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On a covering problem related to the centered Hardy–Littlewood maximal inequality

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Abstract. We find the exact value of the best possible constant associated with a covering problem on the real line.

1. Introduction

As is well known covering lemmas play an essential role in the study of the behavior of maximal operators, especially regarding weak type (1,1) bounds. Related to the uncentered Hardy–Littlewood maximal operator on $\mathbf{R}$ is the well-known covering lemma that says that given a finite collection $\mathcal{F}$ of intervals in $\mathbf{R}$ having union E we can extract two subcollections $\mathcal{F}_1$ and $\mathcal{F}_2$ such that (i) no interval is contained in both $\mathcal{F}_1$ and $\mathcal{F}_2$; (ii) the intervals in $\mathcal{F}_1$ are pairwise disjoint and the intervals in $\mathcal{F}_2$ are pairwise disjoint; and (iii) the union of all intervals in $\mathcal{F}_1 \cup \mathcal{F}_2$ is still E. This easily implies a weak (1,1) bound for the uncentered maximal operator with a constant 2 which actually is best possible and extends to more general measures (see [2]). However this does not give the best possible bound for the *centered* Hardy–Littlewood maximal operator (see [1]). The main point is that the above lemma involves only the *topology* of the real line whereas it has become clear that the best possible weak (1,1) constant C for the centered maximal operator depends heavily on the *geometry* of $\mathbf{R}$ (see [1], [3] and [6] for details on this problem). So it had to be expected that some kind of geometric covering problem should be hidden behind this operator. Indeed in [4] and [5] such a geometric covering problem of a very precise nature has been introduced and used to find the exact value of C which turns out to be $\frac{1}{12}\left(11+\sqrt{61}\right)=1.5675208...$ and so is much closer to $\frac{3}{2}$ than to 2.

The purpose of the present paper is to generalize the above mentioned covering problem, freeing it from the dependence on maximal operators but keeping its most basic ingredients, and to study the corresponding best possible behavior in an attempt to gain a deeper understanding of the geometry of the real line. Roughly speaking in this more general problem intervals of two types are given, let us call them *right intervals* and *left intervals.* Then we can cover certain places having at our disposal all possible nonempty intersections $I \cap J$ of a right interval I with a left interval J. We can break each such intersection $I \cap J$ into as many pieces as we want and translate each of them to cover places. However we are allowed to place any such piece that comes from $I \cap J$ only somewhere between the *left endpoint of the right interval* I and *the right endpoint of the left interval* J. This is our only essential restriction. Then the main point is to estimate the ratio of the total space covered in this way over the total interval mass involved in this covering. In a sense this will measure our capability to cover not just by single intervals but with appropriately displaced *intersections* of pairs of intervals. For example the most obvious such covering is that of the interval $[0,1]$ covered by the intersection of the right interval $[0,1]$ with the left interval $[0,1]$ with corresponding ratio $\frac{1}{2}$. However this does not give the best possible constant. What we are going to prove is that for any such covering the measure of the total space covered is at most $1/\sqrt{3} = 0.5774...$ times the total interval mass involved and moreover this is *best possible.*

To state our main theorem we consider two (countable) families $\mathcal{F}^+$ and $\mathcal{F}^-$ of *labeled* (not necessarily distinct) closed intervals in $\mathbf{R}$. That means that an interval I might appear more than once in say $\mathcal{F}^+$ and to distinguish these occurences we give them different labels. One way to do this formally is to consider $\mathcal{F}^\pm$ as sets of pairs of the form $I=(L,j)$, where $L \subseteq \mathbf{R}$ is a closed interval and j a positive integer called its *label.* However it would be more convenient, without causing any confusion, to just call the elements of $\mathcal{F}^\pm$ *labeled* intervals and in the notation $I \in \mathcal{F}^\pm$, I will mean both the labeled element of $\mathcal{F}^\pm$ and the actual underlying closed interval. Also when we say that two elements $I=(L,j)$ and $I'=(L',j')$ of say $\mathcal{F}^+$ are equal as *labeled* intervals we mean that the corresponding *pairs* are equal so the underlying intervals and the labels coincide. We will call the elements of $\mathcal{F}^+$ *right intervals* and the elements of $\mathcal{F}^-$ *left intervals.*

Next for any measurable $E \subseteq \mathbf{R}$ we will denote its Lebesgue measure by $|E|$ and for any interval $I \subseteq \mathbf{R}$ we will denote by $\mathfrak{l}(I)$ and $\mathfrak{r}(I)$ the left and right endpoints of I, respectively. Also if Λ is a finite collection of intervals we will denote the cardinality of Λ by $|\Lambda|$.

Then our main result is the following theorem.

Theorem 1. *Suppose that we are given two (countable) collections* $\mathcal{F}^+$ *and* $\mathcal{F}^-$ *of labeled closed intervals in* $\mathbf{R}$ *and moreover suppose that for each pair* $(I,J) \in \mathcal{F}^+ \times \mathcal{F}^-$ *we are given a measurable set* $A(I,J) \subseteq [\mathfrak{l}(I), \mathfrak{r}(J)] \subseteq \mathbf{R}$ *such that*

$|A(I,J)|\le|I\cap J|$. *Then we have*

$$\left|\bigcup_{(I,J)\in\mathcal{F}^+\times\mathcal{F}^-} A(I,J)\right|\le\frac{1}{\sqrt{3}}\left(\sum_{I\in\mathcal{F}^+}|I|+\sum_{J\in\mathcal{F}^-}|J|\right) \tag{1.1}$$

and this is best possible.

It is clear that for say finite $\mathcal{F}^+$ and $\mathcal{F}^-$ the *sum* of the lengths $|I\cap J|$ over all pairs (I,J) can be made much larger than the sum of the lengths of all labeled intervals. Hence no estimate similar to (1.1) can hold without having some restriction on the *location* of the sets $A(I,J)$. For example we could for any positive integer n take each of the $\mathcal{F}^+$ and $\mathcal{F}^-$ to consist of n copies of the interval $[0,1]$ and cover $[0,n^2]$ with the n^2 possible intersections of a right with a left labeled interval. This would give n^2 in the left-hand side and $2n$ for the sum in the right-hand side of (1.1). The condition $A(I,J)\subseteq[\mathfrak{l}(I),\mathfrak{r}(J)]$ we have imposed was suggested by the behavior of the centered Hardy–Littlewood maximal operator acting on linear combinations of Dirac deltas (see [4] and [5]). However other types of restrictions would lead to different covering problems with possibly different best constants.

The proof of Theorem 1 involves combinatorial-geometric as well as analytic arguments. In Section 2 the covering problem is discretized and an equivalent problem of more combinatorial nature is introduced. Then proving Theorem 1 is reduced into studying this new problem as stated in Theorem 2. The sharpness of the corresponding estimates is also shown in this section. The proof of Theorem 2 is then given in Sections 3–6.

2. Discretization of the covering problem

In proving Theorem 1 it clearly suffices, using an easy limiting argument, to assume that both families $\mathcal{F}^+$ and $\mathcal{F}^-$ are finite. Next each $A(I,J)$ can be approximated by a compact subset of it which in turn can be approximated by a finite union of closed intervals all of whose endpoints are *rational* numbers. Thus by appropriately perturbing all endpoints of the intervals in the families $\mathcal{F}^+$ and $\mathcal{F}^-$, so that the conditions of Theorem 1 still hold, and then scaling we may assume the following: (i) all endpoints of the intervals in the families $\mathcal{F}^+$ and $\mathcal{F}^-$ are *integers*; and (ii) for each pair of labeled intervals $I\in\mathcal{F}^+$ and $J\in\mathcal{F}^-$ the set $A(I,J)$ is a union of at most $|I\cap J|$ intervals of the form $[m-1,m]$ with m *integer*, each of which is contained in $[\mathfrak{l}(I),\,\mathfrak{r}(J)]$. This is our *discretization of the covering problem.*

Now for each integer m we set

$$\omega_m=[m-1,m], \tag{2.1}$$

call each such ω_m a *place* and in view of the above discretization we give the following definition.

Definition 1. A *covering model* is a triple $\mathcal{T}=(\mathcal{G}^+,\mathcal{G}^-,D)$, where $\mathcal{G}^+$ and $\mathcal{G}^-$ are two labeled families of intervals all of whose endpoints are integers and D is a union of certain places ω_m, together with a one-to-one mapping

$$\Delta \ni p \longmapsto (R_p, L_p, c(p)) \in \mathcal{G}^+ \times \mathcal{G}^- \times \Delta, \tag{2.2}$$

where $\Delta=\{p:\omega_p \subseteq D\}$ such that

$$\omega_p \subseteq [\mathfrak{l}(R_p), \mathfrak{r}(L_p)] \subseteq R_p \cup L_p \quad \text{and} \quad \omega_{c(p)} \subseteq R_p \cap L_p \tag{2.3}$$

whenever $p\in\Delta$. We will say that each such place ω_p is covered by $\omega_{c(p)}$ through the interaction of the right interval R_p with the left interval L_p and denote it by $\omega_p \mapsto (\omega_{c(p)}, R_p, L_p)$.

Remark. Note that D is required to be covered by itself (since we consider only pieces from the intersections $I\cap J\cap D$). This is not a severe restriction as will be explained later.

Given a covering model $\mathcal{T}$ as above we define for any $s\in\Delta$ the integers

$$h_s^+(\mathcal{T}) = |\{I \in \mathcal{G}^+ : \omega_s \subseteq I\}| \quad \text{and} \quad h_s^-(\mathcal{T}) = |\{J \in \mathcal{G}^- : \omega_s \subseteq J\}| \tag{2.4}$$

and

$$h_s(\mathcal{T}) = h_s^+(\mathcal{T}) + h_s^-(\mathcal{T}). \tag{2.5}$$

We will think that over each such place ω_s there exist $h_s(\mathcal{T})$ distinct intervals of length one which we will call *bricks*. Each brick over an ω_s will come from a certain *labeled* interval $I\in\mathcal{G}^+\cup\mathcal{G}^-$. We will say that this brick *corresponds* to I. Hence $h_s^+(\mathcal{T})$ of the bricks over ω_s correspond to right intervals and $h_s^-(\mathcal{T})$ correspond to left intervals. It is clear, by (2.3), that $h_s(\mathcal{T})\geq 1$ for every $s\in\Delta$.

Next we define

$$H(\mathcal{T}) = \sum_{p\in\Delta} h_p(\mathcal{T}), \quad m(\mathcal{T}) = |D| \quad \text{and} \quad \varrho(\mathcal{T}) = \frac{m(\mathcal{T})}{H(\mathcal{T})}. \tag{2.6}$$

Example. Let N be a positive integer, $I_1=[-N,1]$, $I_2=[0,1]$ and $J_1=...=J_N=[0,1]$, considered different as labeled intervals. Also let $D=[-N,1]$ and cover it by declaring that $[-m,-m+1]$ is covered by (I_1,J_m) and $[0,1]$ by (I_2,J_1). Then this produces a covering model with $m(\mathcal{T})=N+1$ and $H(\mathcal{T})=2N+2$ and so $\varrho(\mathcal{T})=\frac{1}{2}$.

Then we have the following theorem.

Theorem 2. *For any covering model $\mathcal{T}$ we have*

$$\sqrt{3}\, m(\mathcal{T}) \le H(\mathcal{T}) \tag{2.7}$$

and this is best possible.

To show that the above theorem is sharp we will now prove the following proposition.

Proposition 1. *For any covering model $\mathcal{T}$ there exists a covering model $\mathcal{T}'$ such that*

$$\varrho(\mathcal{T}') = \frac{2\varrho(\mathcal{T})+1}{3\varrho(\mathcal{T})+2}. \tag{2.8}$$

Proof. Let $\mathcal{T}=(\mathcal{G}^+,\mathcal{G}^-,D)$ and let $F=[\alpha,\beta]$ be any interval containing all intervals in $\mathcal{G}^+\cup\mathcal{G}^-$ (and hence also D). Then let $\mathcal{G}_1^+=\mathcal{G}^+\cup\{J\}$, $\mathcal{G}_1^-=\mathcal{G}^-\cup\{I\}$, where

$$I=[\alpha-H^-(\mathcal{T})-|D|,\beta] \quad \text{and} \quad J=[\alpha,\beta+H^+(\mathcal{T})], \tag{2.9}$$

and let

$$D_1=D\cup[(I\cup J)\setminus F]. \tag{2.10}$$

Then declaring that the interaction of I and J, over D, covers the $|D|$ part of $I\setminus F$, the interactions of I with the intervals of $\mathcal{G}^-$ cover the $H^-(\mathcal{T})$ part of $I\setminus F$, and the interactions of J with the intervals of $\mathcal{G}^+$ cover the interval $J\setminus F$ (of length $H^+(\mathcal{T})$), we conclude that the triple $\mathcal{T}'=(\mathcal{G}_1^+,\mathcal{G}_1^-,D_1)$ is a covering model. Now it is easy to see that $m(\mathcal{T}')=2|D|+H(\mathcal{T})=2m(\mathcal{T})+H(\mathcal{T})$ and $H(\mathcal{T}')=3m(\mathcal{T})+2H(\mathcal{T})$ which easily imply (2.8). This completes the proof. □

The above proposition allows us, starting from the obvious covering model $\mathcal{T}_0=(\{I\},\{I\},I)$, where $I=[0,1]$, to construct a sequence of covering models $\mathcal{T}_0$, $\mathcal{T}_1$, $\mathcal{T}_2,\dots$ such that $\varrho(\mathcal{T}_0)=\frac{1}{2}$ and $\varrho(\mathcal{T}_{k+1})=(2\varrho(\mathcal{T}_k)+1)/(3\varrho(\mathcal{T}_k)+2)$ for any $k\ge0$. It is then easy to see that $\varrho(\mathcal{T}_k)\to1/\sqrt{3}$, as $k\to\infty$, which proves that the estimate (2.7), if true, is sharp.

To show that Theorem 2 implies Theorem 1 let us consider the discretized version of it as explained in the beginning of this section and let

$$Q=\bigcup_{\substack{I\in\mathcal{F}^+\\ J\in\mathcal{F}^-}} A(I,J) \tag{2.11}$$

which is a union of certain places ω_m and then using the assumptions on each $A(I,J)$ we can define, for each such $A(I,J)$, a one-to-one mapping

$$\Delta(I,J)\ni p\longmapsto(I,J,c(p))\in\mathcal{F}^+\times\mathcal{F}^-\times\Delta(I,J), \tag{2.12}$$

where $\Delta(I,J)=\{p:\omega_p\subseteq A(I,J)\}$ such that $\omega_p\subseteq[\mathfrak{l}(I),\mathfrak{r}(J)]$, and $\omega_{c(p)}\subseteq I\cap J$ whenever $p\in\Delta(I,J)$. Of course there might exist p such that $\omega_p\subseteq Q$ appears in more than one of the above mappings. However, for each such p we can choose exactly one such pair (I,J) by defining a one-to-one mapping

$$\Sigma\ni p\longmapsto(R_p,L_p,c(p))\in\mathcal{F}^+\times\mathcal{F}^-\times Z, \tag{2.13}$$

where $\Sigma=\{p:\omega_p\subseteq Q\}$ and Z is the set of all integers satisfying the conditions (2.3). Let now D be the union of all places $\omega_p\subseteq Q$ together with all places ω_q such that $q=c(p)$ for some $p\in\Sigma$. Next for each $\omega_q\subseteq D\setminus Q$ choose a $p\in\Sigma$ with $q=c(p)$ and add $I=\omega_q$ in $\mathcal{F}^+$ as a new labeled interval declaring that the place ω_q is covered by ω_q (i.e. by itself) through the interaction of the new right interval $I=\omega_q$ with the left interval L_p. This produces the new family $\mathcal{G}^+\supseteq\mathcal{F}^+$ and by setting $\mathcal{G}^-=\mathcal{F}^-$ it is easy to see that $\mathcal{T}=(\mathcal{G}^+,\mathcal{G}^-,D)$ is a covering model with

$$m(\mathcal{T})=|Q|+|D\setminus Q| \quad\text{and}\quad H(\mathcal{T})\le\sum_{I\in\mathcal{F}^+}|I|+\sum_{J\in\mathcal{F}^-}|J|+|D\setminus Q|. \tag{2.14}$$

Then the estimate (2.7) in Theorem 2 applied to this covering model easily implies (1.1) in Theorem 1.

To show the sharpness of Theorem 1 it suffices to remark that given any covering model $\mathcal{T}=(\mathcal{G}^+,\mathcal{G}^-,D)$ as in Definition 1 we can, for each pair $(I,J)\in\mathcal{F}^+\times\mathcal{F}^-$, define the set $A(I,J)$ as the union of all ω_p such that $R_p=I$ and $L_p=J$ and that these sets satisfy all the conditions in Theorem 1. Applying this remark to each one of the covering models $\mathcal{T}_0$, $\mathcal{T}_1$, $\mathcal{T}_2,\ldots$ defined after the proof of Proposition 1 we can easily conclude that (1.1) is sharp.

Hence to complete the proof of Theorem 1 it suffices to prove (2.7) for every covering model.

Consider now such a covering model $\mathcal{T}$. By a translation we may assume that all intervals in $\mathcal{G}^+\cup\mathcal{G}^-$ are contained in $[0,N]$, where N is a fixed large integer.

Next we may assume that $\mathcal{T}$ has the property that $\varrho(\mathcal{T})$ is *maximum* among all covering models all of whose intervals are contained in $[0,N]$. Indeed although there can be infinitely many such covering models (since intervals may appear as many times as we wish) observing that we can freely remove any interval from $\mathcal{G}^+\cup\mathcal{G}^-$ that does not meet D, we conclude that essentially there are only finitely many such covering models $\mathcal{T}'$ with $H(\mathcal{T}')\leq 2N$. Since any covering model $\mathcal{T}'$, contained in $[0,N]$, with $H(\mathcal{T}')>2N$ will have $\varrho(\mathcal{T}')<\frac{1}{2}$, a $\mathcal{T}$ with $\varrho(\mathcal{T})$ maximum can be found.

From now on we will fix such a $\mathcal{T}$ and to complete the proof of Theorem 2 it will suffice to show that $3\varrho(\mathcal{T})^2\leq 1$. We may assume that $\varrho(\mathcal{T})>\frac{4}{7}$ otherwise there is nothing to prove.

For any integer $t\geq 1$ we define the sets

$$E_t=E_t(\mathcal{T})=\{\omega_p\subseteq D : h_p(\mathcal{T})=t\} \quad \text{and} \quad E_t^*=\bigcup_{r\geq t} E_r. \tag{2.15}$$

We have the following lemma.

Lemma 1. *Let $\omega_p\subseteq D$. Then ω_p can cover at most $h_p^+(\mathcal{T})h_p^-(\mathcal{T})$ intervals $\omega_q\subseteq D$, but at most $h_p(\mathcal{T})-1$ intervals $\omega_q\in E_1$.*

Proof. Let $h_p^+(\mathcal{T})=a$ and $h_p^-(\mathcal{T})=b$. Then exactly ab pairs of a right interval with a left interval that contain ω_p in their intersection can be formed proving the first statement. For the other intervals among all the right intervals containing ω_p let I be one with minimum left endpoint, and among all the left intervals containing ω_p let J be one with maximum right endpoint (there might be more than one such pair of intervals). Then for any pair $I'\in\mathcal{G}^+$ and $J'\in\mathcal{G}^-$, such that $\omega_p\subseteq I'\cap J'$ and with I' different from I and J' different from J (as always as labeled intervals) we have $[\mathfrak{l}(I'),\mathfrak{r}(J')]\subseteq[\mathfrak{l}(I),\mathfrak{r}(J)]$. Thus it is easy to see that any ω_q covered by the interaction of I' and J' will have $h_q\geq 2$. Hence the possible $\omega_q\in E_1$ covered by ω_p can come only from the $b-1$ interactions of I with the left intervals containing ω_p except for J, from the $a-1$ interactions of J with the right intervals containing ω_p except for I, and from the interaction of I with J. Therefore they are at most $a+b-1=h_p(\mathcal{T})-1$. $\square$

Remark. This lemma, in particular, implies that an ω_p in E_1 does not cover any place, an ω_p in E_2 covers at most one place (and this can happen only if $h_p^+(\mathcal{T})=h_p^-(\mathcal{T})=1$), and an ω_p in E_3 covers at most two places.

3. Rearranging the bricks

An important device used to prove Theorem 2 will be to construct another arrangement (by displacing certain bricks) of the $H(\mathcal{T})$ bricks over D, different from the arrangement that the families $\mathcal{G}^+$ and $\mathcal{G}^-$ determine, that would eliminate the initial E_1 without affecting E_2. This will be done in three steps.

For any arrangement $\mathcal{U}$ of the collection of the $H(\mathcal{T})$ bricks that lie over D we let $h_p(\mathcal{U})$ denote the number of bricks that lie over ω_p in the arrangement $\mathcal{U}$, and define the set

$$E_t(\mathcal{U}) = \{\omega_p \subseteq D : h_p(\mathcal{U}) = t\}. \tag{3.1}$$

Also for any $\mathcal{M} \subseteq \{\omega_1, \dots, \omega_N\}$, $\bigcup \mathcal{M} \subseteq D$ will as usual denote the union of all elements of $\mathcal{M}$.

To proceed further our first step is to construct the arrangement $\mathcal{A}_1$ of the bricks by the following rule: Whenever ω_p is in E_1 there is a unique interval I_p from $\mathcal{G}^+ \cup \mathcal{G}^-$ that contains it. Then clearly ω_p will be covered by some ω_q through the interaction of this I_p with an interval J of the opposite direction. We move the brick that lies over ω_q and corresponds to J from ω_q to over ω_p. Doing this for every $\omega_p \in E_1$ we get $\mathcal{A}_1$ and obtain the following lemma.

Lemma 2. *In the above construction no brick has to be moved from its initial place more than once. The resulting arrangement satisfies $h_s(\mathcal{A}_1) \geq 1$ for every $s \in \Delta$ and moreover $E_1 \subseteq E_2(\mathcal{A}_1)$.*

Proof. To have to move a brick more than once there must exist an ω_q, two distinct (as labeled) intervals I and I' of the same direction and an interval J of the opposite direction, so that ω_q covers something in $I \cap \bigcup E_1$ through (I, J) and something in $I' \cap \bigcup E_1$ through (I', J) and so in particular $\omega_q \in I \cap I' \cap J$. Suppose that I and I' are right intervals and $\mathfrak{l}(I) \leq \mathfrak{l}(I')$. Then since both places so covered must be in $[\mathfrak{l}(I), \mathfrak{r}(J)]$ only places in $I \backslash I'$ can be contained in $I' \cap \bigcup E_1$ and be covered by ω_q through (I', J) since it is easy to see that $\chi_I + \chi_{I'} + \chi_J \geq 2$ on $I' \cap (-\infty, \mathfrak{r}(J)]$. This is a contradiction since the place covered by (ω_q, I', J) is contained in I'. A similar contradiction follows if I and I' are left intervals. Hence no brick has to be moved from its initial place more than once and so the construction can be carried out to doubly cover all E_1 which gives $E_1 \subseteq E_2(\mathcal{A}_1)$. Moreover Lemma 1 implies that from each ω_q at most $h_q(\mathcal{T}) - 1$ bricks can be moved. Hence at least one is not moved proving that $h_s(\mathcal{A}_1) \geq 1$ for every $s \in \Delta$. $\square$

The second step is to consider the set $E_1(\mathcal{A}_1) \cap E_2$. If nonempty then to every $\omega_p \in E_1(\mathcal{A}_1) \cap E_2$ we can uniquely associate two labeled intervals called I_p and J_p of opposite directions such that $\omega_p \subseteq I_p \cap J_p$ and that the brick over it that corresponds to J_p has been moved to cover something in $I_p \cap \bigcup E_1$. Clearly ω_p is not contained in any other labeled interval and $J_p \cap (-\infty, p] \subseteq I_p$ since J_p cannot pass through the $\omega_a \subseteq I_p \cap \bigcup E_1$ that ω_p covers.

Fix now $\omega_p \in E_1(\mathcal{A}_1) \cap E_2$ and for definiteness suppose that I_p is a right interval and thus J_p a left interval. Then it is easy to see that ω_p is covered either from $(\omega_{c(p)}, I_p, F)$, where F is a left interval (which could be J_p) or from $(\omega_{c(p)}, G, J_p)$, where $G \neq I_p$ is a right interval. We consider each of these cases separately:

Case 1. In the first case we easily get $c(p)>a$. Moreover we see that the brick over $\omega_{c(p)}$ that corresponds to F has not been moved in the first step (even if F has places from E_1), since (I_p, F) covers an E_2 and not an E_1. Also as in the proof of the previous lemma no interaction $(\omega_{c(p)}, I, F)$ with I different from I_p, which must have $\mathfrak{l}(I)>\mathfrak{l}(I_p)$ since I cannot pass through ω_a, can cover in $I\cap\bigcup E_1$. We thus move the brick over $\omega_{c(p)}$ that corresponds to F from $\omega_{c(p)}$ to over ω_p.

Case 2. In the second case G cannot pass through ω_p nor the $\omega_a\in E_1$ that ω_p covers. Since $\omega_p\subseteq[\mathfrak{l}(G), \mathfrak{r}(J_p)]\cap\bigcup E_2$ this easily implies that: (i) G lies strictly between ω_a and ω_p and so $h_{c(p)}\geq 3$ (since $\omega_{c(p)}$ is contained in I_p and in $G\cap J_p$) and $c(p)<p$; (ii) except for J_p no other left interval J with $\mathfrak{r}(J)\geq p$ can intersect G; and so (iii) $\omega_{c(p)}$ can cover an E_1 only through the interaction of I_p with some left interval (possibly J_p) or the interaction of J_p with a right interval *different* from G. Hence the brick over $\omega_{c(p)}$ that corresponds to G has not been moved in the first step. We thus move the brick over $\omega_{c(p)}$ that corresponds to G from $\omega_{c(p)}$ to over ω_p.

We work in an analogous symmetrical way if I_p is a left interval, and thus J_p a right interval.

In this way starting from $\mathcal{A}_1$ we apply the above moves for every $\omega_p\in E_1(\mathcal{A}_1)\cap E_2$, and thus obtaining the arrangement $\mathcal{A}_2$.

Lemma 3. *In the combined application of the above two steps no brick has to be moved from its initial place more than once. The resulting arrangement* $\mathcal{A}_2$ *is thus well defined and moreover satisfies* $h_s(\mathcal{A}_2)\geq 1$ *for every* $s\in\Delta$ *and* $E_1\cup(E_1(\mathcal{A}_1)\cap E_2)\subseteq E_2(\mathcal{A}_2)$.

Proof. Suppose that the brick that lies above $\omega_s\subseteq D$ and corresponds to the left interval B (that contains ω_s) has to be moved more than once in the above process. The argument is similar if B is a right interval.

The two cases considered above imply that no brick that has been moved in the first step to form $\mathcal{A}_1$ will be used in the second step. Hence there should exist two right intervals I and I' such that both (ω_s, I, B) and (ω_s, I', B) cover $\omega_q, \omega_{q'}\in E_1(\mathcal{A}_1)\cap E_2$, respectively. Supposing that $\mathfrak{l}(I)\leq\mathfrak{l}(I')$ we conclude that over each place between $\mathfrak{l}(I')$ and s at least two labeled intervals of the *same* direction (that is I and I') pass and thus no such place can belong to $E_1(\mathcal{A}_1)\cap E_2$ (because any place in that set is contained in exactly two intervals of *opposite* directions). This implies that $q'>s$, and so $\omega_{q'}\subseteq B$. If $B=I_{q'}$, then by our construction the brick over ω_s that corresponds to I' and *not to* B would have to be moved in relation to the covering $\omega_{q'}\mapsto(\omega_s, I', B)$. Hence we must have $B=J_{q'}$. Examining our construction again we conclude that in the covering $\omega_{q'}\mapsto(\omega_s, I', J_{q'})$ the brick corresponding to $B=J_{q'}$ would be moved only if we were in Case 1 that is only if

$I'=I_{q'}$. Hence $\omega_{q'}\subseteq I'\cap B$ and $(\omega_{q'},I',B)$ must cover some place in $I'\cap\bigcup E_1$. But this is a contradiction since $\chi_I+\chi_{I'}+\chi_B\geq 2$ on $I'\cap(-\infty,\mathfrak{r}(B)]$. This proves that the construction of $\mathcal{A}_2$ is well defined and clearly the produced arrangement $\mathcal{A}_2$ satisfies $E_1\cup(E_1(\mathcal{A}_1)\cap E_2)\subseteq E_2(\mathcal{A}_2)$.

To prove that $h_s(\mathcal{A}_2)\geq 1$ for every $s\in\Delta$, fix any $\omega_s\subseteq D$ that covers at least a place (and so is contained in at least one right and at least one left interval), let I be one right interval with smallest left endpoint among all right intervals that contain ω_s and let J be one left interval with largest right endpoint among all left intervals that contain ω_s. Then if I' is any right interval different from I and J' is any left interval different from J both containing ω_s we get as before that the interval $[\mathfrak{l}(I'),\mathfrak{r}(J')]$ cannot contain any places in $E_1\cup(E_1(\mathcal{A}_1)\cap E_2)$ and so (ω_s,I',J') will never appear in our construction. Hence bricks will be moved away from ω_s to places in $E_1\cup(E_1(\mathcal{A}_1)\cap E_2)$ only through the involvement of I or J (or both). But then a reasoning similar to that in Lemma 1 shows that at most $h_s(\mathcal{T})-1$ bricks could be moved away from ω_s in the combined application of these two steps, and this completes the proof. □

The third step is to consider the set $E_1(\mathcal{A}_2)\cap E_2$. If nonempty fix any $\omega_q\in E_1(\mathcal{A}_2)\cap E_2$. Then there is exactly one right interval A and exactly one left interval B such that $\omega_q\in A\cap B$ and moreover (ω_q,A,B) covers an $\omega_p\in E_1(\mathcal{A}_1)\cap E_2$ that is $q=c(p)$. Clearly this covering must follow the pattern of Case 1 above since in Case 2 we have seen that $h_{c(p)}\geq 3$. Hence either A or B must be I_p and the other must be an interval of the opposite direction (which could be J_p).

We assume that I_p is a right interval and thus equals A, the construction being symmetrical if I_p is a left interval. As before (ω_p,I_p,J_p) covers an $\omega_a\subseteq I_p\cap\bigcup E_1$. Then let I_q be $A=I_p$ and let J_q denote the other interval B (necessarily a left interval) that contains ω_q. Here also it is clear that ω_q is covered either from $(\omega_{c(q)},I_q,F)$ for a left interval F (that might be equal to J_q) or from $(\omega_{c(q)},G,J_q)$, where G is a right interval different from I_q. We consider each of these cases separately:

Case 1. In the first case arguing as in the proof of Lemma 3 we conclude that given any right interval I' containing $\omega_{c(p)}$ (that clearly must have $\mathfrak{l}(I')>\mathfrak{l}(I_q)$), $(\omega_{c(p)},I',F)$ cannot cover any place in $E_1\cup(E_1(\mathcal{A}_1)\cap E_2)$. Since $(\omega_{c(q)},I_q,F)$ covers $\omega_q\notin E_1\cup(E_1(\mathcal{A}_1)\cap E_2)\subseteq E_2(\mathcal{A}_2)$ the brick over $\omega_{c(q)}$ that corresponds to F has not been moved in the first two steps. We thus move this brick from $\omega_{c(q)}$ to over ω_q.

Case 2. In the second case as before and since $\omega_q\in E_2$ we have that, except for J_q no other left interval J with $\mathfrak{r}(J)\geq q$ can intersect G and so $\omega_{c(q)}$ can cover an E_1 only through the interaction of I_q with some left interval or the interaction of J_q with a right interval different from G. However here we must examine whether

$\omega_{c(q)}$ with G could cover something in $E_1(\mathcal{A}_1)\cap E_2$. If this happened and for some left interval J, $(\omega_{c(q)}, G, J)$ covers ω_a in $E_1(\mathcal{A}_1)\cap E_2$, then obviously $J\neq J_q$ and so $\mathfrak{r}(J)<q$. But since also $\mathfrak{l}(G)>\mathfrak{l}(I_q)$ as in the proof of Lemma 3, $G\cap(-\infty, c(q)]$ does not contain any places from $E_1\cup(E_1(\mathcal{A}_1)\cap E_2)$ and moreover $\chi_G+\chi_J+\chi_{I_q}+\chi_{J_q}\geq 3$ on $[c(q), \mathfrak{r}(J)]$. Since ω_a must be contained in $[\mathfrak{l}(G), \mathfrak{r}(J)]$ this is a contradiction. Hence the brick over $\omega_{c(q)}$ that corresponds to G has not been moved in the first two steps. We thus move this brick from $\omega_{c(q)}$ to over ω_q.

Lemma 4. *In the combined application of the above three steps no brick has to be moved from its initial place more than once. The resulting arrangement $\mathcal{A}$ is thus well defined and satisfies $h_s(\mathcal{A})\geq 1$ for every $s\in\Delta$. In addition $\mathcal{A}$ satisfies*

$$E_1(\mathcal{A})\cap(E_1\cup E_2)=\emptyset. \tag{3.2}$$

Proof. If the first statement does not hold then arguing as in the proof of Lemma 3 and observing that no brick moved in the third step was ever used in the first two steps we may suppose that for some $\omega_s\subseteq D$ there is a left interval B and two right intervals I and I' all containing ω_s such that $\mathfrak{l}(I)\leq\mathfrak{l}(I')$, both (ω_s, I, B) and (ω_s, I', B) cover $\omega_q, \omega_{q'}\in E_1(\mathcal{A}_2)\cap E_2$, respectively, and such that the brick that lies above ω_s and corresponds to B has to be moved in both these coverings. Similarly we conclude that no place between $\mathfrak{l}(I')$ and s can belong to $E_1(\mathcal{A}_2)\cap E_2$. Thus $q'>s$ and $\omega_{q'}\subseteq B$ and again we must have $I'=I_{q'}$ and $B=J_{q'}$. But examining our construction we conclude that $(\omega_{q'}, I', B)$ must cover an $\omega_{p'}\in E_1(\mathcal{A}_1)\cap E_2$ and that $I_{p'}$ is also equal to I'. But this $\omega_{p'}$ lies between $\mathfrak{l}(I')$ and $\mathfrak{r}(B)$ and there is a left interval F such that $\omega_{p'}\subseteq F$ and so $\mathfrak{l}(F)\leq\mathfrak{r}(B)$ and $(\omega_{p'}, I', F)$ covers something in $I'\cap\bigcup E_1$ which clearly contradicts the easy to verify $\chi_I+\chi_{I'}+\chi_B+\chi_F\geq 2$ on $I'\cap(-\infty, \mathfrak{r}(F)]$. This proves that the construction of $\mathcal{A}$ is well defined.

The proof that $h_s(\mathcal{A})\geq 1$ for every $s\in\Delta$ is similar to the proof of the corresponding statement in Lemma 3.

To prove (3.2), since clearly $E_1(\mathcal{A})\cap E_1=\emptyset$, suppose that $\omega_p\in E_1(\mathcal{A})\cap E_2$. Then in view of the above construction, ω_p must cover an $\omega_a\in E_1(\mathcal{A}_2)\cap E_2$ that in turn must cover an $\omega_b\in E_1(\mathcal{A}_1)\cap E_2$ which covers an $\omega_c\in E_1(\mathcal{A}_1)\cap E_1$. It is clear that p, a, b and c are distinct, and, by the remark following Lemma 1, that the covering potential of the set $W=\{\omega_p,\ \omega_a,\ \omega_b,\ \omega_c\}$ is exhausted, meaning that any place covered by an element in W is also contained in W. Hence taking $D_1=D\setminus W$ the triple $\mathcal{T}_1=(\mathcal{G}^+, \mathcal{G}^-, D_1)$ is a covering model with $m(\mathcal{T}_1)=m(\mathcal{T})-4$ and $H(\mathcal{T}_1)=H(\mathcal{T})-7$. Then $\varrho(\mathcal{T}_1)>\varrho(\mathcal{T})$ since $\varrho(\mathcal{T})>\frac{4}{7}$ contradicting the maximality of $\varrho(\mathcal{T})$. This completes the proof. □

Remark. In the above proof the following easy to show fact has been used:

Given the real numbers $0<x<X$ and $0<y<Y$ we have $(X-x)/(Y-y)>X/Y$ if and only if $X/Y>x/y$. This will be used in several places throughout this paper.

The arrangement $\mathcal{A}$ has the following additional properties.

Lemma 5. (i) *For any $\omega_p \in E_1(\mathcal{A})\cap E_3$ both moved bricks over it have been moved in the first step to cover two places in E_1.*

(ii) *There exists no $\omega_p \in E_1(\mathcal{A})\cap E_4^*$ such that either $h_p^+(\mathcal{T})$ or $h_p^-(\mathcal{T})$ is equal to 1.*

Proof. (i) Suppose that one of these bricks has been moved to over ω_{a_1} in E_1 and the other has been moved to over ω_b in $E_1(\mathcal{A}_1)\cap E_2$ from which one brick has been moved to over ω_{a_2} in E_1. Then p, a_1, a_2 and b are clearly distinct and the covering potential of the set $\{\omega_p, \omega_{a_1}, \omega_{a_2}, \omega_b\}$ has been exhausted. Therefore removing $\{\omega_p, \omega_{a_1}, \omega_{a_2}, \omega_b\}$ from D will produce a covering model and we will have reduced $m(\mathcal{T})$ by 4 and $H(\mathcal{T})$ by 7. This leads to a contradiction as in the proof of Lemma 4. From the nine possibilities for the two moved bricks over ω_p all but the one stated in the lemma lead to similar contradictions.

(ii) Suppose that $\omega_p \in E_1(\mathcal{A})$ is such that $h_p^+(\mathcal{T})=1$ and $h_p^-(\mathcal{T})=d\geq 3$. Then ω_p can cover at most d places and since d bricks have been moved from it, it covers exactly d places in E_1 or in an E_2 that covers an E_1 or in an E_2 that covers an E_2 that covers an E_1. Consider the set A that contains ω_p and all E_1's and E_2's involved if all these paths are followed. It is easy to see that A contains exactly d places from E_1 and that the covering potential of the set A is exhausted. Hence removing A from D will produce a covering model and we will have reduced $m(\mathcal{T})$ by $1+d+x$ and $H(\mathcal{T})$ by $1+2d+2x$, where x is the number of E_2's in A. This leads to a similar contradiction since $d+x\geq 3$. □

Having produced the arrangement $\mathcal{A}$ we now consider the set

$$K=E_1(\mathcal{A})\cap E_3. \tag{3.3}$$

For every $\omega_p \in K$ let

$$\mathcal{C}_p=\{\omega_q : q=c^{[m]}(p) \text{ for some } m\geq 0\}, \tag{3.4}$$

where $c^{[m+1]}=c\circ c^{[m]}$ and $c^{[0]}(p)=p$, denote its covering chain (that means ω_p is covered by $\omega_{c(p)}$ which is covered by $\omega_{c^{[2]}(p)}$ etc.) and let

$$\mathcal{B}=\bigcup_{\omega_p\in K} \mathcal{C}_p. \tag{3.5}$$

Clearly $\mathcal{B}$ is a finite set of ω_q's having the property that $c(\mathcal{B})\subseteq\mathcal{B}$. This in particular implies that $(\mathcal{G}^+,\mathcal{G}^-,\bigcup\mathcal{B})$ is a covering model.

Next for any positive integers a and b consider the sets

$$\Lambda_{ab}=\mathcal{B}\cap E_a\cap E_b(\mathcal{A}). \tag{3.6}$$

From the above lemmas we easily obtain that

$$\mathcal{B}=\bigcup_{1\le b\le a}\Lambda_{ab} \quad \text{and} \quad \Lambda_{11}=\Lambda_{21}=\emptyset. \tag{3.7}$$

Now we can prove the following result.

Proposition 2. *Letting $\sum'$ denote summation over all pairs (a,b) of integers such that either $1\le b\le a$ and $a\ge 3$ or $a=b=2$ we have*

$$\varrho(\mathcal{T})\le\frac{\sum'(a-b+1)|\Lambda_{ab}|}{\sum'(2a-b)|\Lambda_{ab}|}. \tag{3.8}$$

In particular $\varrho(\mathcal{T})\le\frac{3}{5}$.

Proof. Since $\mathcal{A}$ has been obtained by moving certain bricks from certain places of D to certain other places of D we easily have

$$H(\mathcal{T})=\sum_{\omega_p\in D}h_p(\mathcal{A}). \tag{3.9}$$

From the construction of $\mathcal{A}$ it easily follows that all moved bricks have gone to different ω_q's and have put (or kept) these ω_q's into $E_2(\mathcal{A})$. Also, since bricks have been moved to places either in E_1 or in E_2 that covers an E_1 or in E_2 that covers an E_2 that covers an E_1, we conclude that no bricks have moved into $\mathcal{B}\cup E_1(\mathcal{A})$.

Next exactly $\sum'(a-b)|\Lambda_{ab}|$ bricks have been moved away from $\mathcal{B}$ and clearly $|\mathcal{B}|=\sum'|\Lambda_{ab}|$. Consider now any $\omega_s\in E_1(\mathcal{A})\setminus\mathcal{B}$. Then $d\ge 3$ bricks have been moved from it (since $K\subseteq\mathcal{B}$). Hence this and the places its bricks have gone to contribute $d+1$ in $m(\mathcal{T})$ and $2d+1$ in $H(\mathcal{T})$ and since $(d+1)/(2d+1)\le\frac{4}{7}<\varrho(\mathcal{T})$ they can be ignored in the upper estimation of $\varrho(\mathcal{T})$. Any other ω_r, except for the ones in $\mathcal{B}\cup E_1(\mathcal{A})$ and the ones in $E_2(\mathcal{A})$ that have received bricks will contribute 1 in $m(\mathcal{T})$ and at least 2 in $H(\mathcal{T})$ and thus can also be ignored. Hence only $\mathcal{B}$ together with all places where bricks from $\mathcal{B}$ have gone to can play a role and there are $\sum'(a-b)|\Lambda_{ab}|$ such places in $E_2(\mathcal{A})$ outside $\mathcal{B}$. Also each $\omega_p\in\Lambda_{ab}$ will contribute b in $H(\mathcal{T})$. Hence

$$\varrho(\mathcal{T})\le\frac{\sum'(a-b)|\Lambda_{ab}|+\sum'|\Lambda_{ab}|}{\sum'2(a-b)|\Lambda_{ab}|+\sum'b|\Lambda_{ab}|} \tag{3.10}$$

and this proves (3.8). In order to prove that $\varrho(\mathcal{T})\leq\frac{3}{5}$ it suffices to observe that $(a-b+1)/(2a-b)\leq\frac{3}{5}$ whenever $1\leq b\leq a$ and $a\geq 3$ or $a=b=2$ (with equality if and only if $a=3$ and $b=1$). This completes the proof. □

To use our basic estimate (3.8) efficiently further considerations are needed. What we want to do is to construct from our initial covering model $\mathcal{T}$ another covering model which will allow us to use the maximality assumption of $\varrho(\mathcal{T})$. In the next section we will start our construction by appropriately choosing certain subintervals of some elements of our initial family $\mathcal{G}^+\cup\mathcal{G}^-$.

4. Selection of certain intervals

Consider first the collection $\mathcal{P}_1$ of all labeled intervals $I\in\mathcal{G}^+\cup\mathcal{G}^-$ such that

$$I\cap\bigcup(E_1\cup(E_1(\mathcal{A}_1)\cap E_2)\cup(E_1(\mathcal{A}_2)\cap E_2))\neq\emptyset \tag{4.1}$$

and let

$$S=\bigcup(E_1\cup(E_1(\mathcal{A}_1)\cap E_2)\cup(E_1(\mathcal{A}_2)\cap E_2))\subseteq D. \tag{4.2}$$

To every $\omega_p\in K$ we associate the exactly three (labeled) intervals $I_p, J_p, F_p\in \mathcal{G}^+\cup\mathcal{G}^-$ that contain ω_p, denoted so that J_p and F_p have the same direction, I_p has opposite direction (so $I_p\in\mathcal{G}^+$ and $J_p, F_p\in\mathcal{G}^-$ or the other way around) and moreover $F_p\subseteq I_p\cup J_p$. This is possible since first of all if all three intervals were of the same direction then ω_p could not cover any place at all. Then using Lemma 5 and the reasoning in the proof of Lemma 2 it is easy to conclude that ω_p covers at least one ω_a in $I_p\cap\bigcup E_1$ and so $I_p\in\mathcal{P}_1$ and neither J_p nor F_p can pass through this ω_a. Since also all three intervals contain ω_p, one of the two intervals of the same direction which we call F_p is contained in the union of I_p with the other. It may happen that also $J_p\subseteq I_p\cup F_p$. In this case and if these two (underlying to the labeled F_p and J_p) intervals are not equal we choose them so that $F_p\subseteq J_p$. In the special case where F_p and J_p coincide as intervals (but not as labeled elements of $\mathcal{G}^+$ or $\mathcal{G}^-$) we pick one of them to be called not only J_p but also J_q *for any other* $\omega_q\in J_p\cap\bigcup K$ (clearly for every such ω_q the two associated intervals of the same direction also coincide). In this way the mapping $K\ni\omega_p\mapsto(I_p, J_p, F_p)$ is well defined and it is easy to see that we always have $J_q=J_p$ for any other $\omega_q\in K$ with $\omega_q\subseteq F_p\cap J_p$.

Noticing that each ω_p covers exactly two places in E_1 we let ω_{l_p} denote the $I_p\cap\bigcup E_1$ place covered by ω_p that is closest to ω_p.

We let $\mathcal{P}$ denote the collection that consists of all labeled intervals in $\mathcal{P}_1$ together with all intervals of the form J_p for some $\omega_p\in K$.

Now for each $I \in \mathcal{P}$ we define the subinterval I^* of I as follows.

If a labeled right interval $I \in \mathcal{G}^+ \cap \mathcal{P}_1$ is not of the form J_p for any $\omega_p \in K$, let

$$I^* = I \cap [\min(S \cap I), +\infty) \tag{4.3}$$

and similarly if a labeled left interval $J \in \mathcal{G}^- \cap \mathcal{P}_1$ is not of the form J_p for any $\omega_p \in K$, let

$$J^* = J \cap (-\infty, \max(S \cap I)]. \tag{4.4}$$

Next if $I \in \mathcal{G}^+ \cap \mathcal{P}$ is of the form J_p for some $\omega_p \in K$, we choose the smallest possible such p and let

$$I^* = I \cap [\min(S \cap I), +\infty) \cap [p-1, +\infty), \tag{4.5}$$

where we set $\min \emptyset = -\infty$. If $J \in \mathcal{G}^- \cap \mathcal{P}$ is of the form J_p for some $\omega_p \in K$, we choose the largest possible such p and let

$$J^* = J \cap (-\infty, \max(S \cap I)] \cap (-\infty, p], \tag{4.6}$$

where we set $\max \emptyset = +\infty$.

Let $\mathcal{P}^*$ denote the collection of all I^* for $I \in \mathcal{P}$. Then we have the following lemma.

Lemma 6. *Let $\omega_q \subseteq D$. Then*

(i) *there is at most one $I \in \mathcal{P} \cap \mathcal{G}^+$ such that $\omega_q \subseteq I^*$ and at most one $J \in \mathcal{P} \cap \mathcal{G}^-$ such that $\omega_q \subseteq J^*$;*

(ii) *if $\omega_q \in E_1(\mathcal{A})$ there are exactly one $I \in \mathcal{P} \cap \mathcal{G}^+$ such that $\omega_q \subseteq I^*$ and exactly one $J \in \mathcal{P} \cap \mathcal{G}^-$ such that $\omega_q \subseteq J^*$.*

Proof. (i) Suppose that A_1 and A_2 are two labeled right intervals in $\mathcal{P}$ such that $\omega_q \subseteq A_1^* \cap A_2^*$ and assume that $\mathfrak{l}(A_1) \leq \mathfrak{l}(A_2)$. Then, using the reasoning in the proof of Lemma 3, there exists no $\mathfrak{l}(A_2) < a < q$ such that $\omega_a \in S$, which implies that A_2 cannot meet S in $(-\infty, q)$. Therefore $A_2 = J_p$ for some $\omega_p \in K$ and since $\omega_q \subseteq A_2^*$ we can choose such a p with $p < q$. But then, since $\mathfrak{l}(A_1) \leq \mathfrak{l}(A_2)$ we have $\omega_p \subseteq A_1$ also and hence A_1 must be equal to F_p. This clearly implies that we must have $\mathfrak{l}(A_1) = \mathfrak{l}(A_2)$ and $A_1 \subseteq A_2$ also, hence A_1 must also be equal to some $J_{p'}$ for some $\omega_{p'}$ with $p' < q$ which in a similar manner gives $A_2 \subseteq A_1$, and hence that $A_1 = A_2$ (this means that the corresponding underlying intervals coincide) which contradicts the consistent way we have picked J_p and $J_{p'}$ in case the underlying intervals of the two labeled intervals of the same direction associated to ω_p coincide. If A_1 and A_2 are both left intervals the argument is similar. This completes the proof of (i).

(ii) It suffices by (i) to find at least one such I and at least one such J. If $\omega_q \in E_1(\mathcal{A}) \cap E_3 = K$, then we may take I_q and J_q for I and J (or J and I). Suppose that $\omega_q \in E_1(\mathcal{A}) \cap E_4^*$. Then by Lemma 5 we have $h_q^+(\mathcal{T}) \geq 2$ and $h_q^-(\mathcal{T}) \geq 2$. Among all right intervals that contain ω_q, let I be one with $\mathfrak{l}(I)$ minimum and I' be one with the next smallest $\mathfrak{l}(I')$, and among all left intervals that contain ω_q let J be one with $\mathfrak{r}(J)$ maximum and J' be one with the next largest $\mathfrak{r}(I')$. Then, as in (i), S does not meet $(\mathfrak{l}(I'), q-1)$ nor $(q, \mathfrak{r}(J'))$ and since $h_q \geq 4$ we have $S \cap [\mathfrak{l}(I'), \mathfrak{r}(J')] = \emptyset$. This (combined with the requirements in Definition 1) easily implies that ω_q can cover places in S only if either I or J (or both) is involved and these places of S must belong to $[\mathfrak{l}(I), \mathfrak{l}(I')] \cup [\mathfrak{r}(J'), \mathfrak{r}(J)]$. Since as in the proof of Lemma 1 this amounts to at most $h_q(\mathcal{T}) - 1$ possible coverings and since $\omega_q \in E_1(\mathcal{A})$ we conclude that all possible interactions over ω_q with I or J participating have actually taken place to cover exactly $h_q(\mathcal{T}) - 1$ places in S. In particular both (ω_q, I, J') and (ω_q, I', J) cover some $\omega_s, \omega_t \in S$, respectively. Since the first triple is allowed to cover only in $[\mathfrak{l}(I), \mathfrak{r}(J')]$ we conclude that $\omega_s \subseteq [\mathfrak{l}(I), \mathfrak{l}(I')]$ from which we easily obtain that $I \in \mathcal{P}_1$, $\min(S \cap I) \leq \mathfrak{l}(I') < q$ and therefore that $\omega_q \subseteq I^*$. In a similar way we get $J \in \mathcal{P}_1$ and $\omega_q \subseteq J^*$ and this completes the proof. $\square$

Actually we have the following lemma.

Lemma 7. *Let $\omega_q \subseteq D$ cover at least one place. Then we can find a right interval I and a left interval J such that* (i) $\omega_q \subseteq I \cap J$; (ii) $\mathfrak{l}(I) \leq \mathfrak{l}(I')$ *for every right interval I' that contains ω_q, and* $\mathfrak{r}(J') \leq \mathfrak{r}(J)$ *for every left interval J' that contains ω_q; and* (iii) *any $A \in \mathcal{P}$ such that $\omega_q \subseteq A^*$ must equal either I or J, as a labeled interval.*

Proof. Since ω_q covers some place, at least a right and at least a left interval must contain it. As in the previous lemma choose the right interval I so that $\mathfrak{l}(I) \leq \mathfrak{l}(I')$ for every right interval I' that contains ω_q but in case more than one such right interval with minimum left endpoint is available, choose I among them so that $I = J_p$ for at least one $\omega_p \in K$ with $\mathfrak{l}(I) < p \leq q$, and choose I arbitrarily among them if no such ω_p exists. Then choose J in a similar way among the left intervals containing ω_q with maximum right endpoint.

It is now easy to see, as in the previous lemma, that any right interval A other than I that contains ω_q cannot intersect S between $\mathfrak{l}(I)$ and ω_q and cannot be any $J_{p'}$ for $\mathfrak{l}(I) < p' \leq q$. Hence even if we had $A \in \mathcal{P}$ it cannot happen that $\omega_q \in A^*$. This completes the proof, the argument for J being similar. $\square$

5. Construction of related covering model

Here we will use the considerations of the previous section to construct a covering model that will be essential in proving Theorem 2. This will be done by truncating some of the chains $\mathcal{C}_p$, where $\omega_p \in K$, appropriately deleting certain subintervals of the intervals of $\mathcal{P}$, and adding certain new intervals of length one, to be called *brick-intervals.*

For this purpose fix an $\omega_p \in K$ and consider its chain

$$\omega_p \longmapsto (\omega_{c(p)}, R_p, L_p) \longmapsto (\omega_{c^{[2]}(p)}, R_{c(p)}, L_{c(p)}) \longmapsto \dots, \tag{5.1}$$

and let $m_p \geq 0$ be the least nonnegative integer (if any) such that there exists at least one (labeled) interval $I \in \{R_{c^{[m_p]}(p)}, L_{c^{[m_p]}(p)}\} \cap \mathcal{P}$ such that $\omega_{c^{[m_p+1]}(p)} \subseteq I^*$ (roughly speaking something from $\mathcal{P}^*$ is used to cover $\omega_{c^{[m_p]}(p)}$). If $m_p < \infty$ we let

$$\bar{\mathcal{C}}_p = \{\omega_q : q = c^{[t]}(p) \text{ for some } 0 \leq t \leq m_p\} \subseteq \mathcal{C}_p \tag{5.2}$$

and otherwise we let $\bar{\mathcal{C}}_p = \mathcal{C}_p$. Now we define

$$\bar{\mathcal{B}} = \bigcup_{\omega_p \in K} \bar{\mathcal{C}}_p \quad \text{and} \quad \bar{D} = \bigcup \bar{\mathcal{B}}, \tag{5.3}$$

and also for any positive integers $a \geq b$ the sets

$$\bar{\Lambda}_{ab} = \bar{\mathcal{B}} \cap E_a \cap E_b(\mathcal{A}) \subseteq \Lambda_{ab}. \tag{5.4}$$

Then we have the following lemma.

Lemma 8. (i) *For any $\omega_p \in K$ all places in $\bar{\mathcal{C}}_p$ lie between ω_{l_p} and ω_p. In particular $\bigcup \bar{\mathcal{C}}_p \subseteq I_p^*$;*

(ii) $\bar{\Lambda}_{31} = \Lambda_{31} = K$ *and* $\bar{\Lambda}_{22} = \emptyset$;

(iii) $\mathcal{B} \cap S = \emptyset$.

Proof. (i) Assume that I_p is a right interval and thus $l_p < p$, the argument being symmetrical in the other case. Suppose $q = c^{[m]}(p)$ is the first time in the chain $\mathcal{C}_p$ such that ω_q lies outside $[l_p - 1, p]$. Then ω_q must cover something between ω_{l_p} and ω_p. Since $\omega_{l_p} \in E_1$ and is contained in I_p, any left interval meeting $(-\infty, l_p)$ must have right endpoint less than l_p, and hence cannot cover anything to the right of ω_{l_p}. Therefore ω_q must lie to the right of ω_p. But the only right interval that meets $(p, +\infty)$ and has left endpoint less than p is I_p and moreover $I_p \cap (p, +\infty) \subseteq I_p^*$. Hence ω_q can cover something between ω_{l_p} and ω_p only through I_p^* which implies that $m > m_p$, and hence that the chain $\mathcal{C}_p$ must have stopped, to form $\bar{\mathcal{C}}_p$, before ω_q.

Therefore $\omega_q \notin \bar{\mathcal{C}}_p$ and this completes the proof, the inclusion following from the easy observation $I_p \cap [l_p, +\infty) \subseteq I_p^*$.

(ii) The first relation is trivial. To prove that $\bar{\Lambda}_{22}=\emptyset$ suppose that $\omega_q \in \bar{\Lambda}_{22}$ and choose an $\omega_p \in K$ such that $\omega_q \in \bar{\mathcal{C}}_p \subseteq \mathcal{C}_p$. By (i), $\omega_q \subseteq I_p^*$ and since all places in $\mathcal{B}$ cover something, there is exactly one other interval J containing ω_q, which must have opposite direction from I_p, and ω_q can cover only through (I_p, J). But then as in (i) we conclude that $\omega_q \notin \bar{\mathcal{C}}_p$, a contradiction.

(iii) Given any $\omega_s \in S$ there is a chain $\omega_a \mapsto \omega_{c(a)} \mapsto ... \mapsto \omega_s$ (of length at most three) from E_1 through E_2's, all of which cover unique places. Hence ω_s cannot belong to any chain starting from an E_3, and so $\omega_s \notin \mathcal{B}$. $\square$

Now we construct the collections $\mathcal{G}_1^+$ and $\mathcal{G}_1^-$ of labeled intervals from our initial collections $\mathcal{G}^+$ and $\mathcal{G}^-$ as follows: The collection $\mathcal{G}_1^+$ is produced from $\mathcal{G}^+$ replacing every right interval $I \in \mathcal{P} \cap \mathcal{G}^+$ by the closure of $I \setminus I^*$ (removing it if $I^*=I$) and leaving any other right interval unaltered. Similarly the collection $\mathcal{G}_1^-$ is produced from $\mathcal{G}^-$ replacing every left interval $J \in \mathcal{P} \cap \mathcal{G}^-$ by the closure of $J \setminus J^*$ (removing it if $J^*=J$) and leaving any other left interval unaltered

Let $h_{1,q}$ denote the number of bricks over $\omega_q \in \bar{\mathcal{B}}$ determined by these new collections $\mathcal{G}_1^+$ and $\mathcal{G}_1^-$ (in the same way as h_p is defined from the collections $\mathcal{G}^+$ and $\mathcal{G}^-$).

Lemma 9. *We have $1 \le h_{1,q} \le h_q - 1$ for every $\omega_q \in \bar{\mathcal{B}}$ and $h_{1,q}=h_q-2$ for every $\omega_q \in \bar{\mathcal{B}} \cap E_1(\mathcal{A})$.*

Proof. The inequalities $h_{1,q} \le h_q - 1$ for every $\omega_q \in \bar{\mathcal{B}}$ and $h_{1,q}=h_q-2$ for every $\omega_q \in \bar{\mathcal{B}} \cap E_1(\mathcal{A})$ follow from Lemma 8(i) and Lemma 6(ii), respectively. Also Lemma 6(i) implies that $h_{1,q} \ge h_q - 2$ for every $\omega_q \in \bar{\mathcal{B}}$. Hence if there exists an $\omega_q \in \bar{\mathcal{B}}$ such that $h_{1,q}=0$, then (since $\mathcal{B} \subseteq E_2^*$) we must have $h_q=2$. Assume then that the two intervals containing ω_q are I and J. Using Lemma 6(i) again we conclude that I and J belong to $\mathcal{P}$, they have opposite directions and (since $h_{1,q}=0$) $\omega_q \subseteq I^* \cap J^*$. Thus ω_q covers exactly one place and this is done through I^* and J^*. But then for any $\omega_p \in K$ such that $\omega_q \in \mathcal{C}_p$, we conclude, as in the proof of Lemma 8, that ω_q would have been removed from $\mathcal{C}_p$ and so cannot belong to $\bar{\mathcal{C}}_p$. This contradicts our assumption $\omega_q \in \bar{\mathcal{B}}$ and so completes the proof. $\square$

Next we let $\mathcal{W}$ denote the set of all $\omega_q \in \bar{\mathcal{B}}$ such that $q=c^{[m_p]}(p)$ for some $\omega_p \in K$ and we place a new brick-interval over each such $\omega_q \in \mathcal{W}$ and give to all these new bricks the color *red* (to remind us that we have stopped certain chains there).

Lemma 10. *The number $|\mathcal{W}|$ of all new red brick-intervals satisfies*

$$|\mathcal{W}| \le {\sum}'(b-1)|\Lambda_{ab}|. \tag{5.5}$$

Proof. Suppose that the place $\omega_s \in \Lambda_{ab} \subseteq \mathcal{B}$ produces exactly $d>0$ red bricks over d places in $\bar{\mathcal{B}}$. To prove (5.5) it clearly suffices to show that $d \le b-1$.

Choose the intervals I and J containing ω_s as in Lemma 7. Moreover it follows as in the proof of Lemma 6 that if ω_s covers something in S through I with a left interval different from J, then $\omega_s \subseteq I^*$ and a similar thing holds for J.

Therefore by Lemma 7, ω_s can cover places in S or produce red bricks over places in $\bar{\mathcal{B}}$ (covered by ω_s) only through interactions where I or J (or both) is involved. Thus as in the reasoning of Lemma 1 the total number of these interactions is at most $h_s(\mathcal{T})-1=a-1$, and since $S \cap \mathcal{B}=\emptyset$ no such interaction can cover both in S and in $\bar{\mathcal{B}}$. We conclude that if x is the total number of places in S covered by ω_s, then $d+x \le a-1$. Then since clearly exactly x bricks have been moved away from ω_s in the process of forming $\mathcal{A}$ it follows that $b=a-x$ and so $d \le b-1$. This completes the proof. □

We will give each red brick-interval a certain direction, thus adding it to either $\mathcal{G}_1^+$ or $\mathcal{G}_1^-$ (but not to both). There are certainly many possible ways to do this but we have the following result.

Lemma 11. *There is a way to give directions to every red brick-interval adding it to either $\mathcal{G}_1^+$ or $\mathcal{G}_1^-$ (but not to both) so that the labeled collections $\bar{\mathcal{G}}^+$ and $\bar{\mathcal{G}}^-$ thus produced have the property that the triple $\overline{\mathcal{T}}=(\bar{\mathcal{G}}^+, \bar{\mathcal{G}}^-, \overline{D})$ is a covering model.*

Proof. Given any $\omega_q \in \mathcal{W} \subseteq \bar{\mathcal{B}}$, Lemma 9 implies that $h_{1,q} \ge 1$. Hence ω_q is contained in at least one interval from $\mathcal{G}_1^+ \cup \mathcal{G}_1^-$. We choose one such interval and denote it by I. Then if $I \in \mathcal{G}_1^+$ (resp. $\mathcal{G}_1^-$) we add the red brick-interval that has been added over ω_q to the family $\mathcal{G}_1^-$ (resp. to $\mathcal{G}_1^+$) and we declare that ω_q is covered by ω_q through the interaction of I with this new added red brick-interval. Doing this for every $\omega_q \in \mathcal{W}$ we produce the collections $\bar{\mathcal{G}}^+$ and $\bar{\mathcal{G}}^-$ and we consider the triple $\overline{\mathcal{T}}=(\bar{\mathcal{G}}^+, \bar{\mathcal{G}}^-, \overline{D})$. Given any $\omega_s \subseteq \overline{D}$ (so $\omega_s \in \bar{\mathcal{B}}$), either $\omega_s \in \mathcal{W}$ in which case it is covered by itself from $\bar{\mathcal{G}}^+$ and $\bar{\mathcal{G}}^-$ by our construction, or $\omega_s \notin \mathcal{W}$. In that case, if $\omega_s \in \bar{\mathcal{C}}_p$ we also have $\omega_{c(s)} \in \bar{\mathcal{C}}_p$, hence $\omega_{c(s)} \subseteq \overline{D}$, and moreover both intervals R_s and L_s involved in the covering of ω_s by $\omega_{c(s)}$ either do not belong to $\mathcal{P}$, or any I of these two that might belong to $\mathcal{P}$ satisfies $\omega_{c(s)} \subseteq \overline{I \setminus I^*}$. Hence in all cases ω_s will be covered from $\overline{\mathcal{T}}$ in essentially the same way as it was covered in our initial covering model $\mathcal{T}$. From these we conclude that $\overline{\mathcal{T}}$ is a also a covering model. □

The covering model $\overline{\mathcal{T}}$ derived from $\mathcal{T}$ will be used to exploit the assumption of the maximality of $\varrho(\mathcal{T})$ and lead to the proof of Theorem 2. For this it is important to estimate $H(\overline{\mathcal{T}})$. This estimation is furnished by the following lemma.

Lemma 12. *We have*

$$H(\overline{\mathcal{T}}) \leq {\sum}'(a-2)|\bar{\Lambda}_{ab}|+2\sum_{a\geq 3}(b-1)|\Lambda_{ab}|+|\Lambda_{22}|. \tag{5.6}$$

Proof. It is clear that $H(\overline{\mathcal{T}})$ is equal to the sum of all $h_{1,q}$, where $\omega_q \in \bar{\mathcal{B}}$, plus the number of all added red brick-intervals. Hence we have

$$H(\overline{\mathcal{T}}) \leq {\sum}'(a-2)|\bar{\Lambda}_{ab}|+\sum_{\substack{a\geq 3\\ b>1}}|\bar{\Lambda}_{ab}|+|\mathcal{W}| \tag{5.7}$$

since, by Lemma 9, $\omega_q \in \bar{\Lambda}_{ab}$ implies that $h_{1,q}=a-2$ for $b=1$ and $h_{1,q} \leq a-1$ for $b>1$ and, by Lemma 8(ii), $\bar{\Lambda}_{22}=\emptyset$. This, (5.5) and the obvious inequalities $|\bar{\Lambda}_{ab}| \leq |\Lambda_{ab}|$ (and $b-1\geq 1$ for $b>1$) easily imply (5.6). □

6. End of the proof of Theorem 2

Using the above we can now complete the proof of Theorem 2 as follows.

Since $\overline{\mathcal{T}}$ is a also a covering model of the form considered in Section 3 (i.e. with all intervals involved being contained in $[0,N]$) we have

$$\varrho(\overline{\mathcal{T}}) \leq \varrho(\mathcal{T}). \tag{6.1}$$

Hence, observing that $m(\overline{\mathcal{T}})={\sum}'|\bar{\Lambda}_{ab}|$ and using (5.6) we have

$${\sum}'|\bar{\Lambda}_{ab}| \leq \varrho(\mathcal{T})\Big({\sum}'(a-2)|\bar{\Lambda}_{ab}|+2\sum_{a\geq 3}(b-1)|\Lambda_{ab}|+|\Lambda_{22}|\Big). \tag{6.2}$$

Let

$$\begin{aligned}
X &= {\sum}'|\bar{\Lambda}_{ab}|,\\
Y &= {\sum}'(a-2)|\bar{\Lambda}_{ab}|+2\sum_{a\geq 3}(b-1)|\Lambda_{ab}|+|\Lambda_{22}|,\\
Z &= 2\sum_{a\geq 3}(b-1)|\Lambda_{ab}|+|\Lambda_{22}|.
\end{aligned}$$

Then inequality (3.8) can be written as

$$\varrho(\mathcal{T}) \leq \frac{2X+Y-Z+\sum_{a\geq 3}a(|\Lambda_{ab}|-|\bar{\Lambda}_{ab}|)-\sum_{a\geq 3}(b-1)|\Lambda_{ab}|+|\Lambda_{22}|}{3X+2Y-2Z+\sum_{a\geq 3}(2a-1)(|\Lambda_{ab}|-|\bar{\Lambda}_{ab}|)-\sum_{a\geq 3}(b-1)|\Lambda_{ab}|+2|\Lambda_{22}|}$$

and Lemma 8(ii) gives

$$\frac{\sum_{a\geq 3} a(|\Lambda_{ab}|-|\bar{\Lambda}_{ab}|)}{\sum_{a\geq 3}(2a-1)(|\Lambda_{ab}|-|\bar{\Lambda}_{ab}|)} \leq \frac{4}{7}$$

so since $\varrho(\mathcal{T})>\frac{4}{7}$ these terms can be ignored and substituting Z we obtain

$$\varrho(\mathcal{T}) \leq \frac{2X+Y-3\sum_{a\geq 3}(b-1)|\Lambda_{ab}|}{3X+2Y-5\sum_{a\geq 3}(b-1)|\Lambda_{ab}|} \leq \frac{2X+Y}{3X+2Y} \leq \frac{2\varrho(\mathcal{T})+1}{3\varrho(\mathcal{T})+2} \tag{6.3}$$

since $X\leq\varrho(\mathcal{T})Y$ and because we know from Proposition 2 that $\varrho(\mathcal{T})\leq\frac{3}{5}<\frac{2}{3}$. Now (6.3) implies that $3\varrho(\mathcal{T})^2\leq 1$ and completes the proof of Theorem 2.

References

1. Aldaz, J. M., Remarks on the Hardy–Littlewood maximal function, *Proc. Roy. Soc. Edinburgh Sect. A* **128** (1998), 1–9.
2. Bernal, A., A note on the one-dimensional maximal function, *Proc. Roy. Soc. Edinburgh Sect. A* **111** (1989), 325–328.
3. Brannan, D. A. and Hayman, W. K., Research problems in complex analysis, *Bull. London Math. Soc.* **21** (1989), 1–35.
4. Melas, A., On the centered Hardy–Littlewood maximal operator, *Trans. Amer. Math. Soc.* **354** (2002), 3263–3273.
5. Melas, A., The best constant for the centered Hardy–Littlewood maximal inequality, to appear in *Ann. of Math.*
6. Trinidad Menarguez, M. and Soria, F., Weak type (1,1) inequalities of maximal convolution operators, *Rend. Circ. Mat. Palermo* **41** (1992), 342–352.

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