VALUE DISTRIBUTION UNDER ANALYTIC MAPPINGS OF ARBITRARY RIEMANN SURFACES

BY

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1. We consider analytic mappings of an open Riemann surface R into a closed Riemann surface S.

The first and second main theorems of the classical Nevanlinna–Ahlfors [3] theory were generalized in 1960 by S. Chern [4] to an R obtained from a closed surface by omitting a finite number of points. He also referred to a forthcoming paper where, in addition, a finite number of disks are removed.

Chern's elegant work stimulated the present author to look into the question: what can be said about arbitrary open Riemann surfaces R? In particular, can Nevanlinna's first and second main theorems and any counterpart of Picard's great theorem be established on them? A priori there seemed to be no basis for a conjecture: it was known from classification theory that when the genus becomes infinite, intuition can no longer be relied upon, and surprises are possible. Moreover, although J. Tamura [12] had singled out a class of meromorphic functions with at most two Picard values, M. Heins [5] had exhibited a parabolic Riemann surface with one boundary component, which carried a meromorphic function with an infinite number of Picard values. This rather seemed to speak against any second main theorem in the general case.

In the present paper we propose a new choice for the proximity function and the characteristic function. The first main theorem, the second main theorem, and the defect and ramification relations can then be given for arbitrary Riemann surfaces

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R (Theorems 1, 2, 3). Our form of the second main theorem is universally valid, without exceptional intervals. Meromorphic functions on an arbitrary R are a special case, and earlier results are included (Corollaries 1–4). In particular, for functions in the plane or the disk we have new simple proofs for classical theorems.

In contrast with Chern's paper, written for differential geometers, our method is elementary, the most sophisticated tool used being Stokes' formula. The method also applies, mutatis mutandis, to open image surfaces S [11].

§ 1. Proximity function

2. Let R be an arbitrary open Riemann surface and S an arbitrary closed Riemann surface. On R we use the symbol z both for the generic point and the local variable. On S we similarly employ ζ. A mapping $\zeta = f(z)$ into S is, by definition, analytic if it is so in terms of local variables.

On S consider points $a_1, \ldots, a_q$. The question is, in essence, how many a_i's can be Picard points. First we need a function to measure the proximity of two points on S.

Take $\zeta_0, \zeta_1 \in S$ different from $a_1, \ldots, a_q$. Construct on $S - \zeta_0 - \zeta_1$ the harmonic function t_0 with $t_0(\zeta) + 2\log|\zeta - \zeta_0|$ harmonic at ζ_0, and $t_0(\zeta) - 2\log|\zeta - \zeta_1|$ harmonic at ζ_1, the former tending to 0 as $\zeta \to \zeta_0$ in a fixed parametric disk. Set

$$s_0(\zeta) = \log(1 + e^{t_0(\zeta)}). \tag{1}$$

For $a \neq \zeta_0$ let $t(\zeta, a)$ be harmonic on $S - a - \zeta_0$ with $t(\zeta, a) + 2\log|\zeta - a|$ harmonic at a, and $t(\zeta, a) - 2\log|\zeta - \zeta_0|$ harmonic at ζ_0, the latter tending to $s_0(a)$ as $\zeta \to \zeta_0$. The function

$$s(\zeta, a) = t(\zeta, a) + s_0(\zeta) \tag{2}$$

becomes logarithmically infinite as $\zeta \to a$ and is bounded below on S. It is the proximity function we set out to construct.

The function is symmetric: for any $a, b \in S$,

$$s(a, b) = s(b, a). \tag{3}$$

This is seen directly by applying Stokes' formula to $t(\zeta, a)$ and $t(\zeta, b)$ along small circles about a, b, ζ_0 and then letting the circles tend to these points.

3. In terms of s we endow S with a conformal metric. The Euclidian area element dS in the parametric ζ-disk is given the area

$$d\omega = \lambda^2 \, dS, \tag{4}$$

where
$$\lambda^2 = \Delta s = \frac{e^{t_0} |\operatorname{grad} t_0|^2}{(1+e^{t_0})^2}. \tag{5}$$

Clearly $d\omega$ is a conformally invariant area element in this metric.

Throughout this paper we denote by $\nu(\varphi)$ the number of zeros of the "function" φ in question. We have
$$\nu(\lambda) = e_S + 2 = 2g, \tag{6}$$

where e_S is the Euler characteristic of S, and g is its genus. In fact, the zeros of λ are those of grad t_0, and this vector forms a differentiable vector field on $S - \zeta_0 - \zeta_1$. By Lefschetz's fixed point theorem the number of its singularities is the Euler characteristic of $S - \zeta_0 - \zeta_1$, that is, $e_S + 2$. This in turn is $2g$. The proof can also be based on the Riemann-Roch theorem or on geometric properties of $t_0 + it_0^*$.

The total area of S is
$$\omega = \int_S d\omega = \int_{-\infty}^{\infty} \int_{\beta_x} \frac{e^{t_0}}{(1+e^{t_0})^2} dt_0^* \, dx = 4\pi, \tag{7}$$

where β_x is the level line $t_0 = x \in (-\infty, \infty)$.

Furthermore
$$-\frac{\Delta \log \lambda}{\lambda^2} = 1. \tag{8}$$

Indeed, the logarithm of the numerator of λ is harmonic, and that of the denominator is the same function s_0 we started with.

In passing we note that if S is open, then we require that t_0 has, in essence, vanishing normal derivative on the boundary (cf. [7]), and our reasoning holds verbatim. As a by-product we have constructed on an arbitrary Riemann surface a metric with finite total area and constant Gaussian curvature.

§ 2. First main theorem

4. On R choose a parametric disk R_0 with boundary β_0 such that $f(\beta_0)$ does not meet $a_1, \ldots, a_q, \zeta_0, \zeta_1$. Let Ω be an adjacent regular region with boundary $\beta_0 \cup \beta_\Omega$. On $\bar{\Omega}$ form the harmonic function u with $u = 0$ on β_0, $u = k(\Omega) = \text{const.}$ on β_Ω, $\int_{\beta_0} du^* = 1$.

For any $h \in [0, k]$ let β_h be the level line $u = h$ and denote by Ω_h the region between β_0 and β_h. By the same reasoning as before, the number of zeros of grad u is the Euler characteristic $e(h)$ of Ω_h:
$$\nu(h, \operatorname{grad} u) = e(h). \tag{9}$$

5. We are now ready to prove the first main theorem. Let a be any point on S and let z_j be its inverse images in Ω_h. Denote their number, counted with multiplicities, by $\nu(h,a)$. Let Δ_j be small disks about z_j with boundaries α_j, oriented clockwise; β_h and β_0 are oriented to leave R_0 to the left. Apply Stokes' formula to $v(z) = h - u(z)$ and $s(f(z), a)$:

$$\int_{\Sigma\alpha_j + \beta_h - \beta_0} v\, ds^* - s\, dv^* = \int_{\Omega_h - \cup\Delta_j} v\, \Delta_z s\, dR. \tag{10}$$

Here dR is the Euclidian area element in the z-disk.

As the α_j shrink to the z_j, the second term in $\int_{\Sigma\alpha_j}$ goes to zero, for the flux of v across α_j vanishes. The flux of s tends to 4π and

$$\int_{\Sigma\alpha_j} \to 4\pi \sum v(z_j) = 4\pi \int_0^h (h-x)\, d\nu(x,a).$$

Integrate by parts and denote this contribution of the α_j's by

$$A(h,a) = 4\pi \int_0^h \nu(x,a)\, dx. \tag{11}$$

It is our *counting function*. Clearly it vanishes for a Picard point a.

In $\int_{\beta_h}$, $v=0$, and we designate this part corresponding to β_h by

$$B(h,a) = \int_{\beta_h} s\, du^*. \tag{12}$$

This we choose as our *proximity function*. It gives the mean proximity to a of the image of β_h.

On the right we have, in terms of $d\omega(f(z))$,

$$C(h) = \int_{\Omega_h} v\, d\omega. \tag{13}$$

This is our *characteristic function*. It is independent of a.

What remain are the integrals along β_0:

$$D(h,a) = B(0,a) + hB'(0,a). \tag{14}$$

Note that this is $O(h)$. The only functions of interest turn out to be those for which $C(h)$ grows more rapidly than h. Thus D is a negligible remainder. This is even more clearly seen if we do not omit R_0 from Ω, in which case there are no integrals along β_0. But for our later results the present set-up is more suitable.

We have established the first main theorem:

THEOREM 1. *For analytic mappings of an arbitrary R into a closed S, and for any $\Omega \subset R$,*

$$A(k,a) + B(k,a) = C(k) + D(k,a). \tag{15}$$

Thus the beautiful balance continues to hold despite infinite genus: any defect in coverage of a point a, such as of a Picard point, is compensated for by a close proximity of $f(\beta_h)$. The functions A and B add up, in essence, to the same C for all a.

6. The characteristic C retains its simple geometric meaning. To see this take $v=1$ in (10). The integrand on the left is then ds^*, on the right, $\Delta_z s\, dR = d\omega(f(z))$, and we have

$$A'(h,a) + B'(h,a) = \int_{\Omega_h} d\omega + D'(h,a).$$

A comparison with (15) gives

$$C'(h) = \int_{\Omega_h} d\omega. \tag{16}$$

We see that the characteristic is the integral of the total area of the multisheeted image of Ω_h over S.

7. We next ask how numerous are the points a with a Picard nature, i.e., with great contributions from B. Do they continue to be exceptional compared with strongly covered points a? To answer this we must estimate B upward.

This process is facilitated by replacing B by the integral of its integral. For heuristic motivation consider the simple case of the exponential function e^z. It maps not only the boundaries of exhausting regions but, naturally, also these regions themselves, with increasing mean proximity to the Picard points 0 and ∞. This phenomenon is universal, and we can replace the conventional curvilinear proximity by areal proximity. To facilitate subsequent computations, we even integrate this.

We introduce for any function $\varphi(h)$ the notations

$$\varphi_1(h) = \int_0^h \varphi(x)\,dx, \qquad \varphi_2(h) = \int_0^h \varphi_1(x)\,dx. \tag{17}$$

Then the new proximity function is B_2, and we can attach subindices "2" to each term in (15).

8. Another simple device that will shorten later reasoning is the following. We add to $a_1, \ldots, a_q$ the $2g$ zeros $a_{q+1}, \ldots, a_{q+2g}$ of λ and set for any function $\psi(h,a)$,

$$\psi(h) = \sum_{1}^{q+2g} \psi(h, a_i). \tag{18}$$

Then (15) gives

$$A_2(h) + B_2(h) = (q + 2g)\, C_2(h) + D_2(h). \tag{19}$$

Our task is to find an upper estimate for $B_2(h)$.

§ 3. Second main theorem

9. Set $\sigma(\zeta) = \exp\left[\sum_1^{q+2g} s(\zeta, a_i) - 2 \log\left(\sum_1^{q+2g} s(\zeta, a_i) + \text{const.}\right)\right]$ with $\sum s + \text{const.} > 0$ on S, and distribute on S the mass $dm = \sigma\, d\omega$. It has singularities at the a_i. The total mass $m = \int_S dm$ is finite, however.

The density $\sigma\lambda^2$ of dm induces in the $u + iu^*$-plane the density $\sigma\mu^2$, where

$$\mu(z) = \lambda(z)\, |f'(z)|\, |\operatorname{grad} u(z)|^{-1}. \tag{20}$$

We use the concavity of the logarithm to obtain

$$B(h) \leqslant \int_{\beta_h} \log \sigma\, du^* + 2 \log\left(B(h) + \text{const.}\right) \tag{21}$$

and decompose the integral into

$$\left.\begin{aligned} F(h) &= \int_{\beta_h} \log\left(\sigma\, \mu^2\right) du^*, \\ G(h) &= -\int_{\beta_h} \log \mu^2\, du^*. \end{aligned}\right\} \tag{22}$$

Then

$$B_2(h) \leqslant F_2(h) + G_2(h) + 2[\log\left(C(h) + O(h)\right)]_2. \tag{23}$$

We shall first estimate F_2, then G_2.

10. We write

$$H(h) = \int_{\beta_h} \sigma \mu^2\, du^* \tag{24}$$

and obtain by the concavity of the logarithm,

$$F(h) \leqslant \log H(h),$$

$$F_1(h) \leqslant h \log\left(\frac{1}{h} H_1(h)\right) = h \log H_1(h) - h \log h,$$

$$F_2(h) < h^2 \log H_2(h) + O(h^2 \log h). \tag{25}$$

To estimate H_2 note that

$$H_1(h) = \int_S \nu(h, a)\, dm(a).$$

In fact, both quantities give the total mass on the multisheeted image of Ω_h over S. On integrating (15) with respect to $dm(a)$ the first term thus gives $4\pi H_2(h)$. Since $s(\zeta, a) = s(a, \zeta)$ is uniformly bounded below on the compact S, $-B(h)$ contributes $0(1)$ and we obtain

$$4\pi H_2(h) < mC(h) + O(h).$$

Substitution into (25) yields

$$F_2(h) < h^2 \log (C(h) + O(h)). \tag{26}$$

11. To obtain $G_2(h)$ we first evaluate

$$G'(h) = -2\int_{\beta_h} d^* \log \mu.$$

Let Γ_j be small disks about the singularities of $\log \mu$, bounded by clockwise oriented γ_j. Then

$$G'(h) = 2\int_{\Sigma \gamma_j - \beta_0} d^* \log \mu - 2\int_{\Omega_h - \cup \Gamma_j} \Delta_z \log \lambda\, dR.$$

By virtue of (20), (9), (4), (8), and (16), we have

$$G'(h) = 4\pi[-\nu(h, \lambda) - \nu(h, f') + e(h)] + 2C'(h) + \text{const.} \tag{27}$$

Triple integration gives $G_2(h)$.

12. In the last term of (23) we replace the integrand in both integrations by its value at the right end point of the interval of integration and obtain the estimate $2h^2 \log (C(h) + O(h))$. We substitute it together with $G_2(h)$ and (26) into (23), this into (19), and set

$$E(h) = 4\pi \int_0^h e(x)\, dx.$$

We have reached our main result, the second main theorem:

THEOREM 2. *For an analytic mapping f of an arbitrary open Riemann surface R into a closed Riemann surface S, any regular subregion $\Omega \subset R$ with $k = k(\Omega)$ gives*

$$(q + e_S)\, C_2(k) < \sum_1^q A_2(k, a_i) - A_2(k, f') + E_2(k) + O(k^3 + k^2 \log C(k)). \tag{28}$$

As a special case we have the second main theorem for meromorphic functions on arbitrary Riemann surfaces.

§ 4. Consequences

13. In the sequel we consider nondegenerate mappings f characterized by

$$\lim_{\Omega\to R} \frac{k^3 + k^2 \log C(k)}{C_2(k)} = 0. \tag{29}$$

The condition is assured by $C(k)/k \to \infty$ and the existence of a constant $0 < \alpha < 1$ with $C(\alpha k)/\log C(k) \to \infty$. (This formulation was suggested to the author by K.V.R. Rao.) The condition is general and is obviously even met by such functions as k^τ and $e^{\tau k}$, $\tau = \text{const.} > 1, 0$, respectively.

14. Using canonical regions Ω we introduce the defect

$$\delta(a) = 1 - \overline{\lim_{\Omega\to R}} \frac{A_2(k,a)}{C_2(k)}, \tag{30}$$

the ramification index

$$\vartheta = \underline{\lim_{\Omega\to R}} \frac{A_2(k,f')}{C_2(k)}, \tag{31}$$

and the Euler index

$$\eta = \underline{\lim_{\Omega\to R}} \frac{E_2(k)}{C_2(k)}. \tag{32}$$

From (28) we obtain the following defect and ramification relation:

THEOREM 3. *For nondegenerate mappings of an arbitrary R into a closed S,*

$$\sum \delta(a) + \vartheta \leqslant \eta - e_S. \tag{33}$$

In particular there can be at most $\eta - e_S$ Picard values.

If S is obtained from a closed surface S_0 by removing n points then, in terms of quantities defined for S_0,

$$\sum \delta(a) + \vartheta \leqslant \eta - e_{S_0} - n. \tag{33'}$$

15. We have the following immediate consequences. First consider the existence of mappings f.

COROLLARY 1. *A necessary condition for a nondegenerate mapping of an arbitrary R into a closed S is that $\eta \geqslant e_S$. For a closed S less n points, η must dominate $e_S + n$.*

For the sphere S or the plane or the punctured plane we have $e_S \geqslant 0$ and there is no restriction. This is compatible with the theorem of Behnke and Stein: every open Riemann surface can be mapped into the punctured plane. For the torus S we again have no restriction but for genus > 1, $C(k)$ cannot grow more rapidly than $E(k)$.

16. Next suppose R has finite Euler characteristic.

COROLLARY 2. *If* $e(R) < \infty$, *then the number* P *of Picard values for a mapping into a closed* S *is*

$$P \leqslant -e_S. \tag{34}$$

In the case where R is obtained from a closed surface by removing a finite number of points this is the Chern result. We see that a finite number of disks or any closed connected sets can as well be removed.

17. Suppose S is the sphere or the plane.

COROLLARY 3. *For a meromorphic or entire function on an arbitrary open Riemann surface,*

$$P \leqslant 2 + \eta \quad \text{or} \quad P \leqslant 1 + \eta, \tag{35}$$

respectively.

The former bound was shown to be sharp in [9, 10] and by B. Rodin in [6]. For $\eta = 0$ we have the Tamura [12] functions with $P \leqslant 2$.

18. As the most special case let R be the finite or infinite disk.

COROLLARY 4. *For meromorphic functions on* $|z| < \varrho \leqslant \infty$,

$$\sum' \delta(a) + \vartheta \leqslant 2. \tag{36}$$

We can now take $u = \log r$ on R and the integration with respect to h covers all of R. By L'Hospital's rule the subscripts "2" in (30)–(32) can be dropped and we have a new proof for the conventional form of the defect relation, simultaneously for the plane and the disk. The second main theorem (28) is valid without exceptional intervals.

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