DISCRETE SERIES FOR SEMISIMPLE LIE GROUPS I

Construction of invariant eigendistributions

BY

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§ 1. Introduction

Let G be a connected semisimple Lie group with a compact Cartan subgroup B, and B^* the character group of B. Let $\mathfrak{g}$ and $\mathfrak{b}$ denote the Lie algebras of G and B respectively. Then every $b^* \in B^*$ defines a linear function $\lambda = \log b^*$ on $\mathfrak{b}_c$ by the relation

$$\langle b^*, \exp H\rangle = e^{\lambda(H)} \quad (H \in \mathfrak{b}).$$

Let W be the Weyl group of $(\mathfrak{g}, \mathfrak{b})$. We say that b^* is regular if $s\lambda \neq \lambda$ for every $s \neq 1$ in W. Let $B^{*\prime}$ denote the set of all regular elements of B^* and define $\mathfrak{Z}$ as in [2 (m), § 1]. Then corresponding to every $b^* \in B^{*\prime}$, we construct in Theorem 3 an invariant eigendistribution Θ_{b^*} of $\mathfrak{Z}$ on G (cf. [2 (h), Theorem 2]). We shall see later in another paper that those irreducible characters of G which correspond to the discrete series (see [2 (a), § 5]) are actually finite linear combinations of these distributions (cf. [2 (h), Theorems 3 and 4]).

The second main result of this paper is contained in Theorem 4 which gives an alternative formula for the distribution Θ_{b^*}. This will be needed for the determination of the contribution of the discrete series to the Plancherel formula of G.

Our method consists in first proving analogous results on $\mathfrak{g}$ and then lifting them to G, roughly speaking, by means of the exponential mapping. Theorem 1 is the $\mathfrak{g}$-analogue of Theorem 4 and its proof depends very much on Theorem 5 of [2 (k)]. Then in § 8 we introduce the notion of a tempered distribution on an open subset of a Euclidean space (see also [2 (c), p. 90]) and prove some elementary results which are then applied in § 14 to certain tempered and invariant eigendistributions on a reductive subalgebra $\mathfrak{z}$ of $\mathfrak{g}$ containing $\mathfrak{b}$. Lemma 28 asserts the uniqueness of such distributions and the existence is proved in Theorem 2 and Lemma 37. Lemma 41 contains the key result required for the reduction of the proof of Theorem 4 from the group to the Lie algebra.

The rest of this paper is devoted to the proofs of Theorems 3 and 4. The uniqueness part of Theorem 3 is relatively easy and follows from Lemma 28. However the problem of existence is more delicate. Lemma 50 contains the main step required in its solution. Lemma 59 gives a rather explicit formula for Θ_{b^*} which will be useful in later work. The main burden of the proof of Theorem 4 rests on Lemma 66.

Let L' be the set of all linear functions λ on $\mathfrak{b}$ of the form $\lambda = \log b^*$ ($b^* \in B^{*\prime}$) and write $\Theta_\lambda = \Theta_{b^*}$. Define $\varpi \in S(\mathfrak{b}_c)$ as in [2 (k), § 11]. Then we show in § 29 that for any $f \in C_c^\infty(G)$, the series

$$\sum_{\lambda \in L'} \varpi(\lambda)\, \Theta_\lambda(f)$$

converges absolutely and its sum represents a distribution T on G. We shall see later that, apart from a constant factor, T is just the contribution of the discrete series to the Plancherel formula of G (cf. [2 (h), Theorem 4]).

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Part I. Theory on the Lie algebra

§ 2. Reduction of Theorem 1 to the semisimple case

We use the notation and terminology of [2 (l)]. Let $\mathfrak{g}$ be a reductive Lie algebra over $\mathbf{R}$, Ω a completely invariant open subset of $\mathfrak{g}$, T a distribution on Ω satisfying the conditions of [2 (l), Theorem 1] and F the corresponding analytic function on $\Omega' = \Omega \cap \mathfrak{g}'$. Then we have seen in [2 (l), § 9] that $\Phi = \nabla_{\mathfrak{g}} F$ extends to a continuous function on Ω.

Let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$. For any function ϕ on Ω' let $\phi_{\mathfrak{h}}$ denote its restriction on $\mathfrak{h} \cap \Omega'$.

LEMMA 1. *Let $D \in \mathfrak{D}(\mathfrak{h}_c)$. Then the function $D\Phi_{\mathfrak{h}}$ is locally bounded*(1) *on* $\mathfrak{h} \cap \Omega$.

Fix a point $H_0 \in \mathfrak{h} \cap \Omega$ and select a positive-definite quadratic form Q on $\mathfrak{h}$. For any $\varepsilon > 0$, consider the set $\mathfrak{h}(\varepsilon)$ of all $H \in \mathfrak{h}$ such that $Q(H - H_0) < \varepsilon^2$. Then if ε is sufficiently small, $\mathfrak{h}(\varepsilon) \subset \Omega$. Moreover the set $\mathfrak{h}'(\varepsilon) = \mathfrak{h}(\varepsilon) \cap \Omega'$ has only a finite number of connected components. It follows from [2 (l), Lemma 2] that $D\Phi_{\mathfrak{h}}$ remains bounded on each connected component of $\mathfrak{h}'(\varepsilon)$ and therefore also on $\mathfrak{h}'(\varepsilon)$. Obviously this implies the statement of the lemma.

(1) This means that $D\Phi_{\mathfrak{h}}$ remains bounded on $C \cap \Omega'$ for any compact subset C of $\mathfrak{h} \cap \Omega$.

COROLLARY. *For any* $D \in \mathfrak{Z}(\mathfrak{g}_c)$, $D\Phi$ *is locally summable on* Ω.

Fix $\mathfrak{h}$ as above. Then by [2 (j), Lemma 14],

$$(D\Phi)_{\mathfrak{h}} = \delta_{\mathfrak{g}/\mathfrak{h}}'(D)\,\Phi_{\mathfrak{h}} = \pi^{-1}(\delta_{\mathfrak{g}/\mathfrak{h}}(D)\circ\pi)\,\Phi_{\mathfrak{h}}.$$

But $\delta_{\mathfrak{g}/\mathfrak{h}}(D)\circ\pi \in \mathfrak{D}(\mathfrak{h}_c)$ by [2 (j), Theorem 1] and therefore we conclude from the above lemma that $\pi(D\Phi)_{\mathfrak{h}}$ is locally bounded on $\mathfrak{h}\cap\Omega$.

Let $m=(n-l)/2$ where $n=\dim\mathfrak{g}$, $l=\operatorname{rank}\mathfrak{g}$. Then $m=d^0\pi$. Let t be an indeterminate and $\eta(X)$ the coefficient of t^l in $\det(t-adX)$ $(X\in\mathfrak{g}_c)$. Then η is an invariant polynomial function on $\mathfrak{g}_c$ and $\eta(H)=(-1)^m\pi(H)^2$ $(H\in\mathfrak{h}_c)$. Moreover it follows from the above result (see the proof of Lemma 3 of [2 (l)]) that $|\eta|^{\frac{1}{2}}|D\Phi|$ is locally bounded on Ω. Therefore since $|\eta|^{-\frac{1}{2}}$ is locally summable on $\mathfrak{g}$ [2 (k), Corollary 2 of Lemma 30], our assertion is now obvious.

Let $\nabla_{\mathfrak{g}}^*$ denote the adjoint of $\nabla_{\mathfrak{g}}$. Then $\nabla_{\mathfrak{g}}^*$ is also an invariant and analytic differential operator on $\mathfrak{g}'$.

LEMMA 2. *Put* $f(x:H)=f(H^x)$ $(x\in G,\ H\in\mathfrak{h})$ *for* $f\in C^\infty(\mathfrak{g})$. *Then*

$$f(H^x;\ \nabla_{\mathfrak{g}}^*) = (-1)^m f(x:H;\ \pi^{-1}\partial(\varpi)\circ\pi^2) \quad (x\in G,\ H\in\mathfrak{h}')$$

where $m=\frac{1}{2}(\dim\mathfrak{g}-\operatorname{rank}\mathfrak{g})$.

Put $\mathfrak{g}_{\mathfrak{h}}=(\mathfrak{h}')^G$. Then $\mathfrak{g}_{\mathfrak{h}}$ is an open subset of $\mathfrak{g}'$. Fix $g\in C_c^\infty(\mathfrak{g}_{\mathfrak{h}})$. Then

$$\int \nabla_{\mathfrak{g}}^* f\cdot g\,dX = \int f\cdot\nabla_{\mathfrak{g}} g\,dX$$

and therefore we conclude from Corollary 1 of Lemma 30 of [2 (k)] that

$$\int \pi(H)^2 f(x^*H;\ \nabla_{\mathfrak{g}}^*)\,g(x^*H)\,dx^*\,dH = \int \pi(H)^2 f(x^*H)\,g(x^*H;\ \nabla_{\mathfrak{g}})\,dx^*\,dH.$$

Now define $\phi(x^*:H)=\phi(x^*H)$ $(x^*\in G^*,\ H\in\mathfrak{h})$ for $\phi=f$ or g. Then it follows from the definition of $\nabla_{\mathfrak{g}}$ [2 (l), Lemma 24] that

$$g(x^*H;\ \nabla_{\mathfrak{g}}) = g(x^*:H;\ \partial(\varpi)\circ\pi).$$

Therefore

$$\int \pi(H)^2 f(x^*H)\,g(x^*H;\ \nabla_{\mathfrak{g}})\,dx^*\,dH = (-1)^m \int \pi(H)^2 f(x^*:H;\ \pi^{-1}\partial(\varpi)\circ\pi^2)\,g(x^*H)\,dx^*\,dH$$

since ϖ is homogeneous of degree m. The differential operator $\pi^{-1}\partial(\varpi)\circ\pi^2$ being invariant under the Weyl group of $(\mathfrak{g}, \mathfrak{h})$, there exists (see the proof of Lemma 24 of [2 (l)]) a unique invariant differential operator D on $\mathfrak{g}_{\mathfrak{h}}$ such that

$$f(x^* H;\, D) = (-1)^m f(x^* : H;\, \pi^{-1}\partial(\varpi)\circ\pi^2)$$

for $x^* \in G^*$, $H \in \mathfrak{h}'$ and $f \in C^\infty(\mathfrak{g})$. Hence it is clear that

$$\int \nabla_{\mathfrak{g}}{}^* f \cdot g\, dX = \int Df \cdot g\, dX.$$

This being true for every $g \in C_c^\infty(\mathfrak{g}_{\mathfrak{h}})$, we conclude that $\Delta_{\mathfrak{g}}{}^* = D$ on $\mathfrak{g}_{\mathfrak{h}}$ and therefore

$$f(H^{x^*};\, \nabla_{\mathfrak{g}}{}^*) = (-1)^m f(x^* : H;\, \pi^{-1}\partial(\varpi)\circ\pi^2)$$

for $x^* \in G^*$, $H \in \mathfrak{h}'$. This is equivalent to the statement of the lemma.

COROLLARY. $f(H^x;\, \nabla_{\mathfrak{g}}{}^*\circ\eta^{-1}) = f(x : H;\, \pi^{-1}\partial(\varpi)) \quad (x \in G,\ H \in \mathfrak{h}')$.

Since $\eta(H) = (-1)^m \pi(H)^2$, this is obvious from Lemma 2.

By Chevalley's theorem [2 (c), Lemma 9], there exists a unique element $p \in I(\mathfrak{g}_c)$ such that $p_{\mathfrak{h}} = (\varpi^{\mathfrak{h}})^2$ for every Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$. (Here we have used the notation of [2 (i), § 8] and [2 (l), Theorem 3].) Put $\Box = \partial(p)$.

LEMMA 3. *Let f be a locally invariant C^∞ function on an open subset U of $\mathfrak{g}'$. Then*

$$(\nabla_{\mathfrak{g}}{}^*\circ\eta^{-1}\circ\nabla_{\mathfrak{g}})f = \Box f.$$

Fix a point $H_0 \in U$ and let $\mathfrak{h}$ be the centralizer of H_0 in $\mathfrak{g}$. Then $\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{g}$ and it follows from the corollary of Lemma 2 that

$$f(H;\, \nabla_{\mathfrak{g}}{}^*\circ\eta^{-1}\circ\nabla_{\mathfrak{g}}) = f_1(H;\, \pi^{-1}\partial(\varpi)) \quad (H \in \mathfrak{h} \cap U)$$

where $f_1 = \nabla_{\mathfrak{g}} f$. However

$$f_1(H) = f(H;\, \partial(\varpi)\circ\pi) \quad (H \in \mathfrak{h} \cap U)$$

from the definition of $\nabla_{\mathfrak{g}}$. Therefore

$$f(H;\, \nabla_{\mathfrak{g}}{}^*\circ\eta^{-1}\circ\nabla_{\mathfrak{g}}) = f(H;\, \pi^{-1}\partial(\varpi^2)\circ\pi).$$

On the other hand since f is locally invariant, we have

$$f(H;\, \Box) = f(H;\, \delta_{\mathfrak{g}/\mathfrak{h}}{}'(\Box)) = f(H;\, \pi^{-1}\partial(\varpi^2)\circ\pi) \quad (H \in \mathfrak{h} \cap U)$$

from [2 (c), Theorem 1] and the definition of $\Box$. This shows that

$$f(H_0;\, \nabla_{\mathfrak{g}}^* \circ \eta^{-1} \circ \nabla_{\mathfrak{g}}) = f(H_0;\, \square)$$

and so the lemma is proved.

COROLLARY. $\square F = (\nabla_{\mathfrak{g}}^* \circ \eta^{-1} \circ \nabla_{\mathfrak{g}}) F = \nabla_{\mathfrak{g}}^*(\eta^{-1}\Phi)$.

This is obvious since F is invariant and $\nabla_{\mathfrak{g}} F = \Phi$.

For any $\varepsilon > 0$ let $\mathfrak{g}(\varepsilon)$ denote the set of all $X \in \mathfrak{g}$ where $|\eta(X)| > \varepsilon^2$. Let u be a measurable function on $\mathfrak{g}'$ which is integrable (with respect to the Euclidean measure dX) on $\mathfrak{g}(\varepsilon)$ for every $\varepsilon > 0$. Then we define (1)

$$\text{p.v.} \int u\, dX = \lim_{\varepsilon \to 0} \int_{\mathfrak{g}(\varepsilon)} u\, dX$$

provided this limit exists and is finite.

THEOREM 1. *For any $f \in C_c^\infty(\Omega)$ we have*

$$\int f \square F\, dX = \text{p.v.} \int \eta^{-1} \nabla_{\mathfrak{g}} f \cdot \Phi\, dX.$$

Since $\square \in \mathfrak{J}(\mathfrak{g}_c)$, it follows from [2 (l), Lemma 16] that $\square F$ is locally summable on Ω. Hence the left side of the above equation is well defined. Now consider the right side. Let V_δ $(0 < \delta \leqslant \delta_0)$ be a family of invariant measurable functions on $\mathfrak{g}$ with the following properties.

1) There exists a number a such that $|V_\delta(X)| \leqslant a$ for $X \in \mathfrak{g}$ and all δ.
2) $V_\delta(X) = 0$ if $|\eta(X)| < \delta^2$ $(X \in \mathfrak{g},\, 0 < \delta \leqslant \delta_0)$.
3) $\lim_{\delta \to 0} V_\delta(X) = 1$ for $X \in \mathfrak{g}'$.

Fix a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ and put $\mathfrak{g}_{\mathfrak{h}} = (\mathfrak{h}')^G$ as before. Then we can choose a real number $c = c(\mathfrak{h}) \neq 0$ such that

$$\int g\, dX = c \int \pi(H)^2 g(x^* H)\, dx^*\, dH$$

for $g \in C_c(\mathfrak{g}_{\mathfrak{h}})$ in the notation of Corollary 1 of [2 (k), Lemma 30]. Since $V_\delta \eta^{-1} \nabla_{\mathfrak{g}} f \cdot \Phi$ vanishes outside a compact subset of $\mathfrak{g}'$, it is obviously integrable on $\mathfrak{g}$. Therefore

$$\int_{\mathfrak{g}_{\mathfrak{h}}} V_\delta \eta^{-1} \nabla_{\mathfrak{g}} f \cdot \Phi\, dX = (-1)^m c \int_{\mathfrak{h}} V_\delta(H) \Phi(H)\, dH \int_{G^*} f(x^* H;\, \nabla_{\mathfrak{g}})\, dx^*$$

(1) p.v. stands for "principal value".

if we recall that $\eta=(-1)^m\pi^2$ on $\mathfrak{h}$. On the other hand it follows from the definition of $\nabla_{\mathfrak{g}}$ that

$$\int_{G^*} f(x^*H; \nabla_{\mathfrak{g}})\, dx^* = \varepsilon_R(H)\,\psi_f(H:\partial(\varpi)) \quad (H \in \mathfrak{h}')$$

in the notation of [2 (k), § 5], Therefore since $\partial(\varpi)^*=(-1)^m\partial(\varpi)$, we get

$$\int_{\mathfrak{g}_{\mathfrak{h}}} V_\delta \eta^{-1}\nabla_{\mathfrak{g}} f\cdot\Phi\, dx = c\int_{\mathfrak{h}} V_{\delta,\mathfrak{h}}\,\varepsilon_R\Phi_{\mathfrak{h}}\partial(\varpi)^*\psi_f\, dH$$

where $V_{\delta,\mathfrak{h}}$ denotes the restriction of V_δ on $\mathfrak{h}$. Since Φ is continuous on Ω, it is clear (see [2 (k), § 15]) that

$$\int |\Phi_{\mathfrak{h}}\partial(\varpi)^*\psi_f|\, dH < \infty.$$

Therefore the following lemma is now obvious.

Lemma 4. *Let $f\in C_c^\infty(\Omega)$. Then*

$$\lim_{\delta\to 0}\int_{\mathfrak{g}_{\mathfrak{h}}} V_\delta\eta^{-1}\nabla_{\mathfrak{g}} f\cdot\Phi\, dX = c\int_{\mathfrak{h}} \varepsilon_R\Phi_{\mathfrak{h}}\partial(\varpi)^*\psi_f\, dH.$$

Select a maximal set $\mathfrak{h}_i$ $(1\leqslant i\leqslant r)$ of Cartan subalgebras of $\mathfrak{g}$ no two of which are conjugate under G. Put $\mathfrak{g}_i=(\mathfrak{h}_i{}')^G$. Then $\mathfrak{g}'$ is the disjoint union of $\mathfrak{g}_1, \mathfrak{g}_2, \ldots, \mathfrak{g}_r$. Fix a Euclidean measure d_iH on $\mathfrak{h}_i$ and put $c_i=c(\mathfrak{h}_i)$, $\Phi_i=\Phi_{\mathfrak{h}_i}$ and $\varpi_i=\varpi^{\mathfrak{h}_i}$. Then we have the following result in the notation of (1) [2 (k), § 16].

Corollary. *For any $f\in C_c^\infty(\Omega)$,*

$$\lim_{\delta\to 0}\int_{\mathfrak{g}} V_\delta\eta^{-1}\nabla_{\mathfrak{g}} f\cdot\Phi\, dX = \sum_{1\leqslant i\leqslant r} c_i\int \varepsilon_{R,i}\Phi_i\partial(\varpi_i)^*\psi_{f,i}\, d_iH = \text{p.v.}\int \eta^{-1}\nabla_{\mathfrak{g}} f\cdot\Phi\, dX.$$

The first equality is obvious from Lemma 4 and the second follows by taking V_δ to be the characteristic function of $\mathfrak{g}(\delta)$.

On the other hand (see the proof of Lemma 3),

$$F(H;\square) = F(H;\pi^{-1}\partial(\varpi)^2\circ\pi) = \Phi(H;\pi^{-1}\partial(\varpi)) \quad (H\in\mathfrak{h}'\cap\Omega).$$

Therefore
$$\int_{\mathfrak{g}_{\mathfrak{h}}} f\cdot\square F\, dX = c\int \varepsilon_R\psi_f\partial(\varpi)\,\Phi_{\mathfrak{h}}\, dH$$

and so it is obvious that Theorem 1 is equivalent to the following lemma.

(1) $\varepsilon_{R,i}$ denotes ε_R for $\mathfrak{h}=\mathfrak{h}_i$.

LEMMA 5. *Let* $f \in C_c^\infty(\Omega)$. *Then*

$$\sum_{1 \leqslant i \leqslant r} c_i \int_{\mathfrak{h}_i} \varepsilon_{R,i} (\psi_{f,i} \partial(\varpi_i) \Phi_i - \partial(\varpi_i)^* \psi_{f,i} \cdot \Phi_i) \, d_i H = 0.$$

We shall now prove Theorem 1 by induction on dim $\mathfrak{g}$. Put

$$J(f) = \int f \Box F \, dX - \text{p.v.} \int \eta^{-1} \nabla_{\mathfrak{g}}^* f \cdot \Phi \, dX$$

$$= \sum_{1 \leqslant i \leqslant r} c_i \int_{\mathfrak{h}_i} \varepsilon_{R,i} (\psi_{f,i} \partial(\varpi_i) \Phi_i - \partial(\varpi_i)^* \psi_{f,i} \cdot \Phi_i) \, d_i H$$

for $f \in C_c^\infty(\Omega)$. Then it follows from [2 (k), § 15] that J is an invariant distribution on Ω. We have to prove that $J = 0$.

Let $\mathfrak{c}$ be the center and $\mathfrak{g}_1$ the derived algebra of $\mathfrak{g}$ and first assume that $\mathfrak{c} \neq \{0\}$. Fix a point $X_0 \in \Omega$. We have to show that $J = 0$ around X_0. Let $X_0 = C_0 + Z_0$ ($C_0 \in \mathfrak{c}$, $Z_0 \in \mathfrak{g}_1$). Select on open and relatively compact neighborhood $\mathfrak{c}_0$ of C_0 in $\mathfrak{c}$ such that $Z_0 + \text{Cl}(\mathfrak{c}_0) \subset \Omega$. Let Ω_1 be the set of all points $Z \in \mathfrak{g}_1$ such that $Z + \text{Cl}(\mathfrak{c}_0) \subset \Omega$. Then Ω_1 is an open and completely invariant neighborhood of Z_0 in $\mathfrak{g}_1$ (see [2 (l), Lemma 9]). It would be sufficient to prove (see [2 (i), Lemma 3]) that

$$J(\alpha \times g) = 0 \quad (\alpha \in C_c^\infty(\mathfrak{c}_0),\ g \in C_c^\infty(\Omega_1)).$$

Fix $\alpha \in C_c^\infty(\mathfrak{c}_0)$ and consider the distributions

$$T_\alpha(g) = T(\alpha \times g), \quad J_\alpha(g) = J(\alpha \times g) \quad (g \in C_c^\infty(\Omega_1))$$

on Ω_1. Then T_α and J_α are both invariant. Put $\mathfrak{U}_1 = \mathfrak{U} \cap I(\mathfrak{g}_{1c})$ where $\mathfrak{U}$ has the same meaning as in [2 (l), Theorem 1]. Then

$$\dim I(\mathfrak{g}_{1c})/\mathfrak{U}_1 \leqslant \dim I(\mathfrak{g}_c)/\mathfrak{U} < \infty$$

and $\partial(\mathfrak{U}_1) T_\alpha = \{0\}$. Hence Theorem 1 of [2 (l)] is also applicable to $(T_\alpha, \mathfrak{g}_1, \Omega_1)$ instead of $(T, \mathfrak{g}, \Omega)$. Put $\Omega_1' = \Omega_1 \cap \mathfrak{g}'$ and fix Euclidean measures dC and dZ on $\mathfrak{c}$ and $\mathfrak{g}_1$ respectively such that $dX = dC\,dZ$ ($X = C + Z$, $C \in \mathfrak{c}$, $Z \in \mathfrak{g}_1$). Let F_α be the analytic function on Ω_1' such that

$$T_\alpha(g) = \int F_\alpha g \, dZ \quad (g \in C_c^\infty(\Omega_1)).$$

Then it is clear that

$$F_\alpha(Z) = \int \alpha(C) F(C + Z) \, dC \quad (Z \in \Omega_1').$$

Put $\Phi_\alpha = \nabla_{\mathfrak{g}_1} F_\alpha$. If $\mathfrak{h}$ is any Cartan subalgebra of $\mathfrak{g}$, it is clear that $\mathfrak{h} = \mathfrak{c} + \mathfrak{h}_1$ where $\mathfrak{h}_1 = \mathfrak{h} \cap \mathfrak{g}_1$. Moreover π and $\partial(\varpi)$ are in $\mathfrak{D}(\mathfrak{h}_{1c})$ and $\square \in \partial(I(\mathfrak{g}_{1c}))$. Hence it follows without difficulty that

$$J_\alpha(g) = \int_{\mathfrak{g}_1} g \square F_\alpha \, dZ - \text{p.v.} \int_{\mathfrak{g}_1} \eta^{-1} \nabla_{\mathfrak{g}_1} g \cdot \Phi_\alpha \, dZ$$

for $g \in C_c^\infty(\mathfrak{g}_1)$. But since $\dim \mathfrak{g}_1 < \dim \mathfrak{g}$, we conclude from the induction hypothesis that $J_\alpha = 0$. This shows that $J(\alpha \times g) = 0$ for $\alpha \in C_c^\infty(\mathfrak{c}_0)$ and $g \in C_c^\infty(\Omega_1)$ and therefore $J = 0$ around X_0.

§ 3. Second reduction

Hence we may now assume that $\mathfrak{g}$ is semisimple and identify $\mathfrak{g}$ with its dual space by means of the Killing form ω of $\mathfrak{g}$. For any $p \in I(\mathfrak{g}_c)$, let p_i denote the restriction of p on $\mathfrak{h}_i$ and put $\pi_i = \pi^{\mathfrak{h}_i}$ $(1 \leqslant i \leqslant r)$. We also identify $\mathfrak{h}_i$ with its dual space by means of ω_i. Then $\varpi_i = \pi_i$. Put $\delta_i(D) = \delta_{\mathfrak{g}/\mathfrak{h}_i}(D)$ $(D \in \mathfrak{J}(\mathfrak{g}_c))$ in the notation of [2 (j), Theorem 1].

LEMMA 6. *Let $D \in \mathfrak{J}(\mathfrak{g}_c)$, $p \in I(\mathfrak{g}_c)$ and $f \in C_c^\infty(\Omega)$. Then*

$$\sum_{1 \leqslant i \leqslant r} c_i \int \varepsilon_{R,i} \, \partial(\omega_i p_i)(\pi_i \psi_{f,i}) \cdot \delta_i(D) \Phi_i \, d_i H$$
$$= \sum_{1 \leqslant i \leqslant r} c_i \int \varepsilon_{R,i} \, \partial(p_i)(\pi_i \psi_{f,i}) \cdot \delta_i(\partial(\omega) \circ D) \Phi_i \, d_i H$$

and

$$\sum_{1 \leqslant i \leqslant r} c_i \int \varepsilon_{R,i} \, \partial(\omega_i) \psi_{f,i} \cdot (\delta_i(D) \circ \pi_i \circ \partial(p_i)) \Phi_i \, d_i H$$
$$= \sum_{1 \leqslant i \leqslant r} c_i \int \varepsilon_{R,i} \, \psi_{f,i} (\delta_i(\partial(\omega) \circ D) \circ \pi_i \circ \partial(p_i)) \Phi_i \, d_i H.$$

We shall prove this in § 4.

COROLLARY 1. *For any $k \geqslant 0$,*

$$\sum_{1 \leqslant i \leqslant r} c_i \int \varepsilon_{R,i} \, \partial(\omega_i^k) \psi_{f,i} \cdot (\delta_i(D) \circ \pi_i \circ \partial(p_i)) \Phi_i \, d_i H$$
$$= \sum_{1 \leqslant i \leqslant r} c_i \int \varepsilon_{R,i} \, \psi_{f,i} (\partial(\omega_i^k) \circ \delta_i(D) \circ \pi_i \circ \partial(p_i)) \Phi_i \, d_i H.$$

Since $\psi_{\partial(\omega) f, i} = \partial(\omega_i) \psi_{f,i}$ and $\delta_i(\partial(\omega^k) \circ D) = \partial(\omega_i^k) \circ \delta_i(D)$, this follows immediately from the second statement of Lemma 6 by induction on k.

COROLLARY 2.

$$\sum_i c_i \int \varepsilon_{R,i}\,\partial(\omega_i^k)\,(\pi_i\psi_{f,i})\cdot\delta_i(D)\,\Phi_i\,d_iH=\sum_i c_i\int \varepsilon_{R,i}\,\pi_i\psi_{f,i}\cdot\delta_i(\partial(\omega^k)\circ D)\,\Phi_i\,d_iH$$

for $k\geq 0$.

This follows from the first statement of Lemma 6 by induction on k.

COROLLARY 3.

$$\sum_i c_i \int \varepsilon_{R,i}(\partial(\omega_i^j)\circ\pi_i\circ\partial(\omega_i^k))\,\psi_{f,i}\cdot\Phi_i\,d_iH=\sum_i\int \varepsilon_{R,i}\,\psi_{f,i}\,(\partial(\omega_i^k)\circ\pi_i\circ\partial(\omega_i^j))\,\Phi_i\,d_iH$$

for $j,\ k\geq 0$.

Apply Corollary 2 to $f_k=\partial(\omega)^k f$ with $D=1$. Then since

$$\psi_{f_k,i}=\partial(\omega_i^k)\,\psi_{f,i},$$

we obtain

$$\sum_i c_i\int \varepsilon_{R,i}(\partial(\omega_i^j)\circ\pi_i\circ\partial(\omega_i^k))\,\psi_{f,i}\cdot\Phi_i\,d_iH$$

$$=\sum_i c_i\int\varepsilon_{R,i}\partial(\omega_i^k)\,\psi_{f,i}\cdot\pi_i\partial(\omega_i^j)\,\Phi_i\,d_iH.$$

Now apply Corollary 1 with $D=1$ and $p=\omega^j$. This gives the required result.

We shall now complete the proof of Lemma 5 and therefore also of Theorem 1. Let Λ_i denote the derivation of $\mathfrak{D}(\mathfrak{h}_{ic})$ given by(1)

$$\Lambda_i\xi=\tfrac{1}{2}\{\partial(\omega_i),\xi\}\quad(\xi\in\mathfrak{D}(\mathfrak{h}_{ic})).$$

Then since π_i is homogeneous of degree m, it is clear that (see [2 (c), p. 99]) that

$$\Lambda_i^m\pi_i=m\,!\;\partial(\pi_i).$$

Therefore

$$\partial(\pi_i)=(m\,!\;2^m)^{-1}\sum_{0\leq k\leq m}C_k^m(-1)^{m-k}\partial(\omega_i^k)\circ\pi_i\circ\partial(\omega_i^{m-k})$$

where C_k^m denotes the usual binomial coefficient. Hence Lemma 5 follows immediately from Corollary 3 above.

§ 4. Third reduction

Fix $D\in\mathfrak{Z}(\mathfrak{g}_c)$ and put

(1) As usual $\{D_1, D_2\}=D_1\circ D_2-D_2\circ D_1$ for two differential operators D_1, D_2.

$$J(f)=\sum_i c_i\int \varepsilon_{R,i}\{\partial(\omega_i)(\pi_i\psi_{f,i})\cdot\delta_i(D)\Phi_i-\pi_i\psi_{f,i}\,\delta_i(\partial(\omega)\circ D)\Phi_i\}\,d_iH$$

and
$$J'(f)=\sum_i c_i\int \varepsilon_{R,i}\{\partial(\omega_i)\psi_{f,i}\cdot\delta_i(D)(\pi_i\Phi_i)-\psi_{f,i}\,\delta_i(\partial(\omega)\circ D)(\pi_i\Phi_i)\}\,d_iH$$

for $f\in C_c^\infty(\Omega)$. Then J and J' are invariant distributions on Ω.

LEMMA 7. *No semiregular element of Ω of noncompact type lies in*

$$(\operatorname{Supp} J)\cup(\operatorname{Supp} J').$$

Assuming this result, we shall now prove Lemma 6. For $p\in I(\mathfrak{g}_c)$ and $f\in C_c^\infty(\Omega)$, define

$$J_p(f)=\sum_i c_i\int \varepsilon_{R,i}\{\partial(\omega_i p_i)(\pi_i\psi_{f,i})\cdot\delta_i(D)\Phi_i-\partial(p_i)(\pi_i\psi_{f,i})\cdot\delta_i(\partial(\omega)\circ D)\Phi_i\}\,d_iH.$$

Then J_p is an invariant distribution on Ω. We shall now show that $J_p=J'=0$.

Fix a point $X_0\in\Omega$ and, for any $\varepsilon>0$, define $U_{X_0}(\varepsilon)$ as in [2 (l), Lemma 14] and put $\Omega(\varepsilon)=\Omega\cap U_{X_0}(\varepsilon)$. Then $\Omega(\varepsilon)$ is an open and completely invariant neighborhood of X_0 in $\mathfrak{g}$. Put

$$\mathfrak{h}_i(\varepsilon)=\mathfrak{h}_i\cap\Omega_i(\varepsilon),\quad \mathfrak{h}_i(0)=\bigcap_{\varepsilon>0}\mathfrak{h}_i(\varepsilon)\quad(1\leqslant i\leqslant r).$$

Then we have seen during the proof of [2 (l), Lemma 13] that $\mathfrak{h}_i(0)$ is a finite set. For every $H\in\mathfrak{h}_i(0)$, select two open, convex neighborhoods U_H, V_H of H in $\mathfrak{h}_i$ such that $\mathrm{Cl}\,U_H\subset V_H\subset\mathfrak{h}_i(1)$ and $V_H\cap V_{H'}=\emptyset$ for $H\neq H'$ $(H,H'\in\mathfrak{h}_i(0))$. Put

$$U_i=\bigcup_{H\in\mathfrak{h}_i(0)}U_H,\quad V_i=\bigcup_{H\in\mathfrak{h}_i(0)}V_H$$

and select $\alpha_H\in C_c^\infty(V_H)$ such that $\alpha_H=1$ on U_H $(H\in\mathfrak{h}_i(0))$. Define

$$\alpha_i=\sum_{H\in\mathfrak{h}_i(0)}\alpha_H$$

and put
$$g_i=c_i\,\varepsilon_{R,i}\,\alpha_i\,\delta_i(D)\Phi_i,\quad g_i'=c_i\,\varepsilon_{R,i}\,\alpha_i\,\delta_i(D)(\pi_i\Phi_i).$$

Then it follows from [2 (j), Theorem 1], [2 (l), Theorem 2] and [2 (l), § 4] that g_i and g_i' are functions of class C^∞ on the closure of each connected component of $\mathfrak{h}_i'(R)$.

Now choose $\varepsilon>0$ so small that $\mathfrak{h}_i(\varepsilon)\subset U_i$ $(1\leqslant i\leqslant r)$. Then if $f\in C_c^\infty(\Omega(\varepsilon))$, it is clear that $\operatorname{Supp}\psi_{f,i}\subset U_i$. Since $\alpha_i=1$ on U_i, it follows that

$$J_p(f)=\sum_i\int\{\partial(\omega_i p_i)\psi_{f,i}\cdot g_i-\partial(p_i)\psi_{f,i}\cdot\partial(\omega_i)g_i\}\,d_iH$$

and
$$J'(f)=\sum_i \int \{\partial(\omega_i)\psi_{f,i}\cdot g_i' - \psi_{f,i}\cdot \partial(\omega_i) g_i'\}\, d_i H$$

for $f \in C_c^\infty(\Omega(\varepsilon))$. Moreover $\Omega(\varepsilon)$ being completely invariant, we can choose an open neighborhood V of X_0 in $\mathfrak{g}$ such that $\mathrm{Cl}(V^G) \subset \Omega(\varepsilon)$. Now $J = J_1$ and therefore it follows from Lemma 7 and [2 (k), Theorem 5] that $J_p = 0$ on V^G for $p \in I(\mathfrak{g}_c)$. Hence $X_0 \notin \mathrm{Supp}\, J_p$. But X_0 was an arbitrary point of Ω. Therefore we conclude that $J_p = 0$. This proves the first statement of Lemma 6.

Similarly by applying [2 (k), Theorem 4] we conclude that $J' = 0$. This gives the second statement of Lemma 6 in the special case $p = 1$. Now fix $p \in I(\mathfrak{g}_c)$ and consider the distribution $T_0 = \partial(p) T$. Then T_0 also satisfies the conditions of [2 (l), Theorem 1] and therefore $T_0 = F_0$ where $F_0 = \partial(p) F$. Put $\Phi_0 = \nabla_{\mathfrak{g}} F_0$ and let Φ_{0i} denote the restriction of Φ_0 on $\mathfrak{h}_i \cap \Omega'$ $(1 \leqslant i \leqslant r)$.

LEMMA 8. $$\Phi_{0i} = \partial(p_i)\Phi_i \quad (1 \leqslant i \leqslant r).$$

Let F_i and F_{0i} respectively denote the restrictions of F and F_0 on $\mathfrak{h}_i \cap \Omega'$. Since F is an invariant function, we know [2 (c), Theorem 1] that

$$F_{0i} = \pi_i^{-1}\partial(p)(\pi_i F_i).$$

Therefore
$$\Phi_{0i} = \partial(\pi_i)(\pi_i F_{0i}) = \partial(p_i \pi_i)(\pi_i F_i) = \partial(p_i)\Phi_i.$$

Now if we apply the result $J' = 0$ to the distribution T_0 (instead of T), we obtain

$$\sum_i c_i \int \varepsilon_{R,i}\{\partial(\omega_i)\psi_{f,i}\cdot \delta_i(D)(\pi_i \Phi_{0i}) - \psi_{f,i}\,\delta_i(\partial(\omega)\circ D)(\pi_i \Phi_{0i})\}\, d_i H = 0$$

for $f \in C_c^\infty(\Omega)$. In view of Lemma 8, this is equivalent to the second assertion of Lemma 6.

§ 5. New expressions for J and J'

Define η as in § 2. Then $\eta \in I(\mathfrak{g}_c)$ and $\eta_i = (-1)^m \pi_i^2$ $(1 \leqslant i \leqslant r)$. Moreover $|\eta|^{\frac12}$ and $|\eta|^{-\frac12}$ are analytic functions on $\mathfrak{g}'$.

LEMMA 9. *Define J and J' as in § 4. Then*

$$J(f) = \mathrm{p.v.}\int \{\partial(\omega)(|\eta|^{\frac12} f)\cdot D(|\eta|^{-\frac12}\Phi) - |\eta|^{\frac12} f(\partial(\omega)\circ D \circ |\eta|^{-\frac12})\Phi\}\, dX$$

and
$$J'(f)=\int\{\partial(\omega)f\cdot D\Phi-f\partial(\omega)(D\Phi)\}\,dX$$
for $f\in C_c^\infty(\Omega)$.

Since η takes only real values on $\mathfrak{g}$, it is obvious that $|\eta_i|^{\frac{1}{2}}=\varepsilon_i\pi_i$ on $\mathfrak{h}_i'$ where ε_i is a locally constant function on $\mathfrak{h}_i'$ such that $\varepsilon_i^4=1$. Since Φ and $|\eta|^{-\frac{1}{2}}$ are invariant functions, it follows from [2 (j), Lemma 14] that

$$D'(|\eta|^{-\frac{1}{2}}\Phi)=\varepsilon_i^{-1}\pi_i^{-1}\delta_i(D')\Phi_i$$

on $\mathfrak{h}_i\cap\Omega'$ for any $D'\in\mathfrak{J}(\mathfrak{g}_c)$.

For any $f\in C_c^\infty(\Omega)$, let g_f denote the function on $\mathfrak{g}'$ given by

$$g_f=\partial(\omega)(|\eta|^{\frac{1}{2}}f)\cdot D(|\eta|^{-\frac{1}{2}}\Phi)-|\eta|^{\frac{1}{2}}f(\partial(\omega)\circ D\circ|\eta|^{-\frac{1}{2}})\Phi.$$

Fix a function $v\in C^\infty(\mathbf{R})$ such that $v(t)=0$ if $|t|\leqslant\frac{1}{2}$ and $v(t)=1$ if $|t|\geqslant 1$ $(t\in\mathbf{R})$. For any $\varepsilon>0$, put $v_\varepsilon(t)=v(\varepsilon^{-2}t)$ and

$$V_\varepsilon(X)=v_\varepsilon(\eta(X))\quad(X\in\mathfrak{g}).$$

Then V_ε is an invariant C^∞ function on $\mathfrak{g}$ and $V_\varepsilon=1$ on $\mathfrak{g}(\varepsilon)$ (in the notation of § 2). Put $f_\varepsilon'=V_\varepsilon f$ and $f_\varepsilon=|\eta|^{\frac{1}{2}}f_\varepsilon'$. It is clear that f_ε and f_ε' are in $C_c^\infty(\Omega)$ and $f=f_\varepsilon'$ on $\mathfrak{g}(\varepsilon)$. Hence

$$\int_{\mathfrak{g}(\varepsilon)}g_f\,dX=\int_{\mathfrak{g}(\varepsilon)}g_{f_\varepsilon'}\,dX$$
$$=\sum_i c_i\int_{\mathfrak{h}_i(\varepsilon)}\varepsilon_{i,R}\,\varepsilon_i^{-1}\{\partial(\omega_i)\psi_{f_\varepsilon,i}\cdot\delta_i(D)\Phi_i-\psi_{f_\varepsilon,i}\,\delta_i(\partial(\omega)\circ D)\Phi_i\}\,d_iH$$

where $\mathfrak{h}_i(\varepsilon)=\mathfrak{h}_i\cap\mathfrak{g}(\varepsilon)$. However it is obvious that

$$\psi_{f_\varepsilon,i}=\varepsilon_i\pi_i\psi_{f,i}$$

on $\mathfrak{h}_i(\varepsilon)$. Therefore

$$\int_{\mathfrak{g}(\varepsilon)}g_f\,dX=\sum_i c_i\int_{\mathfrak{h}_i(\varepsilon)}\varepsilon_{i,R}\{\partial(\omega_i)(\pi_i\psi_{f,i})\cdot\delta_i(D)\Phi_i-\pi_i\psi_{f,i}\,\delta_i(\partial(\omega)\circ D)\Phi_i\}\,d_iH.$$

Making $\varepsilon\to 0$ we get
$$\text{p.v.}\int g_f\,dX=J(f)$$

and this proves the first statement of the lemma.

We know from the corollary of Lemma 1 that the integral

$$\int \{\partial(\omega) f \cdot D\Phi - f\partial(\omega)(D\Phi)\}\, dX$$

is well defined. Moreover since Φ is an invariant function,

$$D'\Phi = \pi_i^{-1}\delta_i(D')(\pi_i \Phi_i) \quad (D' \in \mathfrak{J}(\mathfrak{g}_c))$$

on $\mathfrak{h}_i \cap \Omega'$ and therefore the above integral is equal to $J'(f)$. This proves the second statement of the lemma.

LEMMA 10. *For any $\varepsilon > 0$, define the function V_ε as above and put*

$$J_\varepsilon(f) = \int V_\varepsilon \{\partial(\omega)(|\eta|^{\frac{1}{2}} f) \cdot D(|\eta|^{-\frac{1}{2}}\Phi) - |\eta|^{\frac{1}{2}} f(\partial(\omega) \circ D \circ |\eta|^{-\frac{1}{2}})\Phi\}\, dX$$

and

$$J_\varepsilon'(f) = \int V_\varepsilon \{\partial(\omega) f \cdot D\Phi - f\partial(\omega)(D\Phi)\}\, dX$$

for $f \in C_c^\infty(\Omega)$. Then

$$J(f) = \lim_{\varepsilon \to 0} J_\varepsilon(f), \quad J'(f) = \lim_{\varepsilon \to 0} J_\varepsilon'(f).$$

Put $f_\varepsilon = |\eta|^{\frac{1}{2}} V_{\varepsilon/2} f$. Then $f_\varepsilon \in C_c^\infty(\Omega)$ and $J_\varepsilon(f) = J_\varepsilon(V_{\varepsilon/2} f)$. Hence it follows that

$$J_\varepsilon(f) = \sum_i c_i \int V_{\varepsilon, i}\, \varepsilon_{i,R}\, \varepsilon_i^{-1} \{\partial(\omega_i)\psi_{f_\varepsilon, i} \cdot \delta_i(D)\Phi_i - \psi_{f_\varepsilon, i}\, \delta_i(\partial(\omega) \circ D)\Phi_i\}\, d_i H$$

where $V_{\varepsilon, i}$ is the restriction of V_ε on $\mathfrak{h}_i$. On the other hand, it is clear that

$$\psi_{f_\varepsilon, i}(H) = \varepsilon_i \pi_i(H) \psi_{f, i}(H)$$

if $|\pi_i(H)| \geq \varepsilon/2$ $(H \in \mathfrak{h}_i)$. Hence

$$J_\varepsilon(f) = \sum_i c_i \int V_{\varepsilon, i}\, \varepsilon_{i,R} \{\partial(\omega_i)(\pi_i \psi_{f,i}) \cdot \delta_i(D)\Phi_i - \pi_i \psi_{f,i}\, \delta_i(\partial(\omega) \circ D)\Phi_i\}\, d_i H.$$

The two assertions of the lemma are now obvious.

LEMMA 11. *Put* [1]

$$\Psi_\varepsilon = (|\eta|^{\frac{1}{2}} \{\partial(\omega), V_\varepsilon\} \circ D \circ |\eta|^{-\frac{1}{2}})\Phi,$$

$$\Psi_\varepsilon' = (\{\partial(\omega), V_\varepsilon\} \circ D)\Phi$$

for $\varepsilon > 0$. Then

[1] See footnote 1, p. 250.

$$J_\varepsilon(f) = \int f\Psi_\varepsilon \, dX, \quad J_\varepsilon'(f) = \int f\Psi_\varepsilon' \, dX$$

for $f \in C_c^\infty(\Omega)$.

Since $\operatorname{Supp} V_\varepsilon \subset \mathfrak{g}'$, this follows from Lemma 10 if we observe that

$$(V_\varepsilon \partial(\omega) \circ |\eta|^{\frac{1}{2}})^* = |\eta|^{\frac{1}{2}} \partial(\omega) \circ V_\varepsilon, \quad (V_\varepsilon \partial(\omega))^* = \partial(\omega) \circ V_\varepsilon.$$

Fix a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ and let us use the notation introduced at the beginning of § 2. In particular $V_{\varepsilon,\mathfrak{h}}$ and $\omega_\mathfrak{h}$ denote the restrictions of V_ε and ω on $\mathfrak{h}$.

LEMMA 12. *For any* $\varepsilon > 0$, *we have*

$$(\Psi_\varepsilon)_\mathfrak{h} = (\{\partial(\omega_\mathfrak{h}), V_{\varepsilon,\mathfrak{h}}\} \circ \delta_{\mathfrak{g}/\mathfrak{h}}(D)) \, \Phi_\mathfrak{h}$$

and

$$(\Psi_\varepsilon')_\mathfrak{h} = (\pi^{-1}\{\partial(\omega_\mathfrak{h}), V_{\varepsilon,\mathfrak{h}}\} \circ \delta_{\mathfrak{g}/\mathfrak{h}}(D) \circ \pi) \, \Phi_\mathfrak{h}.$$

$|\eta|^{\frac{1}{2}}$, Φ and V_ε are invariant C^∞ functions on Ω'. Moreover there exists a locally constant function a on $\mathfrak{h}'$ such that $a^4 = 1$ and $|\eta|^{\frac{1}{2}} = a\pi$ on $\mathfrak{h}'$. The required relations now follow easily by a repeated use of [2 (j), Lemma 14] and [2 (c), Theorem 1].

§ 6. Proof of Lemma 7

We now come to the proof of Lemma 7. Fix a semiregular element $H_0 \in \Omega$ of noncompact type and let $\mathfrak{z}$ denote the centralizer of H_0 in $\mathfrak{g}$. Define ζ and $\mathfrak{z}'$ as in [2 (j), § 2] and put $\Omega_\mathfrak{z} = \Omega \cap \mathfrak{z}'$. Then $\Omega_\mathfrak{z}$ is an open and completely invariant neighborhood of H_0 in $\mathfrak{z}$. Fix a Euclidean measure dZ on $\mathfrak{z}$ and define

$$j = \sigma_J, \; j' = \sigma_{J'}, \; j_\varepsilon = \sigma_{J_\varepsilon} \text{ and } j_\varepsilon' = \sigma_{J_\varepsilon'} \quad (\varepsilon > 0)$$

in the notation of [2 (j), Lemma 17] corresponding to $G_0 = G$ and $\mathfrak{z}_0 = \Omega_\mathfrak{z}$. Since $J_\varepsilon = \Psi_\varepsilon$ and $J_\varepsilon' = \Psi_\varepsilon'$ (Lemma 11), it is obvious that

$$j_\varepsilon(\gamma) = \int \gamma(Z) \Psi_\varepsilon(Z) \, dZ, \quad j_\varepsilon'(\gamma) = \int \gamma(Z) \Psi_\varepsilon'(Z) \, dZ$$

for $\gamma \in C_c^\infty(\Omega_\mathfrak{z})$. Moreover

$$j(\gamma) = \lim_{\varepsilon \to 0} j_\varepsilon(\gamma), \quad j'(\gamma) = \lim_{\varepsilon \to 0} j_\varepsilon'(\gamma) \quad (\gamma \in C_c^\infty(\Omega_\mathfrak{z}))$$

from Lemma 10.

Now we use the notation of [2 (k), § 7]. In particular σ is the center of $\mathfrak{z}$ and $\mathfrak{a} = \mathbf{R}H' + \sigma$, $\mathfrak{b} = \mathbf{R}(X' - Y') + \sigma$ are two Cartan subalgebras of $\mathfrak{z}$. Fix Euclidean measures $d\sigma$, $d\mathfrak{a}$, $d\mathfrak{b}$ on σ, $\mathfrak{a}$, $\mathfrak{b}$ respectively such that

$$d\mathfrak{a} = dt\,d\sigma, \quad d\mathfrak{b} = d\phi\,d\sigma$$

where $t = \alpha/2$ and $\phi = (-1)^{\frac{1}{2}}\beta/2$ in the notation of [2 (k), Lemma 13]. Then $d\sigma$ can be so normalized that (see [2 (e), Lemma 3])

$$\int \gamma\,dZ = \frac{1}{2}\int_{\mathfrak{a}^+} \alpha J_\gamma^{\mathfrak{a}}\,d\mathfrak{a} + (-1)^{\frac{1}{2}}\frac{1}{2}\int_{\mathfrak{b}} \beta J_\gamma^{\mathfrak{b}}\,d\mathfrak{b}$$

for $\gamma \in C_c^\infty(\mathfrak{z})$. Here

$$J_\gamma^{\mathfrak{a}}(H) = J_{\mathfrak{a}}(\gamma : H) \quad (H \in \mathfrak{a}),$$

$$J_\gamma^{\mathfrak{b}}(H) = J_{\mathfrak{b}}(\gamma : H) \quad (H \in \mathfrak{b}'')$$

in the notation of [2 (k), Lemma 14], $\mathfrak{a}^+$ is the set of all points H in $\mathfrak{a}$ where $\alpha(H) > 0$ and $\mathfrak{b}''$ is the set of those $H \in \mathfrak{b}$ where $\beta(H) \neq 0$. Therefore since Ψ_ε is an invariant C^∞ function on Ω, it is clear that

$$j_\varepsilon(\gamma) = \frac{1}{2}\int_{\mathfrak{a}^+} \alpha J_\gamma^{\mathfrak{a}}(\Psi_\varepsilon)_{\mathfrak{a}}\,d\mathfrak{a} + (-1)^{\frac{1}{2}}\frac{1}{2}\int_{\mathfrak{b}} \beta J_\gamma^{\mathfrak{b}}(\Psi_\varepsilon)_{\mathfrak{b}}\,d\mathfrak{b}$$

for $\gamma \in C_c^\infty(\Omega_\mathfrak{z})$. Now apply Lemma 12 and observe that $\operatorname{Supp} V_{\varepsilon,\mathfrak{h}} \subset \mathfrak{h} \cap \mathfrak{g}'$ and

$$(\partial(\omega_{\mathfrak{h}}) \circ V_{\varepsilon,\mathfrak{h}})^* = V_{\varepsilon,\mathfrak{h}}\,\partial(\omega_{\mathfrak{h}}) \quad (\mathfrak{h} = \mathfrak{a} \text{ or } \mathfrak{b}).$$

Then it follows that

$$j_\varepsilon(\gamma) = \frac{1}{2}\int_{\mathfrak{a}^+} V_{\varepsilon,\mathfrak{a}}\{\partial(\omega_{\mathfrak{a}})(\alpha J_\gamma^{\mathfrak{a}}) \cdot \Phi_{0,\mathfrak{a}} - \alpha J_\gamma^{\mathfrak{a}} \cdot \partial(\omega_{\mathfrak{a}})\Phi_{0,\mathfrak{a}}\}\,d\mathfrak{a}$$
$$+ (-1)^{\frac{1}{2}}\frac{1}{2}\int_{\mathfrak{b}} V_{\varepsilon,\mathfrak{b}}\{\partial(\omega_{\mathfrak{b}})(\beta J_\gamma^{\mathfrak{b}}) \cdot \Phi_{0,\mathfrak{b}} - \beta J_\gamma^{\mathfrak{b}} \cdot \partial(\omega_{\mathfrak{b}})\Phi_{0,\mathfrak{b}}\}\,d\mathfrak{b}$$

where $\Phi_{0,\mathfrak{h}} = \delta_{\mathfrak{g}/\mathfrak{h}}(D)\Phi_{\mathfrak{h}}$ ($\mathfrak{h} = \mathfrak{a}$ or $\mathfrak{b}$). Hence it is obvious that

$$j(\gamma) = \frac{1}{2}\int_{\mathfrak{a}^+} \{\partial(\omega_{\mathfrak{a}})(\alpha J_\gamma^{\mathfrak{a}}) \cdot \Phi_{0,\mathfrak{a}} - \alpha J_\gamma^{\mathfrak{a}}\partial(\omega_{\mathfrak{a}})\Phi_{0,\mathfrak{a}}\}\,d\mathfrak{a}$$
$$+ (-1)^{\frac{1}{2}}\frac{1}{2}\int_{\mathfrak{b}} \{\partial(\omega_{\mathfrak{b}})(\beta J_\gamma^{\mathfrak{b}}) \cdot \Phi_{0,\mathfrak{b}} - \beta J_\gamma^{\mathfrak{b}}\partial(\omega_{\mathfrak{b}})\Phi_{0,\mathfrak{b}}\}\,d\mathfrak{b}$$

for $\gamma \in C_c^\infty(\Omega_\mathfrak{z})$. Now $\omega_{\mathfrak{a}} = \omega_\sigma + |\alpha|^{-2}\alpha^2$ where ω_σ is the restriction of ω on σ. Similarly $\omega_{\mathfrak{b}} = \omega_\sigma + |\beta|^{-2}\beta^2$. Hence (see [2 (k), Lemma 21]) it follows that

$$j(\gamma) = \frac{1}{2|\alpha|^2}\int_{\mathfrak{a}^+} \partial(\alpha)\{\partial(\alpha)(\alpha J_\gamma^{\mathfrak{a}}) \cdot \Phi_{0,\mathfrak{a}} - \alpha J_\gamma^{\mathfrak{a}}\partial(\alpha)\Phi_{0,\mathfrak{a}}\}\,d\mathfrak{a}$$
$$+ \frac{(-1)^{\frac{1}{2}}}{2|\beta|^2}\int_{\mathfrak{b}} \partial(\beta)\{\partial(\beta)(\beta J_\gamma^{\mathfrak{b}}) \cdot \Phi_{0,\mathfrak{b}} - \beta J_\gamma^{\mathfrak{b}}\partial(\beta)\Phi_{0,\mathfrak{b}}\}\,d\mathfrak{b}.$$

Now $d\mathfrak{a}=d\sigma\,dt$, $d\mathfrak{b}=d\sigma\,d\phi$ and

$$\partial(\alpha)=\tfrac{1}{2}|\alpha|^2\partial/\partial t,\quad \partial(\beta)=\tfrac{1}{2}(-1)^{\frac{1}{2}}|\beta|^2\partial/\partial\phi$$

since
$$H'=2|\alpha|^{-2}H_\alpha,\quad X'-Y'=-2(-1)^{\frac{1}{2}}|\beta|^{-2}H_\beta$$

in the notation of [2 (k), § 7]. Therefore

$$j(\gamma)=-\tfrac{1}{4}\int_\sigma\{\partial(\alpha)(\alpha J_\gamma^{\mathfrak{a}})\cdot\Phi_{0,\mathfrak{a}}-\alpha J_\gamma^{\mathfrak{a}}\cdot\partial(\alpha)\Phi_{0,\mathfrak{a}}\}^+\,d\sigma$$
$$+\tfrac{1}{4}\int_\sigma\{\partial(\beta)(\beta J_\gamma^{\mathfrak{b}})\cdot\Phi_{0,\mathfrak{b}}-\beta J_\gamma^{\mathfrak{b}}\,\partial(\beta)\Phi_{0,\mathfrak{b}}\}_-{}^+\,d\sigma.$$

Here

$$u_{\mathfrak{a}}{}^+(H)=\lim_{t\to+0}u_{\mathfrak{a}}(H+tH'),\quad u_{\mathfrak{b}}{}^\pm(H)=\lim_{\phi\to+0}u_{\mathfrak{b}}(H\pm\phi(X'-Y'))\quad(H\in\sigma;\ t,\ \phi\in\mathbf{R})$$

for two functions $u_{\mathfrak{a}}$ and $u_{\mathfrak{b}}$ on $\mathfrak{a}$ and $\mathfrak{b}$ respectively and $(u_{\mathfrak{b}})_-{}^+=u_{\mathfrak{b}}{}^+-u_{\mathfrak{b}}{}^-$. Since $\alpha=\beta=0$ on σ and $|\alpha|^2=|\beta|^2$ [2 (k), Lemma 13], it follows that

$$j(\gamma)=\tfrac{1}{4}|\alpha|^2\int_\sigma\{(J_\gamma^{\mathfrak{b}}\Phi_{0,\mathfrak{b}})_-{}^+-(J_\gamma^{\mathfrak{a}}\Phi_{0,\mathfrak{a}})^+\}\,d\sigma.$$

However $\Phi_{0,\mathfrak{a}}$ and $\Phi_{0,\mathfrak{b}}$ are continuous functions on $\mathfrak{a}\cap\Omega_{\mathfrak{z}}$ and $\mathfrak{b}\cap\Omega_{\mathfrak{z}}$ respectively and $\Phi_{0,\mathfrak{a}}=\Phi_{0,\mathfrak{b}}$ on $\sigma\cap\Omega_{\mathfrak{z}}$ [2 (l), Lemma 18]. Therefore

$$j(\gamma)=\tfrac{1}{4}|\alpha|^2\int_\sigma\{(J_\gamma^{\mathfrak{b}})_-{}^+-J_\gamma^{\mathfrak{a}}\}\Phi_{0,\mathfrak{a}}\,d\sigma\quad(\gamma\in C_c^\infty(\Omega_{\mathfrak{z}})).$$

But $(J_\gamma^{\mathfrak{b}})_-{}^+=J_\gamma^{\mathfrak{a}}$ on σ [2 (k), § 19]. Hence $j=0$ on $\Omega_{\mathfrak{z}}$.

Now put (1) $\pi_\alpha=\alpha^{-1}\pi^{\mathfrak{a}}$, $\pi_\beta=\beta^{-1}\pi^{\mathfrak{b}}$ and

$$\Phi_{\mathfrak{h}}'=\delta_{\mathfrak{g}/\mathfrak{h}}(D)(\pi^{\mathfrak{h}}\Phi_{\mathfrak{h}})$$

for $\mathfrak{h}=\mathfrak{a}$ or $\mathfrak{b}$. Then if $\gamma\in C_c^\infty(\Omega_{\mathfrak{z}})$, we have

$$j_\varepsilon'(\gamma)=\frac{1}{2}\int_{\mathfrak{a}^+}\alpha J_\gamma^{\mathfrak{a}}(\Psi_\varepsilon')_{\mathfrak{a}}d\mathfrak{a}+(-1)^{\frac{1}{2}}\frac{1}{2}\int_{\mathfrak{b}}\beta J_\gamma^{\mathfrak{b}}(\Psi_\varepsilon')_{\mathfrak{b}}\,d\mathfrak{b}$$
$$=\frac{1}{2}\int_{\mathfrak{a}^+}V_{\varepsilon,\mathfrak{a}}\{\partial(\omega_{\mathfrak{a}})(\pi_\alpha{}^{-1}J_\gamma^{\mathfrak{a}})\cdot\Phi_{\mathfrak{a}}'-\pi_\alpha{}^{-1}J_\gamma^{\mathfrak{a}}\partial(\omega_{\mathfrak{a}})\Phi_{\mathfrak{a}}'\}\,d\mathfrak{a}$$
$$+\tfrac{1}{2}(-1)^{\frac{1}{2}}\int_{\mathfrak{b}}V_{\varepsilon,\mathfrak{b}}\{\partial(\omega_{\mathfrak{b}})(\pi_\beta{}^{-1}J_\gamma^{\mathfrak{b}})\cdot\Phi_{\mathfrak{b}}'-\pi_\beta{}^{-1}J_\gamma^{\mathfrak{b}}\partial(\omega_{\mathfrak{b}})\Phi_{\mathfrak{b}}'\}\,d\mathfrak{b}$$

(1) We assume, as we may, that $(\pi^{\mathfrak{a}})^\nu=\pi^{\mathfrak{b}}$ in the notation of [2 (k). § 7].

from Lemma 12. Hence

$$
\begin{aligned}
j'(\gamma) &= \frac{1}{2|\alpha|^2}\int_{\mathfrak{a}^+} \partial(\alpha)\{\partial(\alpha)(\pi_\alpha^{-1}J_\gamma^{\mathfrak{a}})\cdot\Phi_{\mathfrak{a}}' - \pi_\alpha^{-1}J_\gamma^{\mathfrak{a}}\partial(\alpha)\Phi_{\mathfrak{a}}'\}\,d\mathfrak{a}\\
&\quad + \frac{(-1)^{\frac{1}{2}}}{2|\beta|^2}\int_{\mathfrak{b}} \partial(\beta)\{\partial(\beta)(\pi_\beta^{-1}J_\gamma^{\mathfrak{b}})\cdot\Phi_{\mathfrak{b}}' - \pi_\beta^{-1}J_\gamma^{\mathfrak{b}}\partial(\beta)\Phi_{\mathfrak{b}}'\}\,d\mathfrak{b}\\
&= -\tfrac{1}{4}\int_\sigma \{\partial(\alpha)(\pi_\alpha^{-1}J_\gamma^{\mathfrak{a}})\cdot\Phi_{\mathfrak{a}}' - \pi_\alpha^{-1}J_\gamma^{\mathfrak{a}}\partial(\alpha)\Phi_{\mathfrak{a}}'\}^+\,d\sigma\\
&\quad + \tfrac{1}{4}\int_\sigma \{\partial(\beta)(\pi_\beta^{-1}J_\gamma^{\mathfrak{b}})\cdot\Phi_{\mathfrak{b}}' - \pi_\beta^{-1}J_\gamma^{\mathfrak{b}}\partial(\beta)\Phi_{\mathfrak{b}}'\}_-^{\,+}\,d\sigma.
\end{aligned}
$$

Now $\pi_\alpha^{-1}J_\gamma^{\mathfrak{a}}$ is a C^∞ function on $\mathfrak{a}$ which is invariant under the Weyl reflexion s_α. Hence $\partial(\alpha)(\pi_\alpha^{-1}J_\gamma^{\mathfrak{a}})=0$ on σ. Moreover, $\partial(\beta)(\pi_\beta^{-1}J_\gamma^{\mathfrak{b}})$ is a continuous function on $\mathfrak{b}$ by [2 (k), Theorem 1] and $\partial(\alpha)\Phi_{\mathfrak{a}}'$, $\partial(\beta)\Phi_{\mathfrak{b}}'$ are continuous functions on $\mathfrak{a}\cap\Omega_{\mathfrak{z}}$, $\mathfrak{b}\cap\Omega_{\mathfrak{z}}$ respectively and they are equal on $\sigma\cap\Omega_{\mathfrak{z}}$ [2 (l), Lemma 18]. Finally $\Phi_{\mathfrak{b}}'$ is an analytic function on $\mathfrak{b}\cap\Omega_{\mathfrak{z}}$ [2 (l), Theorem 2]. Hence

$$
\begin{aligned}
j'(\gamma) &= \tfrac{1}{4}\int_\sigma \pi_\alpha^{-1}J_\gamma^{\mathfrak{a}}\partial(\alpha)\Phi_{\mathfrak{a}}'\,d\sigma - \tfrac{1}{4}\int_\sigma (\pi_\beta^{-1}J_\gamma^{\mathfrak{b}})_-^{\,+}\partial(\beta)\Phi_{\mathfrak{b}}'\,d\sigma\\
&= \tfrac{1}{4}\int_\sigma \{\pi_\alpha^{-1}J_\gamma^{\mathfrak{a}} - (\pi_\beta^{-1}J_\gamma^{\mathfrak{b}})_-^{\,+}\}\,\partial(\beta)\Phi_{\mathfrak{b}}'\,d\sigma.
\end{aligned}
$$

But[1] $\pi_\alpha=\pi_\beta$ and $J_\gamma^{\mathfrak{a}}=(J_\gamma^{\mathfrak{b}})_-^{\,+}$ on σ. Therefore $j'=0$ on $\Omega_{\mathfrak{z}}$. In view of [2 (j), Lemma 17] this completes the proof of Lemma 7.

§ 7. A consequence of Theorem 1

We now return to the notation of § 2 so that $\mathfrak{g}$ is again reductive. For any $p\in I(\mathfrak{g}_c)$, let p_i denote the projection of p in $I(\mathfrak{h}_{ic})$ (see [2 (j), § 8]).

LEMMA 13. *Fix $p\in I(\mathfrak{g}_c)$. Then*

$$\sum_{1\le i\le r} c_i\int \varepsilon_{R,i}\{\partial(\varpi_i p_i)\psi_{f,i}\cdot\Phi_i - \psi_{f,i}\partial(\varpi_i p_i)^*\Phi_i\}\,d_iH=0$$

for $f\in C_c^\infty(\Omega)$.

We note that $\partial(\varpi_i)^*=(-1)^m\partial(\varpi_i)$. Therefore applying Lemma 5 to $\partial(p)f$, instead of f, we get

---

(1) See footnote 1, p. 257.

$$\sum_i c_i \int \varepsilon_{i,R}\, \partial(\varpi_i p_i)\, \psi_{f,i} \cdot \Phi_i\, d_i H = \sum_i c_i \int \varepsilon_{i,R}\, \partial(p_i)\, \psi_{f,i}\, \partial(\varpi_i)^* \Phi_i\, d_i H = (-1)^m \int \partial(p) f \cdot \Box F\, dX.$$

But it follows from the corollary of [2 (l), Lemma 16] that

$$\int \partial(p) f \cdot \Box F\, dX = \int f \cdot \Box(\partial(p)^* F)\; dX.$$

Hence we conclude from Lemma 8 and [2 (j), Lemma 13] that

$$(-1)^m \int f \Box(\partial(p)^* F)\, dX = \sum_i c_i \int \varepsilon_{i,R}\, \psi_{f,i}\, \partial(p_i \varpi_i)^* \Phi_i\, d_i H.$$

The statement of Lemma 13 is now obvious.

§ 8. Some elementary facts about tempered distributions

Let E be a vector space over $\mathbf{R}$ of finite dimension. Define $S(E_c)$, $P(E_c)$ and $\mathfrak{D}(E_c)$ as usual (see [2 (j), § 3]). Let U be an open subset of E and T a distribution on U. We say that T is tempered if we can choose $D_i \in \mathfrak{D}(E_c)$ $(1 \leqslant i \leqslant r)$ such that

$$|T(f)| \leqslant \sum_i \sup |D_i f| \quad (f \in C_c^\infty(U)).$$

It is clear that if T is tempered, the same holds for DT for any $D \in \mathfrak{D}(E_c)$.

Fix a Euclidean measure dX on E and let g be a locally summable function on U. Then g will be said to be tempered (on U) if the distribution

$$f \to \int fg\; dX \quad (f \in C_c^\infty(U))$$

on U is tempered.

Introduce a Euclidean norm $\| \;\|$ on E.

LEMMA 14. *Let g be a measurable function on U such that*

$$\sup_{X \in U} |g(X)|\, (1 + \|X\|)^{-m} < \infty$$

for some $m \geqslant 0$. Then g is tempered.

We can choose $r \geqslant 0$ such that

$$c_1 = \int_E (1 + \|X\|)^{-r}\, dX < \infty.$$

Put $c_2 = \sup_{X \in U} |g(X)| (1 + \|X\|)^{-m}$. Then

$$\left| \int g f dX \right| \leqslant c_1 c_2 \nu_{m+r}(f) \quad (f \in C_c^\infty U))$$

where
$$\nu_{m+r}(f) = \sup_{X \in U} |f(X)| (1 + \|X\|)^{m+r}.$$

Since $X \to \|X\|^2$ is a quadratic form on E, it is now clear that g is tempered.

A subset V of E is called full if $tX \in V$ whenever $X \in V$ and $t \geqslant 1$.

LEMMA 15. *Let V be a non-empty, open and full subset of E. Put*

$$g(X) = \sum_{1 \leqslant i \leqslant r} p_i(X) e^{\lambda_i(X)} \quad (X \in E)$$

where $\lambda_1, \ldots, \lambda_r$ are distinct linear functions on E_c and $p_i \in P(E_c)$ $(p_i \neq 0)$. Then g is tempered on V if and only if(1)

$$\Re \lambda_i(X) \leqslant 0$$

for all $X \in V$ and $1 \leqslant i \leqslant r$.

We recall that $S(E_c)$ is the algebra of polynomial functions on the dual space E_c' of E_c. Fix $p \in S(E_c)$ and $\lambda \in E_c'$. Then

$$\partial(p) \circ e^\lambda = e^\lambda \partial(p_\lambda)$$

where p_λ is the polynomial function $\mu \to p(\lambda + \mu)$ $(\mu \in E_c')$. Therefore if $q \in P(E_c)$ and $\partial(p)(e^\lambda q) = 0$, we conclude that $\partial(p_\lambda) q = 0$. Now assume that $q \neq 0$ and let q_0 be the homogeneous component of q of the highest degree. Then it is clear that $p_\lambda(0) q_0 = 0$ and therefore $p(\lambda) = 0$. We shall need this fact presently.

Let us now turn to the proof of Lemma 15. If $\Re \lambda_i(X) \leqslant 0$ for $X \in V$ and $1 \leqslant i \leqslant r$, it follows from Lemma 14 that g is tempered on V. To prove the converse we use induction on r.

So let us assume that g is tempered on V. It would be enough to show that $\Re \lambda_1(X) \leqslant 0$ for $X \in V$. First suppose that $r \geqslant 2$. Then $\lambda_1 \neq \lambda_r$ and therefore we can choose $q \in S(E_c)$ such that $q(\lambda_r) = 0$ while $q(\lambda_1) \neq 0$. Put $p = q^d$ where $d > d^0 p_r$. Then

$$\partial(p)(e^{\lambda_i} p_i) = p_i' e^{\lambda_i} \quad (1 \leqslant i \leqslant r)$$

where $p_i' = \partial(p_{\lambda_i}) p_i$. Since $p(\lambda_1) = q(\lambda_1)^d \neq 0$, it follows from what we have seen above, that $p_1' \neq 0$. On the other hand $p_{\lambda_r} = (q_{\lambda_r})^d$ and $q_{\lambda_r}(0) = q(\lambda_r) = 0$. Therefore since $d > d^0 p_r$, it is obvious that $p_r' = 0$. Hence

(1) $\Re c$ denotes the real part of a complex number c.

$$\partial(p)g = \sum_{1 \leqslant i < r} p_i' e^{\lambda_i}.$$

Now $\partial(p)g$ is also tempered on V and $p_1' \neq 0$. Therefore we conclude from the induction hypothesis that $\Re\lambda_1(X) \leqslant 0$ for $X \in V$.

Thus it remains to consider the case $r=1$. Fix $X_1 \in V$ and write λ and p instead of λ_1 and p_1 respectively. Then we have to prove that $\Re\lambda(X_1) \leqslant 0$. If $X_1 = 0$, this is obvious. So let us assume that $X_1 \neq 0$. Choose a linear subspace F of E complementary to $\mathbf{R}X_1$ and an open convex neighborhood U of zero in F such that $X_1 + U \subset V$. Then

$$tX_1 + U = t(X_1 + t^{-1}U) \subset t(X_1 + U) \subset V$$

for $t \geqslant 1$. Let J denote the open interval $(1, \infty)$ in $\mathbf{R}$. Fix $\alpha \in C_c^\infty(U)$ and for any $\beta \in C_c^\infty(J)$, consider the function $\gamma_\beta \in C_c^\infty(V)$ given by

$$\gamma_\beta(tX_1 + X_2) = \beta(t)\,\alpha(X_2) \quad (t \in \mathbf{R},\ X_2 \in F).$$

Put
$$\sigma(\beta) = \int \gamma_\beta g\, dX = \int \beta(t)\,\alpha(X_2)\, g(tX_1 + X_2)\, dt\, dX_2$$

where dX_2 is the Euclidean measure on F normalized in such a way that $dX = dt\, dX_2$ for $X = tX_1 + X_2$. Then

$$\sigma(\beta) = \int e^{ct} q(t)\,\beta(t)\, dt \quad (\beta \in C_c^\infty(J))$$

where $c = \lambda(X_1)$ and
$$q(t) = \int p(tX_1 + X_2)\,\alpha(X_2)\, e^{\lambda(X_2)}\, dX_2.$$

Since $p \neq 0$, α can obviously be so selected that $q \neq 0$. Moreover since g is tempered on V, it is easy to see that σ is a tempered distribution on J. Hence it would be sufficient to prove the following lemma.

LEMMA 16. *Fix $c \in \mathbf{C}$, $t_0 \in \mathbf{R}$ and let $q \neq 0$ be a (complex-valued) polynomial function on $\mathbf{R}$. Then if the function $q(t)\, e^{ct}$ $(t \in \mathbf{R})$ is tempered on the open interval $J = (t_0, \infty)$, we can conclude that $\Re c \leqslant 0$.*

Put
$$T(\beta) = \int \beta(t)\, q(t)\, e^{ct}\, dt \quad (\beta \in C_c^\infty(J))$$

and $D = d/dt$. Let
$$T_0 = (D - c)^d T$$

where $d = d^0 q$. Then T_0 is also a tempered distribution on J. But

$$(D-c)^d(qe^{ct}) = e^{ct}D^d q = ae^{ct}$$

where a is a nonzero constant. Hence it would be enough to consider the case when $q=1$. Then T being tempered, we can choose a number $A \geqslant 0$ and an integer $r \geqslant 0$ such that

$$\left|\int \alpha(t)\, e^{ct} dt\right| \leqslant A \sum_{0\leqslant m,n\leqslant r} \sup |t^m D^n \alpha|$$

for $\alpha \in C_c^\infty(J)$. Let $c = 2c_1 + (-1)^{\frac{1}{2}} c_2$ where $c_i \in \mathbf{R}$ $(i=1,2)$. We have to show that $c_1 \leqslant 0$. So let us assume that $c_1 > 0$. Put

$$\alpha(t) = \beta(t)\, e^{-c't}$$

where $c' = c_1 + (-1)^{\frac{1}{2}} c_2$ and $\beta \in C_c^\infty(J)$. Then

$$|D^n \alpha| = e^{-c_1 t} |(D-c')^n \beta|.$$

Therefore we can select a number $A_1 \geqslant 0$ such that

$$\left|\int \beta(t)\, e^{c_1 t} dt\right| \leqslant A_1 \sum_{0\leqslant m,n\leqslant r} \sup e^{-c_1 t} |t^m D^n \beta|$$

for all $\beta \in C_c^\infty(J)$.

Now fix a function $f \in C^\infty(\mathbf{R})$ such that 1) $0 \leqslant f \leqslant 1$, 2) $f(t)=0$ if $t \leqslant 0$ and 3) $f(t)=1$ if $t \geqslant 1$. For any $M > t_0 + 2$, define

$$\beta_M(t) = f(t - t_0 - 1)\, f(M + 1 - t) \quad (t \in \mathbf{R}).$$

Then $\beta_M \in C_c^\infty(J)$ and
$$\int \beta_M(t)\, e^{c_1 t} dt \geqslant \int_{t_0+2}^{M} e^{c_1 t} dt.$$

On the other hand
$$\sup |e^{-c_1 t} t^m D^n \beta_M| \leqslant a_m b_n$$

where
$$a_m = \sup_{t \geqslant t_0} |t^m e^{-c_1 t}|, \quad b_n = 2^n \max_{0\leqslant k\leqslant n} \sup |D^k f|^2.$$

Therefore
$$\left|\int \beta_M(t)\, e^{c_1 t} dt\right| \leqslant A' \sum_{0\leqslant m\leqslant r} a_m \sum_{0\leqslant n\leqslant r} b_n = B \quad \text{(say)}.$$

This proves that
$$B \geqslant \int_{t_0+2}^{M} e^{c_1 t} dt.$$

But as $M \to +\infty$, the right side tends to $+\infty$ giving a contradiction. This completes the proof.

Let U be an open subset of E and $\mathsf{C}(U)$ the space of all C^∞ functions f on U such that

$$\nu_D(f) = \sup |Df| < \infty$$

for all $D \in \mathfrak{D}(E_c)$. The seminorms ν_D $(D \in \mathfrak{D}(E_c))$ define the structure of a locally convex space on $\mathcal{C}(U)$.

It is well known (see [3, p. 93]) that the inclusion mapping of $C_c^\infty(E)$ into $\mathcal{C}(E)$ is continuous and the image is dense in $\mathcal{C}(E)$. Hence tempered distributions on E are the same as continuous linear functions on $\mathcal{C}(E)$.

§ 9. Proof of Lemma 17

We now return to the notation of § 2. Let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$ and define g as in [2 (l), Theorem 2].

LEMMA 17. *Suppose T is tempered on Ω. Then we can choose an integer $q \geqslant 0$ such that $\pi^q g$ is tempered on $\Omega \cap \mathfrak{h}'$.*

Let A be the Cartan subgroup of G corresponding to $\mathfrak{h}$ and $x \to x^*$ the natural projection of G on $G^* = G/A$. The group W_G operates on G^* on the right in the usual way (see [2 (l), § 9]). Fix an invariant measure dx^* on G^* and a function $\alpha_0 \in C_c^\infty(G^*)$ such that

$$\int \alpha_0(x^*)\, dx^* = 1.$$

Put

$$\alpha(x^*) = [W_G]^{-1} \sum_{s \in W_G} \alpha_0(x^* s).$$

Select a compact set C in G such that $\operatorname{Supp} \alpha \subset C^*$ and $C^* s = C^*$ for $s \in W_G$ and, for any $\beta \in C_c^\infty(\mathfrak{h}')$, define a function $f_\beta \in C_c^\infty(\mathfrak{g})$ as follows.

$$f_\beta(x^* H) = [W_G]^{-1} \alpha(x^*) \sum_{s \in W_G} \beta(s^{-1} H)$$

for $x^* \in C^*$ and $H \in \operatorname{Supp} \beta$ and $\operatorname{Supp} f_\beta \subset (\operatorname{Supp} \beta)^{C^*}$.

Now define $f(x : X) = f(X^x)$ as usual $(x \in G,\ X \in \mathfrak{g})$ for any $f \in C^\infty(\mathfrak{g})$. Fix $D \in \mathfrak{D}(\mathfrak{g}_c)$. Then

$$f(xH; D) = f(x : H; D^{x^{-1}}) \quad (H \in \mathfrak{h})$$

and it is clear that

$$D^{x^{-1}} = \sum_{1 \leqslant i \leqslant r} a_i(x)\, D_i \quad (x \in G)$$

where $a_1, \ldots, a_r$ are analytic functions on G and $D_1, \ldots, D_r$ are linearly independent elements in $\mathfrak{D}(\mathfrak{g}_c)$. Hence

$$f(xH; D) = \sum_i a_i(x)\, f(x : H; D_i).$$

On the other hand if $\mathfrak{q}=[\mathfrak{h},\mathfrak{g}]$, we can choose (see [2 (j), § 2]) an integer $m\geqslant 0$ and elements $q_{ij}\in\mathfrak{S}_+(\mathfrak{q}_c)$, $\xi_{ij}\in\mathfrak{D}(\mathfrak{h}_c)$ $(1\leqslant j\leqslant N)$ such that

$$f(x:H;D_i)=\pi(H)^{-m}\sum_j f(x;q_{ij}:H;\xi_{ij})\quad(1\leqslant i\leqslant r)$$

for $f\in C^\infty(\mathfrak{g})$, $x\in G$ and $H\in\mathfrak{h}'$. Put $\alpha(x)=\alpha(x^*)$ $(x\in G)$. Then if $x\in C$ and $H\in\mathfrak{h}'$, we get

$$f_\beta(xH;D)=\pi(H)^{-m}\sum_{i,j}a_i(x)\,\alpha(x;q_{ij})\,\beta_0(H;\xi_{ij})$$

where $$\beta_0(H)=[W_G]^{-1}\sum_{s\in W_G}\beta(s^{-1}H).$$

Since C is compact, it is obvious that

$$\sup|Df_\beta|\leqslant B\sum_{i,j}\sup|\pi^{-m}\xi_{ij}\beta_0|\quad(\beta\in C_c^\infty(\mathfrak{h}')),$$

where B is a constant which depends only on D. Thus we have obtained the following result.

LEMMA 18. *For any $D\in\mathfrak{D}(\mathfrak{g}_c)$, we can choose an integer $m\geqslant 0$ and a finite number of elements $\xi_i\in\mathfrak{D}(\mathfrak{h}_c)$ $(1\leqslant i\leqslant N)$ such that*

$$\sup|Df_\beta|\leqslant\sum_{1\leqslant i\leqslant N}\sup|\pi^{-m}\xi_i\beta|$$

for all $\beta\in C_c^\infty(\mathfrak{h}')$.

Now we come to the proof of Lemma 17. Since T is tempered, there exist $D_i\in\mathfrak{D}(\mathfrak{g}_c)$ $(1\leqslant i\leqslant r)$ such that

$$|T(f)|\leqslant\sum_i\sup|D_i f|$$

for all $f\in C_c^\infty(\Omega)$. Therefore by Lemma 18, we can choose an integer $m_0\geqslant 0$ and elements $\xi_j\in\mathfrak{D}(\mathfrak{h}_c)$ $(1\leqslant j\leqslant N)$ such that

$$|T(f_\beta)|\leqslant\sum_{1\leqslant i\leqslant r}\sup|D_i f_\beta|\leqslant\sum_{1\leqslant j\leqslant N}\sup|\pi^{-m_0}\xi_j\beta|$$

for $\beta\in C_c^\infty(\Omega\cap\mathfrak{h}')$. On the other hand

$$T(f_\beta)=\int f_\beta F\,dx=c\int\varepsilon_R\psi_{f_\beta}g\,dH$$

where $c=c(\mathfrak{h})$ in the notation of § 2. Moreover

$$\psi_{f_\beta}(H)=\varepsilon_R(H)\pi(H)\int f_\beta(x^*H)\,dx^*=\varepsilon_R(H)\pi(H)\beta_0(H)\quad(H\in\mathfrak{h}').$$

Hence
$$T(f_\beta)=c\int \pi\beta_0 g\,dH=c\int \pi\beta g\,dH$$

if we take into account the fact that $g^s=\varepsilon(s)g$ $(s\in W_G)$. Put $\gamma=\pi^{m-1}\beta$ $(m\geqslant 1)$. Then

$$\left|\int \beta\pi^m g\,dH\right|=|c^{-1}T(f_\gamma)|\leqslant |c|^{-1}\sum_j \sup|\pi^{-m_0}\xi_j(\pi^{m-1}\beta)|$$

for $\beta\in C_c^\infty(\Omega\cap\mathfrak{h}')$. If m is sufficiently large, it is clear that $\pi^{-m_0}\xi_j\circ\pi^{m-1}\in\mathfrak{D}(\mathfrak{h}_c)$. This shows that $\pi^m g$ is tempered on $\Omega\cap\mathfrak{h}'$.

Fix a Euclidean norm $\|X\|$ $(X\in\mathfrak{g})$ on $\mathfrak{g}$ and for any Cartan subalgebra $\mathfrak{h}$ define $g^{\mathfrak{h}}$ as in [2 (l), Theorem 3].

LEMMA 19. *Suppose for every Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ we can choose numbers $a\geqslant 0$ and $m\geqslant 0$ such that*
$$|g^{\mathfrak{h}}(H)|\leqslant a(1+\|H\|)^m$$
for $H\in\Omega\cap\mathfrak{h}'(R)$. Then T is tempered.

We use the notation of Lemma 5 and put $g_i=g^{\mathfrak{h}_i}$ $(1\leqslant i\leqslant r)$. Then

$$T(f)=\sum_i c_i\int \varepsilon_{R,i}\,g_i\,\psi_{f,i}\,d_iH\quad (f\in C_c^\infty(\Omega)).$$

Therefore we can choose $c\geqslant 0$ and an integer $M\geqslant 0$ such that

$$|T(f)|\leqslant c\sum_i \sup_{\mathfrak{h}_i'}\,(1+\|H\|)^M|\psi_{f,i}(H)|$$

for $f\in C_c^\infty(\Omega)$. Our assertion now follows immediately from [2 (d), Theorem 3].

§ 10. An auxiliary result

Let $\mathfrak{g}$ be a reductive Lie algebra over $\mathbf{C}$, $\mathfrak{h}$ a Cartan subalgebra of $\mathfrak{g}$ and W the Weyl group of $(\mathfrak{g},\mathfrak{h})$.

LEMMA 20. *Let λ be a linear function on $\mathfrak{h}$ and α a root of $(\mathfrak{g},\mathfrak{h})$. Suppose $s\lambda=\lambda-c\alpha$ for some $s\in W$ and $c\neq 0$ in $\mathbf{C}$. Then* (1) $s\lambda=s_\alpha\lambda$.

For the proof we may obviously assume that $\mathfrak{g}$ is semisimple. Let $\mathfrak{F}$ be the real vector space consisting of all linear functions μ on $\mathfrak{h}$ such that (2) $\mu(H_\beta)\in\mathbf{R}$ for every

(1) As usual s_α denotes the Weyl reflexion corresponding to α.

(2) H_β has the same meaning as in [2 (k), § 4].

root β. Fix an order in $\mathfrak{F}$ and first assume that $\lambda \in \mathfrak{F}$. Then $\sigma\lambda \in \mathfrak{F}$ for every $\sigma \in W$. Select $\sigma_0 \in W$ such that $\sigma_0\lambda \geqslant \sigma\lambda$ for all $\sigma \in W$. Then if we put $\lambda' = \sigma_0\lambda$, $s' = \sigma_0 s \sigma_0^{-1}$ and $\alpha' = \sigma_0\alpha$, we obviously get $s'\lambda' = \lambda' - c\alpha'$. Moreover the relation $s\lambda = s_\alpha\lambda$ is equivalent to $s'\lambda' = s_{\alpha'}\lambda'$. Hence without loss of generality, we may assume that $\lambda \geqslant \sigma\lambda$ for all $\sigma \in W$. Since λ and $s\lambda$ are both in $\mathfrak{F}$, it is clear that $c \in \mathbf{R}$. Replacing α by $-\alpha$, if necessary, we may assume that $\alpha > 0$. Then $c > 0$ since $\lambda \geqslant s\lambda$. Now consider

$$s_\alpha s\lambda = s_\alpha\lambda + c\alpha = \lambda - c'\alpha$$

where $c' = 2(\lambda(H_\alpha)/\alpha(H_\alpha)) - c$. We claim $c' = 0$. For otherwise $c' > 0$ since $s_\alpha s\lambda \leqslant \lambda$. Moreover

$$s_\alpha\lambda = \lambda - (c + c')\alpha = s\lambda - c'\alpha.$$

Therefore $$s^{-1}s_\alpha\lambda = \lambda - c's^{-1}\alpha, \quad s^{-1}\lambda = \lambda + cs^{-1}\alpha.$$

Since c and c' are both positive, it follows that at least one of the two elements $s^{-1}s_\alpha\lambda$, $s^{-1}\lambda$ is higher than λ in our order. But this contradicts the condition that $\lambda \geqslant \sigma\lambda$ for all $\sigma \in W$. Hence $c' = 0$. This shows that $s_\alpha s\lambda = \lambda$ and therefore $s\lambda = s_\alpha\lambda$.

Now consider the general case. Then $\lambda = \lambda_R + (-1)^{\frac{1}{2}}\lambda_I$ and $c = a + (-1)^{\frac{1}{2}}b$ where $\lambda_R, \lambda_I \in \mathfrak{F}$ and $a, b \in \mathbf{R}$. The relation $s\lambda = \lambda - c\alpha$ implies that

$$s\lambda_R = \lambda_R - a\alpha, \quad s\lambda_I = \lambda_I - b\alpha.$$

Hence if $ab \neq 0$, we get $s\lambda_R = s_\alpha\lambda_R$, $s\lambda_I = s_\alpha\lambda_I$ from the above proof. Therefore $s\lambda = s_\alpha\lambda$ in this case. Now suppose $a \neq 0$, $b = 0$. Then $s\lambda_R = s_\alpha\lambda_R$ and $s\lambda_I = \lambda_I$ again from the above proof. Let W_0 be the subgroup of all $\sigma \in W$ such that $\sigma\lambda_I = \lambda_I$. For $\mu_1, \mu_2 \in \mathfrak{F}$, let $\langle \mu_1, \mu_2 \rangle$ denote the usual scalar product defined by means of the Killing form of $\mathfrak{g}$ so that

$$\langle \mu_1, \mu_2 \rangle = \sum_\beta \mu_1(H_\beta)\mu_2(H_\beta)$$

where β runs over all roots of $(\mathfrak{g}, \mathfrak{h})$. Then $\langle \sigma\mu_1, \mu_2 \rangle = \langle \mu_1, \sigma^{-1}\mu_2 \rangle$ for $\sigma \in W$. Hence

$$\langle s\lambda_R, \lambda_I \rangle = \langle \lambda_R, s^{-1}\lambda_I \rangle = \langle \lambda_R, \lambda_I \rangle.$$

But $\lambda_R - s\lambda_R = a\alpha$ and $a \neq 0$. Therefore $\langle \alpha, \lambda_I \rangle = 0$ and this implies that $s_\alpha\lambda_I = \lambda_I$. Hence $s\lambda = s_\alpha\lambda$. The case $a = 0$, $b \neq 0$ can be reduced to the one above by replacing λ by $(-1)^{\frac{1}{2}}\lambda$.

We shall need the above result for the proof of Lemma 26.

§ 11. Proof of Lemma 21

We return to the notation of § 2. So $\mathfrak{g}$ is a reductive Lie algebra over $\mathbf{R}$ and $\mathfrak{g}_1=[\mathfrak{g},\mathfrak{g}]$. Let $\mathfrak{a}$ be a Cartan subalgebra of $\mathfrak{g}$ and $\mathfrak{a}_R$ the set of all points of $\mathfrak{a}_1=\mathfrak{a}\cap\mathfrak{g}_1$ where every root of $(\mathfrak{g},\mathfrak{a})$ takes a real value. Similarly let $\mathfrak{a}_I$ be the set of those points of $\mathfrak{a}$ where all roots of $(\mathfrak{g},\mathfrak{a})$ take pure imaginary values. Define $\theta, \mathfrak{k}, \mathfrak{p}$ and K as in [2 (m), § 16] corresponding to $\mathfrak{a}$. Then it is clear that $\mathfrak{a}_R=\mathfrak{a}\cap\mathfrak{p}$, $\mathfrak{a}_I=\mathfrak{a}\cap\mathfrak{k}$ and therefore $\mathfrak{a}=\mathfrak{a}_R+\mathfrak{a}_I$, where the sum is direct.

Define $\mathfrak{a}'(R)$ as usual (see [2 (k), § 4]) and fix a connected component $\mathfrak{a}_R{}^+$ of $\mathfrak{a}'(R)\cap\mathfrak{a}_R$. Let P_R be the set of all real roots of $(\mathfrak{g},\mathfrak{a})$ which take only positive values on $\mathfrak{a}_R{}^+$. We can introduce compatible orders (see [2 (d), p. 195]) in the spaces of real-valued linear functions on $\mathfrak{a}_R$ and $\mathfrak{a}_R+(-1)^{\frac{1}{2}}\mathfrak{a}_I$ in such a way that all roots in P_R are positive. Let P be the set of all positive roots of $(\mathfrak{g},\mathfrak{a})$ under this order.

Let $\mathfrak{m}$ be the centralizer of $\mathfrak{a}_I$ in $\mathfrak{g}$. Then $\mathfrak{m}$ is reductive in $\mathfrak{g}$ (see [2 (m), Cor. 3 of Lemma 26]) and it is obvious that P_R is the set of all positive roots of $(\mathfrak{m},\mathfrak{a})$.

LEMMA 21. *Suppose* $\mathfrak{g}$ *has a Cartan subalgebra* $\mathfrak{h}$ *such that every root of* $(\mathfrak{g},\mathfrak{h})$ *is imaginary. Then* $\mathfrak{a}_R$ *is a Cartan subalgebra of* $\mathfrak{m}_1=[\mathfrak{m},\mathfrak{m}]$ *and* $\mathfrak{a}_I$ *is the center of* $\mathfrak{m}$.

We can choose $x\in G$ such that $\mathfrak{h}^x\subset\mathfrak{k}$ (see [2 (d), § 8]). Since $\mathfrak{h}^x$ is maximal abelian in $\mathfrak{k}$ and $\mathfrak{a}_I\subset\mathfrak{k}$, we can select $k\in K$ such that $\mathfrak{h}^{kx}\supset\mathfrak{a}_I$. Hence without loss of generality we may suppose that $\mathfrak{a}_I\subset\mathfrak{h}\subset\mathfrak{k}$.

Let Q be the set of all positive roots of $(\mathfrak{g},\mathfrak{h})$ and Q_0 the subset consisting of those $\beta\in Q$ which vanish identically on $\mathfrak{a}_I$. Then it is clear that

$$\mathfrak{m}_c=\mathfrak{h}_c+\sum_{\beta\in Q_0}(\mathbf{C}X_\beta+\mathbf{C}X_{-\beta})$$

in the usual notation (see [(2 (k), § 4]). Since $\mathfrak{h}\subset\mathfrak{k}$, both $\mathfrak{k}$ and $\mathfrak{p}$ are stable under $\operatorname{ad}\mathfrak{h}$ and therefore, for any root γ, X_γ lies either in $\mathfrak{k}_c$ or in $\mathfrak{p}_c$. Hence it is obvious that

$$[\mathfrak{m}_c,\mathfrak{m}_c]\supset[\mathfrak{h}_c,\mathfrak{m}_c]=\sum_{\beta\in Q_0}(\mathbf{C}X_\beta+\mathbf{C}X_{-\beta})\supset\mathfrak{m}_c\cap\mathfrak{p}_c.$$

This shows that $$\mathfrak{m}_1\supset\mathfrak{m}\cap\mathfrak{p}\supset\mathfrak{a}_R.$$

On the other hand let $\mathfrak{c}_\mathfrak{m}$ denote the center of $\mathfrak{m}$ and put $l=\operatorname{rank}\mathfrak{g}$. Since $\mathfrak{a}\subset\mathfrak{m}$, it is clear that

$$l=\operatorname{rank}\mathfrak{m}=\dim\mathfrak{c}_\mathfrak{m}+\operatorname{rank}\mathfrak{m}_1.$$

But $\mathfrak{a}_I\subset\mathfrak{c}_\mathfrak{m}$, $\mathfrak{a}_R\subset\mathfrak{m}_1$ and $\dim\mathfrak{a}_I+\dim\mathfrak{a}_R=\dim\mathfrak{a}=l$. Therefore we conclude that $\mathfrak{c}_\mathfrak{m}=\mathfrak{a}_I$ and $\mathfrak{a}_R$ is a Cartan subalgebra of $\mathfrak{m}_1$.

Select a fundamental system $(\alpha_1, \ldots, \alpha_m)$ of positive roots of $(\mathfrak{m}, \mathfrak{a})$ and let W_R be the subgroup (1) of $W(\mathfrak{g}/\mathfrak{a})$ generated by (2) s_α for $\alpha \in P_R$. Then $s_{\alpha_1}, \ldots, s_{\alpha_m}$ generate W_R and $m = \dim \mathfrak{a}_R$ from Lemma 21.

LEMMA 22. *Let μ be a linear function on $\mathfrak{a}_c$ which takes only real values on $\mathfrak{a}_R + (-1)^{\frac{1}{2}} \mathfrak{a}_I$ and suppose $\mu \geqslant s_{\alpha_i}\mu$ $(1 \leqslant i \leqslant m)$. Then $\mu \geqslant s\mu$ for $s \in W_R$ and $\mu(H) \geqslant 0$ for $H \in \mathfrak{a}_R^+$.*

Define linear functions μ_j $(1 \leqslant j \leqslant m)$ on $\mathfrak{a}$ as follows.

$$s_{\alpha_i}\mu_j = \mu_j - \delta_{ij}\alpha_j \quad (1 \leqslant i \leqslant m)$$

and $\mu_j = 0$ on $\mathfrak{a}_I$. Then $\mu_j \geqslant s\mu_j$ $(s \in W_R)$ and $\mu_j(H) \geqslant 0$ for $H \in \mathfrak{a}_R{}^+$ (see [2 (g), p. 280]). Let

$$s_{\alpha_i}\mu = \mu - c_i\alpha_i \quad (1 \leqslant i \leqslant m)$$

where $c_i \in \mathbf{R}$. Then $c_i \geqslant 0$. Put $\mu_0 = \sum_j c_j \mu_j$. Then $\mu = \mu_0$ on $\mathfrak{a}_R$. Therefore it is clear that

$$\mu - s\mu = \mu_0 - s\mu_0 \geqslant 0 \quad (s \in W_R)$$

and $\mu(H) = \mu_0(H) \geqslant 0$ for $H \in \mathfrak{a}_R{}^+$.

§ 12. Recapitulation of some elementary facts

Fix a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ and let $j = j_\mathfrak{h}$ denote the Chevalley isomorphism $p \rightarrow p_\mathfrak{h}$ of $I(\mathfrak{g}_c)$ onto $I(\mathfrak{h}_c)$ [2 (j), § 9]. Let λ be a linear function on $\mathfrak{h}_c$. Since every element q of $S(\mathfrak{h}_c)$ is a polynomial function on the dual space of $\mathfrak{h}_c$, we can consider its value $q(\lambda)$ at λ. Let $\chi_\lambda = \chi_\lambda^\mathfrak{h}$ denote the homomorphism $p \rightarrow p_\mathfrak{h}(\lambda)$ $(p \in I(\mathfrak{g}_c))$ of $I(\mathfrak{g}_c)$ into $\mathbf{C}$.

Let $W = W(\mathfrak{g}/\mathfrak{h})$. Then W operates on $\mathfrak{D}(\mathfrak{h}_c)$. We say that λ is regular if $s\lambda = \lambda^s \neq \lambda$ for $s \neq 1$ in W. It is well known that λ is singular or regular according as $\varpi(\lambda) = 0$ or not. Moreover if λ' is another linear function on $\mathfrak{h}_c$, then $\chi_\lambda = \chi_{\lambda'}$ if and only if $\lambda' = s\lambda$ for some $s \in W$.

Conversely let $\chi \neq 0$ be a homomorphism of $I(\mathfrak{g}_c)$ into $\mathbf{C}$. Then $\xi : q \rightarrow \chi(j^{-1}(q))$ $(q \in I(\mathfrak{h}_c))$ is a homomorphism of $I(\mathfrak{h}_c)$ into $\mathbf{C}$. Since $S(\mathfrak{h}_c)$ is a finite module over $I(\mathfrak{h}_c)$ (see [2 (c), Lemma 11]), ξ can be extended to a homomorphism of $S(\mathfrak{h}_c)$. Hence there exists a linear function λ on $\mathfrak{h}_c$ such that $\xi(q) = q(\lambda)$ for all $q \in I(\mathfrak{h}_c)$. This shows

(1) $W(\mathfrak{g}/\mathfrak{a})$ denotes the Weyl group of $(\mathfrak{g}, \mathfrak{a})$.

(2) See footnote 1, p. 265.

that $\chi = \chi_\lambda$. Moreover, as we have seen above, λ is unique up to an operation of W. We say that χ is regular if λ is regular. Put $p_0 = j^{-1}(\varpi^2)$. Then

$$\chi(p_0) = \varpi(\lambda)^2.$$

Hence χ is regular if and only if $\chi(p_0) \neq 0$. We note that p_0 is actually independent of $\mathfrak{h}$ and therefore the concept of the regularity of χ does not depend on the choice of $\mathfrak{h}$.

Let $\mathfrak{a}$, $\mathfrak{b}$ be two Cartan subalgebras of $\mathfrak{g}$ and y an element of the connected complex adjoint group G_c of $\mathfrak{g}_c$ such that $\mathfrak{b}_c = (\mathfrak{a}_c)^y$. Then y defines an isomorphism $D \to D^y$ of $\mathfrak{D}(\mathfrak{a}_c)$ onto $\mathfrak{D}(\mathfrak{b}_c)$.

LEMMA 23. *Let λ be a linear function on $\mathfrak{a}_c$. Then*

$$\chi_\lambda{}^{\mathfrak{a}} = \chi_{\lambda^y}{}^{\mathfrak{b}}.$$

This follows from the obvious fact that $j_{\mathfrak{b}}(p) = (j_{\mathfrak{a}}(p))^y$ for $p \in I(\mathfrak{g}_c)$.

LEMMA 24. *Let U be a non-empty open connected subset of $\mathfrak{h}$ and λ a regular linear function on $\mathfrak{h}_c$. Suppose g is an analytic function on U such that $\partial(q)g = q(\lambda)g$ for all $q \in I(\mathfrak{h}_c)$. Then there exist unique complex numbers $c_s\,(s \in W)$ such that*

$$g(H) = \sum_{s \in W} \varepsilon(s)\, c_s\, e^{\lambda(s^{-1}H)} \quad (H \in U).$$

For a proof see [2 (c), p. 102].

§ 13. Proof of Lemma 26

Let $\mathfrak{z}$ be a subalgebra of $\mathfrak{g}$ such that 1) $\mathfrak{z}$ is reductive in $\mathfrak{g}$ and 2) rank $\mathfrak{z}$ = rank $\mathfrak{g}$. Let $\Omega_{\mathfrak{z}}$ be an open and completely invariant subset of $\mathfrak{z}$ and χ a regular homomorphism of $I(\mathfrak{g}_c)$ into $\mathbf{C}$. Let Ξ denote the analytic subgroup of G corresponding to $\mathfrak{z}$ and define the isomorphism $p \to p_{\mathfrak{z}}$ of $I(\mathfrak{g}_c)$ into $I(\mathfrak{z}_c)$ as in [2 (j), § 9]. Consider a distribution $T_{\mathfrak{z}}$ on $\Omega_{\mathfrak{z}}$ such that

1) $T_{\mathfrak{z}}$ is invariant under Ξ,
2) $\partial(p_{\mathfrak{z}})\, T_{\mathfrak{z}} = \chi(p)\, T_{\mathfrak{z}}$ for all $p \in I(\mathfrak{g}_c)$.

Fix a Euclidean measure dZ on $\mathfrak{z}$ and let $\Omega_{\mathfrak{z}}'$ denote the set of those points of $\Omega_{\mathfrak{z}}$ which are regular in $\mathfrak{z}$. Then by [2 (j), Lemma 19] and [2 (l), Theorem 1], $T_{\mathfrak{z}}$ coincides with an analytic function on $\Omega_{\mathfrak{z}}'$. We denote by $T_{\mathfrak{z}}(Z)$ the value of this function at any point $Z \in \Omega_{\mathfrak{z}}'$.

Let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{z}$, P the set of all positive roots of $(\mathfrak{g}, \mathfrak{h})$ and $P_{\mathfrak{z}}$ the subset consisting of all positive roots of $(\mathfrak{z}, \mathfrak{h})$. Put

$$\pi_{\mathfrak{z}} = \prod_{\alpha \in P_{\mathfrak{z}}} \alpha, \quad g_{\mathfrak{z}}(H) = \pi_{\mathfrak{z}}(H) T_{\mathfrak{z}}(H) \quad (H \in \Omega_{\mathfrak{z}}' \cap \mathfrak{h}).$$

Let $\mathfrak{h}'(\mathfrak{z}:R)$ denote the set of those points of $\mathfrak{h}$ where no real root in $P_{\mathfrak{z}}$ takes the value zero. Then by [2 (l), Theorem 2], $g_{\mathfrak{z}}$ extends to an analytic function on $\Omega_{\mathfrak{z}} \cap \mathfrak{h}'(\mathfrak{z}:R)$. Put $W = W(\mathfrak{g}/\mathfrak{h})$ and select a linear function λ on $\mathfrak{h}_c$ such that $\chi = \chi_\lambda$ in the notation of § 12.

LEMMA 25. *There exist locally constant functions c_s $(s \in W)$ on $\Omega_{\mathfrak{z}} \cap \mathfrak{h}'(\mathfrak{z}:R)$ such that*

$$g_{\mathfrak{z}} = \sum_{s \in W} \varepsilon(s) c_s e^{s\lambda}$$

on $\Omega_{\mathfrak{z}} \cap \mathfrak{h}'(\mathfrak{z}:R)$.

Since $(\partial(p_{\mathfrak{z}}) - \chi_\lambda(p)) T = 0$ $(p \in I(\mathfrak{g}_c))$, it follows (see the proof of [2 (l), Lemma 1]) that

$$(\partial(q) - q(\lambda)) g_{\mathfrak{z}} = 0 \quad (q \in I(\mathfrak{h}_c)).$$

Hence our assertion is an immediate consequence of Lemma 24.

Put $\zeta(Z) = \det(\operatorname{ad} Z)_{\mathfrak{g}/\mathfrak{z}}$ $(Z \in \mathfrak{g})$ and fix an element $H_0 \in \mathfrak{z}$ such that $\zeta(H_0) \neq 0$. Then the centralizers of H_0 in $\mathfrak{z}$ and $\mathfrak{g}$ are the same. Hence H_0 is semiregular in $\mathfrak{z}$ if and only if it is so in $\mathfrak{g}$. Now assume $H_0 \in \Omega_{\mathfrak{z}}$, $\zeta(H_0) \neq 0$ and H_0 is semiregular of noncompact type. We shall now use the notation of [2 (k), § 7] without further comment. Then it is clear that $\mathfrak{a}$ and $\mathfrak{b}$ are Cartan subalgebras of $\mathfrak{z}$. Put $W = W(\mathfrak{g}/\mathfrak{a})$ and choose a linear function λ on $\mathfrak{a}_c$ such that $\chi = \chi_\lambda^{\mathfrak{a}}$. Define G_c as in § 12 and let Ξ_c denote its complex-analytic subgroup corresponding to $\operatorname{ad} \mathfrak{z}_c$. Then it is clear that the element ν of [2 (k), § 7] lies in Ξ_c. We assume that the orders of roots are so chosen that (1) $(\varpi^{\mathfrak{a}})^\nu = \varpi^{\mathfrak{b}}$ and $(\pi_{\mathfrak{z}}^{\mathfrak{a}})^\nu = \pi_{\mathfrak{z}}^{\mathfrak{b}}$. Then it follows from Lemma 24 that

$$\partial(\varpi^{\mathfrak{a}}) g_{\mathfrak{z}}^{\mathfrak{a}} = \varpi^{\mathfrak{a}}(\lambda) \sum_{s \in W} c_s^{\mathfrak{a}} e^{s\lambda}$$

on $\Omega_{\mathfrak{z}} \cap \mathfrak{a}'(\mathfrak{z}:R)$ and
$$\partial(\varpi^{\mathfrak{b}}) g_{\mathfrak{z}}^{\mathfrak{b}} = \varpi^{\mathfrak{a}}(\lambda) \sum_{s \in W} c_s^{\mathfrak{b}} \exp((s\lambda)^\nu)$$

on $\Omega_{\mathfrak{z}} \cap \mathfrak{b}'(\mathfrak{z}:R)$. Here $c_s^{\mathfrak{h}}$ are locally constant functions on $\Omega_{\mathfrak{z}} \cap \mathfrak{h}'(\mathfrak{z}:R)$ $(\mathfrak{h} = \mathfrak{a}$ or $\mathfrak{b})$. Put

$$c_s^{\pm\alpha}(H_0) = \lim_{t \to +0} c_s^{\mathfrak{a}}(H_0 \pm tH')$$

and note that $H_0 \in \Omega_{\mathfrak{z}} \cap \mathfrak{b}'(\mathfrak{z}:R)$.

LEMMA 26. *For any* (2) $s \in W$,

(1) Here the notation is obvious (cf. [2 (l), Theorem 3]).

(2) See footnote 1, p. 265.

$$c_s^{\alpha}(H_0)+c_{s_\alpha s}{}^{\alpha}(H_0)=c_s^{-\alpha}(H_0)+c_{s_\alpha s}{}^{-\alpha}(H_0)=c_s^{\mathfrak{b}}(H_0)+c_{s_\alpha s}{}^{\mathfrak{b}}(H_0).$$

Put $\sigma=\mathfrak{a}\cap\mathfrak{b}$, $\pi_\alpha=\alpha^{-1}\pi^{\mathfrak{a}}$, $\pi_\beta=\beta^{-1}\pi^{\mathfrak{b}}$ and let U be an open and convex neighborhood of H_0 in $\Omega_{\mathfrak{z}}$. We assume that U is so small that π_α and π_β never take the value zero on $U\cap\mathfrak{a}$ and $U\cap\mathfrak{b}$ respectively. Since $c_s^{\mathfrak{a}}$ $(s\in W)$ is locally constant on $\Omega_{\mathfrak{z}}\cap\mathfrak{a}'(\mathfrak{z}:R)$, it is clear that

$$c_s^{\mathfrak{a}}(H)=c_s^{\pm\alpha}(H_0)\quad(H\in U\cap\mathfrak{a}')$$

according as $\alpha(H)$ is positive or negative. Similarly $c_s^{\mathfrak{b}}(H)=c_s^{\mathfrak{b}}(H_0)$ for $H\in U\cap\mathfrak{b}$. Moreover $\varpi^{\mathfrak{a}}(\lambda)\neq 0$ since λ is regular. Therefore if we apply [2 (l), Lemma 18] with $D=\partial(\varpi^{\mathfrak{a}})$ and recall that ν leaves σ pointwise fixed, we get

$$\sum_{s\in W}c_s^{\alpha}(H_0)\exp(\lambda(s^{-1}H))$$
$$=\sum_{s\in W}c_s^{-\alpha}(H_0)\exp(\lambda(s^{-1}H))=\sum_{s\in W}c_s^{\mathfrak{b}}(H_0)\exp(\lambda(s^{-1}H))\quad(H\in U\cap\sigma).$$

Let μ_s denote the restriction of $s\lambda$ on σ.

LEMMA 27. *Suppose s_1, s_2 are two distinct elements in W. Then $\mu_{s_1}=\mu_{s_2}$ if and only if $s_2=s_\alpha s_1$.*

Since s_α leaves σ pointwise fixed, it is clear that $\mu_{s_1}=\mu_{s_2}$ if $s_2=s_\alpha s_1$. Conversely suppose $\mu_{s_1}=\mu_{s_2}$. Then it is obvious that $s_2\lambda-s_1\lambda=c\alpha$ for some $c\in\mathbf{C}$. Since λ is regular, $c\neq 0$. Therefore it follows from Lemma 20 that $s_1^{-1}s_2\lambda=s_\gamma\lambda$ where $\gamma=s_1^{-1}\alpha$. But then $s_\gamma=s_1^{-1}s_\alpha s_1$ and therefore $s_2\lambda=s_\alpha s_1\lambda$. Since λ is regular, this implies that $s_2=s_\alpha s_1$.

Now if we take into account the elementary fact that the exponentials of distinct linear functions on σ are linearly independent, Lemma 26 follows immediately from the relations proved above.

§ 14. Tempered and invariant eigendistributions

Let $\mathfrak{c}$ be the center and $\mathfrak{g}_1$ the derived algebra of $\mathfrak{g}$. Fix a number $c>0$ and put $\mathfrak{g}_0=\mathfrak{c}_0+\mathfrak{g}_1(c)$. Here $\mathfrak{c}_0$ is a nonempty, open, connected subset of $\mathfrak{c}$ and $\mathfrak{g}_1(c)$ is defined as in [2 (m), § 3]. Then $\mathfrak{g}_0$ is a completely invariant open set in $\mathfrak{g}$.

Now take $\Omega_{\mathfrak{z}}=\mathfrak{z}\cap\mathfrak{g}_0$ and assume that there exists a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{z}$ and a linear function λ on $\mathfrak{h}_c$ such that 1) every root of $(\mathfrak{g},\mathfrak{h})$ is imaginary, 2) λ takes only pure imaginary values on $\mathfrak{h}$ and 3) $\chi=\chi_\lambda{}^{\mathfrak{h}}$ in the notation of § 12. Since

$(\mathfrak{g}, \mathfrak{h})$ has no real roots and $\mathfrak{g}_0 \cap \mathfrak{h}$ is connected, we conclude from Lemma 25 that

$$g_{\mathfrak{z}}^{\mathfrak{h}}(H) = \sum_{s \in W(\mathfrak{g}/\mathfrak{h})} \varepsilon(s) c_s \exp(\lambda(s^{-1}H)) \quad (H \in \mathfrak{g}_0 \cap \mathfrak{h})$$

where $c_s \in \mathbf{C}$. Let C denote the additive subgroup of $\mathbf{C}$ generated by c_s $(s \in W(\mathfrak{g}/\mathfrak{h}))$. Fix a Cartan subalgebra $\mathfrak{a}$ of $\mathfrak{z}$ and a connected component $\mathfrak{a}^+$ of $\mathfrak{g}_0 \cap \mathfrak{a}'(\mathfrak{z}:R)$. Select a linear function $\lambda_{\mathfrak{a}}$ on $\mathfrak{a}_c$ such that $\chi_{\lambda_{\mathfrak{a}}}{}^{\mathfrak{a}} = \chi$. Then by Lemma 25 there exist unique complex numbers $c_s(\mathfrak{a}^+)$ such that

$$g_{\mathfrak{z}}^{\mathfrak{a}} = \sum_{s \in W(\mathfrak{g}/\mathfrak{a})} \varepsilon(s) c_s(\mathfrak{a}^+) \exp(s\lambda_{\mathfrak{a}})$$

on $\mathfrak{a}^+$.

Lemma 28. *Suppose $T_{\mathfrak{z}}$ is tempered on $\mathfrak{z}_0 = \mathfrak{z} \cap \mathfrak{g}_0$. Then for a given $s \in W(\mathfrak{g}/\mathfrak{a})$, $c_s(\mathfrak{a}^+) = 0$ unless* (1)

$$\Re \lambda_{\mathfrak{a}}(s^{-1}H) \leqslant 0$$

for all $H \in \mathfrak{a}^+$. Moreover $c_s(\mathfrak{a}^+) \in C$.

Corollary. *Under the above conditions $g_{\mathfrak{z}}^{\mathfrak{h}} = 0$ implies $T_{\mathfrak{z}} = 0$.*

This is obvious from the lemma since $C = \{0\}$ if $g_{\mathfrak{z}}^{\mathfrak{h}} = 0$.

Fix a real quadratic form Q on $\mathfrak{g}$ such that 1) $Q(X) = \operatorname{tr}(\operatorname{ad} X)^2$ for $X \in \mathfrak{g}_1$, 2) Q is negative-definite on $\mathfrak{c}$ and 3) $\mathfrak{g}_1$ and $\mathfrak{c}$ are orthogonal under Q. Let U be any subspace of $\mathfrak{g}$ such that the restriction of Q on U is nondegenerate. Then we denote by $i_U(Q)$ the index of Q on U (see the proof of Lemma 12 of [2 (k)]).

Since $\mathfrak{c} \subset \mathfrak{a}$, it is obvious that the restriction of Q on $\mathfrak{a}$ is nondegenerate. We shall prove Lemma 28 by induction on $i_{\mathfrak{a}}(Q)$. Let $l = \operatorname{rank} \mathfrak{g}$. It is obvious that $i_{\mathfrak{a}}(Q) \geqslant -l$. Now if $i_{\mathfrak{a}}(Q) = -l$, it follows that all roots of $(\mathfrak{g}, \mathfrak{a})$ (and therefore also of $(\mathfrak{z}, \mathfrak{a})$) are imaginary. Hence (see [2 (d), p. 237]) $\mathfrak{a}$ is conjugate to $\mathfrak{h}$ under Ξ and so our assertion is obvious in this case. Therefore we may assume that $i_{\mathfrak{a}}(Q) > -l$ so that $\mathfrak{a}_R \neq \{0\}$. Since $\mathfrak{c}_0$ is connected, it is clear that

$$\mathfrak{a}^+ = \mathfrak{a}_I \cap \mathfrak{g}_0 + \mathfrak{a}_R{}^+(\mathfrak{z})$$

where $\mathfrak{a}_R{}^+(\mathfrak{z})$ is a connected component of $\mathfrak{a}'(\mathfrak{z}:R) \cap \mathfrak{a}_R$.

Lemma 29. *Let $\mathfrak{a}_R(\mathfrak{z})$ be the set of points in $\mathfrak{a} \cap [\mathfrak{z}, \mathfrak{z}]$ where every root of $(\mathfrak{z}, \mathfrak{a})$ takes real values. Similarly let $\mathfrak{a}_I(\mathfrak{z})$ be the set of those points of $\mathfrak{a}$ where all roots of $(\mathfrak{z}, \mathfrak{a})$ take pure imaginary values. Then $\mathfrak{a}_R(\mathfrak{z}) = \mathfrak{a}_R$ and $\mathfrak{a}_I(\mathfrak{z}) = \mathfrak{a}_I$.*

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(1) See footnote 1, p. 260.

It is obvious that $\mathfrak{a}_I(\mathfrak{z}) \supset \mathfrak{a}_I$. Moreover we may assume without loss of generality that $\mathfrak{a}_I(\mathfrak{z}) \subset \mathfrak{h}$ (see the proof of Lemma 21). Fix $H \in \mathfrak{a}_I(\mathfrak{z})$. Since every root of $(\mathfrak{g}, \mathfrak{h})$ is imaginary, it is clear that every eigenvalue of $\operatorname{ad} H$ is pure imaginary. Hence $H \in \mathfrak{a}_I$. This proves that $\mathfrak{a}_I(\mathfrak{z}) = \mathfrak{a}_I$. Let $\mathfrak{m}$ be the centralizer of $\mathfrak{a}_I$ in $\mathfrak{g}$ and put $\mathfrak{m}_\mathfrak{z} = \mathfrak{m} \cap \mathfrak{z}$. Then it follows from Lemma 21 that

$$\mathfrak{a}_R(\mathfrak{z}) = \mathfrak{a} \cap [\mathfrak{m}_\mathfrak{z}, \mathfrak{m}_\mathfrak{z}] \subset \mathfrak{a} \cap [\mathfrak{m}, \mathfrak{m}] = \mathfrak{a}_R.$$

Since $\mathfrak{a} = \mathfrak{a}_R + \mathfrak{a}_I = \mathfrak{a}_R(\mathfrak{z}) + \mathfrak{a}_I(\mathfrak{z})$ and both sums are direct (see § 11), we conclude that $\mathfrak{a}_R(\mathfrak{z}) = \mathfrak{a}_R$.

Let $P_R(\mathfrak{z})$ be the set of all real roots of $(\mathfrak{z}, \mathfrak{a})$ which take only positive values on $\mathfrak{a}_R{}^+(\mathfrak{z})$. Then $P_R(\mathfrak{z})$ can be regarded as the set of all positive roots of $(\mathfrak{m}_\mathfrak{z}, \mathfrak{a})$ and if $m = \dim \mathfrak{a}_R$, we can choose a fundamental system $(\alpha_1, \ldots, \alpha_m)$ of roots in $P_R(\mathfrak{z})$ (see § 11). Let $W_R(\mathfrak{z}/\mathfrak{a})$ be the subgroup of $W(\mathfrak{g}/\mathfrak{a})$ generated by s_α $(\alpha \in P_R(\mathfrak{z}))$. Then $W_R(\mathfrak{z}/\mathfrak{a})$ is also generated by s_{α_i} $(1 \leqslant i \leqslant m)$ and $\mathfrak{a}_R{}^+(\mathfrak{z})$ is exactly the set of those $H \in \mathfrak{a}_R$ where $\alpha_i(H) > 0$ $(1 \leqslant i \leqslant m)$.

Now fix i and choose $H_R \in \mathfrak{a}_R$ such that $\alpha_i(H_R) = 0$, $\alpha_j(H_R) > 0$ $(j \neq i,\ 1 \leqslant j \leqslant m)$ and $\alpha(H_R) \neq 0$ for any real root $\alpha \neq \pm \alpha_i$ of $(\mathfrak{g}, \mathfrak{a})$. Then $H_R \in \mathrm{Cl}(\mathfrak{a}_R{}^+(\mathfrak{z}))$ and we can obviously choose a connected component $\mathfrak{a}_R{}^+$ of $\mathfrak{a}'(R) \cap \mathfrak{a}_R$ such that 1) $\mathfrak{a}_R{}^+ \subset \mathfrak{a}_R{}^+(\mathfrak{z})$ and 2) $H_R \in \mathrm{Cl}(\mathfrak{a}_R{}^+)$. Define P and P_R as in § 11 corresponding to $\mathfrak{a}_R{}^+$ and select $H_I \in \mathfrak{a}_I \cap \mathfrak{g}_0$ in such a way that $\alpha(H_I) \neq 0$ for $\alpha \in P$ unless $\alpha \in P_R$. This is obviously possible. Then it is clear that $H_0 = H_I + H_R \in \mathrm{Cl}(\mathfrak{a}^+)$ and the only root in P which vanishes at H_0 is α_i. Therefore H_0 is semiregular in $\mathfrak{z}$ and $\zeta(H_0) \neq 0$. Define ν and $\mathfrak{b}$ as in § 13. Then $\mathfrak{b}$ is a Cartan subalgebra of $\mathfrak{z}$ and, as we have seen during the proof of [2 (k), Lemma 12], $i_\mathfrak{b}(Q) = i_\mathfrak{a}(Q) - 2$. Therefore the induction hypothesis is applicable to $\mathfrak{b}$ and so it follows from Lemma 26 that

$$c_s(\mathfrak{a}^+) + c_{s_{\alpha_i} s}(\mathfrak{a}^+) \in C \quad (s \in W(\mathfrak{g}/\mathfrak{a})).$$

Now fix $s \in W(\mathfrak{g}/\mathfrak{a})$. Then it follows from Lemma 23 that we can choose $y \in G_c$ such that $\mathfrak{a}_c = (\mathfrak{h}_c)^y$ and $s\lambda_\mathfrak{a} = \lambda^y$. Let $(\beta_1, \ldots, \beta_r)$ be a maximal set of linearly independent roots of $(\mathfrak{g}, \mathfrak{h})$. Since λ takes only pure imaginary values on $\mathfrak{h}$, we can choose $a_j \in \mathbf{R}$ such that

$$\lambda - \sum_{1 \leqslant j \leqslant r} a_j \beta_j = 0$$

on $\mathfrak{h}_1 = \mathfrak{h} \cap \mathfrak{g}_1$. Hence

$$s\lambda_\mathfrak{a} = \lambda^y = \sum_j a_j \beta_j{}^y$$

on $\mathfrak{a}_1 = \mathfrak{a} \cap \mathfrak{g}_1$. Since $\beta_j{}^y$ is a root of $(\mathfrak{g}, \mathfrak{a})$, it follows that $s\lambda_\mathfrak{a}$ takes only real values on $\mathfrak{a}_R$. Moreover $\lambda = \lambda^y$ on $\mathfrak{c}$ and so $\lambda_{\mathfrak{a}}(s^{-1}H)$ is pure imaginary for $H \in \mathfrak{a}_I$. Fix a non-empty open subset U of $\mathfrak{a}_I$ such that 1) $U \subset \mathfrak{a}_I \cap \mathfrak{g}_0$ and 2) all the roots of $(\mathfrak{g}, \mathfrak{a})$ which take the value zero on U, are real. Also fix a connected component $\mathfrak{a}_R^+$ of $\mathfrak{a}_R^+(\mathfrak{z}) \cap \mathfrak{a}'(R)$. Then it is clear that

$$U + \mathfrak{a}_R^+ \subset \mathfrak{a}' \cap \mathfrak{z}_0.$$

Since $T_{\mathfrak{z}}$ is tempered on $\mathfrak{z}_0$, it follows from Lemma 17 that $(\pi_{\mathfrak{z}}^{\mathfrak{a}})^q g_{\mathfrak{z}}^{\mathfrak{a}}$ is tempered on $\mathfrak{a}' \cap \mathfrak{z}_0$ for some $q \geqslant 0$. Fix a function $\gamma \in C_c^\infty(U)$ and put

$$g_\gamma(H) = \int_U \gamma(H_I)\,(\pi_{\mathfrak{z}}^{\mathfrak{a}}(H+H_I))^q g_{\mathfrak{z}}^{\mathfrak{a}}(H+H_I)\,dH_I \quad (H \in \mathfrak{a}_R^+)$$

where dH_I is a Euclidean measure on $\mathfrak{a}_I$. Then it is obvious that g_γ is tempered on $\mathfrak{a}_R^+$. Let μ_s and ν_s respectively denote the restrictions of $s\lambda_{\mathfrak{a}}$ $(s \in W(\mathfrak{g}/\mathfrak{a}))$ on $\mathfrak{a}_R$ and $\mathfrak{a}_I$. Then it is clear that

$$g_\gamma(H) = \sum_{s \in W(\mathfrak{g}/\mathfrak{a})} \varepsilon(s)\, c_s(\mathfrak{a}^+)\, e^{\mu_s(H)} \int \gamma(H_I)\,(\pi_{\mathfrak{z}}^{\mathfrak{a}}(H+H_I))^q e^{\nu_s(H_I)}\,dH_I$$

for $H \in \mathfrak{a}_R^+$. Fix $s_0 \in W(\mathfrak{g}/\mathfrak{a})$ and suppose $\mu_{s_0}(H) > 0$ for some $H \in \mathfrak{a}_R^+$. Let W_0 be the set of all $s \in W(\mathfrak{g}/\mathfrak{a})$ such that $\mu_s = \mu_{s_0}$. Then it follows from Lemma 15 that

$$\sum_{s \in W_0} \varepsilon(s)\, c_s(\mathfrak{a}^+) \int \gamma(H_I)\,(\pi_{\mathfrak{z}}^{\mathfrak{a}}(H+H_I))^q e^{\nu_s(H_I)}\,dH_I = 0.$$

This being true for every $\gamma \in C_c^\infty(U)$, we conclude that

$$\sum_{s \in W_0} \varepsilon(s)\, c_s(\mathfrak{a}^+)\, e^{\nu_s} = 0.$$

But since $\mu_s = \mu_{s_0}$ $(s \in W_0)$, it follows that

$$\sum_{s \in W_0} \varepsilon(s)\, c_s(\mathfrak{a}^+)\, e^{s\lambda_{\mathfrak{a}}} = 0.$$

However $\lambda_{\mathfrak{a}}$ being regular, this implies that $c_s(\mathfrak{a}^+) = 0$ $(s \in W_0)$. Therefore in particular $c_{s_0}(\mathfrak{a}^+) = 0$. Since $\mathfrak{a}_R^+$ was an arbitrary component of $\mathfrak{a}_R^+(\mathfrak{z}) \cap \mathfrak{a}'(R)$, the first assertion of Lemma 28 is now obvious.

It remains to show that $c_s(\mathfrak{a}^+) \in C$ for all $s \in W(\mathfrak{g}/\mathfrak{a})$. Suppose this is false. Let W_1 be the set of all $s \in W(\mathfrak{g}/\mathfrak{a})$ such that $c_s(\mathfrak{a}^+) \notin C$. We have seen above that

$$c_s(\mathfrak{a}^+) + c_{s_{\alpha_i}s}(\mathfrak{a}^+) \in C \quad (s \in W(\mathfrak{g}/\mathfrak{a}),\ 1 \leqslant i \leqslant m).$$

Therefore $s_{\alpha_i}s \in W_1$ whenever $s \in W_1$. This shows that W_1 is a union of cosets of the form $W_R(\mathfrak{z}/\mathfrak{a})\,s$.

Introduce compatible orders on the spaces of real-valued linear functions on $\mathfrak{a}_R$ and $\mathfrak{a}_R+(-1)^{\frac{1}{2}}\mathfrak{a}_I$ corresponding to some connected component $\mathfrak{a}_R^+$ of $\mathfrak{a}_R^+(\mathfrak{z})\cap\mathfrak{a}'(R)$ (see § 11). We have seen that $s\lambda_\mathfrak{a}$ $(s\in W(\mathfrak{g}/\mathfrak{a}))$ takes only real values on $\mathfrak{a}_R+(-1)^{\frac{1}{2}}\mathfrak{a}_I$. Choose $\sigma\in W_1$ such that $\mu=\sigma\lambda_\mathfrak{a}\geqslant s\lambda_\mathfrak{a}$ for all $s\in W_1$. Then $\mu\geqslant s\mu$ for all $s\in W_R(\mathfrak{z}/\mathfrak{a})$ and therefore we conclude from Lemma 22 (applied to $(\mathfrak{z},\mathfrak{a})$) that $\mu(H)\geqslant 0$ for $H\in\mathfrak{a}_R^+(\mathfrak{z})$. However $c_\sigma(\mathfrak{a}^+)\neq 0$ since $\sigma\in W_I$. Therefore it follows from the above proof that $\mu(H)\leqslant 0$ for $H\in\mathfrak{a}_R^+(\mathfrak{z})$. This shows that $\mu=0$ on $\mathfrak{a}_R$ and therefore $s_{\alpha_i}\mu=\mu$ $(1\leqslant i\leqslant m)$. But since $\lambda_\mathfrak{a}$ is regular and $m=\dim\mathfrak{a}_R\geqslant 1$, this is impossible. The proof of Lemma 28 is now complete.

§ 15. Proof of Lemma 30

We keep to the notation of § 14. Let $\mathfrak{z}_1,\mathfrak{z}_2$ be two subalgebras of $\mathfrak{g}$ and $\mathfrak{h}$ a Cartan subalgebra of $\mathfrak{g}$ such that:

1) $\mathfrak{z}_i$ is reductive in $\mathfrak{g}$ $(i=1, 2)$ and $\mathfrak{z}_1\supset\mathfrak{z}_2\supset\mathfrak{h}$.
2) Every root of $(\mathfrak{g},\mathfrak{h})$ is imaginary.
3) If $\mathfrak{a}$ is any Cartan subalgebra of $\mathfrak{z}_2$, then every real root of $(\mathfrak{z}_1,\mathfrak{a})$ is also a root of $(\mathfrak{z}_2,\mathfrak{a})$.

Define χ as in § 14.

Let T_i be a tempered distribution on $\mathfrak{z}_i\cap\mathfrak{g}_0$ such that

$$\partial(p_{\mathfrak{z}_i})T_i=\chi(p)T_i\quad(p\in I(\mathfrak{g}_c),\ i=1,2).$$

Consider the set P of positive roots of $(\mathfrak{g},\mathfrak{h})$ and let P_i denote the subset of those $\beta\in P$ which are roots of $(\mathfrak{z}_i,\mathfrak{h})$ $(i=1, 2)$. Then $P\supset P_1\supset P_2$. Put $\pi_i=\prod_{\alpha\in P_i}\alpha$. Then it is clear that π_1/π_2 is a polynomial function on $\mathfrak{h}_c$ which is invariant under the Weyl reflexions s_α for $\alpha\in P_2$. Therefore by Chevalley's theorem [2(c), Lemma 9] there exists a unique invariant polynomial function η_0 on $\mathfrak{z}_2$ which coincides with π_1/π_2 on $\mathfrak{h}$.

Put $\mathfrak{g}_0'=\mathfrak{g}_0\cap\mathfrak{g}'$ where $\mathfrak{g}'$ denotes, as before, the set of all regular elements of $\mathfrak{g}$.

LEMMA 30. *Suppose $T_2=\eta_0T_1$ pointwise on $\mathfrak{h}\cap\mathfrak{g}_0'$. Then $T_2=\eta_0T_1$ pointwise on $\mathfrak{z}_2\cap\mathfrak{g}_0'$.*

Let $\mathfrak{a}$ be a Cartan subalgebra of $\mathfrak{z}_2$. It would be enough to show that $T_2=\eta_0T_1$ pointwise on $\mathfrak{a}\cap\mathfrak{g}_0'$. We shall do this by induction on $i_\mathfrak{a}(Q)$ as in § 14. Let Ξ_2 be the analytic subgroup of G corresponding to $\mathfrak{z}_2$. If $i_\mathfrak{a}(Q)=-l$, then $\mathfrak{a}$ is conjugate to $\mathfrak{h}$ under Ξ_2 and so our assertion is obvious. Hence we may assume that $i_\mathfrak{a}(Q)>-l$ so that $m=\dim\mathfrak{a}_R\geqslant 1$.

We use the notation of § 14 corresponding to $\mathfrak{z}=\mathfrak{z}_1, \mathfrak{z}_2$. In particular $\mathfrak{g}_{\mathfrak{z}_i}{}^{\mathfrak{a}}$ is defined corresponding to T_i and we put $g_i{}^{\mathfrak{a}}=g_{\mathfrak{z}_i}{}^{\mathfrak{a}}$, $\pi_i{}^{\mathfrak{a}}=\pi_{\mathfrak{z}_i}{}^{\mathfrak{a}}$ $(i=1,2)$. It follows from our assumptions on $\mathfrak{z}_1, \mathfrak{z}_2$ that

$$\mathfrak{a}'(\mathfrak{z}_1:R)=\mathfrak{a}'(\mathfrak{z}_2:R).$$

Fix a connected component $\mathfrak{a}_R{}^+(\mathfrak{z}_2)$ of $\mathfrak{a}'(\mathfrak{z}_2:R)\cap\mathfrak{a}_R$ and let $P_R(\mathfrak{z}_2)$ be the set of all real roots of $(\mathfrak{z}_2, \mathfrak{a})$ which take only positive values on $\mathfrak{a}_R{}^+(\mathfrak{z}_2)$. Select the fundamental system $(\alpha_1, \ldots, \alpha_m)$ of roots in $P_R(\mathfrak{z}_2)$ as in § 14.

Choose a linear function $\lambda_{\mathfrak{a}}$ on $\mathfrak{a}_c$ such that $\chi=\chi_{\lambda_{\mathfrak{a}}}{}^{\mathfrak{a}}$. Then by Lemma 25 there exist complex numbers $c_s(i)$ $(s\in W(\mathfrak{g}/\mathfrak{a}))$ such that

$$g_i{}^{\mathfrak{a}}=\sum_{s\in W(\mathfrak{g}/\mathfrak{a})}\varepsilon(s)\,c_s(i)\,e^{s\lambda_{\mathfrak{a}}}\quad(i=1,2)$$

on $\mathfrak{a}^+=\mathfrak{g}_0\cap\mathfrak{a}_I+\mathfrak{a}_R{}^+(\mathfrak{z}_2)$. It is obvious that

$$\eta_0=a\,\pi_1{}^{\mathfrak{a}}/\pi_2{}^{\mathfrak{a}}$$

on $\mathfrak{a}$ where a is a constant $(a=\pm 1)$. Therefore it would be sufficient to show that $g_2{}^{\mathfrak{a}}=a g_1{}^{\mathfrak{a}}$ on $\mathfrak{a}^+$.

Fix j $(1\leqslant j\leqslant m)$. Then (see § 14) we can select an element $H_0\in \mathrm{Cl}(\mathfrak{a}^+)$ such that 1) $\alpha_j(H_0)=0$ and 2) $\alpha(H_0)\neq 0$ for any root $\alpha\neq\pm\alpha_j$ of $(\mathfrak{g}, \mathfrak{a})$. It is clear that H_0 is semiregular in each of the three algebras $\mathfrak{z}_1$, $\mathfrak{z}_2$ and $\mathfrak{g}$. Define ν and $\mathfrak{b}$ as in § 13. Then $\mathfrak{b}\subset\mathfrak{z}_2$ and $i_{\mathfrak{b}}(Q)=i_{\mathfrak{a}}(Q)-2$. Hence our induction hypothesis is applicable to $\mathfrak{b}$ and so it follows from Lemma 26 that

$$c_s(2)+c_{s_{\alpha_j}s}(2)=a\{c_s(1)+c_{s_{\alpha_j}s}(1)\}$$

for $s\in W(\mathfrak{g}/\mathfrak{a})$.

In order to complete the proof we have to show that $c_s(2)=ac_s(1)$ for all $s\in W(\mathfrak{g}/\mathfrak{a})$. Suppose this is false. Let W_1 be the set of all $s\in W(\mathfrak{g}/\mathfrak{a})$ such that $c_s(2)\neq ac_s(1)$. Then it follows from the above result that if $s\in W_1$, the same holds for $s_{\alpha_j}s$ $(1\leqslant j\leqslant m)$. Define $W_R(\mathfrak{z}_2/\mathfrak{a})$ as in § 14. Then W_1 is a union of cosets of the form $W_R(\mathfrak{z}_2/\mathfrak{a})s$. Fix a connected component $\mathfrak{a}_R{}^+$ of $\mathfrak{a}_R{}^+(\mathfrak{z}_2)\cap\mathfrak{a}'(R)$ and define an order in the space $\mathfrak{F}$ of real-valued linear functions on $\mathfrak{a}_R+(-1)^{\frac{1}{2}}\mathfrak{a}_I$ corresponding to $\mathfrak{a}_R{}^+$ as in § 11. We have seen in § 14 that $s\lambda_{\mathfrak{a}}\in\mathfrak{F}$ for all $s\in W(\mathfrak{g}/\mathfrak{a})$. Choose $\sigma\in W_1$ such that $\mu=\sigma\lambda_{\mathfrak{a}}\geqslant s\lambda_{\mathfrak{a}}$ for all $s\in W_1$. Then $\mu\geqslant s\mu$ for $s\in W_R(\mathfrak{z}_2/\mathfrak{a})$. Therefore by Lemma 22, $\mu(H)\geqslant 0$ for $H\in\mathfrak{a}_R{}^+$. On the other hand since $\sigma\in W_1$, it is clear that $c_\sigma(1)$ and $c_\sigma(2)$ cannot both be zero. Therefore it follows from Lemma 28 that $\mu(H)\leqslant 0$ for $H\in\mathfrak{a}_R{}^+$. But this implies that $\mu=0$ on $\mathfrak{a}_R$ and therefore $s_{\alpha_j}\mu=\mu$ $(1\leqslant j\leqslant m)$. Since $m\geqslant 1$ and $\lambda_{\mathfrak{a}}$ is regular, this is impossible and thus Lemma 30 is proved.

We continue our assumption that $\mathfrak{h}_I=\mathfrak{h}$ and define θ as in [2 (m), § 16] corresponding to $\mathfrak{h}$.

LEMMA 31. *Let $\mathfrak{z}_1$ be a subalgebra of $\mathfrak{g}$ such that $\theta(\mathfrak{z}_1)=\mathfrak{z}_1$ and $\mathfrak{z}_1\supset\mathfrak{h}$. Fix an element $H_1\in\mathfrak{h}$ and let $\mathfrak{z}_2$ be the centralizer of H_1 in $\mathfrak{z}_1$. Then $\mathfrak{z}_1$, $\mathfrak{z}_2$ satisfy all the conditions required above.*

Since $\theta=1$ on $\mathfrak{h}$, it is clear that $\theta(\mathfrak{z}_i)=\mathfrak{z}_i\supset\mathfrak{h}$ and hence $\mathfrak{z}_i$ $(i=1,2)$ is reductive in $\mathfrak{g}$ (see [2 (d), Lemma 10]). Let $\mathfrak{a}$ be a Cartan subalgebra of $\mathfrak{z}_2$. Then we know from Lemma 29 that $\mathfrak{a}_R(\mathfrak{z}_i)=\mathfrak{a}_R$ and $\mathfrak{a}_I(\mathfrak{z}_i)=\mathfrak{a}_I$ $(i=1,2)$. Let $\mathfrak{m}$ be the centralizer of $\mathfrak{a}_I$ in $\mathfrak{z}_2$. Since $H_1\in\mathfrak{a}_I$, $\mathfrak{m}$ is also the centralizer of $\mathfrak{a}_I$ in $\mathfrak{z}_1$. Therefore the real roots of $(\mathfrak{z}_i,\mathfrak{a})$ are the same as the roots of $(\mathfrak{m},\mathfrak{a})$ (see § 11). This proves the lemma.

§ 16. The distribution T_λ

Let $\mathfrak{b}$ be a Cartan subalgebra of $\mathfrak{g}$ and assume that every root of $(\mathfrak{g},\mathfrak{b})$ is imaginary. Consider the space $\mathfrak{F}$ of all linear functions on $\mathfrak{b}_c$ which take only pure imaginary values on $\mathfrak{b}$. Define π, ϖ and $W=W(\mathfrak{g}/\mathfrak{b})$ as usual (corresponding to $\mathfrak{b}$) and let $\mathfrak{F}'$ be the set of all $\lambda\in\mathfrak{F}$ where $\varpi(\lambda)\neq 0$. Consider the subgroup $W_k=W_k(\mathfrak{g}/\mathfrak{b})$ of W generated by s_β corresponding to the compact roots β of $(\mathfrak{g},\mathfrak{b})$ (see [2 (k), § 4]). Then $W_k=W_G$ (see Cor. 2 of [2 (m), Lemma 6]) in the notation of [2 (k), § 4].

THEOREM 2. *For any $\lambda\in\mathfrak{F}'$, there exists a unique distribution T_λ on $\mathfrak{g}$ with the following properties:*

1) *T_λ is invariant and tempered.*
2) $\partial(p)\,T_\lambda=p_{\mathfrak{b}}(\lambda)\,T_\lambda\quad(p\in I(\mathfrak{g}_c))$.
3) $T_\lambda(H)=\pi(H)^{-1}\sum_{s\in W_k}\varepsilon(s)\,e^{\lambda(s^{-1}H)}\quad(H\in\mathfrak{b}')$.

The uniqueness is obvious from the corollary of Lemma 28. Hence only the existence requires proof.

First assume that $\mathfrak{g}$ is semisimple. We identify $\mathfrak{g}_c$ and $\mathfrak{b}_c$ with their respective duals by means of the Killing form of $\mathfrak{g}$ (see [2 (j), § 6]). Fix a Euclidean measure dX on $\mathfrak{g}$ and put

$$\hat{f}(Y)=\int f(X)\exp\left((-1)^{\frac{1}{2}}B(X,Y)\right)dX\quad(Y\in\mathfrak{g})$$

for $f \in \mathcal{C}(\mathfrak{g})$. (As usual $B(X, Y) = \mathrm{tr}(\mathrm{ad}\, X\ \mathrm{ad}\, Y)$ for $X, Y \in \mathfrak{g}_c$.) Moreover for any $H_0 \in \mathfrak{b}'$ define

$$\tau_{H_0}(f) = \psi_{\hat{f}}(H_0) = \pi(H_0) \int_{G^*} \hat{f}(x^* H_0)\, dx^* \quad (f \in \mathcal{C}(\mathfrak{g}))$$

in the notation of [2 (k), § 5] (for $\mathfrak{h} = \mathfrak{b}$). Then we know from [2 (d), Theorem 3] that the integral is absolutely convergent and τ_{H_0} is an invariant and tempered distribution on $\mathfrak{g}$ which satisfies (see [2 (d), p. 226]) the differential equations

$$\partial(p)\, \tau_{H_0} = p((-1)^{\frac{1}{2}} H_0)\, \tau_{H_0} \quad (p \in I(\mathfrak{g}_c)).$$

Fix $H_0 \in \mathfrak{b}'$ and let $\mathfrak{T}_{H_0}$ denote the space of all invariant and tempered distributions T on $\mathfrak{g}$ such that

$$\partial(p)\, T = p((-1)^{\frac{1}{2}} H_0)\, T \quad (p \in I(\mathfrak{g}_c)).$$

For any $T \in \mathfrak{T}_{H_0}$, let g_T denote the analytic function on $\mathfrak{b}$ (see [2 (l), Theorem 2]) given by

$$g_T(H) = \pi(H)\, T(H) \quad (H \in \mathfrak{b}').$$

Then by Lemma 25,

$$g_T(H) = \sum_{s \in W} \varepsilon(s)\, c_s(T) \exp\,((-1)^{\frac{1}{2}} B(sH_0, H)) \quad (H \in \mathfrak{b})$$

where $c_s(T)$ are uniquely determined complex numbers. It is clear that $g_T{}^t = \varepsilon(t) g_T$ and therefore $c_{ts}(T) = c_s(T)$ for $t \in W_G = W_k$ and $s \in W$. On the other hand the linear mapping $T \to g_T$ is injective from the corollary of Lemma 28. Hence it is obvious that

$$\dim \mathfrak{T}_{H_0} \leqslant [W : W_k].$$

On the other hand it is clear that $\tau_{sH_0} \in \mathfrak{T}_{H_0}\ (s \in W)$. Put $r = [W : W_k]$ and select $s_i \in W$ $(1 \leqslant i \leqslant r)$ such that

$$W = \bigcup_{1 \leqslant i \leqslant r} W_k s_i.$$

Write $\tau_i = \tau_{s_i H_0}$. Then we claim that $\tau_1, \ldots, \tau_r$ are linearly independent over $\mathbf{C}$. Put

$$\sigma_i(f) = \psi_f(s_i H_0) \quad (f \in \mathcal{C}(\mathfrak{g})).$$

Since $f \to \hat{f}$ is a topological mapping of $\mathcal{C}(\mathfrak{g})$ onto itself, it would be enough to verify that the tempered distributions $\sigma_1, \ldots, \sigma_r$ are linearly independent. Since $s_i H_0$ is semisimple, the orbit $(s_i H_0)^G$ is closed in $\mathfrak{g}$ (see [1, p. 523]). Therefore it follows from the definition of σ_i that

$$\mathrm{Supp}\ \sigma_i = (s_i H_0)^G.$$

Now we claim that $(s_i H_0)^G \cap (s_j H_0)^G = \emptyset$ if $i \neq j$. For otherwise $s_i H_0 = (s_j H_0)^x$ for some $x \in G$. Since H_0 is regular, this implies that $s_i = s s_j$ for some $s \in W_G = W_k$. But this is impossible from the definition of $(s_1, \ldots, s_r)$. This shows that the sets Supp σ_i are disjoint and non-empty and therefore the distributions σ_i $(1 \leqslant i \leqslant r)$ are linearly independent.

So it is now obvious that dim $\mathfrak{T}_{H_0} = r$ and $\tau_1, \ldots, \tau_r$ is a base for $\mathfrak{T}_{H_0}$. Let a_s $(s \in W)$ be given complex numbers such that $a_{ts} = a_s$ $(t \in W_k)$. Then it follows from the above result that we can choose a unique element $T \in \mathfrak{T}_{H_0}$ such that $a_s = c_s(T)$. Hence, in particular, there exists a distribution T in $\mathfrak{T}_{H_0}$ such that

$$g_T(H) = \sum_{s \in W_k} \varepsilon(s) \exp\left((-1)^{\frac{1}{2}} B(sH_0, H)\right).$$

This proves Theorem 2 when $\mathfrak{g}$ is semisimple.

Now we come to the general case. Define $\mathfrak{g}_1$ and $\mathfrak{c}$ as before (see § 2), put $\mathfrak{b}_1 = \mathfrak{b} \cap \mathfrak{g}_1$ and let λ_1 denote the restriction of λ on $\mathfrak{b}_{1c}$. Fix Euclidean measures dC and dZ on $\mathfrak{c}$ and $\mathfrak{g}_1$ respectively such that $dX = dC\,dZ$ for $X = C + Z$ $(C \in \mathfrak{c},\ Z \in \mathfrak{g}_1)$. Since $\mathfrak{g}_1$ is semisimple, there exists, from the above proof, an invariant and tempered distribution T_1 on $\mathfrak{g}_1$ such that $\partial(p)\,T_1 = p_{\mathfrak{b}}(\lambda)\,T_1$ $(p \in I(\mathfrak{g}_{1c}))$ and

$$\pi(H)\,T_1(H) = \sum_{s \in W_k} \varepsilon(s) \exp\left(\lambda_1(s^{-1}H)\right) \quad (H \in \mathfrak{b}_1 \cap \mathfrak{g}').$$

Put
$$T_\lambda(f) = T_1(f_1) \quad (f \in C_c^\infty(\mathfrak{g}))$$

where
$$f_1(Z) = \int f(Z + C)\, e^{\lambda(C)}\, dC \quad (Z \in \mathfrak{g}_1).$$

Since λ takes only pure imaginary values on $\mathfrak{c}$, it is clear that T_λ satisfies all the conditions of Theorem 2.

Fix a Cartan subalgebra $\mathfrak{a}$ of $\mathfrak{g}$ and an element $y \in G_c$ such that $(\mathfrak{b}_c)^y = \mathfrak{a}_c$. For any $\lambda \in \mathfrak{F}'$, define the analytic function $g_\lambda^{\mathfrak{a}}$ on $\mathfrak{a}'(R)$ corresponding to T_λ as usual so that

$$g_\lambda^{\mathfrak{a}}(H) = \pi^{\mathfrak{a}}(H)\,T_\lambda(H) \quad (H \in \mathfrak{a}').$$

Fix a connected component $\mathfrak{a}^+$ of $\mathfrak{a}'(R)$. Then by Lemmas 25 and 28,

$$g_\lambda^{\mathfrak{a}} = \sum_{s \in W} \varepsilon(s)\, c(s:\lambda:\mathfrak{a}^+)\, e^{(s\lambda)^y}$$

on $\mathfrak{a}^+$ where $c(s:\lambda:\mathfrak{a}^+) \in \mathbf{Z}$.

LEMMA 32. *For fixed* $s \in W$ *and* $\mathfrak{a}^+$, *the integer* $c(s:\lambda:\mathfrak{a}^+)$ $(\lambda \in \mathfrak{F}')$ *depends only on the connected component of* λ *in* $\mathfrak{F}'$.

In view of the last part of the proof of Theorem 2, it is clear that it would be sufficient to consider the case when $\mathfrak{g}$ is semisimple. Define τ_{H_0} as above for $H_0 \in \mathfrak{b}'$. Then by [2 (l), Theorem 1] there exists an analytic function F_{H_0} on $\mathfrak{g}'$ such that

$$\tau_{H_0}(f) = \psi_{\hat{f}}(H_0) = \int F_{H_0}(X) f(X)\, dX \quad (f \in C_c^\infty(\mathfrak{g})).$$

We know from Lemma 25 that

$$\pi^{\mathfrak{a}}(H) F_{H_0}(H) = \sum_{s \in W} \varepsilon(s)\, a_s(H_0) \exp((-1)^{\frac{1}{2}} B(sH_0, y^{-1}H))$$

for $H \in \mathfrak{a}^{+\prime} = \mathfrak{a}^+ \cap \mathfrak{g}'$. Here $a_s(H_0)$ are uniquely determined complex numbers. Moreover we know from [2 (d), pp. 229–231] that a_s, regarded as functions on $\mathfrak{b}'$, are locally constant. By considering, in particular, the case $\mathfrak{a} = \mathfrak{b}$, we get

$$\pi(H) F_{H_0}(H) = \sum_{s \in W} \varepsilon(s)\, b_s(H_0) \exp((-1)^{\frac{1}{2}} B(sH_0, H))$$

for $H, H_0 \in \mathfrak{b}'$. Here b_s are certain locally constant functions on $\mathfrak{b}'$.

Now define $s_1 = 1, s_2, \ldots, s_r$ as in the proof of Theorem 2 and put

$$b_{ij}(H_0) = b_{s_i s_j^{-1}}(s_j H_0) \quad (1 \leqslant i, j \leqslant r,\ H_0 \in \mathfrak{b}').$$

Fix $H_0 \in \mathfrak{b}'$. Since $b_{ts}(H_0) = b_s(H_0)$ $(t \in W_k)$ and $\tau_{s_i H_0}$ $(1 \leqslant i \leqslant r)$ are linearly independent, it follows from the proof of Theorem 2 that the matrix $(b_{ij}(H_0))_{1 \leqslant i, j \leqslant r}$ is non-singular. Let $(b^{ij}(H_0))_{1 \leqslant i, j \leqslant r}$ denote its inverse. Put $b^j = b^{j1}$ and

$$T_{H_0} = \sum_{1 \leqslant j \leqslant r} b^j(H_0)\, \tau_{s_j H_0} \quad (H_0 \in \mathfrak{b}').$$

Then it is obvious that $T_{H_0} \in \mathfrak{T}_{H_0}$ (in the notation of the proof of Theorem 2) and

$$\pi(H) T_{H_0}(H) = \sum_{s \in W_k} \varepsilon(s) \exp((-1)^{\frac{1}{2}} B(sH_0, H)) \quad (H \in \mathfrak{b}')$$

for $H_0 \in \mathfrak{b}'$. Hence it follows from Theorem 2 that $T_{H_0} = T_\lambda$ where λ is the element of $\mathfrak{F}'$ given by $\lambda(H) = (-1)^{\frac{1}{2}} B(H_0, H)$ $(H \in \mathfrak{b})$. Therefore

$$g_\lambda^{\mathfrak{a}}(H) = \sum_{1 \leqslant j \leqslant r} b^j(H_0)\, \pi^{\mathfrak{a}}(H) F_{s_j H_0}(H) \quad (H \in \mathfrak{a}')$$

and this shows that

$$c(s:\lambda:\mathfrak{a}^+) = \sum_{1 \leqslant j \leqslant r} \varepsilon(s_j)\, b^j(H_0)\, a_{ss_j^{-1}}(s_j H_0) \quad (s \in W).$$

Since b^j and a_s are locally constant on $\mathfrak{b}'$, the assertion of the lemma is now obvious.

$\mathfrak{F}^+$ being any connected component of $\mathfrak{F}'$, we denote by $c(s:\mathfrak{F}^+:\mathfrak{a}^+)$ the integer $c(s:\lambda:\mathfrak{a}^+)$ $(\lambda \in \mathfrak{F}^+)$. Put

$$\phi_\lambda = \varpi(\lambda)^{-1}\nabla_{\mathfrak{g}} F_\lambda \quad (\lambda \in \mathfrak{F}')$$

where F_λ is the analytic function on $\mathfrak{g}'$ corresponding to T_λ and $\nabla_{\mathfrak{g}}$ is defined as before (see § 2).

LEMMA 33.
$$\phi_\lambda = \sum_{s\in W} c(s:\mathfrak{F}^+:\mathfrak{a}^+)\, e^{(s\lambda)^y}$$
on $\mathfrak{a}^+$ *for* $\lambda \in \mathfrak{F}^+$.

This is obvious from the definition of $\nabla_{\mathfrak{g}}$ and the above formula for $g_\lambda{}^{\mathfrak{a}}$.

For any $s \in W$ define an element $s^y \in W(\mathfrak{g}/\mathfrak{a})$ as follows:

$$(sH)^y = s^y H^y \quad (H \in \mathfrak{b}_c).$$

Then $s \to s^y$ is an isomorphism of $W(\mathfrak{g}/\mathfrak{a})$ whose inverse we denote by $t \to t^{y^{-1}}$ $(t \in W(\mathfrak{g}/\mathfrak{a}))$. Define the subgroup $W_G(\mathfrak{g}/\mathfrak{a})$ of $W(\mathfrak{g}/\mathfrak{a})$ as usual (see [2 (k), § 4]). We have seen above that $W_k = W_G = W_G(\mathfrak{g}/\mathfrak{b})$.

COROLLARY. *Fix* $s \in W$, $t \in W_G(\mathfrak{g}/\mathfrak{a})$ *and* $u \in W_k$. *Then*

$$c(t^{y^{-1}} s u^{-1} : u\,\mathfrak{F}^+ : t\,\mathfrak{a}^+) = c(s:\mathfrak{F}^+:\mathfrak{a}^+).$$

Fix $\lambda \in \mathfrak{F}^+$. Then it is clear from Theorem 2 that $T_{u\lambda} = \varepsilon(u) T_\lambda$ and therefore $\phi_{u\lambda} = \phi_\lambda$. Moreover ϕ_λ is invariant under G and therefore its restriction on $\mathfrak{a}$ is invariant under $W_G(\mathfrak{g}/\mathfrak{a})$. Our assertion is an immediate consequence of these facts.

§ 17. Application of Theorem 1 to T_λ

Now we use the notation of § 2 and assume that $\mathfrak{h}_1 = \mathfrak{b}$. Let $m_i(R)$ denote the number of positive real roots of $(\mathfrak{g}, \mathfrak{h}_i)$ $(1 \leqslant i \leqslant r)$ and put $m = \frac{1}{2}(\dim \mathfrak{g} - \operatorname{rank} \mathfrak{g})$. For any $\lambda \in \mathfrak{F}'$, let $\phi_{\lambda,i}$ denote the restriction of ϕ_λ on $\mathfrak{h}_i$.

Define numbers $c_i > 0$ by the relation

$$\int_{\mathfrak{g}} f(X)\, dX = \sum_{1\leqslant i\leqslant r} c_i(-1)^{m_i(I)} \int \varepsilon_{R,i}\,\pi_i\,\psi_{f,i}\, d_i H \quad (f \in C_c^\infty(\mathfrak{g}))$$

where $m_i(I)$ is the number of positive imaginary roots of $(\mathfrak{g}, \mathfrak{h}_i)$ (see [2 (k), Cor. 1 of Lemma 30]). Also put $dH = d_1 H$.

LEMMA 34. *For any* $f \in C_c^\infty(\mathfrak{g})$ *and* $\lambda \in \mathfrak{F}'$,

$$c_1[W_k]\int_{\mathfrak{b}} \partial(\varpi)\,\psi_f e^{\lambda}\,dH = c_1\int_{\mathfrak{b}} \partial(\varpi)\,\psi_f \sum_{s\in W_k} e^{s\lambda}\,dH$$

$$= \varpi(\lambda)\,T_\lambda(f) - \sum_{2\leqslant i\leqslant r}(-1)^{m_i(R)} c_i \int_{\mathfrak{h}_i} \varepsilon_{R,i}\,\partial(\varpi_i)\,\psi_{f,i}\cdot\phi_{\lambda,i}\,d_iH.$$

Since the number of positive complex roots of $(\mathfrak{g}, \mathfrak{h}_i)$ is even (see the proof of Lemma 9 of [2 (k)]), it follows that

$$m_i(R) + m_i(I) \equiv m \bmod 2.$$

Hence
$$(-1)^m\int_{\mathfrak{g}} f(X)\,dX = \sum_{1\leqslant i\leqslant r} c_i(-1)^{m_i(R)}\int \varepsilon_{R,i}\,\pi_i\,\psi_{f,i}\,d_iH \qquad (f\in C_c^\infty(\mathfrak{g})).$$

Moreover $\partial(\varpi)\psi_f$ is invariant under $W_k = W_G$ (see [2 (k), § 6]) and

$$\phi_{\lambda,1} = \sum_{s\in W_k} e^{s\lambda}.$$

Therefore our assertion follows from Theorem 1 and the corollary of Lemma 4, if we take into account the fact that $\square F_\lambda = \varpi(\lambda)^2 F_\lambda$.

Fix a connected component $\mathfrak{F}^+$ of $\mathfrak{F}'$. Then for any $\mu \in \mathrm{Cl}(\mathfrak{F}^+)$, we define a distribution $T_{\mu,\mathfrak{F}^+} = T_\mu^+$ as follows:

$$T_\mu^+(f) = \lim_{\lambda\to\mu} T_\lambda(f) \qquad (f\in C_c^\infty(\mathfrak{g}))$$

where $\lambda\in\mathfrak{F}^+$. Put $g_{\lambda,i} = g_\lambda^{\mathfrak{h}_i}$. Then

$$T_\lambda(f) = (-1)^m \sum_{1\leqslant i\leqslant r}(-1)^{m_i(R)} c_i\int \varepsilon_{R,i}\,\psi_{f,i}\,g_{\lambda,i}\,d_iH$$

and so it is obvious that the above limit exists and

$$T_\mu^+(f) = (-1)^m \sum_{1\leqslant i\leqslant r}(-1)^{m_i(R)} c_i\int \varepsilon_{R,i}\,\psi_{f,i}\,g_{\mu,i}^{\ +}\,d_iH$$

where $g_{\mu,i}^{\ +}$ is defined as follows. Fix i and put $\mathfrak{a} = \mathfrak{h}_i$. Then

$$g_{\mu,i}^{\ +} = \lim_{\lambda\to\mu} g_\lambda^{\mathfrak{a}} = \sum_{s\in W}\varepsilon(s)\,c(s:\mathfrak{F}^+:\mathfrak{a}^+)\,e^{(s\mu)^\nu}$$

on any connected component $\mathfrak{a}^+$ of $\mathfrak{a}'(R)$. We know from Lemma 28 that $c(s:\mathfrak{F}^+:\mathfrak{a}^+) = 0$ $(s\in W)$ unless $\Re(s\lambda)^\nu(H) \leqslant 0$ for all $H\in\mathfrak{a}^+$ and $\lambda\in\mathfrak{F}^+$. Therefore it is clear from the above formulas and Lemma 19 that T_μ^+ is an invariant and tempered distribution on $\mathfrak{g}$. Since $\partial(p)T_\lambda = p_{\mathfrak{b}}(\lambda)T_\lambda$, it follows immediately by going over to the limit that

$$\partial(p)T_\mu{}^+ = p_{\mathfrak{b}}(\mu)T_\mu{}^+ \quad (p \in I(\mathfrak{g}_c)).$$

For any Cartan subalgebra $\mathfrak{a}$ of $\mathfrak{g}$ define the function $(\phi_\mu{}^+)_{\mathfrak{a}}$ on $\mathfrak{a}'(R)$ by

$$(\phi_\mu{}^+)_{\mathfrak{a}} = \sum_{s\in W} c(s:\mathfrak{F}^+:\mathfrak{a}^+)\, e^{(s\mu)^y}$$

on $\mathfrak{a}^+$.

LEMMA 35. $$|(\phi_\mu{}^+)_{\mathfrak{a}}| \leqslant \sum_{s\in W} |c(s:\mathfrak{F}^+:\mathfrak{a}^+)|$$

on $\mathfrak{a}^+$.

Fix $H\in\mathfrak{a}^+$. Then if $\lambda\in\mathfrak{F}^+$, it follows from Lemma 28 that

$$|\phi_\lambda(H)| \leqslant \sum_{s\in W} |c(s:\mathfrak{F}^+:\mathfrak{a}^+)|.$$

Our assertion now follows by letting λ tend to μ.

For $\mathfrak{a}=\mathfrak{h}_i$ we denote the function $(\phi_\mu{}^+)_{\mathfrak{a}}$ by $\phi_{\mu,i}{}^+$.

LEMMA 36. *For any* $f\in C_c^\infty(\mathfrak{g})$,

$$c_1[W_k]\int_{\mathfrak{b}} \partial(\varpi)\,\psi_f\, e^\mu\, dH = \varpi(\mu)\,T_\mu{}^+(f) - \sum_{2\leqslant i\leqslant r} (-1)^{m_i(R)}\, c_i \int \varepsilon_{R,i}\,\partial(\varpi_i)\,\psi_{f,i}\cdot\phi_{\mu,i}{}^+\, d_iH.$$

Take a variable element $\lambda\in\mathfrak{F}^+$ which converges to μ. Then our assertion follows immediately from Lemma 34 by taking limits.

§ 18. Proof of Lemma 41

As in § 14, let $\mathfrak{z}$ be a subalgebra of $\mathfrak{g}$ such that 1) $\mathfrak{z}\supset\mathfrak{b}$ and 2) $\mathfrak{z}$ is reductive in $\mathfrak{g}$. Fix a Euclidean measure dZ on $\mathfrak{z}$ and let $W_k(\mathfrak{z}/\mathfrak{b})$ be the subgroup of $W(\mathfrak{z}/\mathfrak{b})$ generated by the Weyl reflexions corresponding to the compact roots of $(\mathfrak{z}, \mathfrak{b})$. Then $W(\mathfrak{z}/\mathfrak{b})\subset W$ and $W_k(\mathfrak{z}/\mathfrak{b})\subset W_k$. Define $\varpi_{\mathfrak{z}}$ and $\pi_{\mathfrak{z}}$ as in § 13 for $\mathfrak{h}=\mathfrak{b}$.

LEMMA 37. *Let a_s $(s\in W_k)$ be continuous functions*(1) *on $\mathfrak{F}$ such that $a_{ts}=a_s$ for $t\in W_k\cap W(\mathfrak{z}/\mathfrak{b})$. Then for any $\lambda\in\mathfrak{F}'$, there exists a unique distribution $T_{\mathfrak{z},\lambda}$ on $\mathfrak{z}$ such that:*

1) $T_{\mathfrak{z},\lambda}$ *is invariant and tempered.*
2) $\partial(p_{\mathfrak{z}})\,T_{\mathfrak{z},\lambda} = p_{\mathfrak{b}}(\lambda)\,T_{\mathfrak{z},\lambda} \quad (p\in I(\mathfrak{g}_c))$.
3) $\pi_{\mathfrak{z}}\,T_{\mathfrak{z},\lambda} = \sum_{s\in W_k} \varepsilon(s)\,a_s(\lambda)\,e^{s\lambda}$ *pointwise on* $\mathfrak{b}'$.

(1) For most applications a_s will be constants.

The uniqueness is obvious from the corollary of Lemma 28. The existence is proved as follows. Applying Theorem 2 to $(\mathfrak{z}, \mathfrak{b})$ instead of $(\mathfrak{g}, \mathfrak{b})$, we conclude that there exists a unique invariant and tempered distribution τ_λ on $\mathfrak{z}$ such that $\partial(p)\tau_\lambda = p_{\mathfrak{b}}(\lambda)\tau_\lambda$ $(p \in I(\mathfrak{z}_c))$ and

$$\pi_{\mathfrak{z}}\tau_\lambda = \sum_{s \in W_k(\mathfrak{z}/\mathfrak{b})} \varepsilon(s)\, e^{s\lambda}$$

pointwise on $\mathfrak{b}'$. Put

$$T_{\mathfrak{z},\lambda} = [W_k(\mathfrak{z}/\mathfrak{b})]^{-1} \sum_{s \in W_k} \varepsilon(s)\, a_s(\lambda)\, \tau_{s\lambda} = \sum_{s \in W_k(\mathfrak{z}/\mathfrak{b}) \backslash W_k} \varepsilon(s)\, a_s(\lambda)\, \tau_{s\lambda}$$

where the second sum is over a complete system of representatives. Then it is obvious that $T_{\mathfrak{z},\lambda}$ fulfills all the conditions of the lemma.

COROLLARY.

$$T_{\mathfrak{z},\lambda} = [W_k(\mathfrak{z}/\mathfrak{b})]^{-1} \sum_{s \in W_k} \varepsilon(s)\, a_s(\lambda)\, \tau_{s\lambda} = \sum_{s \in W_k(\mathfrak{z}/\mathfrak{b}) \backslash W_k} \varepsilon(s)\, a_s(\lambda)\, \tau_{s\lambda}.$$

Fix a connected component $\mathfrak{F}^+$ of $\mathfrak{F}'$ and for any $\mu \in \mathrm{Cl}\,\mathfrak{F}^+$ define $T_{\mathfrak{z},\mu}{}^+$ and $\tau_\mu{}^+ = \tau_{\mu,\mathfrak{F}^+}$ by means of the limits

$$T_{\mathfrak{z},\mu}{}^+(f) = \lim_{\lambda \to \mu} T_{\mathfrak{z},\lambda}(f), \quad \tau_\mu{}^+(f) = \lim_{\lambda \to \mu} \tau_\lambda(f) \quad (f \in C_c^\infty(\mathfrak{z}))$$

where $\lambda \in \mathfrak{F}^+$. We have seen in § 17 that $\tau_\mu{}^+$ is a tempered distribution and therefore it follows from the above corollary that the same holds for $T_{\mathfrak{z},\mu}{}^+$. In fact the following lemma is now obvious.

LEMMA 38. $$T_{\mathfrak{z},\mu}{}^+ = \sum_{s \in W_k(\mathfrak{z}/\mathfrak{b}) \backslash W_k} \varepsilon(s)\, a_s(\mu)\, \tau_{s\mu, s\mathfrak{F}^+}.$$

Let P and $P_{\mathfrak{z}}$ respectively be the sets of all positive roots of $(\mathfrak{g}, \mathfrak{b})$ and $(\mathfrak{z}, \mathfrak{b})$ and let $P_{\mathfrak{g}/\mathfrak{z}}$ denote the complement of $P_{\mathfrak{z}}$ in P. Put

$$\pi_{\mathfrak{g}/\mathfrak{z}} = \prod_{\alpha \in P_{\mathfrak{g}/\mathfrak{z}}} \alpha, \quad \varpi_{\mathfrak{g}/\mathfrak{z}} = \prod_{\alpha \in P_{\mathfrak{g}/\mathfrak{z}}} H_\alpha.$$

It is clear that $\pi_{\mathfrak{z}}^2$, $\pi_{\mathfrak{g}/\mathfrak{z}}$ and $\varpi_{\mathfrak{g}/\mathfrak{z}}$ are all invariant under $W(\mathfrak{z}/\mathfrak{b})$. Hence by Chevalley's theorem [2 (c), Lemma 9], we can choose an invariant polynomial function $\eta_{\mathfrak{z}}$ on $\mathfrak{z}_c$ and an element $q_{\mathfrak{g}/\mathfrak{z}} = q \in I(\mathfrak{z}_c)$ such that $\eta_{\mathfrak{z}} = (-1)^r \pi_{\mathfrak{z}}^2$ on $\mathfrak{b}$ and $q_{\mathfrak{b}} = \varpi_{\mathfrak{g}/\mathfrak{z}}$. (Here r is the number of roots in $P_{\mathfrak{z}}$.) Let $\mathfrak{z}'$ be the set of all $Z \in \mathfrak{z}$ where $\eta_{\mathfrak{z}}(Z) \neq 0$ and define the invariant differential operator $\nabla_{\mathfrak{z}}$ on $\mathfrak{z}'$ as usual (see [2 (l), § 9]). Fix $\lambda \in \mathfrak{F}'$. Then we know [2 (l), Lemma 25] that there exists a continuous function $S_{\mathfrak{z},\lambda}$ on $\mathfrak{z}$ such that

$$S_{\mathfrak{z},\lambda} = \varpi(\lambda)^{-1} \nabla_{\mathfrak{z}} (\partial(q_{\mathfrak{g}/\mathfrak{z}}) T_{\mathfrak{z},\lambda})$$

pointwise on $\mathfrak{z}'$.

LEMMA 39. *Fix* $\lambda \in \mathfrak{F}'$. *Then* (1)

$$\varpi(\lambda) T_{\mathfrak{z},\lambda}(f) = \text{p.v.} \int \eta_{\mathfrak{z}}^{-1} S_{\mathfrak{z},\lambda} \nabla_{\mathfrak{z}} (\partial(q_{\mathfrak{g}/\mathfrak{z}})^* f) \, dZ \quad (f \in C_c^\infty(\mathfrak{z})),$$

in the notation of Theorem 1.

Put $\Box_{\mathfrak{z}} = \partial(q_1)$ where q_1 is the unique element in $I(\mathfrak{z}_c)$ such that $(q_1)_{\mathfrak{b}} = \varpi_{\mathfrak{z}}^2$. Then $(q_1 q^2)_{\mathfrak{b}} = \varpi^2$. Hence if Q is the unique element in $I(\mathfrak{g}_c)$ such that $Q_{\mathfrak{b}} = \varpi^2$ and $Q_{\mathfrak{z}}$ is the projection (see [2 (j), § 8]) of Q on $\mathfrak{z}$, it is obvious that $Q_{\mathfrak{z}} = q_1 q^2$. Therefore

$$(\Box_{\mathfrak{z}} \circ \partial(q^2)) T_{\mathfrak{z},\lambda} = \partial(Q_{\mathfrak{z}}) T_{\mathfrak{z},\lambda} = Q_{\mathfrak{b}}(\lambda) T_{\mathfrak{z},\lambda} = \varpi(\lambda)^2 T_{\mathfrak{z},\lambda}.$$

Hence if $T = (\Box_{\mathfrak{z}} \circ \partial(q)) T_{\mathfrak{z},\lambda}$, it follows from Theorem 1 that

$$\varpi(\lambda)^2 T_{\mathfrak{z},\lambda}(f) = T(\partial(q)^* f) = \varpi(\lambda) \left\{ \text{p.v.} \int \eta_{\mathfrak{z}}^{-1} S_{\mathfrak{z},\lambda} (\nabla_{\mathfrak{z}} \circ \partial(q)^*) f \, dZ \right\}$$

for $f \in C_c^\infty(\mathfrak{z})$. Since $\varpi(\lambda) \neq 0$, this implies the assertion of the lemma.

Let $\mathfrak{a}$ be a Cartan subalgebra of $\mathfrak{z}$ and $S_\lambda^{\mathfrak{a}}$ the restriction of $S_{\mathfrak{z},\lambda}$ on $\mathfrak{a}$. Then it follows from the definitions of $\nabla_{\mathfrak{z}}$ and q and [2 (c), Lemma 8] that (2)

$$S_\lambda^{\mathfrak{a}} = \varpi(\lambda)^{-1} \partial(\varpi^y) (\pi_{\mathfrak{z}}^{\mathfrak{a}} T_{\mathfrak{z},\lambda})$$

pointwise on $\mathfrak{a}'$.

On the other hand let $\mathfrak{F}_{\mathfrak{z}}'$ be the set of all $\lambda \in \mathfrak{F}$ where $\varpi_{\mathfrak{z}}(\lambda) \neq 0$ and $\mathfrak{F}_{\mathfrak{z}}^+$ a connected component of $\mathfrak{F}_{\mathfrak{z}}'$. Fix a connected component $\mathfrak{a}^+$ of $\mathfrak{a}'(\mathfrak{z}:R)$ (see § 13). Then corresponding to Lemma 32 and the corollary of Lemma 33 we have the following result for $\mathfrak{z}$.

LEMMA 40. *There exist integers* $c_{\mathfrak{z}}(s:\mathfrak{F}_{\mathfrak{z}}^+:\mathfrak{a}^+)$ $(s \in W(\mathfrak{z}/\mathfrak{a}))$ *such that*

$$\pi_{\mathfrak{z}}^{\mathfrak{a}} \tau_\lambda = \sum_{s \in W(\mathfrak{z}/\mathfrak{a})} \varepsilon(s) c_{\mathfrak{z}}(s:\mathfrak{F}_{\mathfrak{z}}^+:\mathfrak{a}^+) e^{s\lambda^y}$$

on $\mathfrak{a}^+ \cap \mathfrak{z}'$ *for* $\lambda \in \mathfrak{F}_{\mathfrak{z}}^+$. *Moreover*

$$c_{\mathfrak{z}}(st^y : t^{-1}\mathfrak{F}_{\mathfrak{z}}^+ : \mathfrak{a}^+) = c_{\mathfrak{z}}(s:\mathfrak{F}_{\mathfrak{z}}^+:\mathfrak{a}^+)$$

for $t \in W_k(\mathfrak{z}/\mathfrak{b})$.

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(1) As usual, the star denotes the adjoint here.

(2) Here y is an element in the complex analytic subgroup Ξ_c of G_c corresponding to $\mathrm{ad}\,\mathfrak{z}_c$ such that $(\mathfrak{b}_c)^y = \mathfrak{a}_c$. We also assume that $P_{\mathfrak{z}}^y$ is the set of positive roots of $(\mathfrak{z}, \mathfrak{a})$.

Now write $c_{\mathfrak{z}}(s:\mathfrak{F}^+:\mathfrak{a}^+)=c_{\mathfrak{z}}(s:\mathfrak{F}_{\mathfrak{z}}^+:\mathfrak{a}^+)$ for any connected component $\mathfrak{F}^+$ of $\mathfrak{F}'$ which is contained in $\mathfrak{F}_{\mathfrak{z}}^+$. Then it follows from the corollary of Lemma 37 that

$$\pi_{\mathfrak{z}}{}^{\mathfrak{a}} T_{\mathfrak{z},\lambda}=\sum_{t\in W_k(\mathfrak{z}/\mathfrak{b})\backslash W_k}\varepsilon(t)\,a_t(\lambda)\sum_{s\in W(\mathfrak{z}/\mathfrak{a})}\varepsilon(s)\,c_{\mathfrak{z}}(s:t\mathfrak{F}^+:\mathfrak{a}^+)\,e^{s(t\lambda)^y}$$

on $\mathfrak{a}^+\cap\mathfrak{z}'$ for any λ lying in a connected component $\mathfrak{F}^+$ of $\mathfrak{F}'$. Therefore it follows from the above formula for $S_\lambda{}^{\mathfrak{a}}$ that

$$S_\lambda{}^{\mathfrak{a}}=\sum_{t\in W_k(\mathfrak{z}/\mathfrak{b})\backslash W_k}a_t(\lambda)\sum_{s\in W(\mathfrak{z}/\mathfrak{a})}c_{\mathfrak{z}}(s:t\mathfrak{F}^+:\mathfrak{a}^+)\,e^{s(t\lambda)^y}$$

on $\mathfrak{a}^+\cap\mathfrak{z}'$.

Now fix $\mu\in\mathrm{Cl}(\mathfrak{F}^+)$. Then as λ tends to μ $(\lambda\in\mathfrak{F}^+)$, it is clear that the functions $S_\lambda{}^{\mathfrak{a}}$ converge uniformly on every compact subset of $\mathfrak{a}$. Hence we conclude (see Lemma 69 of § 30) that the functions $S_{\mathfrak{z},\lambda}$ converge uniformly on every compact subset of $\mathfrak{z}$. We denote the limit function by $S_{\mathfrak{z},\mu}{}^+$. It is obviously continuous and invariant.

LEMMA 41. $$S_{\mathfrak{z},\mu}{}^+=\sum_{t\in W_k(\mathfrak{z}/\mathfrak{b})\backslash W_k}a_t(\mu)\sum_{s\in W(\mathfrak{z}/\mathfrak{a})}c_{\mathfrak{z}}(s:t\mathfrak{F}^+:\mathfrak{a}^+)\,e^{s(t\mu)^y}$$

on $\mathfrak{a}^+\cap\mathfrak{z}'$. *Moreover*

$$\varpi(\mu)\,T_{\mathfrak{z},\mu}{}^+(f)=\mathrm{p.v.}\int\eta_{\mathfrak{z}}{}^{-1}S_{\mathfrak{z},\mu}{}^+\nabla_{\mathfrak{z}}(\partial(q_{\mathfrak{g}/\mathfrak{z}})^*f)\,dZ$$

for $f\in C_c{}^\infty(\mathfrak{z})$.

The first statement is obvious from the above formula for $S_\lambda{}^{\mathfrak{a}}$ and the second follows from Lemma 39 if we take into account the corollary of Lemma 4.

COROLLARY. $S_{\mathfrak{z},\mu}{}^+=0$ *if* $a_t(\mu)=0$ $(t\in W_k)$.

Now suppose $\mathfrak{z}_1,\mathfrak{z}_2$ and η_0 are as in Lemma 30 (with $\mathfrak{h}=\mathfrak{b}$). Then since

$$\pi_{\mathfrak{z}_1}T_{\mathfrak{z}_1,\lambda}=\pi_{\mathfrak{z}_2}T_{\mathfrak{z}_2,\lambda}=\sum_{s\in W_k}\varepsilon(s)\,e^{s\lambda}$$

pointwise on $\mathfrak{b}'$ for $\lambda\in\mathfrak{F}'$, it follows from Lemma 30 that

$$T_{\mathfrak{z}_2,\lambda}=\eta_0\,T_{\mathfrak{z}_1,\lambda}$$

pointwise on $\mathfrak{g}'\cap\mathfrak{z}_2$. Fix a Cartan subalgebra $\mathfrak{a}$ of $\mathfrak{z}_2$ and an element y in the complex analytic subgroup Ξ_{2c} of G_c corresponding to $\mathrm{ad}\,\mathfrak{z}_{2c}$, such that $\mathfrak{b}_c{}^y=\mathfrak{a}_c$. We may assume that P^y is the set of all positive roots of $(\mathfrak{g},\mathfrak{a})$. Then

$$S_{\mathfrak{z}_i,\lambda} = \varpi(\lambda)^{-1}\nabla_{\mathfrak{z}_i}(\partial(q_{\mathfrak{g}/\mathfrak{z}_i})T_{\mathfrak{z}_i,\lambda}) = \varpi(\lambda)^{-1}\partial(\varpi^{y})F_\lambda$$

pointwise on $\mathfrak{a}'$ where

$$F_\lambda(H) = \pi_{\mathfrak{z}_1}{}^{\mathfrak{a}}(H)T_{\mathfrak{z}_1,\lambda}(H) = \pi_{\mathfrak{z}_2}{}^{\mathfrak{a}}(H)T_{\mathfrak{z}_2,\lambda}(H) \quad (H \in \mathfrak{a}').$$

This shows that $S_{\mathfrak{z}_1,\lambda} = S_{\mathfrak{z}_2,\lambda}$ on $\mathfrak{z}_2$ and therefore we get the following result by taking limits.

LEMMA 42. *Fix* $\mu \in \mathrm{Cl}(\mathfrak{F}^+)$. *Then*

$$T_{\mathfrak{z}_2,\mu}{}^+ = \eta_0 T_{\mathfrak{z}_1,\mu}{}^+$$

pointwise on $\mathfrak{g}' \cap \mathfrak{z}_2$ *and*

$$S_{\mathfrak{z}_1\mu}{}^+ = S_{\mathfrak{z}_2\mu}{}^+$$

on $\mathfrak{z}_2$.

We now return to the notation of Lemma 41 and write $T_{\mathfrak{z},\mu,\mathfrak{F}^+} = T_{\mathfrak{z},\mu}{}^+$ whenever it is convenient to do so. Let Ξ be the analytic subgroup of G corresponding to $\mathfrak{z}$ and $\mathfrak{h}_1, \mathfrak{h}_2, \ldots, \mathfrak{h}_r$ a maximal set of Cartan subalgebras of $\mathfrak{z}$ no two of which are conjugate under Ξ. Fix a Euclidean measure $d_i H$ on $\mathfrak{h}_i$ and define $\psi_{\mathfrak{z},f,i}$ $(f \in C_c^\infty(\mathfrak{z}))$ as in Lemma 5 for $(\mathfrak{z}, \mathfrak{h}_i)$ instead of $(\mathfrak{g}, \mathfrak{h}_i)$.

LEMMA 43. *Assume that the functions* a_t $(t \in W_k)$ *remain bounded on* $\mathfrak{F}$. *Then there exists a number* $C \geq 0$ *with the following property. Let* $\mathfrak{F}^+$ *be a connected component of* $\mathfrak{F}'$ *and* μ *a point in* $\mathrm{Cl}(\mathfrak{F}^+)$. *Then*

$$|T_{\mathfrak{z},\mu,\mathfrak{F}^+}(f)| \leq C \sum_{1\leq i\leq r} \int_{\mathfrak{h}_i} |\psi_{\mathfrak{z},f,i}|\, d_i H.$$

Let $\mathfrak{a}$ be a Cartan subalgebra of $\mathfrak{z}$. It follows from Lemmas 28 and 40 that

$$|\pi_{\mathfrak{z}}{}^{\mathfrak{a}}(H)\tau_\lambda(H)| \leq \sum_{s\in W(\mathfrak{z}/\mathfrak{a})} |c_{\mathfrak{z}}(s:\mathfrak{F}^+:\mathfrak{a}^+)|$$

for $H \in \mathfrak{a}^+ \cap \mathfrak{z}'$ and $\lambda \in \mathfrak{F}^+$. Put

$$g_\lambda{}^{\mathfrak{a}}(H) = \pi_{\mathfrak{z}}{}^{\mathfrak{a}}(H)T_{\mathfrak{z},\lambda}(H) \quad (H \in \mathfrak{a}').$$

Then, in view of the corollary of Lemma 37, we can choose a number $a \geq 0$ such that

$$|g_\lambda{}^{\mathfrak{a}}(H)| \leq a$$

for $H \in \mathfrak{a}'$ and $\lambda \in \mathfrak{F}'$. Now put $g_{\lambda,i} = g_\lambda{}^{\mathfrak{h}_i}$ $(1 \leq i \leq r)$. Then, as we have seen in § 2, there exist real numbers $c_1, \ldots, c_r$ such that

$$T_{\mathfrak{z},\lambda}(f)=\sum_{1\leq i\leq r} c_i \int \psi_{\mathfrak{z},f,i}\, g_{\lambda,i}\, \varepsilon_{\mathfrak{z},R,i}\, d_i H$$

for all $f\in C_c^\infty(\mathfrak{z})$ and $\lambda\in\mathfrak{F}'$. (Here $\varepsilon_{\mathfrak{z},R,i}$ is a locally constant function on $\mathfrak{h}_i'$ whose values are ± 1.) Therefore

$$|T_{\mathfrak{z},\lambda}(f)|\leq C\sum_{1\leq i\leq r}\int |\psi_{\mathfrak{z},f,i}|\, d_i H$$

where $C=a\max_i |c_i|$. The statement of the lemma now follows by letting λ tend to μ $(\lambda\in\mathfrak{F}^+)$.

Part II. Theory on the group

§ 19. Statement of Theorem 3

We keep to the notation of § 16 and assume, moreover, that G is acceptable. Let B be the Cartan subgroup of G corresponding to $\mathfrak{b}$. Then B is connected and therefore abelian (see [2 (m), Cor. 5 of Lemma 26]). Let B^* denote the character group of B. For any $b^*\in B^*$, we denote by $\langle b^*, b\rangle$ the value of the character b^* at a point $b\in B$. It is obvious that there exists a unique element $\lambda\in\mathfrak{F}$ such that

$$\langle b^*, \exp H\rangle = e^{\lambda(H)} \quad (H\in\mathfrak{b}).$$

We shall denote λ by $\log b^*$. b^* is called singular or regular according as $\varpi(\lambda)=0$ or not. We have seen that $W_k=W_G$. Now W_G operates on B as usual (see [2 (m), § 20]) and therefore, by duality, also on B^*. Then

$$\langle (b^*)^s, b\rangle = \langle b^*, b^{s^{-1}}\rangle \quad (b^*\in B^*,\ b\in B)$$

and $\log (b^*)^s = s\,(\log b^*)$ $(s\in W_G)$.

Define $\mathfrak{Z}$ as in [2 (m), § 6] and let $z\to p_z$ $(z\in\mathfrak{Z})$ denote the canonical isomorphism of $\mathfrak{Z}$ onto $I(\mathfrak{g}_c)$ (see [2 (m), § 12]). For $b^*\in B^*$, define

$$\chi_{b^*}(z)=\chi_\lambda^{\mathfrak{b}}(p_z) \quad (z\in\mathfrak{Z})$$

(in the notation of § 12) for $\lambda=\log b^*$. Then χ_{b^*} is a homomorphism of $\mathfrak{Z}$ into $\mathbf{C}$.

Let t be an indeterminate and l the rank of G. For any $x\in G$, we denote by $D(x)$ the coefficient of t^l in $\det(t+1-\mathrm{Ad}(x))$. Then D is an analytic function on G. As usual let G' denote the set of all regular elements in G (see [2 (m), § 3]). Fix a Haar measure dx on G and let Θ be a distribution on G. We say that Θ is an invariant eigendistribution of $\mathfrak{Z}$ if 1) $\Theta^x=\Theta$ $(x\in G)$ and 2) there exists a homomorphism

χ of $\mathfrak{Z}$ into $\mathbf{C}$ such that $z\Theta=\chi(z)\Theta$ for all $z\in\mathfrak{Z}$. In view of [2 (m), Theorem 2], we can speak of the value $\Theta(x)$ of such a distribution at a point $x\in G'$.

Let $B^{*\prime}$ denote the set of all regular elements in B^* and put $\Delta=\Delta_B$ in the notation of [2 (m), § 19].

THEOREM 3. *Fix an element $b^*\in B^{*\prime}$. Then there exists exactly one invariant eigendistribution Θ on G such that:*

1) $$z\Theta=\chi_{b^*}(z)\Theta \quad (z\in\mathfrak{Z});$$

2) $$\sup_{x\in G'}|D(x)|^{\frac{1}{2}}|\Theta(x)|<\infty;$$

3) $$\Theta=\Delta^{-1}\sum_{s\in W_G}\varepsilon(s)(b^*)^s \textit{ pointwise on } B'=B\cap G'.$$

§ 20. Proof of the uniqueness

In order to obtain the uniqueness in Theorem 3, it is sufficient to prove the following result.

LEMMA 44. *Fix $b^*\in B^{*\prime}$ and let Θ be an invariant eigendistribution of $\mathfrak{Z}$ on G such that:*

1) $$z\Theta=\chi_{b^*}(z)\Theta \quad (z\in\mathfrak{Z});$$

2) $$\sup_{x\in G'}|D(x)|^{\frac{1}{2}}|\Theta(x)|<\infty;$$

3) $$\Theta=0 \textit{ pointwise on } B'.$$

Then $\Theta=0$.

Fix a semisimple element $a\in G$. In view of [2 (m), Lemma 7], it would be sufficient to prove that $a\notin\operatorname{Supp}\Theta$. We now use the notation of [2 (m), § 4] and put $\sigma=|\nu_a|^{\frac{1}{2}}\sigma_\Theta$ in the notation of [2 (m), Lemma 15]. Since $z\Theta=\chi_{b^*}(z)\Theta$, we conclude from [2 (m), Lemma 22] that

$$\mu_{\mathfrak{g}/\mathfrak{z}}(z)\sigma=\chi_{b^*}(z)\sigma \quad (z\in\mathfrak{Z}).$$

Define $\mathfrak{g}_0=\mathfrak{c}_0+\mathfrak{g}_1(c)$ as in § 14 where $\mathfrak{c}_0$ is an open and convex neighborhood of zero in $\mathfrak{c}$. Then $\mathfrak{g}_0$ is an open and completely invariant neighborhood of zero in $\mathfrak{g}$ and if $\mathfrak{c}_0$ and c are sufficiently small, the exponential mapping of $\mathfrak{g}$ into G is univalent and regular on $\mathfrak{g}_0$ (see [2 (m), § 9]). Put $\mathfrak{z}_0=\mathfrak{g}_0\cap\mathfrak{z}$.

Now first assume that $a\in B$ and let Z_G denote the center of G. Then since

B/Z_G is compact [2 (m), § 16], every eigenvalue of $\mathrm{Ad}(a)$ has absolute value 1. Hence if c is sufficiently small, it is obvious that no eigenvalue of $(\mathrm{Ad}(a \exp Z))_{\mathfrak{g}/\mathfrak{z}}$ can be 1 for $Z \in \mathfrak{z}_0$. This shows that $\exp \mathfrak{z}_0 \subset \Xi'$. Let τ denote the distribution on $\mathfrak{z}_0$ obtained from σ by applying the procedure of [2 (m), § 10] to $\mathfrak{z}$ (in place of $\mathfrak{g}$). Since

$$\mu_{\mathfrak{g}/\mathfrak{z}}(z)\,\sigma = \chi_{b*}(z)\,\sigma \quad (z \in \mathfrak{Z}),$$

it follows from the corollary of [2 (m), Lemma 24] and the definition of $\mu_{\mathfrak{g}/\mathfrak{z}}$ [2 (m), § 12] that

$$\partial(p_{\mathfrak{z}})\,\tau = \chi(p)\,\tau \quad (p \in I(\mathfrak{g}_c)),$$

where $\chi = \chi_\lambda^{\mathfrak{b}}$ and $\lambda = \log b^*$. Now $\mathfrak{b} \subset \mathfrak{z}$ since $a \in B$. Therefore $T_{\mathfrak{z}} = \tau$ satisfies all the conditions of § 13. Let $\mathfrak{z}_0'$ be the set of those elements of $\mathfrak{z}_0$ which are regular in $\mathfrak{z}$. Then we know from [2 (m), Lemma 32] that

$$\tau(Z) = \xi_{\mathfrak{z}}(Z)\,|\nu_a\,(\exp Z)|^{\frac{1}{2}}\,\Theta\,(a \exp Z) \quad (Z \in \mathfrak{z}_0').$$

Let $\mathfrak{a}$ be a Cartan subalgebra of $\mathfrak{z}$ and A the corresponding Cartan subgroup of G. It is easy to verify that

$$|D(a \exp H)| = |\pi_{\mathfrak{z}}^{\mathfrak{a}}(H)\,\xi_{\mathfrak{z}}(H)|^2\,|\nu_a\,(\exp H)|$$

for $H \in \mathfrak{a}$ and therefore

$$|\pi_{\mathfrak{z}}^{\mathfrak{a}}(H)\,\tau\,(H)| = |D(a \exp H)|^{\frac{1}{2}}\,|\Theta\,(a \exp H)|$$

for $H \in \mathfrak{a}' \cap \mathfrak{z}_0$. Hence we conclude from Lemma 19 and condition 2) that τ is a tempered distribution on $\mathfrak{z}_0$. Moreover if we take $\mathfrak{a} = \mathfrak{b}$, it follows from condition 3) that $\tau = 0$ pointwise on $\mathfrak{z}_0 \cap \mathfrak{b}'$. Therefore (see the corollary of Lemma 29), $\tau = 0$ on $\mathfrak{z}_0$. This, in turn, implies that $\Theta = 0$ pointwise on $a \exp \mathfrak{z}_0' = G' \cap (a \exp \mathfrak{z}_0)$. But $V = (a \exp \mathfrak{z}_0)^G$ is open in G [2 (m), Lemma 14]. Hence $\Theta = 0$ on V.

Now we drop the assumption that $a \in B$. Define $\theta, \mathfrak{k}, \mathfrak{p}$ and K as in [2 (m), § 16] corresponding to $\mathfrak{h} = \mathfrak{b}$. Then $B \subset K$ [2 (m), Cor. 5 of Lemma 26]. Let $\mathfrak{a}$ be any Cartan subalgebra of $\mathfrak{z}$. We can choose $x \in G$ such that $\theta(\mathfrak{a}^x) = \mathfrak{a}^x$ and $\mathfrak{a}^x \cap \mathfrak{k} \subset \mathfrak{b}$ (see Lemma 45 below). Let A be the Cartan subgroup of G corresponding to $\mathfrak{h} = \mathfrak{a}^x$. Then $a^x \in A$. Let $a^x = a_0 \exp H$ where $a_0 \in A \cap K$ and $H \in \mathfrak{h} \cap \mathfrak{p}$ (see [2 (m), Cor. 4 of Lemma 26]). Since K is connected and K/Z_G is compact, we can choose $k \in K$ such that $b = a_0^k \in B$. Then

$$a^{kx} = b \exp Z_0$$

where $Z_0 = H^k \in \mathfrak{p} \subset [\mathfrak{g}, \mathfrak{g}]$. Let $\mathfrak{z}_b$ denote the centralizer of b in $\mathfrak{g}$. It is obvious that

$Z_0 \in \mathfrak{z}_\mathfrak{b}$. Moreover since $Z_0 \in \mathfrak{p}$, all the eigenvalues of ad Z_0 are real [2 (i), Lemma 27]. Hence by applying the result obtained above to b, we conclude that

$$a^{kx} = b \exp Z_0 \notin \operatorname{Supp} \Theta.$$

Therefore since Θ is invariant, it follows that $a \notin \operatorname{Supp} \Theta$. This proves the lemma.

§ 21. Some elementary facts about Cartan subgroups

Let $\mathfrak{a}$ be a Cartan subalgebra of $\mathfrak{g}$ and A the corresponding Cartan subgroup of G. Define $\mathfrak{a}_R$ and $\mathfrak{a}_I$ as in § 11.

LEMMA 45. *Let A_I be the subgroup of all $a \in A$ such that all eigenvalues of* $\mathrm{Ad}(a)$ *have absolute values* 1. *Then $(a, H) \rightarrow a \exp H$ $(a \in A_I, H \in \mathfrak{a}_R)$ is a topological mapping of $A_I \times \mathfrak{a}_R$ onto A. Moreover for any $a \in A_I$, we can choose $x \in G$ such that* 1) $a^x \in B$, 2) $\theta(\mathfrak{a}^x) = \mathfrak{a}^x$ *and* 3) $(\mathfrak{a}_I)^x \subset \mathfrak{b}$. *Finally, x may be selected to lie in K if $\theta(\mathfrak{a}) = \mathfrak{a}$.*

It follows from [2 (b), p. 100] that we can choose $y \in G$ such that $\theta(\mathfrak{a}^y) = \mathfrak{a}^y$. Then $(\mathfrak{a}_I)^y$ is an abelian subspace of $\mathfrak{k}$. Since $\mathfrak{b}$ is maximal abelian in $\mathfrak{k}$ and K/Z_G is compact, we can choose $k \in K$ such that $(\mathfrak{a}_I)^{ky} \subset \mathfrak{b}$. Replacing $\mathfrak{a}$ by $\mathfrak{a}^{ky}$, we can now obviously assume that $\theta(\mathfrak{a}) = \mathfrak{a}$ and $\mathfrak{a}_I \subset \mathfrak{b}$. Then the first statement follows from the results of [2 (m), § 16]. Moreover it is clear that $A_I = A \cap K \subset K = B^K$. Fix $a \in A_I$ and choose $k \in K$ such that $b = a^k \in B$. Let $\mathfrak{z}$ be the centralizer of b in $\mathfrak{g}$ and Ξ the analytic subgroup of G corresponding to $\mathfrak{z}$. Then $\mathfrak{a}^k$ and $\mathfrak{b}$ are two Cartan subalgebras of $\mathfrak{z}$ and $\mathfrak{a}_I^k + \mathfrak{b} \subset \mathfrak{z} \cap \mathfrak{k}$. Since $\mathfrak{b}$ is maximal abelian in $\mathfrak{z} \cap \mathfrak{k}$, we can choose $\xi \in \Xi \cap K$ such that $(\mathfrak{a}_I)^{\xi k} \subset \mathfrak{b}$. Put $x = \xi k$. Then $a^x = b^\xi = b$ and $(\mathfrak{a}_I)^x \subset \mathfrak{b}$. Moreover since $x \in K$, it is clear that $\theta(\mathfrak{a}^x) = \mathfrak{a}^x$. The last statement follows from the fact that we can take $y = 1$ if $\theta(\mathfrak{a}) = \mathfrak{a}$.

COROLLARY. *An element a of G lies in B^G if and only if* 1) *a is semisimple and* 2) *all eigenvalues of* $\mathrm{Ad}(a)$ *have absolute value* 1.

Since $B \subset K$, it is obvious that any $a \in B^G$ fullfills these two conditions. Conversely suppose these conditions hold. Then by 1), a is contained in some Cartan subgroup A of G [2 (m), Cor. of Lemma 5]. Therefore by 2) $a \in A_I$. But then $a \in B^G$ by Lemma 45.

We write $A_R = \exp \mathfrak{a}_R$. By Lemma 45, every $h \in A$ can be written uniquely in the form $h = h_1 h_2$ $(h_1 \in A_I, h_2 \in A_R)$. We call h_1 and h_2 the components of h in A_I and A_R respectively.

§ 22. Proof of the existence

We now come to the proof of the existence of Θ in Theorem 3. In view of later applications, we shall consider a somewhat more general situation.

Fix a connected component $\mathfrak{F}^+$ of $\mathfrak{F}'$ and a point $b^* \in B^*$ such that

$$\lambda = \log b^* \in \mathrm{Cl}(\mathfrak{F}^+).$$

Select an open convex neighborhood $\mathfrak{c}_0$ of zero in $\mathfrak{c}$ and define

$$\mathfrak{g}_0 = \mathfrak{c}_0 + \mathfrak{g}_1(c) \quad (0 < c \leqslant \pi = 3.14 \ldots)$$

as in § 14. We assume that $\mathfrak{c}_0$ is so small that the exponential mapping of $\mathfrak{g}$ into G is univalent and regular on $\mathfrak{g}_0$ (see [2 (m), § 9]).

Fix $b \in B$ and let $\mathfrak{z} = \mathfrak{z}_b$ denote the centralizer of b in $\mathfrak{g}$. Define $T_b{}^+ = T_{\mathfrak{z},\lambda}{}^+$ and $S_b{}^+ = S_{\mathfrak{z},\lambda}{}^+$ in the notation of § 18 corresponding to the constants $a_s = \langle (b^*)^s, b\rangle$ $(s \in W_G)$. (Here we have to observe that $b^t = b$ for $t \in W_k \cap W(\mathfrak{z}/\mathfrak{b})$ and therefore $a_{ts} = a_s$.)

Let $\Xi = \Xi(b)$ denote the analytic subgroup of G corresponding to $\mathfrak{z}$. Put $\mathfrak{z}_0 = \mathfrak{g}_0 \cap \mathfrak{z}$ and $\Xi_0(b) = \Xi_0 = \exp \mathfrak{z}_0$. Then Ξ_0 is an open and completely invariant subset of Ξ [2 (m), Lemma 8]. As usual define the function $\xi_{\mathfrak{z}}$ on $\mathfrak{z}$ by

$$\xi_{\mathfrak{z}}(Z) = |\det\{(e^{\mathrm{ad}\, Z/2} - e^{-\mathrm{ad}\, Z/2})/\mathrm{ad}\, Z\}|^{\frac{1}{2}} \quad (Z \in \mathfrak{z}).$$

Then $\xi_{\mathfrak{z}}$ is analytic and nowhere zero on $\mathfrak{z}_0$. Put

$$\Phi_b{}^+ (\exp Z) = \xi_{\mathfrak{z}}(Z)^{-1} T_b{}^+(Z) \quad (Z \in \mathfrak{g}_0 \cap \mathfrak{z}')$$

where $\mathfrak{z}'$ is the set of those elements of $\mathfrak{z}$ which are regular in $\mathfrak{z}$. Then $\Phi_b{}^+$ is a locally summable function on $\Xi_0(b)$.

Define the homomorphism $\mu_b = \mu_{\mathfrak{g}/\mathfrak{z}}$ as in [2 (m), § 12].

LEMMA 46. $$\mu_b(z)\,\Phi_b{}^+ = \chi_{b^*}(z)\,\Phi_b{}^+ \quad (z \in \mathfrak{Z})$$
as a distribution on $\Xi_0(b)$.

This follows immediately from the corollary of [2 (m), Lemma 24] (applied to $\mathfrak{z}$) and the fact (see § 18) that $\partial(p_{\mathfrak{z}})\, T_b{}^+ = p_{\mathfrak{b}}(\lambda)\, T_b{}^+$ for $p \in I(\mathfrak{g}_c)$.

We have seen in [2 (m), § 22] that there exists an invariant analytic function D_b on $\mathfrak{z}$ such that

$$\Delta(b \exp H) = \pi_{\mathfrak{z}}(H)\, D_b(H) \quad (H \in \mathfrak{b}).$$

Put $\Xi_0{}''(b) = \Xi_0(b) \cap (b^{-1}G')$ and let $\mathfrak{z}''$ be the set of all points $Z \in \mathfrak{z}'$ where $D_b(Z) \neq 0$. Then it is clear that $\Xi_0{}''(b) = \exp(\mathfrak{g}_0 \cap \mathfrak{z}'')$. Put

$$\Theta_b^+ (\exp Z) = D_b(Z)^{-1} T_b^+(Z) \quad (Z \in \mathfrak{g}_0 \cap \mathfrak{z}'').$$

Then Θ_b^+ is an analytic function on $\Xi_0''(b)$. Similarly define

$$\Psi_b^+ (\exp Z) = S_b^+(Z) \quad (Z \in \mathfrak{z}_0).$$

Then Ψ_b^+ is a continuous function on $\Xi_0(b)$.

Define $$\nu_b(y) = \det(\mathrm{Ad}(by)^{-1} - 1)_{\mathfrak{g}/\mathfrak{z}} \quad (y \in \Xi)$$

as in [2 (m), § 14].

LEMMA 47. *Let* $'\mathfrak{z}$ *be the set of all points* $Z \in \mathfrak{z}$ *where* $\nu_b(\exp Z) \neq 0$. *Then there exists a locally constant function* ε_b *on* $'\mathfrak{z}$ *such that* $\varepsilon_b^4 = 1$ *and*

$$\xi_{\mathfrak{z}}(Z)\,|\nu_b(\exp Z)|^{\frac{1}{2}} = \varepsilon_b(Z)\, D_b(Z) \quad (Z \in {}'\mathfrak{z}).$$

It would be enough to verify that

$$\xi_{\mathfrak{z}}(Z)^4 \nu_b(\exp Z)^2 = D_b(Z)^4$$

for $Z \in \mathfrak{z}$. Since both sides are analytic functions on $\mathfrak{z}$ which are invariant under Ξ, it would be enough to do this when Z varies in some non-empty open subset of $\mathfrak{b}$. Hence our assertion follows from [2 (m), Lemma 33].

COROLLARY. $$|\nu_a(\exp Z)|^{\frac{1}{2}}\, \Theta_b^+ (\exp Z) = \varepsilon_b(Z)\, \Phi_b^+ (\exp Z)$$
for $Z \in \mathfrak{g}_0 \cap \mathfrak{z}''$.

This is obvious.

Put $\mathfrak{z}_0'' = \mathfrak{g}_0 \cap \mathfrak{z}''$ and let u be an element in G such that $\mathfrak{b}^u = \mathfrak{b}$.

LEMMA 48. *We have the relations*

$$\Theta_{b^u}^+ (\exp Z^u) = \Theta_b^+ (\exp Z), \quad \Psi_{b^u}^+ (\exp Z^u) = \Psi_b^+ (\exp Z)$$
for $Z \in \mathfrak{z}_0''$.

Since $\mathfrak{z}^u$ is the centralizer of b^u in $\mathfrak{g}$, it is clear that $\Theta_{b^u}^+ (\exp Z^u)$ and $\Psi_{b^u}^+ (\exp Z^u)$ are defined for $Z \in \mathfrak{z}_0''$. Let t be an element in W_G such that $H^u = tH$ for $H \in \mathfrak{b}$. It is obvious that $\pi_{\mathfrak{z}^u} = \gamma \pi_{\mathfrak{z}}^t$ where $\gamma = \pm 1$. Therefore since

$$\Delta(b^u \exp H^u) = \varepsilon(t)\, \Delta(b \exp H) \quad (H \in \mathfrak{b}),$$

it follows that $D_{b^u}(H^u) = \varepsilon(t)\, \gamma\, D_b(H)$. But the function

$$Z \to D_{\mathfrak{b}^u}(Z^u) - \varepsilon(t)\gamma D_{\mathfrak{b}}(Z) \quad (Z \in \mathfrak{z})$$

is obviously analytic and invariant under Ξ. Hence we can conclude that

$$D_{\mathfrak{b}^u}(Z^u) = \varepsilon(t)\gamma D_{\mathfrak{b}}(Z) \quad (Z \in \mathfrak{z}).$$

Now for any $\mu \in \mathfrak{F}^+$, let $T_{\mathfrak{z},\mu}$ be the distribution of Lemma 37 corresponding to the constants $a_s = \langle (\mathfrak{b}^*)^s, \mathfrak{b}\rangle$ $(s \in W_G)$. Similarly define $T_{\mathfrak{z}^u,\mu}$ on $\mathfrak{z}^u$ corresponding to the constants $a_s = \langle (\mathfrak{b}^*)^s, \mathfrak{b}^u\rangle$. Then

$$\pi_{\mathfrak{z}^u}(uH)\,T_{\mathfrak{z}^u,\mu}(uH) = \sum_{s \in W_G} \varepsilon(s)\langle (\mathfrak{b}^*)^s, \mathfrak{b}^u\rangle e^{s\mu(uH)}$$

$$= \varepsilon(t)\sum_{s \in W_G} \varepsilon(s)\langle (\mathfrak{b}^*)^s, \mathfrak{b}\rangle e^{s\mu(H)} = \varepsilon(t)\,\pi_{\mathfrak{z}}(H)\,T_{\mathfrak{z},\mu}(H) \quad (H \in \mathfrak{h}').$$

Hence

$$T_{\mathfrak{z}^u,\mu}(uH) = \varepsilon(t)\gamma T_{\mathfrak{z},\mu}(H) \quad (H \in \mathfrak{h}').$$

Now consider the distribution

$$T_\mu' : f \to \int f(Z)\,T_{\mathfrak{z}^u,\mu}(uZ)\,dZ \quad (f \in C_c^\infty(\mathfrak{z}))$$

on $\mathfrak{z}$. It is obviously invariant and tempered. Moreover it is clear that $p_{\mathfrak{z}^u} = (p_{\mathfrak{z}})^u$ for $p \in I(\mathfrak{g}_c)$. Let dZ' denote the Euclidean measure on $\mathfrak{z}^u$ which corresponds to dZ under the mapping $Z' = Z^u$ $(Z \in \mathfrak{z})$. Then

$$T_\mu'(\partial(p_{\mathfrak{z}})^* f) = \int f(u^{-1}Z'; \partial(p_{\mathfrak{z}})^*)\,T_{\mathfrak{z}^u,\mu}(Z')\,dZ'$$

$$= \int f'(Z'; \partial(p_{\mathfrak{z}^u})^*)\,T_{\mathfrak{z}^u,\mu}(Z')\,dZ'$$

$$= p_{\mathfrak{b}}(\mu)\int f'(Z')\,T_{\mathfrak{z}^u,\mu}(Z')\,dZ' = p_{\mathfrak{b}}(\mu)\,T_\mu'(f)$$

for $p \in I(\mathfrak{g}_c)$ and $f \in C_c^\infty(\mathfrak{z})$. Here f' denotes the function $Z' \to f(u^{-1}Z')$ $(Z' \in \mathfrak{z}^u)$ in $C_c^\infty(\mathfrak{z}^u)$. Hence it follows from the uniqueness assertion of Lemma 37 that

$$T_\mu' = \varepsilon(t)\gamma T_{\mathfrak{z},\mu}.$$

Therefore

$$T_{\mathfrak{z}^u,\mu}(f') = \varepsilon(t)\gamma T_{\mathfrak{z},\mu}(f)$$

and by making μ tend to λ, we conclude that

$$T_{\mathfrak{b}^u}^+(f') = \varepsilon(t)\gamma T_{\mathfrak{b}}^+(f) \quad (f \in C_c^\infty(\mathfrak{z})).$$

This proves that

$$T_{b^u}{}^+(Z^u) = \varepsilon(t)\,\gamma\,T_b{}^+(Z) \quad (Z \in \mathfrak{z}').$$

The first assertion of the lemma is now obvious.

Define $\nabla_{\mathfrak{z}}$, $\nabla_{\mathfrak{z}^u}$ and $\varpi_{\mathfrak{g}/\mathfrak{z}}$, $\varpi_{\mathfrak{g}/\mathfrak{z}^u}$ as in § 18. It is clear that

$$\nabla_{\mathfrak{z}^u} f' = (\nabla_{\mathfrak{z}} f)'$$

for $f \in C_c^\infty(\mathfrak{z}')$. On the other hand $\varpi_{\mathfrak{g}/\mathfrak{z}^u} = \varepsilon(t)\,\gamma\,(\varpi_{\mathfrak{g}/\mathfrak{z}})^t$. Therefore it is clear that

$$q_{\mathfrak{g}/\mathfrak{z}^u} = \varepsilon(t)\,\gamma\,(q_{\mathfrak{g}/\mathfrak{z}})^u$$

in the notation of § 18. Hence

$$\varpi(\mu)\,S_{\mathfrak{z}^u,\mu}(Z^u) = T_{\mathfrak{z}^u,\mu}(Z^u; \nabla_{\mathfrak{z}^u} \circ \partial(q_{\mathfrak{g}/\mathfrak{z}^u})) = T_{\mathfrak{z},\mu}(Z; \nabla_{\mathfrak{z}} \circ \partial(q_{\mathfrak{g}/\mathfrak{z}})) = \varpi(\mu)\,S_{\mathfrak{z},\mu}(Z)$$

for $Z \in \mathfrak{z}'$ and $\mu \in \mathfrak{F}^+$. This shows that

$$S_{\mathfrak{z}^u,\mu}(Z^u) = S_{\mathfrak{z},\mu}(Z)$$

and so by making μ tend to λ, we deduce that

$$S_{b^u}{}^+(Z) = S_b{}^+(Z) \quad (Z \in \mathfrak{z}).$$

Obviously this implies the second assertion of the lemma.

COROLLARY. *Let x be an element in G such that $b^x \in B$. Then*

$$\Theta_{b^x}{}^+(\exp Z^x) = \Theta_b{}^+(\exp Z),$$

$$\Psi_{b^x}{}^+(\exp Z^x) = \Psi_b{}^+(\exp Z) \quad (Z \in \mathfrak{z}_0'').$$

Since $b^x \in B$, it is clear that $B^{x^{-1}} \subset \Xi$. Hence $\mathfrak{b}$ and $\mathfrak{b}^{x^{-1}}$ are two fundamental Cartan subalgebras of $\mathfrak{z}$ and therefore we can choose $y \in \Xi$ such that $\mathfrak{b}^{yx^{-1}} = \mathfrak{b}$ (see [2 (d), p. 237]). Put $u = xy^{-1}$. Then $x = uy$ and $b^x = b^u$. Therefore

$$\Theta_{b^x}{}^+(\exp Z^x) = \Theta_{b^u}{}^+(\exp Z^{uy}) = \Theta_b{}^+(\exp Z^y)$$

by Lemma 48. Similarly

$$\Psi_{b^x}{}^+(\exp Z^x) = \Psi_b{}^+(\exp Z^y) \quad (Z \in \mathfrak{z}_0'').$$

Since $\Theta_b{}^+$ and $\Psi_b{}^+$ are obviously invariant under Ξ, we get the required assertion.

Since $b^x = b^u$, we have obtained the following result during the above proof.

LEMMA 49. *If two elements of B are conjugate under G, then they are also conjugate under the normalizer of B in G.*

Now fix $a \in B^G$, define $\mathfrak{z}_a$ and $\Xi(a)$ as usual (see [2 (m), § 4]) and put $\Xi_0(a) = \exp(\mathfrak{g}_0 \cap \mathfrak{z}_a)$, $\Xi_0''(a) = \Xi_0(a) \cap (a^{-1} G')$. Choose $x \in G$ such that $a^x \in B$ and define

$$\Theta_a^+(y) = \Theta_{a^x}^+(y^x) \quad (y \in \Xi_0''(a))$$

and

$$\Psi_a^+(y) = \Psi_{a^x}^+(y^x) \quad (y \in \Xi_0(a)).$$

It follows from the corollary of Lemma 48 that these definitions are independent of the choice of x.

We now define two functions Θ^+ and Ψ^+ on G' as follows. Fix $h \in G'$ and let $\mathfrak{a}$ be the centralizer of h in $\mathfrak{g}$ and A the corresponding Cartan subgroup of G. Define A_I and A_R as in § 21 and let $h = h_1 h_2$ $(h_1 \in A_I, h_2 \in A_R)$. Since every eigenvalue of ad H is real for $H \in \mathfrak{a}_R$ and since h is regular, it is clear that $h_2 \in \Xi_0''(h_1)$. We define

$$\Theta^+(h) = \Theta_{h_1}^+(h_2), \quad \Psi^+(h) = \Psi_{h_1}^+(h_2).$$

(Observe that $A_I \subset B^G$ from the corollary of Lemma 45.) If $x \in G$, it is obvious that

$$\Theta^+(h^x) = \Theta_{h_1^x}^+(h_2^x) = \Theta_{h_1}^+(h_2) = \Theta^+(h).$$

Similarly $\Psi^+(h^x) = \Psi^+(h)$. This shows that Θ^+ and Ψ^+ are invariant under G. We intend to prove that they are analytic on G'.

LEMMA 50. *Fix $b \in B$. Then there exists a number $c_b > 0$ with the following property. Let $\mathfrak{z}_b(c_b)$ be the set of all $Z \in \mathfrak{z}_b$ such that*(1) $|\operatorname{Im} \mu| < c_b$ *for every eigenvalue μ of $(\operatorname{ad} Z)_{\mathfrak{g}/\mathfrak{z}_b}$. Then*

$$\Theta_b^+(\exp Z) = \Theta^+(b \exp Z), \qquad \Psi_b^+(\exp Z) = \Psi^+(b \exp Z)$$

for all $Z \in \mathfrak{g}_0 \cap \mathfrak{z}_b(c_b)$ such that $b \exp Z \in G'$.

It is obvious that if c_b is sufficiently small, $\nu_b(\exp Z) \neq 0$ for $Z \in \mathfrak{z}_b(c_b)$. Let $\mathfrak{z}_b'(c_b)$ be the set of those elements of $\mathfrak{z}_b(c_b)$ which are regular in $\mathfrak{z}_b$. Then for any $Z \in \mathfrak{g}_0 \cap \mathfrak{z}_b(c_b)$, the two conditions $b \exp Z \in G'$ and $Z \in \mathfrak{g}_0 \cap \mathfrak{z}_b'(c_b)$ are obviously equivalent. Hence, in particular,

$$\mathfrak{g}_0 \cap \mathfrak{z}_b'(c_b) \subset \mathfrak{g}_0 \cap \mathfrak{z}_b''.$$

Fix $Z_0 \in \mathfrak{g}_0 \cap \mathfrak{z}_b'(c_b)$ and let $\mathfrak{a}$ be the centralizer of Z_0 in $\mathfrak{z}_b$. Then $\mathfrak{a}$ is a Cartan subalgebra of $\mathfrak{g}$. Since $b = \theta(b)$, $\mathfrak{z}_b$ is stable under θ and therefore, by Lemmas 29 and 45, we can choose $y \in \Xi(b)$ such that $\mathfrak{a}^y$ is stable under θ and $(\mathfrak{a}_I)^y \subset \mathfrak{b}$. Put $H_0 = Z_0^y$.

(1) Im μ denotes, as usual, the imaginary part of a complex number μ.

Since Θ_b^+, Θ^+, Ψ_b^+ and Ψ^+ are all invariant under $\Xi(b)$, it would be enough to verify that

$$\Theta_b^+(\exp H_0) = \Theta^+(b \exp H_0), \quad \Psi_b^+(\exp H_0) = \Psi^+(b \exp H_0).$$

So we may assume that $Z_0 = H_0$, $y = 1$, $\theta(\mathfrak{a}) = \mathfrak{a}$ and $\mathfrak{a}_I = \mathfrak{a} \cap \mathfrak{k} \subset \mathfrak{b}$.

Let $H_0 = H_1 + H_2$ where $H_1 \in \mathfrak{a}_I$, $H_2 \in \mathfrak{a}_R$. Then $h = b \exp H_0 = h_1 h_2$ where $h_1 = b \exp H_1 \in A_I$ and $h_2 = \exp H_2$. (A is, as before, the Cartan subgroup of G corresponding to $\mathfrak{a}$.) It is clear that $H_1 \in \mathfrak{z}_b(c_b)$ and therefore $\nu_b(\exp H_1) \neq 0$. Hence $\mathfrak{z}_{h_1} \subset \mathfrak{z}_b$. Now put $\mathfrak{z}_1 = \mathfrak{z}_b$, $\mathfrak{z}_2 = \mathfrak{z}_{h_1}$. Then $\mathfrak{z}_2$ is the centralizer of H_1 in $\mathfrak{z}_1$ so that Lemma 31 is applicable.

For $\mu \in \mathfrak{F}'$, define the distributions $T_{i,\mu} = T_{\mathfrak{z}_i,\mu}$ and $S_{i,\mu} = S_{\mathfrak{z}_i,\mu}$ on $\mathfrak{z}_i$ $(i = 1, 2)$ as in Lemma 37 corresponding to the constants $a_s = \langle (b^*)^s, b \rangle$ $(s \in W_G)$. For any $f \in C_c^\infty(\mathfrak{z}_2)$, define $f_{H_1}(Z) = f(Z - H_1)$ $(Z \in \mathfrak{z}_2)$ and put

$$T_{2,\mu}'(f) = T_{2,\mu}(f_{H_1}), \quad S_{2,\mu}'(f) = S_{2,\mu}(f_{H_1}).$$

Then
$$\pi_{\mathfrak{z}_2}(H)\, T_{2,\mu}'(H) = \sum_{s \in W_G} \varepsilon(s) \langle (b^*)^s, b \rangle e^{s\mu(H+H_1)} \quad (H \in \mathfrak{b}').$$

Moreover H_1 lies in the center of $\mathfrak{z}_2$ and

$$\langle (b^*)^s, h_1 \rangle = \langle (b^*)^s, b \rangle e^{s\lambda(H_1)} \quad (s \in W_G).$$

Now suppose μ tends to λ $(\mu \in \mathfrak{F}^+)$. Then it follows from Lemma 38 that

$$\lim_{\mu \to \lambda} T_{2,\mu}'(f) = T_{h_1}^+(f)$$

and similarly (see the corollary of Lemma 41)

$$\lim_{\mu \to \lambda} S_{2,\mu}'(f) = S_{h_1}^+(f) \quad (f \in C_c^\infty(\mathfrak{z}_2)).$$

Define
$$T_i^+ = T_{\mathfrak{z}_i,\lambda}^+, \quad S_i^+ = S_{\mathfrak{z}_i,\lambda}^+ \quad (i = 1, 2)$$

in the notation of § 18. Then it is clear from the above result that

$$T_{h_1}^+(f) = T_2^+(f_{H_1}), \quad S_{h_1}^+(f) = S_2^+(f_{H_1}) \quad (f \in C_c^\infty(\mathfrak{z}_2)).$$

Moreover $T_b^+ = T_1^+$, $S_b^+ = S_1^+$ by definition. Hence

$$\Theta^+(h) = \Theta_{h_1}^+(h_2) = D_{h_1}(H_2)^{-1} T_{h_1}^+(H_2) = D_{h_1}(H_2)^{-1} T_2^+(H_1 + H_2).$$

On the other hand $T_2^+ = \eta_0 T_1^+$ pointwise on $\mathfrak{g}' \cap \mathfrak{z}_2$ by Lemma 42 and

$$\Theta_b^+(\exp H_0) = D_b(H_0)^{-1} T_1^+(H_0).$$

Hence it would be enough to verify that $D_b(H)\eta_0(H) = D_{h_1}(H - H_1)$ for $H \in \mathfrak{a}$. Put

$$v(Z) = D_{h_1}(Z - H_1) - D_b(Z)\eta_0(Z) \quad (Z \in \mathfrak{z}_2).$$

Then v is an analytic function on $\mathfrak{z}_2$ which is invariant under $\Xi_2 = \Xi(h_1)$. So it would be enough to show that $v = 0$ on $\mathfrak{b}'$. But it follows from the definition of D_{h_1}, D_b and η_0 that

$$v(H) = \pi_{\mathfrak{z}_2}(H - H_1)^{-1}\Delta(h_1 \exp(H - H_1))$$
$$- \pi_{\mathfrak{z}_1}(H)^{-1}\Delta(b \exp H)\,\pi_{\mathfrak{z}_1}(H)\,\pi_{\mathfrak{z}_2}(H)^{-1} = 0 \quad (H \in \mathfrak{b}'),$$

since $\pi_{\mathfrak{z}_2}(H - H_1) = \pi_{\mathfrak{z}_2}(H)$. This proves the first statement of the lemma.

On the other hand,

$$\Psi^+(h) = \Psi_{h_1}^+(h_2) = S_{h_1}^+(H_2) = S_2^+(H_1 + H_2)$$
$$= S_1^+(H_1 + H_2) = S_b^+(H_0) = \Psi_b^+(\exp H_0)$$

since $S_1^+ = S_2^+$ on $\mathfrak{z}_2$ from Lemma 42. This proves the second statement.

COROLLARY. *Θ^+ and Ψ^+ are both analytic on G'. Moreover Ψ^+ can be extended to a continuous function on G.*

Let Ω be the set of all points $x_0 \in G$ with the following property. There exists an open neighborhood U of x_0 in G such that Θ^+ and Ψ^+ are both analytic on $U \cap G'$ and Ψ^+ extends to a continuous function on U. We have to verify that $\Omega = G$. Clearly Ω is an open and invariant subset of G. Therefore, in view of [2 (m), Lemma 7], it would be sufficient to verify that every semisimple element of G lies in Ω.

Fix a semisimple element $a \in G$. Then we can choose (see the corollary of [2 (m), Lemma 5]) a Cartan subgroup A of G containing a. Let $a = a_1 a_2$ where $a_1 \in A_I$, $a_2 \in A_R$. By Lemma 45 we can choose $x \in G$ such that $b = a_1^x \in B$. Since Ω is invariant, it would be enough to verify that $a^x \in \Omega$. Hence we may assume that $x = 1$ and $a = ba_2$ where $b = a_1 \in A_I \cap B$. Now put $V = \exp(\mathfrak{g}_0 \cap \mathfrak{z}_b(c_b)) \subset \Xi(b)$ in the notation of Lemma 50. Then V is an open neighborhood of 1 in $\Xi(b)$ and

$$\Theta^+(by) = \Theta_b^+(y), \quad \Psi^+(by) = \Psi_b^+(y)$$

for $y \in V' = V \cap (b^{-1}G')$. Moreover we note that Ψ_b^+ is continuous on V, $a_2 \in V$ and $\nu_b(a_2) \neq 0$.

Now let $x \to x^*$ denote the natural mapping of G on $G^* = G/\Xi(b)$ and fix open neighborhoods V_0 and G_0^* of a_2 and 1^* in V and G^* respectively. If V_0 and G_0^* are sufficiently small, we can choose an analytic mapping ϕ of G_0^* into G such that:

1) $(\phi(x^*))^* = x^* \quad (x^* \in G_0^*)$.

2) The mapping $\psi: (x^*, y) \to (by)^{\phi(x^*)}$ of $G_0^* \times V_0$ into G is univalent and regular.

This is evidently possible (see [2 (m), Lemma 14]). Put $U = \psi(G_0^* \times V_0)$. Then U is an open neighborhood of $a = ba_2$ in G and ψ defines an analytic diffeomorphism of $G_0^* \times V_0$ onto U. Put $V_0' = V_0 \cap V'$ and $U' = U \cap G'$. Then it is obvious that $\psi(G_0^* \times V_0') = U'$. Since Θ^+ and Ψ^+ are invariant functions, it is clear that

$$\Theta^+(\psi(x^*, y)) = \Theta^+(by) = \Theta_b^+(y), \qquad \Psi^+(\psi(x^*, y)) = \Psi^+(by) = \Psi_b^+(y)$$

for $x^* \in G_0^*$ and $y \in V_0'$. However Θ_b^+ and Ψ_b^+ are both analytic on V'. Therefore it follows that Θ^+ and Ψ^+ are analytic on U'. Similarly since Ψ_b^+ is continuous on V, we conclude that Ψ^+ can be extended to a continuous function on U. This proves the corollary.

Define the character ξ_ϱ of B as in [2 (m), § 18].

LEMMA 51. *Let Z_G be the center of G. Then*

$$\Theta^+(zx) = \xi_\varrho(z)^{-1} \langle b^*, z \rangle \Theta^+(x), \qquad \Psi^+(zx) = \langle b^*, z \rangle \Psi^+(x)$$

for $z \in Z_G$ and $x \in G'$.

Fix $h \in G'$ and let $\mathfrak{a}$ be the centralizer of h in $\mathfrak{g}$ and A the corresponding Cartan subgroup of G. Then $h = h_1 h_2$ $(h_1 \in A_I, h_2 \in A_R)$ and we can choose $y \in G$ such that $h_1^y \in B$ (Lemma 45). The required result holds for $x = h$ if and only if it holds for $x = h^y$. Hence we may assume that $y = 1$ and therefore $h_1 \in B$. Then

$$\Theta^+(zh) = \Theta_{zh_1}^+(h_2) = D_{zh_1}(H_2)^{-1} T_{zh_1}^+(H_2),$$
$$\Psi^+(zh) = \Psi_{zh_1}^+(h_2) = S_{zh_1}^+(H_2) \quad (z \in Z_G)$$

where(1) $H_2 = \log h_2 \in \mathfrak{a}_R$. Now h_1 and zh_1 have the same centralizer $\mathfrak{z}$ in $\mathfrak{g}$ and so it is obvious from the definitions of the various distributions that

$$T_{zh_1}^+ = \langle b^*, z \rangle T_{h_1}^+, \quad S_{zh_1}^+ = \langle b^*, z \rangle S_{h_1}^+.$$

On the other hand $$\Delta(zb) = \xi_\varrho(z) \Delta(b) \quad (b \in B).$$

(1) As usual log denotes the inverse of the exponential mapping of $\mathfrak{a}_R$ onto A_R.

Therefore it is clear that

$$D_{zh_1}(Z) = \xi_\varrho(z)\, D_{h_1}(Z) \quad (Z \in \mathfrak{z})$$

and now our assertions follow immediately.

LEMMA 52. *Let A be a Cartan subgroup of G and put $A' = A \cap G'$. Then*

$$\sup_{h \in A'} |\Delta_A(h)\, \Theta^+(h)| < \infty,$$

in the notation of [2 (m), § 19].

Since A_I/Z_G is compact, it would, in view of Lemma 51, be enough to prove the following result.

LEMMA 53. *For any $a \in A_I$, we can choose an open neighborhood U of* 1 *in A such that $U \supset A_R$ and*

$$\sup_{h \in U'} |\Delta_A(ah)\, \Theta^+(ah)| < \infty.$$

Here $U' = U \cap a^{-1}A'$.

By Lemma 45 we can select $x \in G$ such that $a^x \in B$. Hence, in view of the invariance of Θ^+, we may assume, without loss of generality, that $a \in B$. Then from Lemma 50,

$$\Theta^+(a \exp Z) = \Theta_a^{\,+}(\exp Z) = D_a(Z)^{-1}\, T_a^{\,+}(Z)$$

for all $Z \in \mathfrak{g}_0 \cap \mathfrak{z}_a(c_a)$ such that $a \exp Z \in G'$. Let $\mathfrak{a}$ be the Lie algebra of A. Then $\mathfrak{a} \subset \mathfrak{z}_a$. Put $\mathfrak{a}_0 = \mathfrak{a} \cap \mathfrak{g}_0 \cap \mathfrak{z}_a(c_a)$ and $U = \exp \mathfrak{a}_0$. Then $U \supset A_R$ and if $a \exp H \in G'$ $(H \in \mathfrak{a}_0)$, it is clear that

$$|\Delta_A(a \exp H)\Theta^+(a \exp H)| = |\Delta_A(a \exp H)\Theta_a^{\,+}(\exp H)| = |\pi_{\mathfrak{z}_a}(H)\, T_a^{\,+}(H)|$$

from the corollary of Lemma 47 and [2 (m), Lemma 33]. Hence if we take into account Lemmas 28, 38 and 40, we get

$$\sup_{h \in U'} |\Delta_A(ah)\, \Theta^+(ah)| < \infty.$$

LEMMA 54. *Θ^+ is locally summable on G and*

$$\sup_{x \in G'} |D(x)|^{\frac{1}{2}}\, |\Theta^+(x)| < \infty.$$

Moreover

$$z\,\Theta^+ = \chi_{b*}(z)\, \Theta^+ \quad (z \in \mathfrak{Z})$$

as a distribution on G.

Since there are only a finite number of non-conjugate Cartan subgroups of G, it follows from Lemma 52 that

$$\sup_{x \in G'} |D(x)|^{\frac{1}{2}} |\Theta^+(x)| < \infty.$$

Therefore Θ^+ is locally summable on G from [2 (m), Lemma 53].

Now fix $z \in \mathfrak{Z}$ and consider the distribution

$$T = z\,\Theta^+ - \chi_{b^*}(z)\,\Theta^+$$

on G. We have to show that $T=0$. In view of [2 (m), Lemma 7], it would be enough to verify that no semisimple element of G lies in Supp T.

Fix a semisimple element $h \in G$. Then h lies in some Cartan subgroup A of G [2 (m), Cor. of Lemma 5]. Let $h = h_1 h_2$ $(h_1 \in A_I,\ h_2 \in A_R)$. Then again by Lemma 45, there exists $x \in G$ such that $h_1^x \in B$. T being invariant, it would be sufficient to prove that $h^x \notin \operatorname{Supp} T$. Hence replacing (h, A) by (h^x, A^x), we may assume that $a = h_1 \in B$. Let σ_T and σ_{Θ^+} be the distributions on $\Xi'(a)$ corresponding to T and Θ^+ respectively under [2 (m), Lemma 15]. Then

$$\sigma_T = |\nu_a|^{-\frac{1}{2}} \mu_a(z)\,(|\nu_a|^{\frac{1}{2}}\,\sigma_{\Theta^+}) - \chi_{b^*}(z)\,\sigma_{\Theta^+}$$

by [2 (m), Lemma 22] where $\mu_a = \mu_{\mathfrak{g}/\mathfrak{z}_a}$ as in Lemma 46. Let θ_a denote the function $y \to \Theta^+(ay)$ on $\Xi'(a)$. Then it follows from [2 (i), Cor. 2 of Theorem 1] that θ_a is locally summable and therefore $\sigma_{\Theta^+} = \theta_a$ from the definition of σ_{Θ^+}. Hence it follows from Lemma 50 and the corollary of Lemma 47 that the distribution $|\nu_a|^{\frac{1}{2}}\sigma_{\Theta^+}$ coincides on $V = \exp(\mathfrak{g}_0 \cap \mathfrak{z}_a(c_a))$ with the locally summable function $\varepsilon_a(0)\,\Phi_a^{\ +}$. Therefore we conclude from Lemma 46 that $\sigma_T = 0$ on V. Since V is an open subset of $\Xi'(a)$ containing h_2, we conclude [2 (m), Lemma 15] that $T=0$ around $h = ah_2$. This proves Lemma 54.

LEMMA 55. $$\Theta^+(b) = \Delta(b)^{-1} \sum_{s \in W_G} \varepsilon(s) \langle (b^*)^s, b \rangle$$

for $b \in B'$.

Fix $b \in B'$. Then $\mathfrak{z}_b = \mathfrak{b}$ and therefore $D_b(H) = \Delta(b \exp H)$ and

$$T_b^{\ +}(H) = \sum_{s \in W_G} \varepsilon(s) \langle (b^*)^s, b \rangle\, e^{\lambda(s^{-1}H)} \quad (H \in \mathfrak{b}).$$

Hence $$\Theta^+(b) = \Theta_b^{\ +}(1) = D_b(0)^{-1} T_b^{\ +}(0) = \Delta(b)^{-1} \sum_{s \in W_G} \varepsilon(s) \langle (b^*)^s, b \rangle.$$

This shows that Θ^+ satisfies all the conditions of Theorem 3. Therefore in view of Lemma 44, the proof of Theorem 3 is now complete.

§ 23. Further properties of Θ

Let $\mathfrak{a}$ be a Cartan subalgebra of $\mathfrak{g}$ and A the corresponding Cartan subgroup of G. Put $\mathfrak{a}_R' = \mathfrak{a}_R \cap \mathfrak{a}'(R)$ and $A_R' = A_R \cap A'(R)$ in the notation of [2 (m), § 19]. Let A^+ be a connected component of $A'(R)$. Then it is obvious that $A^+ = A_I^+ A_R^+$ where A_I^+ is a connected component of A_I and $A_R^+ \subset A_R$.

Let us assume that $\theta(\mathfrak{a}) = \mathfrak{a}$. Then by Lemma 45 we can choose $k \in K$ such that $(A_I^+)^k \subset B$. Hence we may suppose that $A_I^+ \subset B$. Let $\mathfrak{z}$ denote the centralizer of A_I^+ in $\mathfrak{g}$. Then $\mathfrak{a}$ and $\mathfrak{b}$ are both Cartan subalgebras of $\mathfrak{z}$. Consider the complex-analytic subgroup Ξ_c of G_c corresponding to $\operatorname{ad}\mathfrak{z}_c$. We can choose $y \in \Xi_c$ such that $\mathfrak{b}_c^y = \mathfrak{a}_c$. Put $W(A^+) = W(\mathfrak{z}/\mathfrak{a})$. Since $\mathfrak{a}_I$ lies in the center of $\mathfrak{z}$, every root of $(\mathfrak{z}, \mathfrak{a})$ is real. Hence (see [2 (k), Lemma 6]) every element of $W(A^+)$ is induced on $\mathfrak{a}$ by some element of the analytic subgroup Ξ of G corresponding to $\mathfrak{z}$. Let W_Ξ be the subgroup of those elements of $W(\mathfrak{z}/\mathfrak{b})$ which can be induced on $\mathfrak{b}$ by some element of Ξ. Then $W_\Xi = W_k(\mathfrak{z}/\mathfrak{b})$ in the notation of § 18.

Put $\mathfrak{w}(A_I^+) = W_G \cap W(\mathfrak{z}/\mathfrak{b})$ and write $\mathfrak{w} = \mathfrak{w}(A_I^+)$ for simplicity.

LEMMA 56. *Suppose t_1, t_2 are two elements in W_G such that*

$$t_1^y \in W(A^+)\, t_2^y.$$

Then $t_1 \in \mathfrak{w}\, t_2$.

Put $t = t_1 t_2^{-1}$. Then $t \in (W(A^+))^{y^{-1}} \cap W_G = W(\mathfrak{z}/\mathfrak{b}) \cap W_G = \mathfrak{w}$.

COROLLARY. *Let $r = [W_G : \mathfrak{w}]$ and $t_1, \ldots, t_r$ a complete set of representatives in W_G for $\mathfrak{w}\backslash W_G$. Then the elements st_i^y $(s \in W(A^+),\ 1 \leqslant i \leqslant r)$ are all distinct.*

This is obvious from the above lemma.

LEMMA 57. *Fix an element $b^* \in B^{*\prime}$ and define Θ as in Theorem 3. Then there exist unique complex numbers $c_{b^*}(s:t:A^+)$ $(s \in W(A^+), t \in W_G)$ such that*

1) $c_{b^*}(su^y : u^{-1}t : A^+) = c_{b^*}(s:t:A^+) \quad (u \in \mathfrak{w})$,

2) $\Delta_A(h_1 h_2)\,\Theta(h_1 h_2) = \sum_{t \in \mathfrak{w}\backslash W_G} \varepsilon(t) \langle (b^*)^t, h_1 \rangle \sum_{s \in W(A^+)} \varepsilon(s)\, c_{b^*}(s:t:A^+) \exp(s(t\lambda)^y(H_2))$

for $h_1 \in A_I^+$, $h_2 \in A_R^+$. Here $\lambda = \log b^*$ and [1] $H_2 = \log h_2$.

Let $H_1 \in \mathfrak{a}_I$ and $H_2 \in \mathfrak{a}_R$. Then since s^{-1} and y^{-1} leave H_1 fixed, it is clear that

$$\langle (b^*)^t, \exp H_1 \rangle \exp(s(t\lambda)^y(H_2)) = \exp(s(t\lambda)^y(H_1 + H_2))$$

(1) See footnote 1, p. 299.

for $s \in W(A^+)$ and $t \in W_G$. Since λ is regular, the uniqueness is obvious from the corollary of Lemma 56. On the other hand the existence is seen as follows. We use the notation of § 18. Put $\mathfrak{a}^+ = \mathfrak{a}_I + \log A_R^+$. Then $\mathfrak{a}^+$ is a connected component of $\mathfrak{a}'(\mathfrak{z}:R)$ (see § 13).

LEMMA 58. *Put*

$$c(s:\mathfrak{F}^+:A^+) = \sum_{t \in \mathfrak{w}/W_\Xi} c_{\mathfrak{z}}(st^y : t^{-1}\mathfrak{F}^+ : \mathfrak{a}^+)$$

for $s \in W(A^+)$ *and any connected component* $\mathfrak{F}^+$ *of* $\mathfrak{F}'$. *Then*

$$c_{b^*}(s:t:A^+) = c(s:t\mathfrak{F}^+:A^+) \quad (s \in W(A^+), t \in W_G)$$

where $\mathfrak{F}^+$ *is the component of* $\log b^*$ *in* $\mathfrak{F}'$ $(b^* \in B^{*\prime})$.

In view of Lemma 40, the definition of $c(s:\mathfrak{F}^+:A^+)$ is legitimate and it is obvious that

$$c(su^y : u^{-1}\mathfrak{F}^+ : A^+) = c(s:\mathfrak{F}^+:A^+) \quad (u \in \mathfrak{w}).$$

Therefore it would be sufficient to prove the following result.

LEMMA 59. *Fix* $b^* \in B^*$ *and a connected component* $\mathfrak{F}^+$ *of* $\mathfrak{F}'$ *such that* $\lambda = \log b^* \in \mathrm{Cl}\,\mathfrak{F}^+$ *and define* Θ^+, Ψ^+ *as in* § 22 *corresponding to* b^* *and* $\mathfrak{F}^+$. *Then*

$$\Delta_A(h_1 h_2)\,\Theta^+(h_1 h_2) = \sum_{t \in \mathfrak{w}\backslash W_G} \varepsilon(t) \langle (b^*)^t, h_1 \rangle \sum_{s \in W(A^+)} \varepsilon(s)\, c(s:t\mathfrak{F}^+:A^+) \exp(s(t\lambda)^y(H_2)),$$

and

$$\Psi^+(h_1 h_2) = \sum_{t \in \mathfrak{w}\backslash W_G} \langle (b^*)^t, h_1 \rangle \sum_{s \in W(A^+)} c(s:t\mathfrak{F}^+:A^+) \exp(s(t\lambda)^y(H_2))$$

for $h_1 \in A_I^+$ *and* $h_2 \in A_R^+$. *Here* $H_2 = \log h_2$ *as before.*

Fix a point $b_0 \in A_I^+$. Then we can choose $H_0 \in \mathfrak{a}_I$ arbitrarily near zero such that 1) $\nu_{b_0}(\exp H_0) \neq 0$ and 2) every root of $(\mathfrak{z}_{b_0}, \mathfrak{a})$ which vanishes at H_0 is real. Put $b = b_0 \exp H_0$. Then $b \in A_I^+$ and it is obvious that $\mathfrak{z}_b = \mathfrak{z}$. This shows that the set V of those points $b \in A_I^+$ for which $\mathfrak{z}_b = \mathfrak{z}$, is dense in A_I^+. Fix a point $b \in V$. Then from Lemma 50,

$$\Theta^+(b \exp Z) = \Theta_b^+(\exp Z) = D_b(Z)^{-1} T_b^+(Z), \quad \Psi^+(b \exp Z) = \Psi_b^+(\exp Z) = S_b^+(Z)$$

for all $Z \in \mathfrak{g}_0 \cap \mathfrak{z}_b(c_b)$ such that $b \exp Z \in G'$. Put $U = \mathfrak{a}^+ \cap \mathfrak{g}_0 \cap \mathfrak{z}_b(c_b)$ and let U' be the set of all points $H \in U$ where $\Delta_A(b \exp H) \neq 0$. Recall that P is the set of all positive roots of $(\mathfrak{g}, \mathfrak{b})$. Then we may assume, without loss of generality, that P^y is the set of all positive roots of $(\mathfrak{g}, \mathfrak{a})$. Then it is clear that

$$D_b(\exp H) = \Delta_A(b \exp H)\, \pi_{\mathfrak{z}}^{\mathfrak{a}}(H)^{-1}$$

and therefore

$$\Delta_A(b \exp H)\, \Theta^+(b \exp H) = \pi_{\mathfrak{z}}^{\mathfrak{a}}(H)\, T_b^+(H) \quad (H \in U').$$

On the other hand it follows from Lemmas 38 and 40 that

$$\pi_{\mathfrak{z}}^{\mathfrak{a}}(H)\, T_b^+(H) = \sum_{t \in W_\Xi \backslash W_G} \varepsilon(t) \langle (b^*)^t, b\rangle \sum_{s \in W(A^+)} \varepsilon(s)\, c_{\mathfrak{z}}(s : t\mathfrak{F}^+ : \mathfrak{a}^+) \exp(s(t\lambda)^y(H))$$

for $H \in U'$. Now suppose $H = H_1 + H_2$ $(H_1 \in \mathfrak{a}_I, H_2 \in \mathfrak{a}_R)$. Since s^{-1} and y^{-1} leave $\mathfrak{a}_I$ pointwise fixed, it is clear that

$$\langle (b^*)^t, b\rangle \exp(s(t\lambda)^y(H)) = \langle (b^*)^t, h_1\rangle \exp(s(t\lambda)^y(H_2))$$

for $s \in W(A^+)$ and $t \in W_G$. Here $h_1 = b \exp H_1$. Therefore since the function

$$h \to \Delta_A(h)\, \Theta^+(h) \quad (h \in A^+ \cap A')$$

extends to an analytic function on A^+ (see [2 (m), Lemma 31]), it is obvious that

$$\Delta_A(h_1 h_2)\, \Theta^+(h_1 h_2) = \sum_{t \in W_\Xi \backslash W_G} \varepsilon(t) \langle (b^*)^t, h_1\rangle \sum_{s \in W(A^+)} \varepsilon(s)\, c_{\mathfrak{z}}(s : t\mathfrak{F}^+ : \mathfrak{a}^+) \exp(s(t\lambda)^y(H_2))$$

for $h_1 \in A_I^+, h_2 \in A_R^+$. Our first assertion now follows immediately if we take into account Lemma 40.

Similarly we conclude from Lemmas 50 and 41 that

$$\Psi^+(b \exp H) = S_b^+(H) = \sum_{t \in W_\Xi \backslash W_G} \langle (b^*)^t, b\rangle \sum_{s \in W(A^+)} c_{\mathfrak{z}}(s : t\mathfrak{F}^+ : \mathfrak{a}^+) \exp(s(t\lambda)^y(H))$$

for $H \in U'$. Since Ψ^+ extends to a continuous function on G (see the corollary of Lemma 50), this relation holds for all $H \in U$. Now $\log A_R^+ \subset U$ and V is dense in A_I^+. Therefore the second assertion of the lemma is now obvious.

LEMMA 60. *$c(s : \mathfrak{F}^+ : A^+) = 0$ unless $\Re \mu^y(s^{-1}H) \leq 0$ for every $\mu \in \mathfrak{F}^+$ and $H \in \mathfrak{a}^+$.*

This is obvious from Lemma 58 and Lemma 28.

COROLLARY. *There exists a number C (independent of b^* and $\mathfrak{F}^+$) such that*

$$|D(x)|^{\frac{1}{2}} |\Theta^+(x)| \leq C \quad (x \in G')$$

and
$$|\Psi^+(x)| \leqslant C \quad (x \in G)$$
in the above notation.

Let $C(A^+)$ denote the maximum of $|c_3(s:\mathfrak{F}^+:\mathfrak{a}^+)|$ for all $s \in W(A^+)$ and all $\mathfrak{F}^+$. Then it follows from Lemmas 58, 59 and 60 that

$$|\Delta_A(h)\,\Theta^+(h)| \leqslant [\mathfrak{w}:W_\Xi]\,[W_G:\mathfrak{w}]\,[W(A^+)]\,C(A^+) \leqslant [W]^2\,C(A^+) \quad (h \in A' \cap A^+)$$

where $W = W(\mathfrak{g}/\mathfrak{b})$ as usual. Similarly

$$|\Psi^+(h)| \leqslant [W]^2\,C(A^+) \quad (h \in A^+).$$

It is clear that $C(zA^+) = C(A^+)$ for $z \in Z_G$. Therefore since A/Z_G and $\mathfrak{a}'(R)$ both have only a finite number of connected components,

$$C(A) = [W]^2 \operatorname*{Sup}_{A^+} C(A^+) < \infty.$$

Here A^+ runs over all connected components of $A'(R)$. This shows that

$$|\Delta_A(h)\,\Theta^+(h)| \leqslant C(A) \quad (h \in A')$$

and
$$|\Psi^+(h)| \leqslant C(A) \quad (h \in A).$$

But then since G has only a finite number of non-conjugate Cartan subgroups, our assertion is obvious.

§ 24. The distribution $\Theta_\lambda{}^*$

Put $L = \log B^*$. Then L is a closed additive subgroup of $\mathfrak{F}$ which is, in fact, a lattice if B is compact. For any $\lambda \in L$, let ξ_λ denote the corresponding element of B^* so that $\xi_\lambda(\exp H) = e^{\lambda(H)}$ $(H \in \mathfrak{b})$. Fix $\lambda \in L$ and a connected component $\mathfrak{F}^+$ of $\mathfrak{F}'$ such that $\lambda \in \mathrm{Cl}\,\mathfrak{F}^+$. Then we denote by $\Theta_{\lambda,\mathfrak{F}^+}$ and $\Psi_{\lambda,\mathfrak{F}^+}$ respectively, the distributions Θ^+ and Ψ^+ of § 22 for $b^* = \xi_\lambda$. In particular if $\lambda \in L' = L \cap \mathfrak{F}'$, the component $\mathfrak{F}^+$ is uniquely determined and so in this case we denote them simply by Θ_λ and Ψ_λ.

Now fix $\lambda \in L'$ and suppose that $s\lambda \in L$ for every $s \in W = W(\mathfrak{g}/\mathfrak{b})$. Then we intend to study the distribution
$$\Theta_\lambda{}^* = \sum_{s \in W} \varepsilon(s)\,\Theta_{s\lambda}$$
more closely. Let us return to the notation of § 23 and define

$$\xi_{t,\lambda}(h_1 h_2) = \xi_{t\lambda}(h_1)\,\exp\left((t\lambda)^\nu(\log h_2)\right) \quad (t \in W)$$

for $h_1 \in A_I{}^+$ and $h_2 \in A_R$. Let $\mathfrak{m}$ be the centralizer of $\mathfrak{a}_R$ in $\mathfrak{g}$ and put

$$W_0 = W(\mathfrak{m}/\mathfrak{a})^{y^{-1}}.$$

Since $\mathfrak{a}_I$ lies in the center of $\mathfrak{z}$ and $\mathfrak{a}_R$ in the center of $\mathfrak{m}$, it is clear that $W(\mathfrak{z}/\mathfrak{a})$ and $W(\mathfrak{m}/\mathfrak{a})$ commute (as subgroups of $W(\mathfrak{g}/\mathfrak{a})$). Therefore $W(\mathfrak{z}/\mathfrak{b})$ and W_0 also commute in W.

LEMMA 61. *For any connected component* $\mathfrak{F}^+$ *of* $\mathfrak{F}'$, *define*

$$c^*(t:\mathfrak{F}^+:A^+) = [W_G:\mathfrak{w}] \sum_{s \in W(\mathfrak{z}/\mathfrak{b})} c(s^y : s^{-1} t \mathfrak{F}^+ : A^+) \quad (t \in W).$$

Then

$$c^*(u^{-1}t:\mathfrak{F}^+:A^+) = c^*(t:\mathfrak{F}^+:A^+)$$

for $u \in W_0$ *and* $t \in W$. *Moreover,*

$$\Delta_A \Theta_\lambda{}^* = \sum_{t \in W} \varepsilon(t)\, c^*(t:\mathfrak{F}^+:A^+)\, \xi_{t,\lambda}$$

on A^+. *Here* $\mathfrak{F}^+$ *is the component of* $\mathfrak{F}'$ *containing* λ.

Fix $u \in W_0$ and $t \in W$. Since u and $W(\mathfrak{z}/\mathfrak{b})$ commute, it is clear that

$$\begin{aligned} c^*(u^{-1}t:\mathfrak{F}^+:A^+) &= [W_G:\mathfrak{w}] \sum_{s \in W(\mathfrak{z}/\mathfrak{b})} c(s^y : u^{-1} s^{-1} t \mathfrak{F}^+ : A^+) \\ &= [W_G:W_\Xi] \sum_{s \in W(\mathfrak{z}/\mathfrak{b})} c_{\mathfrak{z}}(s^y : u^{-1} s^{-1} t \mathfrak{F}^+ : \mathfrak{a}^+) \end{aligned}$$

from Lemma 58. Define $\mathfrak{F}_{\mathfrak{z}}'$ as in § 18 and for fixed $s \in W(\mathfrak{z}/\mathfrak{b})$ and $t \in W$, let $\mathfrak{F}_{\mathfrak{z}}{}^+$ be the unique connected component of $\mathfrak{F}_{\mathfrak{z}}'$ containing $s^{-1}t\mathfrak{F}^+$. Since u^{-1} leaves every root of $(\mathfrak{z}, \mathfrak{b})$ fixed, it is clear that $u^{-1}\mathfrak{F}_{\mathfrak{z}}{}^+ = \mathfrak{F}_{\mathfrak{z}}{}^+$. Hence

$$c_{\mathfrak{z}}(s^y : u^{-1} s^{-1} t \mathfrak{F}^+ : \mathfrak{a}^+) = c_{\mathfrak{z}}(s^y : \mathfrak{F}_{\mathfrak{z}}{}^+ : \mathfrak{a}^+) = c_{\mathfrak{z}}(s^y : s^{-1} t \mathfrak{F}^+ : \mathfrak{a}^+)$$

in the notation of § 18. This implies the first assertion of the lemma.

Now let $\mathfrak{F}^+$ be the component of $\mathfrak{F}'$ containing λ. Then it follows from Lemma 59 that

$$\Delta_A \Theta_\lambda{}^* = \sum_{u \in W} \varepsilon(u) \sum_{t \in \mathfrak{w} \backslash W_G} \varepsilon(t) \sum_{s \in W(\mathfrak{z}/\mathfrak{b})} \varepsilon(s)\, c(s^y : tu \mathfrak{F}^+ : A^+)\, \xi_{stu,\lambda}$$

on A^+. From this the second assertion of the lemma follows immediately.

Now assume that G_c is an acceptable complexification (see [2 (m), § 18]) of G and G is the real analytic subgroup of G_c corresponding to $\mathfrak{g}$. Let A_c and B_c be the Cartan subgroups of G_c corresponding to $\mathfrak{a}_c$ and $\mathfrak{b}_c$ respectively. Then W operates on

B_c and therefore also on B. Hence L is invariant under W. Similarly $W(\mathfrak{m}/\mathfrak{a})$ operates on A_c. Since it maps $\mathfrak{a}_I$ into itself and leaves $\mathfrak{a}_R$ pointwise fixed, it leaves every point in $A_I \cap \exp(-1)^{\frac{1}{2}}\mathfrak{a}_R$ fixed and maps $A_I^0 = \exp \mathfrak{a}_I$ into itself. Therefore (see [2 (m), Lemma 50]) $W(\mathfrak{m}/\mathfrak{a})$ operates on A and maps A_I^+ into itself. Now if $u \in W_0$ then $s = u^y \in W(\mathfrak{m}/\mathfrak{a})$ and

$$\xi_{ut,\lambda}(h_1 h_2) = \xi_{ut\lambda}(h_1) \exp((t\lambda)^y(\log h_2)) = \xi_{t,\lambda}((h_1 h_2)^{s^{-1}}) \quad (t \in W)$$

for $h_1 \in A_I^+$, $h_2 \in A_R$. Hence we obtain the following result from Lemma 61.

LEMMA 62. *Under the above conditions*

$$\Delta_A(h)\,\Theta_\lambda^*(h) = \sum_{t \in W_0 \backslash W} \varepsilon(t)\, c^*(t : \mathfrak{F}^+ : A^+) \sum_{s \in W(\mathfrak{m}/\mathfrak{a})} \varepsilon(s)\, \xi_{t,\lambda}(h^s)$$

for $h \in A^+$.

Let P_+ be the set of all positive roots of $(\mathfrak{g}, \mathfrak{a})$ which do not vanish identically on $\mathfrak{a}_R$. Put $\sigma = \frac{1}{2}\sum_{\alpha \in P_+} \alpha$ and

$$\Delta_+(h) = e^{\sigma(\log h_2)} \prod_{\alpha \in P_+} (1 - \xi_\alpha(h^{-1})) \quad (h \in A)$$

in the notation of [2 (m), § 18]. Here h_2 is the component of h in A_R.

COROLLARY. $$\sup_{h \in A'} |\Delta_+(h)\,\Theta_\lambda^*(h)| < \infty.$$

In view of Lemma 51, it is enough to show that $\Delta_+(h)\,\Theta_\lambda^*(h)$ remains bounded for $h \in A^+ \cap A'$. In order to do this we can obviously assume that the set of positive roots of $(\mathfrak{g}, \mathfrak{a})$ is chosen as in [2 (m), § 27]. Define M, Δ_M and ξ_ϱ as in [2 (m), § 27]. Then

$$\Delta_A(h) = \Delta_M(h)\,\Delta_+(h) \quad (h \in A).$$

Moreover, it follows from Lemma 60, that

$$c(s^y : s^{-1}t\mathfrak{F}^+ : A^+) = 0 \quad (s \in W(\mathfrak{z}/\mathfrak{b}),\ t \in W)$$

unless $(t\lambda)^y(H_2) \leqslant 0$ for $H_2 \in \log A_R^+$. Therefore it is clear from Lemmas 61 and 62 that

$$|\Delta_+(h)\,\Theta_\lambda^*(h)| \leqslant \sum_{t \in W_0 \backslash W} |c^*(t : \mathfrak{F}^+ : A^+)|\, |\Delta_M(h_1)^{-1} \sum_{s \in W(\mathfrak{m}/\mathfrak{a})} \varepsilon(s)\, \xi_{t\lambda}(h_1^s)|$$

for $h \in A' \cap A^+$ where h_1 is the component of h in A_I^+. Now choose

$$a \in A_I^+ \cap \exp(-1)^{\frac{1}{2}}\mathfrak{a}_R$$

such that $A_I^+ = aA_I^0$, Then (see [2 (m), § 23])

$$\Delta_M(ah) = \xi_\varrho(a)\,\Delta_M(h)$$

and

$$\sum_{s\in W(\mathfrak{m}/\mathfrak{a})} \varepsilon(s)\,\xi_{t\lambda}((ah)^s) = \xi_{t\lambda}(a) \sum_{s\in W(\mathfrak{m}/\mathfrak{a})} \varepsilon(s)\,\xi_{t\lambda}(h^s)$$

for $h \in A_I^0$ and $t \in W$. Therefore our assertion is obvious from [2 (b), Cor. 2, p. 139].

§ 25. Statement of Theorem 4

Fix a Haar measure dx on G and consider the distribution

$$\Theta^+(f) = \int f\,\Theta^+ dx \quad (f \in C_c^\infty(G))$$

as in Lemma 54. For any $\varepsilon > 0$, get $G(\varepsilon)$ denote the set of all $x \in G$ where $|D(x)| > \varepsilon^2$. Suppose u is a measurable function on G' which is integrable (with respect to dx) on $G(\varepsilon)$ for every $\varepsilon > 0$. Then we define [1]

$$\text{p.v.} \int u\,dx = \lim_{\varepsilon \to 0} \int_{G(\varepsilon)} u\,dx$$

provided this limit exists and is finite.

THEOREM 4. *Define* Θ^+ *and* Ψ^+ *as in* § 22 *and put* $\lambda = \log b^*$. *Then*

$$\varpi(\lambda)\,\Theta^+(f) = \text{p.v.} \int D^{-1}\Psi^+ \nabla_G f\,dx$$

where ∇_G *has the same meaning as in* [2 (m), § 20].

Before proceeding with the proof, we need some formulas on integrals (cf. § 2). Let $\mathfrak{a}_1 = \mathfrak{b}, \mathfrak{a}_2, \ldots, \mathfrak{a}_r$ be a maximal set of Cartan subalgebras of $\mathfrak{g}$ no two of which are conjugate under G. Let A_i be the Cartan subgroup of G corresponding to $\mathfrak{a}_i$. Put $G_i^* = G/A_{i0}$ where A_{i0} is the center of A_i and fix a Haar measure $d_i a$ on A_i and an invariant measure $d_i x^*$ on G_i^*. Also let

$$\Delta_i(a) = \Delta_{A_i}(a) \quad (a \in A_i)$$

in the usual notation (see [2 (m), § 19]).

LEMMA 63. *There exist numbers* $c_i > 0$ $(1 \leqslant i \leqslant r)$ *such that*

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(1) See footnote 1, p. 246.

$$\int f(x)\,dx = \sum_{1 \leqslant i \leqslant r} c_i \int_{G_i^* \times A_i} |\Delta_i(a)|^2 f(a^{x^*})\, d_i x^*\, d_i a$$

for $f \in C_c(G)$ *in the notation of* [2 (m), § 22].

Put $G_i = A_i^G \cap G'$. Then G' is the disjoint union of $G_1, \dots, G_r$ and our assertion is an immediate consequence of [2 (m), Lemma 41].

§ 26. A simple property of the function Δ

Let $\mathfrak{a}$ be a Cartan subalgebra of $\mathfrak{g}$ and A the corresponding Cartan subgroup of G. Suppose a is an element of A and α a root of $(\mathfrak{g}, \mathfrak{a})$. We say that a and α *commute* if $\xi_\alpha(a) = 1$ in the notation of [2 (m), § 19].

Put $m = \frac{1}{2}(\dim \mathfrak{g} - \operatorname{rank} \mathfrak{g})$ as in § 2. Then m is the number of positive roots of $(\mathfrak{g}, \mathfrak{a})$. For any $a \in A$, define the integer $m(R:a) \geqslant 0$ as follows. Let $a = a_1 a_2$ $(a_1 \in A_I, a_2 \in A_R)$. Then $m(R:a)$ is the number of positive real roots of $(\mathfrak{g}, \mathfrak{a})$ which commute with a_1. If α is a real root, $\alpha(H) = 0$ for $H \in \mathfrak{a}_I$. Hence it is clear that $m(R:a)$ depends only on the connected component of a_1 in A_I. Therefore the function $m(R): a \to m(R:a)$ is locally constant on A.

LEMMA 64. $\quad \operatorname{conj} \Delta_A(a) = (-1)^{m + m(R:a)} \Delta_A(a) \quad (a \in A).$

This result is obviously independent of the choice of positive roots. Hence we may select compatible orders on the spaces of real linear functions on $\mathfrak{a}_R$ and $\mathfrak{a}_R + (-1)^{\frac{1}{2}} \mathfrak{a}_I$ respectively and assume that P is the set of positive roots of $(\mathfrak{g}, \mathfrak{a})$ in this order. Let η denote the conjugation of $\mathfrak{g}_c$ with respect to $\mathfrak{g}$. Then it is clear that if α is a root, the same holds for $\eta\alpha$ and

$$\xi_{\eta\alpha}(a) = \operatorname{conj} \xi_\alpha(a) \quad (a \in A).$$

Let P_R, P_I and P_c respectively denote the sets of real, imaginary and complex roots in P (see [2 (k), § 4]). We now use the notation of [2 (m), § 19]. Then

$$\Delta(a) = \xi_\varrho(a)\, \Delta_I'(a)\, \Delta_+'(a)$$

where

$$\Delta_I'(a) = \prod_{\alpha \in P_I} (1 - \xi_\alpha(a)^{-1}), \quad \Delta_+'(a) = \prod_{\alpha \in P_+} (1 - \xi_\alpha(a)^{-1}) \quad (a \in A)$$

and $P_+ = P_R \cup P_c$. Since P_+ is invariant under η, it is clear that $\Delta_+'(a)$ is real. On the other hand, $\eta\alpha = -\alpha$ for $\alpha \in P_I$. Therefore

$$\operatorname{conj} \Delta_I'(a) = (-1)^{m(I)} \xi_{2\varrho_I}(a)\, \Delta_I'(a)$$

where $m(I)$ is the number of roots in P_I and $\varrho_I = \frac{1}{2}\sum_{\alpha\in P_I}\alpha$. Now suppose $a = a_1 a_2$ $(a_1 \in A_I,\ a_2 \in A_R)$. Then conj $\xi_\varrho(a) = \xi_\varrho(a_1^{-1}a_2)$ and $\xi_{2\varrho_I}(a_2) = 1$. Hence

$$\begin{aligned}\text{conj } \Delta(a) &= (-1)^{m(I)}\xi_\varrho(a_1^{-1}a_2)\,\xi_{2\varrho_I}(a_1)\,\Delta_I'(a)\,\Delta_+'(a)\\ &= (-1)^{m(I)}\xi_{2\varrho}(a_1)^{-1}\xi_{2\varrho_I}(a_1)\,\Delta(a) = (-1)^{m(I)}\xi_{2\varrho_+}(a_1)^{-1}\Delta(a)\end{aligned}$$

where $\varrho_+ = \frac{1}{2}\sum_{\alpha\in P_+}\alpha$. Now if $\alpha \in P_c$ then the same holds for $\eta\alpha$ and $\eta\alpha \neq \alpha$. Moreover,

$$\xi_\alpha(a_1)\,\xi_{\eta\alpha}(a_1) = |\xi_\alpha(a_1)|^2 = 1.$$

Hence
$$\xi_{2\varrho_+}(a_1) = \xi_{2\varrho_R}(a_1)$$

where $\varrho_R = \frac{1}{2}\sum_{\alpha\in P_R}\alpha$. But for any $\alpha \in P_R$, $\xi_\alpha(a_1)$ is both real and unimodular. Therefore it is ± 1. Hence

$$\xi_{2\varrho_R}(a_1) = \prod_{\alpha\in P_R}\xi_\alpha(\alpha_1) = (-1)^q$$

where q is the number of roots $\alpha \in P_R$ such that $\xi_\alpha(a_1) = -1$. But then $q + m(R:a)$ is the total number of roots in P_R. We have seen above that the roots in P_c occur in pairs. Hence

$$q + m(R:a) + m(I) \equiv m \bmod 2.$$

This shows that
$$q + m(I) \equiv m + m(R:a) \bmod 2$$

and therefore
$$\text{conj } \Delta(a) = (-1)^{m(I)+q}\Delta(a) = (-1)^{m+m(R:a)}\Delta(a).$$

This proves the lemma.

§ 27. Reduction of Theorem 4 to Lemma 66

We now come to Theorem 4. Suppose V_ε $(0 < \varepsilon \leqslant \varepsilon_0)$ is a family of measurable functions on G such that (cf. § 2)

1) $\quad 0 \leqslant V_\varepsilon \leqslant 1$ and $\lim_{\varepsilon\to 0} V_\varepsilon(x) = 1$ for $x \in G'$,

2) $\quad V$ is invariant under G.

3) $\quad V_\varepsilon(x) = 0$ if $|D(x)| < \varepsilon^2 \quad (x \in G)$.

Fix $f \in C_c^\infty(G)$ and define $F_{f,i}$, $\varepsilon_{R,i}$ and ϖ_i on A_i $(1 \leqslant i \leqslant r)$ as in [2 (m), § 22] and let $m_i(R)$ be the locally constant function on A_i introduced in § 26. Since

$$D(a) = (-1)^m \Delta_i(a)^2 \quad (a \in A_i),$$

it is obvious from Lemmas 63 and 64 that

$$\int V_\varepsilon D^{-1}\Psi^+ \nabla_G f\,dx = \sum_i c_i \int V_{\varepsilon,i}(-1)^{m_i(R)} \varepsilon_{R,i} \Psi_i^+ \varpi_i F_{f,i}\, d_i a$$

where $V_{\varepsilon,i}$ and Ψ_i^+ respectively denote the restrictions of V_ε and Ψ^+ on A_i. Therefore the following lemma is now obvious (cf. Lemma 4) from [2 (f), Theorem 2].

LEMMA 65. *Fix $f \in C_c^\infty(G)$. Then*

$$\lim_{\varepsilon\to 0}\int V_\varepsilon D^{-1}\Psi^+ \nabla_G f\,dx = \text{p.v.}\int D^{-1}\Psi^+\nabla_G f\,dx = \sum_{1\leqslant i\leqslant r} c_i \int (-1)^{m_i(R)} \varepsilon_{R,i}\Psi_i^+\varpi_i F_{f,i}\,d_i a.$$

Now put
$$T(f) = \varpi(\lambda)\,\Theta^+(f) - \text{p.v.}\int D^{-1}\Psi^+\nabla_G f\,dx$$

for $f \in C_c^\infty(G)$. It follows from [2 (f), Theorem 2] and the above lemma that T is an invariant distribution on G. We have to show that $T=0$. Hence it is sufficient by [2 (m), Lemma 7] to verify that no semisimple element of G lies in Supp T.

Fix a function $v \in C^\infty(\mathbf{R})$ such that $0 \leqslant v \leqslant 1$, $v(t)=0$ if $|t| \leqslant \frac{1}{2}$ and $v(t)=1$ if $|t| \geqslant 1$ $(t \in \mathbf{R})$. For any $\varepsilon > 0$, put

$$V_\varepsilon(x) = v(2^{-1}\varepsilon^{-2}D(x)) \quad (x \in G).$$

Then it follows from Lemma 65 that

$$\lim_{\varepsilon\to 0}\int D^{-1}V_\varepsilon\Psi^+\nabla_G f\,dx = \text{p.v.}\int D^{-1}\Psi^+\nabla_G f\,dx.$$

Put
$$T_\varepsilon(f) = \varpi(\lambda)\,\Theta^+(f) - \int D^{-1}V_\varepsilon\Psi^+\nabla_G f\,dx \quad (f \in C_c^\infty(G))$$

for $\varepsilon > 0$. As usual let ∇_G^* denote the adjoint of ∇_G on G'. Since $D^{-1}V_\varepsilon\Psi^+$ is a C^∞ function on G whose support is contained in G', if follows that the distribution T_ε is, in fact, a locally summable function given by the formula

$$T_\varepsilon = \varpi(\lambda)\,\Theta^+ - \nabla_G^*(D^{-1}V_\varepsilon\Psi^+).$$

Moreover,
$$T(f) = \lim_{\varepsilon\to 0} T_\varepsilon(f) \quad (f \in C_c^\infty(G)).$$

Fix a semisimple element $a \in G$. Then a is contained in some Cartan subgroup A of G and $a = a_1 a_2$ where $a_1 \in A_I$, $a_2 \in A_R$. Let $\mathfrak{a}$ be the Lie algebra of A. By Lemma 45, we can choose $x \in G$ such that $\theta(\mathfrak{a}^x) = \mathfrak{a}^x$, $(\mathfrak{a}_I)^x \subset \mathfrak{b}$ and $a_1^x \in B$. Since T is invariant under G, it would be enough to verify that $a^x \notin \text{Supp}\, T$. Hence replacing

(a, A) by (a^x, A^x), we may assume that $\theta(\mathfrak{a})=\mathfrak{a}$, $\mathfrak{a}_I\subset\mathfrak{b}$ and $a_1\in B$. Then $a=b\exp H_0$ where $b=a_1\in A_I\cap B$ and $H_0=\log a_2\in\mathfrak{a}_R$. Define $\mathfrak{z}_b(c_b)$ as in Lemma 50. Then $\mathfrak{z}_0=\mathfrak{z}_b(c_b)\cap\mathfrak{g}_0$ is an open and completely invariant neighborhood of zero in $\mathfrak{z}=\mathfrak{z}_b$ and $H_0\in\mathfrak{z}_0$. Put $\Xi=\Xi(b)$ and $\Xi_0=\exp\mathfrak{z}_0$. Then Ξ_0 is an open and completely invariant neighborhood of 1 in Ξ (see [2 (m), Lemma 8]) and $\exp H_0\in\Xi_0$. Let σ and σ_ε be the distributions on Ξ_0 corresponding to T and T_ε respectively under [2 (m), Lemma 15]. It would be sufficient to verify that $\sigma=0$. It is obvious (see [2 (i), Cor. 2 of Theorem 1]) that σ_ε is the locally summable function

$$y\to T_\varepsilon(by)\quad(y\in\Xi_0)$$

on Ξ_0 and therefore

$$\sigma(g)=\lim_{\varepsilon\to0}\sigma_\varepsilon(g)=\varpi(\lambda)\int g(y)\,\Theta^+(by)\,dy-\lim_{\varepsilon\to0}\int g(y)\,\Psi^+(by;\,\nabla_G{}^*\circ D^{-1}V_\varepsilon)\,dy$$

for $g\in C_c^\infty(\Xi_0)$. (Here dy is the Haar measure on Ξ.) Let τ' be the distribution on $\mathfrak{z}_0$ which corresponds to σ under the process described in [2 (m), § 10]. Then by Lemma 50,

$$\tau'(f)=\varpi(\lambda)\int\xi_\mathfrak{z}(Z)f(Z)\,\Theta_b{}^+(\exp Z)\,dZ-\lim_{\varepsilon\to0}\int\xi_\mathfrak{z}(Z)\,\Psi^+(b\exp Z;\,\nabla_G{}^*\circ D^{-1}V_\varepsilon)\,dZ$$

for $f\in C_c^\infty(\mathfrak{z}_0)$ and it would be sufficient to verify that $\tau'=0$.

Put

$$S_\varepsilon{}^+(Z)=V_\varepsilon(b\exp Z)\,\Psi^+(b\exp Z)=V_\varepsilon(b\exp Z)\,S_b{}^+(Z)\quad(Z\in\mathfrak{z}_0)$$

in the notation of § 22.

LEMMA 66. *We have* (1)

$$\Psi^+(b\exp Z;\,\nabla_G{}^*\circ D^{-1}V_\varepsilon)=D_b(Z)^{-1}S_\varepsilon{}^+(Z;\partial(q_{\mathfrak{g}/\mathfrak{z}})\circ\nabla_\mathfrak{z}{}^*\circ\eta_\mathfrak{z}{}^{-1})$$

for $Z\in\mathfrak{z}_0$ in the notation of § 22 and Lemma 41.

Assuming this for a moment, we shall first finish the proof of Theorem 4. Put $\tau=\xi_\mathfrak{z}{}^{-1}D_b\tau'$ and recall that

$$\Theta_b{}^+(\exp Z)=D_b(Z)^{-1}T_b{}^+(Z)$$

by definition (see § 22). Hence if we write $q=q_{\mathfrak{g}/\mathfrak{z}}$, we get

$$\tau(f)=\varpi(\lambda)\,T_b{}^+(f)-\lim_{\varepsilon\to0}\int f(Z)\,S_\varepsilon{}^+(Z;\partial(q)\circ\nabla_\mathfrak{z}{}^*\circ\eta_\mathfrak{z}{}^{-1})\,dZ$$

(1) See footnote 1, p. 285.

for $f \in C_c^\infty(\mathfrak{z}_0)$. But since S_ε^+ is a C^∞ function on $\mathfrak{z}_0$ and $\eta_\mathfrak{z}$ is nowhere zero on its support, it is clear that

$$\int f(Z)\, S_\varepsilon^+(Z; \partial(q)\circ\nabla_\mathfrak{z}^*\circ\eta_\mathfrak{z}^{-1})\, dZ = \int \eta_\mathfrak{z}^{-1}(\nabla_\mathfrak{z}\circ\partial(q)^*)\, f\cdot S_\varepsilon^+\, dZ.$$

Now as $\varepsilon \to 0$ the right side obviously tends (see Lemma 4) to the limit

$$\text{p.v.}\int \eta_\mathfrak{z}^{-1}(\nabla_\mathfrak{z}\circ\partial(q)^*)\, f\cdot S_b^+\, dZ.$$

Hence
$$\tau(f) = \varpi(\lambda)\, T_b^+(f) - \text{p.v.}\int \eta_\mathfrak{z}^{-1}(\nabla_\mathfrak{z}\circ\partial(q)^*)\, f\cdot S_b^+\, dZ = 0$$

by Lemma 41. This proves Theorem 4.

§ 28. Proof of Lemma 66

We have still to prove Lemma 66. This requires some preparation. Fix a Cartan subgroup A of G and define ϖ_A, Δ_A as in [2 (m), § 20]. Also put $A' = A \cap G'$ as usual.

LEMMA 67. *The differential operator ∇_G^* on G' is invariant under G and*

$$f(h; \nabla_G^*) = (-1)^m \Delta_A(h)^{-1} f(h; \varpi_A\circ\Delta_A^2) \quad (h \in A')$$

for $f \in C^\infty(G')$.

Since ∇_G is invariant, it is obvious that the same holds for ∇_G^*. Fix $h_0 \in A'$ and an open and relatively compact neighborhood U of h_0 in A'. Then $V = U^G$ is an open neighborhood of h_0 in G. Put $\Delta = \Delta_A$ and let us use the notation of [2 (m), Lemma 41]. Then if $g \in C_c^\infty(V)$, it is clear that

$$\int g\,\nabla_G^* f\, dx = \int \nabla_G g\cdot f\, dx = c\int_A |\Delta(h)|^2\, dh \int_{G^*} g(h^{x^*}; \nabla_G)\, f(h^{x^*})\, dx^*$$
$$= c\int_{A\cap V} |\Delta(h)|^2\, dh \int_{G^*} g(x^*: h; \varpi_A\circ\Delta)\, f(x^*: h)\, dx^*$$

where $g(x^*: h) = g(h^{x^*})$ and $f(x^*: h) = f(h^{x^*})$ $(h \in A \cap V, x^* \in G^*)$. On the other hand

$$|\Delta|^2 = (-1)^{m+m(R)}\Delta^2$$

from Lemma 64 and it is obvious that

$$A \cap V = \bigcup_{s\in W_A} U^s$$

in the notation of [2 (m), § 20]. Hence $A \cap V$ is relatively compact in A'. Therefore (see [2 (f), Theorem 1]) there exists a compact set Ω^* in G^* such that $h^{x^*} \notin \operatorname{Supp} g$ for $h \in A \cap V$ and $x^* \in G^*$ unless $x^* \in \Omega^*$. Hence it is obvious that

$$\int g \nabla_G^* f\, dx = c(-1)^m \int_{A \cap V} |\Delta(h)|^2\, dh \int_{G^*} g(h^{x^*}) f(x^*: h; \Delta^{-1} \varpi_A \circ \Delta^2)\, dx^*.$$

On the other hand, there exists (see [2 (m), § 20]) a unique differential operator ∇' on $G_A = (A')^G$ such that

$$\beta(h^x; \nabla') = \beta(x: h; \Delta^{-1} \varpi_A \circ \Delta^2)$$

for $x \in G$ and $h \in A'$. Here β is any C^∞ function on G_A and $\beta(x:h) = \beta(h^x)$. Therefore

$$\int g \nabla_G^* f\, dx = c(-1)^m \int_A |\Delta(h)|^2\, dh \int_{G^*} g(h^{x^*}) f(h^{x^*}; \nabla')\, dx^* = (-1)^m \int g \nabla' f\, dx.$$

This shows that $\nabla_G^* = (-1)^m \nabla'$ on V and therefore

$$f(h_0; \nabla_G^*) = (-1)^m f(h_0; \Delta^{-1} \varpi_A \circ \Delta^2).$$

Thus the lemma is proved.

Now in Lemma 66, both sides are C^∞ functions on $\mathfrak{z}_0$ which are invariant under Ξ. Therefore it would be enough to show that they are equal on $\mathfrak{a}_0' = \mathfrak{a}' \cap \mathfrak{z}_0$ for any Cartan subalgebra $\mathfrak{a}$ of $\mathfrak{z}$. Fix $\mathfrak{a}$ and let A denote the corresponding Cartan subgroup of G. Since

$$V_\varepsilon(b \exp Z) \Psi^+(b \exp Z) = S_\varepsilon^+(Z) \quad (Z \in \mathfrak{z}_0)$$

and $D(a) = (-1)^m \Delta_A(a)^2$ $(a \in A)$, it follows from Lemma 67 that

$$\Psi^+(b \exp H; \nabla_G^* \circ D^{-1} V_\varepsilon) = \Delta_A(b \exp H)^{-1} S_\varepsilon^+(H; \partial(\varpi_A)) \quad (H \in \mathfrak{a}_0').$$

Let G_c denote, as before, the (connected) adjoint group of $\mathfrak{g}_c$ and Ξ_c the complex-analytic subgroup corresponding to $\operatorname{ad} \mathfrak{z}_c$. Select $y \in \Xi_c$ such that $\mathfrak{b}_c^y = \mathfrak{a}_c$. P being the set of positive roots of $(\mathfrak{g}, \mathfrak{b})$, we may assume that P^y is the set of all positive roots of $(\mathfrak{g}, \mathfrak{a})$. Then it is clear that

$$\Delta_A(b \exp H) = \pi_{\mathfrak{z}}^{\mathfrak{a}}(H) D_b(H) \quad (H \in \mathfrak{a}).$$

Hence $$D_b(H) \Psi^+(b \exp H; \nabla_G^* \circ D^{-1} V_\varepsilon) = S_\varepsilon^+(H; (\pi_{\mathfrak{z}}^{\mathfrak{a}})^{-1} \partial(\varpi_A)) \quad (H \in \mathfrak{a}_0').$$

Put $q = q_{\mathfrak{g}/\mathfrak{z}}$ and let $q_{\mathfrak{a}}$ denote the projection of q in $S(\mathfrak{a}_c)$ (see [2 (j), § 8]). Then

$$\varpi_A = \varpi^y = (\varpi_{\mathfrak{g}/\mathfrak{z}} \varpi_{\mathfrak{z}})^y = q_{\mathfrak{a}} \varpi_{\mathfrak{z}}^y$$

in the notation of § 18. Therefore since S_ε^+ is invariant under Ξ, it follows from the corollary of Lemma 2 and [2 (c), Theorem 1] that

$$S_\varepsilon^+(H; \partial(q)\circ\nabla_\mathfrak{z}^*\circ\eta_\mathfrak{z}^{-1}) = S_\varepsilon^+(H; (\pi_\mathfrak{z}^\mathfrak{a})^{-1}\partial(\varpi_A)) \quad (H \in \mathfrak{a}_0').$$

This proves Lemma 66.

§ 29. Some convergence questions

We use the notation introduced at the beginning of § 24. Put

$$\chi_\lambda = \chi_{b^*} \text{ for } \lambda = \log b^* \ (b^* \in B^*).$$

LEMMA 68. *Let p be a (complex-valued) polynomial function on $\mathfrak{F}$. Then we can choose an element $z \in \mathfrak{Z}$ with the following property. If $\mathfrak{F}^+$ is a connected component of $\mathfrak{F}'$ and $\lambda \in L \cap \mathrm{Cl}\,\mathfrak{F}^+$, then*

$$|p(\lambda)\,\Theta_{\lambda,\mathfrak{F}^+}(f)| \leqslant \sum_{1\leqslant i\leqslant r} c_i \int_{A_i} |F_{zf,i}|\, d_i a \quad (f \in C_c^\infty(G)).$$

Here the notation is the same as in Lemma 65.

Define $\mathfrak{c}$ and $\mathfrak{g}_1$ as in § 14 and let ω_1 be the Casimir operator corresponding to $\mathfrak{g}_1$ (see [2 (b), p. 140]). Then $\omega_1 \in \mathfrak{Z}$. Put $\omega_0 = \omega_1 - (H_1^2 + \dots + H_s^2)$ where $H_1, \dots, H_s$ is a base for $\mathfrak{c}$ over $\mathbf{R}$. Then a simple calculation shows that $\chi_\lambda(\omega_0) = \|\lambda\|^2 - c$ $(\lambda \in L)$, where c is a real number (independent of λ) and $\mu \to \|\mu\|$ $(\mu \in \mathfrak{F})$ is a Euclidean norm on $\mathfrak{F}$. Put $\omega = 1 + c + \omega_0$. Then $\chi_\lambda(\omega) = 1 + \|\lambda\|^2$ $(\lambda \in \mathfrak{F})$ and ω is a self-adjoint differential operator in $\mathfrak{Z}$. Now fix $\mathfrak{F}^+$ and write $\Theta_\lambda^+ = \Theta_{\lambda,\mathfrak{F}^+}$ $(\lambda \in L^+ = L \cap \mathrm{Cl}\,\mathfrak{F}^+)$. Then

$$\Theta_\lambda^+(\omega^q f) = \chi_\lambda(\omega^q)\,\Theta_\lambda^+(f) = (1 + \|\lambda\|^2)^q\,\Theta_\lambda^+(f)$$

for any integer $q \geqslant 0$ $(\lambda \in L^+,\ f \in C_c^\infty(G))$. Define C as in the corollary of Lemma 60. Then it follows from Lemma 63 that

$$|\Theta_\lambda^+(f)| \leqslant \sum_i c_i \int_{A_i} |\Delta_i(a)\,\Theta_\lambda^+(a)\,F_{f,i}(a)|\, d_i a \leqslant C \sum_i c_i \int_{A_i} |F_{f,i}|\, d_i a \quad (f \in C_c^\infty(G)).$$

Replacing f by $\omega^q f$, we get

$$(1 + \|\lambda\|^2)^q\,|\Theta_\lambda^+(f)| \leqslant \sum_{1\leqslant i\leqslant r} c_i \int |F_{zf,i}|\, d_i a$$

where $z = C\omega^q$. The assertion of the lemma is now obvious.

Now L is a closed additive subgroup of $\mathfrak{F}$. Let $d\lambda$ denote the Haar measure of L. It is clear from Lemmas 57 and 58 that for a fixed $f \in C_c^\infty(G)$, $\Theta_\lambda^+(f)$ $(\lambda \in L^+)$ is a measurable function of $\lambda \in L^+$.

COROLLARY 1. *For any $p \in S(\mathfrak{b}_c)$, we can choose $z \in \mathfrak{Z}$ such that*

$$\int_{L^+} |p(\lambda)\,\Theta_\lambda^+(f)|\,d\lambda \leqslant \sum_{1 \leqslant i \leqslant r} c_i \int_{A_i} |F_{zf,i}|\,d_i a$$

for all $f \in C_c^\infty(G)$.

We can obviously choose an integer $q \geqslant 0$ such that

$$\alpha = \int_L (1 + \|\lambda\|)^{-q}\,d\lambda < \infty.$$

On the other hand, by the above lemma, we can select $z_0 \in \mathfrak{Z}$ such that

$$(1 + \|\lambda\|)^q\,|p(\lambda)\,\Theta_\lambda^+(f)| \leqslant \sum_i c_i \int |F_{z_0 f,i}|\,d_i a$$

for $\lambda \in L^+$ and $f \in C_c^\infty(G)$. Hence we can take $z = \alpha z_0$.

Define Θ_λ for $\lambda \in L'$ as in § 24 and let us agree to the convention that $\varpi(\lambda)\,\Theta_\lambda = 0$ if $\varpi(\lambda) = 0$ $(\lambda \in L)$.

COROLLARY 2. *Put*

$$T(f) = \int_L \varpi(\lambda)\,\Theta_\lambda(f)\,d\lambda \quad (f \in C_c^\infty(G)).$$

Then T is an invariant distribution on G and, in fact, we can choose $z \in \mathfrak{Z}$ such that

$$|T(f)| \leqslant \sum_{1 \leqslant i \leqslant r} c_i \int_{A_i} |F_{zf,i}|\,d_i a$$

for all $f \in C_c^\infty(G)$.

The second statement follows from Corollary 1 above and the rest is obvious from [2 (f), Theorem 2].

Now assume B is compact. Then L is discrete and therefore

$$T(f) = \sum_{\lambda \in L} \varpi(\lambda)\,\Theta_\lambda(f).$$

Put $q = \frac{1}{2}\dim(G/K)$. Then q is an integer (see [2 (k), Lemma 18]) and we shall see in another paper that there exists a number $c > 0$ such that $(-1)^q c\,T$ is precisely the contribution of the discrete series (see [2 (a), § 5]) to the Plancherel formula of G (see [2 (h), Theorem 4]). The proof of this fact depends on Theorem 4.

§ 30. Appendix

Let $\mathfrak{g}$ be a reductive Lie algebra over $\mathbf{R}$ and Ω a completely invariant open subset of $\mathfrak{g}$.

LEMMA 69. *Let F_k $(k \geqslant 1)$ be a sequence of continuous and invariant functions on Ω. Then the following two conditions are equivalent.*

1) *For any Cartan subalgebra $\mathfrak{a}$ of $\mathfrak{g}$, F_k converges uniformly on every compact subset of $\mathfrak{a} \cap \Omega$.*

2) *F_k converges uniformly on every compact subset of Ω.*

Obviously 2) implies 1). So let us assume that 1) holds. Let Ω_0 be the set of all elements $X_0 \in \Omega$ with the following property. There exists an open neighborhood U of X_0 in Ω such that F_k converges uniformly on U. It would be sufficient to show that $\Omega_0 = \Omega$. Clearly Ω_0 is open and invariant. Therefore in view of [2 (l), Cor. 2 of Lemma 8], we have only to verify that every semisimple point of Ω lies in Ω_0.

Fix a semisimple element $H_0 \in \Omega$ and an open and relatively compact neighborhood U of H_0 in Ω. It would obviously be enough to show that F_k converges uniformly on $U' = U \cap \mathfrak{g}'$

Let $\mathfrak{a}_1, \dots, \mathfrak{a}_r$ be a complete set of Cartan subalgebras of $\mathfrak{g}$ no two of which are conjugate under G. Then $V_i = \mathrm{Cl}(\mathfrak{a}_i \cap U^G)$ is a compact subset of $\mathfrak{a}_i \cap \Omega$ (see [2 (k), Lemma 23]). Now fix $X \in U'$. Then $X = H^x$ where $x \in G$ and $H \in V_i$ for some i. Hence

$$F_j(X) - F_k(X) = F_j(H) - F_k(H) \quad (j, k \geqslant 1).$$

However, the sequence F_k converges uniformly on $\bigcup_{1 \leqslant i \leqslant r} V_i$ by 1) and so the required result follows immediately.

References

[1]. BOREL, A., & HARISH-CHANDRA, Arithmetic subgroups of algebraic groups. *Ann. of Math.*, 75 (1962), 485–535.

[2]. HARISH-CHANDRA, (a) Representations of semisimple Lie groups, VI. *Amer. J. Math.*, 78 (1956), 564–628.

(b) The characters of semisimple Lie groups. *Trans. Amer. Math. Soc.*, 83 (1956), 98–163.

(c) Differential operators on a semisimple Lie algebra. *Amer. J. Math.*, 79 (1957), 87–120.

(d) Fourier transforms on a semisimple Lie algebra, I. *Amer. J. Math.*, 79 (1957), 193–257.

(e) Fourier transforms on a semisimple Lie algebra, II. *Amer. J. Math.* 79 (1957), 653–686.
(f) A formula for semisimple Lie groups. *Amer. J. Math.*, 79 (1957), 733–760.
(g) Spherical functions on a semisimple Lie group, I. *Amer. J. Math.*, 80 (1958), 241–310.
(h) Invariant eigendistributions on semisimple Lie groups. *Bull. Amer. Math. Soc.*, 69 (1963), 117–123.
(i) Invariant distributions on Lie algebras. *Amer. J. Math.*, 86 (1964), 271–309.
(j) Invariant differential operators and distributions on a semisimple Lie algebra. *Amer. J. Math.*, 86 (1964), 534–564.
(k) Some results on an invariant integral on a semisimple Lie algebra. *Ann. of Math.*, 80 (1964), 551–593.
(l) Invariant eigendistributions on a semisimple Lie algebra. *Pub. Math. I.H.E.S.*, No. 27.
(m) Invariant eigendistributions on a semisimple Lie group. *Trans. Amer. Math. Soc.* To appear.

[3]. SCHWARTZ L., *Théorie des distributions*, II. Paris, Hermann, 1951.

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