*(Q) = \infty$. We conclude that $f^*(Q) \in E_{b'}$ as desired.

It follows that f^* must map each circle of E_b onto a circle of $E_{b'}$. Let $C_1, C_2, \ldots, C_n, \ldots$ and $C'_1, C'_2, \ldots, C'_n, \ldots$ be the circles of E_b and $E_{b'}$, respectively, ordered according to increasing radii, let S_n be the bounded component of $\partial D_b - C_n$, and let S'_n be the image of S_n under f^*. Then S'_n and $\bar{S}'_n$ must contain exactly $n-1$ and n circles of $E_{b'}$, respectively, and hence S'_n is the bounded component of $\partial D_{b'} - C'_n$. In particular, this means that f^* maps the plane annulus

$$A_n = \{x = (r, \theta, x_3): \quad c_{n-1} + 1 < r < c_n - 1, \quad x_3 = 0\}$$

onto the plane annulus

$$A'_n = \{x = (r, \theta, x_3): \; c'_{n-1} + 1 < r < c'_n - 1, \; x_3 = 0\}$$

for $n = 1, 2, \ldots$, and from (11.16) we obtain

$$(b'_n + 1) \leqslant K(f^*)\,(b_n + 1), \qquad (b_n + 1) \leqslant K(f^*)\,(b'_n + 1). \tag{11.21}$$

Theorem 4.3 and (11.19) imply that

$$K(f^*) \leqslant K(f)^2 < 2, \tag{11.22}$$

and combining (11.21) and (11.22) yields

$$(b'_n + 1) < 2(b_n + 1), \;\; (b_n + 1) < 2(b'_n + 1) \tag{11.23}$$

for $n = 1, 2, \ldots$. Finally since b_n and b'_n take on only the values 0 and 1, (11.23) implies that $b_n = b'_n$ for all n, and hence that $b = b'$ by virtue of (11.15) and its counterpart for b'. This contradicts the hypothesis that $b \neq b'$. Hence (11.13) must hold with $c = \log M$, and the proof for Theorem 11.3 is complete.

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